The Standard Model

The Guideline

Our purpose in theoretical physics is not to describe the world as we find it, but to explain – in terms of a few fundamental principles – why the world is the way it is.

Steven Weinberg

1. Plan and Introduction

These lecture notes cover the Standard Model (SM).

The SM is constructed from first principles.

We begin by demonstrating that essentially only particles with spin/helicity 0, 1/2 ,1, 3/2 , 2 can describe matter and interactions, using spacetime symmetries, soft theorems, gauge redundancies, Ward identities, and perturbative unitarity.

The remaining freedom lies in the choice of the Yang-Mills gauge group and matter representations.

Effective field theories (EFTs) are a central theme throughout the course, with the 4-Fermi interactions and chiral perturbation theory serving as key examples.

Both gravity and the SM itself are treated as EFTs, specifically as the SMEFT (Standard Model Effective Field Theory).

Key phenomenological aspects of the SM are covered, including the Higgs mechanism, Yukawa couplings, the CKM matrix, the GIM mechanism, neutrino oscillations, running couplings, and asymptotic freedom.

The discussion of anomalies and their non-trivial cancellations in the SM is detailed.

Simple examples of calculations, such as scattering amplitudes and decay rates, are provided.

1.1 INTRODUCTION

The Standard Model is one of the greatest scientific achievements of all time.

It consistently describes all known fundamental particles and their interactions with the exception of gravity that is still properly described at low energies.

In this sense we can now explain any fundamental physical phenomenon at the smallest distances that can be probed experimentally with spectacular success.

We may therefore claim that, so far, the Standard Model is the fundamental theory of Nature.

The Standard Model is the most successful application of quantum field theory when it comes to experimental verification.

It is the final conclusion of many decades of intense research both on the theoretical and experimental side.

Its structure was finally completed after the celebrated discovery of the Higgs particle in 2012.

Over the decades since its ingredients were combined, thousands of measurements have been made at energies $E \leq 1$ TeV, all consistent with the Standard Model.

The Standard Model describes the physics of the building blocks of all visible matter: spin 1/2 quarks and leptons interacting via three fundamental forces, each mediated by spin 1 particles known as gauge bosons.

Electrically charged particles feel the electromagnetic force by exchanging photons as described by Quantum Electrodynamics (QED).

The electromagnetic interactions are of long range due to the fact that photons are massless.

In contrast, the short range weak force is responsible for certain radioactive decays such as the neutron β-decay and plays a crucial role in the thermonuclear interactions within stars.

The mediators of this interaction are the massive W and Z bosons.

Their large mass is responsible for the short range of the interaction.

The strong force binds quarks into nucleons (protons and neutrons) and indirectly nucleons into nuclei; the carriers of the strong force are appropriately called the gluons.

Leptons, such as electrons and neutrinos do not feel the strong force.

Particles made out of quarks are called hadrons which can be either baryons (made up of 3 quarks) and mesons (a quark-antiquark pair).

The aim of these lectures is threefold.

First, to let you appreciate all the twists and turns that drove scientists in the past century to discover and develop the Standard Model.

A historical perspective is important to appreciate the magnitude of the achievements, but also the surprises and human drama that came with the development of new ideas in particle physics.

Most importantly, it demonstrates that research is not a straight line of well developed arguments as usually presented in textbooks and lecture notes, but rather a windy road with occasionally unforeseen twists and turns before a proper understanding emerges.

The second goal is for you to internalize that, despite the Standard Model being only one in an infinite number of possible field theories, its structure is extraordinary rigid and compelling.

Following general principles within our basic theories, Special Relativity[1] and Quantum Mechanics, we take a constructive approach arguing that it is essentially unavoidable that elementary particles are determined by unitary representations of the Poincaré group limiting their nature to only a handful of possibilities, namely spins 0, 1/2,1, 3/2, 2 out of an infinite number of possible spins.

Further, we will see why gauge invariance defining the three non-gravitational interactions is only a redundancy needed to properly describe the interactions.

Lastly, even though Field Theory is the basic formalism to describe interactions, the fundamental objects are actually the particles themselves, whereas fields are a necessary tool to describe local interactions among particles.

The third target of these lectures is for you to get familiar with the physical details of the Standard Model and be able to reproduce some of the key calculations and results that led to its successful completion.

Throughout the lectures emphasis will be given to basic principles and potential loopholes that may be important to guide us towards the unknown physics beyond the Standard Model.

That is, we provide crucial methods to build new theories (or models) of nature – a skill that is vital for any theoretical physicist.

We follow the guideline as in the quote of Steven Weinberg earlier with the key word being to explain rather than describe.

We emphasize the explanatory power of the Standard Model towards all the experiments, but also towards some of the approximate or accidental symmetries such as isospin, flavor, baryon number, etc.

Just like the Standard Model provides a UV description of Effective Field Theories (EFTs) such as the Fermi theory of weak interactions and the pion dynamics of Yukawa, it should itself only be regarded as an EFT once gravity or other UV physics is included.

The lectures only assume a basic knowledge of group theory(as we have done) and a first exposure to quantum field theory although an effort is made to be as self-contained as possible(a second exposure).

Most concepts we covered in QFT will be repeated and presented in new ways and I hope your understanding level wil thereby skyrocket!

Subjects such as path integrals, quantization of non-abelian gauge theories and renormalization group are only briefly summarized since we just finished a set of lectures about that stuff.

The presentation aims at preparing you to think about the fundamental ideas underlying the Standard Model.

1.2 Summary and Motivation

Let us start with a first overall glimpse at the Standard Model just to introduce the concepts that will be developed during the subsequent lectures

1.2.1 A Brief Introduction to the Standard Model

Definition:

The Standard Model (SM) is a construction (or model or theory) that describes all the known elementary particles and their interactions in terms of relativistic quantum field theories.

Ingredients:

(1) *Spacetime*. The spacetime is (3 + 1)-dimensional Minkowski spacetime with (global) symmetry group

$$G_{\mathrm{gl}} = \mathbb{R}^{3,1} \rtimes \mathrm{O}(3,1) \tag{1.2.1}$$

that is the semi-direct product of spacetime translations and the Lorentz group, corresponding to the Poincaré group of special relativity.

(2) *Matter*. The particle content can be classified by the spin s (or helicity), i.e., there are the Higgs H with s = 0 as well as 3 families of quarks and leptons with s = 1/2, see also table 1.1.

Name	Label	SU(3)$_C$, SU(2)$_L$, U(1)$_Y$	Spin/Helicity
Quarks	$Q^i_L = \left(u^i_L, d^i_L\right)$	$\left(\mathbf{3}, \mathbf{2}, +\frac{1}{6}\right)$	$\frac{1}{2}$
	u^i_R	$\left(\bar{\mathbf{3}}, \mathbf{1}, \frac{2}{3}\right)$	$\frac{1}{2}$
	d^i_R	$\left(\bar{\mathbf{3}}, \mathbf{1}, -\frac{1}{3}\right)$	$\frac{1}{2}$
Leptons	$L^i_L = \left(\nu^i_L, e^i_L\right)$	$\left(\mathbf{1}, \mathbf{2}, -\frac{1}{2}\right)$	$\frac{1}{2}$
	e^i_R	$(\mathbf{1}, \mathbf{1}, -1)$	$\frac{1}{2}$
	ν^{i*}_R	$(\mathbf{1}, \mathbf{1}, 0)$	$\frac{1}{2}$
Higgs	H	$\left(\mathbf{1}, \mathbf{2}, +\frac{1}{2}\right)$	0
Gluons	g_α	$\left(\mathbf{8}, \mathbf{1}, 0\right)$	1
W/Z-Bosons	$W^\pm, Z^0$	$\left(\mathbf{1}, \mathbf{3}, 0\right)$	1
Photon	γ	$\left(\mathbf{1}, \mathbf{1}, 0\right)$	1
Graviton*	$h_{\mu\nu}$	$\left(\mathbf{1}, \mathbf{1}, 0\right)$	2

Table 1.1: Particle content of the Standard Model and the corresponding group representations. The right-handed neutrino and the graviton are included here for completeness with the understanding that the couplings of the graviton to all other particles can be studied as long as the energies are small enough in terms of an effective QFT.

(3) *Interactions*. The interactions are given by 3 gauge interactions with associated gauge bosons of spin s = 1 and, in general, also gravitational interactions transmitted by a particle of spin s = 2 known as the graviton.

The gauge forces are encoded in the gauge (or local) symmetry group

$$G_{\mathrm{SM}} = \underbrace{\underbrace{\mathrm{SU}(3)_C}_{\text{color}}}_{\text{strong}} \times \underbrace{\underbrace{\mathrm{SU}(2)_L}_{\text{left}} \times \underbrace{\mathrm{U}(1)_Y}_{\text{hypercharge}}}_{\text{electroweak}} \tag{1.2.2}$$

where the subindex C refers to color with $\mathrm{SU}(3)_C$ determining the strong interactions.

The strong force binds quarks into nucleons and nucleons into nuclei; the carrier of the strong force is appropriately called the gluon.

The L in $\mathrm{SU}(2)_L$ refers to left-handed in the sense that weak interactions only act on left-handed particles.

Finally Y refers to hypercharge.

The gauge group in (1.2.2) is broken by the Higgs boson through a non-zero vacuum expectation value, to a subgroup, namely

$$G_{\mathrm{SM}} \xrightarrow{\mathrm{SSB}} \underbrace{\mathrm{SU}(3)_C}_{\mathrm{QCD}} \times \underbrace{\mathrm{U}(1)_{EM}}_{\mathrm{QED}} \, . \tag{1.2.3}$$

In the process referred to as spontaneous symmetry breaking (SSB), the corresponding gauge bosons of the broken group ($W^{\pm}$ and Z^0) receive a mass, but there is also a remaining massless boson corresponding to the unbroken $U(1)_{EM}$ with EM standing for electromagnetic.

This is the familiar photon of the electromagnetic interactions.

To reiterate, after the breaking of the symmetry, only the gluons and photons remain massless.

Photons are free to move but gluons together with quarks are confined within the particles of strong interactions such as protons and neutrons.

The representations of the particles involved are summarized in table 1.1.

We work in a particular representation where $\alpha \in U(1)_Y$ acts on $\psi \in C$ in such a way that $\psi \to \alpha^{6iy}\psi$, i.e., weak hypercharges appear in integer multiples of 1/6.

Keep in mind that different definitions are commonly used in the literature!

(4) *Three families*. For the quarks and leptons, there are 3 distinct families coming with the same copies of the representation:

$$\text{Leptons:} \quad \begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \qquad \text{Quarks:} \quad \begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} t \\ b \end{pmatrix}.$$

Only the first family (with electron, its neutrino and up and down quarks) are enough to make the matter we know.

The second (muon, its neutrino, charm and strange quarks) and third (tau-lepton, its neutrino, top and bottom quarks) are more massive and the corresponding particles are unstable having the particles of the first family as end results of their decay.

It is remarkable that these simple ingredients are enough to account for the structure of the Universe as we know it including every single experience and measurement we make.

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Some comments:

- Chirality. Since the right-handed quarks u_R and d_R transform under the trivial representation of $SU(2)L$ in table 1.1, that is to say they are $SU(2)_L$-singlets, they do not feel the $SU(2)_L$-interactions.

  This is why we call the Standard Model a chiral gauge theory.

  Physically, this implies that left- and right-handed fermions feel certain gauge interactions differently, i.e., they couple non-democratically to the mediators (gauge fields) of a given force.

  Mathematically speaking, left- and right-handed fermions transform in different $SU(2)_L$-representations.

  Thus, the weak interaction is not parity invariant under exchange of left- and right-handed particles.

- *Charge quantization*. The electric charge is defined as

$$Q = T_3 + Y \tag{1.2.4}$$

  where T_3 is the third generator of $SU(2)_L$ which is the diagonal matrix with entries $(1/2,-1/2)$ and Y the hypercharge under $U(1)_Y$.

  For instance to compute the electric charge of the left-handed electron we observe that its hypercharge is $Y = -1/2$ and its value of T_3 is $-1/2$ giving $Q = -1$.

For the right-handed electron the correspondingT3 value is zero because it is a singlet and then charge and hypercharge are the same (here +1 for the positron).

Computing the electric charges of quarks give multiples of 1/3 instead of integers as we are familiar for electrons and protons.

- *Consistency conditions*. We observe that the assignment of these numbers such as hypercharge and the different representations of the Standard Model particles is not arbitrary.

For instance it is easy to verify the following conditions for the hypercharge

$$\sum_{Left} Y - \sum_{Right} Y = \sum_{Left} Y^3 - \sum_{Right} Y^3 = 0 \,. \tag{1.2.5}$$

Also the total number of particles transforming as a **3** of $SU(3)_C$ equals the number of $\bar{\mathbf{3}}$ ($\#\mathbf{3} = \#\bar{\mathbf{3}}$) and the total number of $SU(2)_L$ doublets (#2) is even.

Any modification of these numbers would render the theory mathematically inconsistent.

This will be crucial in ensuring anomaly cancellation within the Standard Model.

If the Standard Model was not chiral, these conditions would be trivially satisfied.

It is the chiral structure of the Standard Model that makes it subject to potential inconsistencies and therefore makes it more interesting when they are satisfied.

- *Coleman-Mandula theorem*. The total symmetry of the Standard Model is given by a direct product between a spacetime and an internal symmetry

$$\text{spacetime}\otimes\text{internal gauge} = (\mathbb{R}^{3,1} \rtimes \mathrm{SO}(3,1))\otimes(\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y)\,. \tag{1.2.6}$$

  The Coleman-Mandula theorem states that this structure is the most general6 for the full symmetry group, i.e., a direct product of the Poincaré group and an internal (gauge) group.

- *Gravity as an effective field theory*. We can treat gravity only as what is called an effective QFT (EFT) by working with energies E well below the *Planck scale*:

$$E \ll M_P = \sqrt{\frac{\hbar c}{G}} \sim 10^{18}\mathrm{GeV}\,. \tag{1.2.7}$$

  At energies $E \sim M_P$, quantum effects of gravity become important and the EFT has to be replaced by a more fundamental theory that is ultra-violet complete.

  But for energies well below $E \ll M_P$ working with quantum aspects of gravity as an EFT are predictable and reliable.

  We will have to say more about the role of gravitational interactions within the Standard Model later.

- *Accidental symmetries*. There are accidental symmetries known as Baryon number B and Lepton number L.

  That is, the total number of baryons, such as the neutron and proton, and the total number of leptons such as the electron and neutrino are conserved in every interaction.

- *Approximate symmetries*. The three families of quarks and leptons in which the members of each family behave the same as the other families except that the particles are heavier for each generation (e.g., the muon is like a heavier copy of the electron, the top quark of the up quark, etc.) implies that there are approximate symmetries known as flavor symmetries such as $SU(3)_f$ known as the eightfold way, see Fig. 1.2.

  This flavor $SU(3)_f$ should not be confused with the color $SU(3)_C$ which is the symmetry describing the strong interactions.

- *Phases of the Standard Model*. The Standard Model is relatively simple, although not the simplest model we can imagine.

  Actually, it is rich enough to illustrate the 3 main phases of gauge theories: The Coulomb phase for $U(1)_{EM}$ meaning that the corresponding gauge boson, the photon, moves freely; the confining phase for $SU(3)_C$ meaning that the interactions are so strong that the corresponding gauge bosons, the gluons, and the quarks are confined within hadrons; and the Higgs phase for the weak interactions in $SU(2)_L \times U(1)_Y$ meaning that the corresponding force is short range since the gauge bosons W, Z are heavy after symmetry breaking.

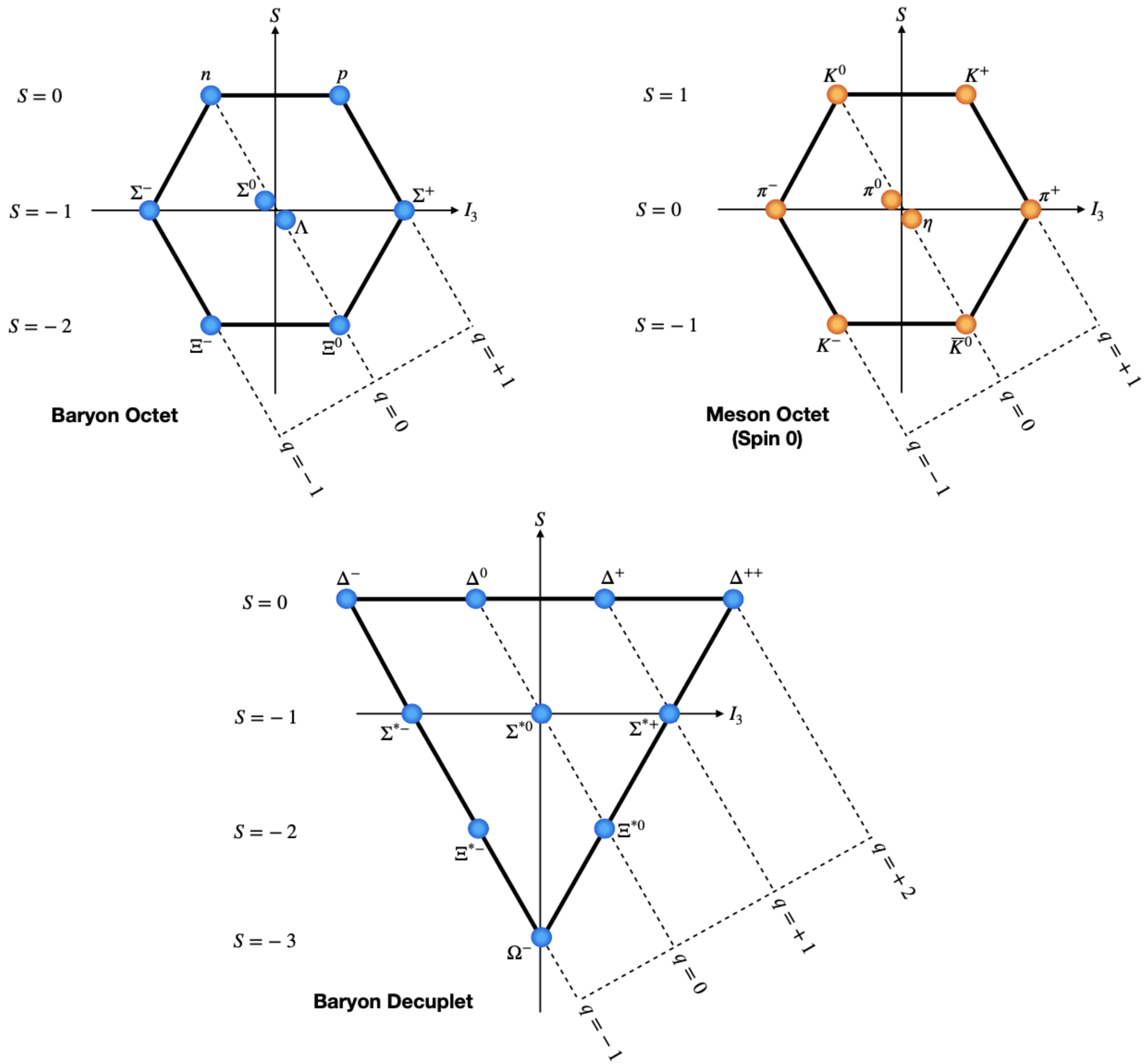

Figure 1.2: The *eightfold way*. The weight lattice of hadrons in their respective $SU(3)_f$ representations with the baryon octet (top left) corresponding to **8** of $SU(3)_f$, pseudoscalar meson octet (top right) again corresponding to **8** of $SU(3)_f$ and the baryon decuplet (bottom) corresponding to **10** of $SU(3)_f$. Here, S denotes 'strangeness' and I_3 isospin. The prediction of the Ω^- particle and its subsequent discovery lead strong credibility to this approximate symmetry.

Why do I like the SM so much?

As you now see…….

- It is fundamental.
- Robustness.
- It is true!
- It is the best test of validity of QFT.
- Cosmology.
- It is incomplete!

So let us begin our journey.

2. SPACETIME SYMMETRIES

Importance

If it were not for these symmetries, the work of science would have to be redone in every new laboratory and in every passing moment.

Steven Weinberg

In this part, we review basic techniques for constructing suitable representations of the Poincaré group – the symmetry group of (Minkowski) spacetime – which defines Special Relativity and is relevant for constructing the Standard Model.

We begin with a general discussion of the Poincaré algebra and its properties before introducing spinor representations of the Lorentz group SO(3, 1).

Subsequently, we will detail Wigner's classification of irreducible representations of the Poincaré group which define for us elementary particles.

We discuss in detail how these particles have to transform under discrete spacetime transformations which allows us to count all of the relevant polarization states.

Among others, we will find that massless particles with helicity ≥ 0 have always just two degrees of freedom which will severely constrain the field theories to be studied in the reminder of these lectures.

2.1 Poincaré symmetry and spinors

For the vast majority of this lecture, we will be interested in describing relativistic processes involving particles moving in (3 + 1)-dimensional Minkowski space $\mathbb{R}^{3,1}$.

Any theory describing such phenomena must respect the symmetry inherited from the spacetime geometry – heuristically, physical processes should not depend on the observer's initial frame.

This is the *principle of relativity* stating that the laws of physics are the same in all viable frames of reference.

For particle physics in $\mathbb{R}^{3,1}$, the spacetime symmetry group in question is the *Poincaré group* $\mathcal{P}(3, 1)$.

The Poincaré group corresponds to the basic symmetries of special relativity, it acts on the Minkowski spacetime coordinates x^μ via

$$x^\mu \quad \mapsto \quad x'^\mu = \underbrace{\Lambda^\mu{}_\nu}_{\text{Lorentz}} \, x^\nu \; + \; \underbrace{a^\mu}_{\text{translation}} \,, \qquad \mu, \nu = 0, 1, 2, 3\,. \tag{2.1.1}$$

It is sometimes convenient to use a shorthand notation $\{\cdot|\cdot\}$ for such a transformation where

$$x'^\mu = \{\Lambda|a\}\, x^\mu \equiv \Lambda^\mu{}_\nu \, x^\nu + a^\mu\,. \tag{2.1.2}$$

Formally, the Poincaré group $\mathcal{P}(3, 1)$ or ISO(3, 1) is the isometry group of Minkowski spacetime.

It is a semidirect product of spacetime translations and the transformations corresponding to the Lorentz group of special relativity

$$\mathcal{P}(3,1) = \mathbb{R}^{3,1} \rtimes \mathrm{O}(3,1)\,, \tag{2.1.3}$$

which is just a fancy way of saying that $\mathcal{P}$ leaves $\mathbb{R}^{3,1}$ invariant and every Poincaré transformation can be decomposed into the product of the form

$$\{\Lambda|a\} = \{\mathbb{1}_{4\times4}|a\}\{\Lambda|0\}\,. \tag{2.1.4}$$

The Lorentz transformations belong to the orthogonal O(3, 1) group that leaves the metric tensor

$$\eta_{\mu\nu} = \mathrm{diag}(1,\ -1,\ -1,\ -1) \tag{2.1.5}$$

in the line element

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{2.1.6}$$

invariant, i.e.,

$$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \qquad \text{or} \qquad \Lambda^T\,\eta\,\Lambda = \eta\,. \tag{2.1.7}$$

From this equation we can easily see that

$$\det\Lambda = \pm1 \tag{2.1.8}$$

and taking the 00 component

$$\left(\Lambda^0{}_0\right)^2 - \left(\Lambda^1{}_0\right)^2 - \left(\Lambda^2{}_0\right)^2 - \left(\Lambda^3{}_0\right)^2 = 1 \qquad \Longrightarrow \qquad |\Lambda^0{}_0| \geq 1\,. \tag{2.1.9}$$

Therefore the Lorentz group has 4 disconnected components according to the signs of $\det \Lambda$ and $\Lambda^0{}_0$.

We will mostly discuss those transformations Λ connected to the identity, i.e., the *proper orthochronous group* $SO(3,1)^\uparrow$ for which $\det \Lambda = 1$ (proper) and $\Lambda^0{}_0 \geq 1$ (orthochronous).

All O(3, 1) transformations can be obtained by combining the $SO(3,1)^\uparrow$ transformations with:

$$\{\mathbb{1},\, \Lambda_P,\, \Lambda_T,\, \Lambda_{PT}\} \tag{2.1.10}$$

where

- $\mathbb{1}$ is the identity matrix,
- $\Lambda_P = \mathrm{diag}(1,-1,-1,-1)$ is the *parity* transformation,
- $\Lambda_T = \mathrm{diag}(-1,1,1,1)$ is *time reversal*, and
- $\Lambda_{PT} = \Lambda_P \times \Lambda_T$ is combined parity and time reversal.

These four elements form a group known as *Klein's four-group*.

From now on we will concentrate on those transformations connected to the identity and drop the arrow on SO(3, 1) for simplifying the notation.

For the same reason, we work from now on with the proper orthochronous Poincaré group and simply write $\mathcal{P}(3,1) \equiv \mathcal{P}_+^\uparrow = \mathbb{R}^{3,1} \supsetneq SO(3,1)^\uparrow$.

2.1.1 The Poincaré Algebra

Let us consider infinitesimal Poincaré transformations $\{\Lambda|a\} \in \mathcal{P}(3,1)$ for which

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu\,; \qquad a^\mu = \epsilon^\mu \qquad \omega^\mu{}_\nu,\, \epsilon^\mu \ll 1\,. \tag{2.1.11}$$

Plugging this back in (2.1.7), we find

$$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\rho\sigma} + \omega_{\sigma\rho} + \omega_{\rho\sigma} + \mathcal{O}(\omega^2) \overset{!}{=} \eta_{\rho\sigma}\,. \tag{2.1.12}$$

In mathematics, $\overset{!}{=}$ denotes a useful, important, but unexpected equality.

To linear order, we deduce that

$$\omega_{\sigma\rho} = -\omega_{\rho\sigma} \tag{2.1.13}$$

is an anti-symmetric tensor which has 6 free parameters.

Together with the 4 translations ϵ^μ, a general Poincaré transformations must have 10 parameters.

This is the dimensionality of the Poincaré group.

In order to determine the algebra we can exponentiate the group elements.

We will do this considering the action of operators acting on the Hilbert space $\mathscr{H}$ relevant in quantum mechanics.

A Poincaré transformation will be represented by a unitary operator U(Λ, a) acting on the Hilbert space vectors $|\Psi>$ ϵ $\mathscr{H}$.

$$|\Psi\rangle \to U(\Lambda, a)|\Psi\rangle \qquad U = U^\dagger \,. \tag{2.1.14}$$

Near the identity, we can expand to linear order

$$U(1+\omega, \epsilon) = \mathbb{1} - \frac{\mathrm{i}}{2}\omega_{\mu\nu}M^{\mu\nu} + i\epsilon_\mu P^\mu \tag{2.1.15}$$

where $M^{\mu\upsilon} = -M^{\upsilon\mu}$ and P^μ are the generators of the group.

Since U is unitary, both $M^{\mu\upsilon}$ and P^μ are Hermitian.

As usual in group theory, the above can be used to determine the algebra satisfied by the generators.

First, since translations commute, their generators also commute with each other

$$[P^\mu\,,\, P^\nu] = 0\,. \tag{2.1.16}$$

Let us now consider the commutator $[P^\sigma\,, M^{\mu\upsilon}]$ by analyzing how P transforms under Lorentz transformations.

For this we can consider the dual role of P^{μ}.

On the one hand, it is a vector that should transform (to leading order in $\omega_{\mu\upsilon}$) as

$$\begin{aligned} P^{\sigma} &\to \Lambda^{\sigma}{}_{\rho} P^{\rho} \\ &= \left(\delta^{\sigma}{}_{\rho} + \omega^{\sigma}{}_{\rho}\right) P^{\rho} \\ &= P^{\sigma} + \frac{1}{2}\left(\omega_{\alpha\rho} - \omega_{\rho\alpha}\right) \eta^{\sigma\alpha} P^{\rho} \\ &= P^{\sigma} + \frac{1}{2}\omega_{\alpha\rho}\left(\eta^{\sigma\alpha} P^{\rho} - \eta^{\sigma\rho} P^{\alpha}\right) \end{aligned} \tag{2.1.17}$$

On the other hand, P^{σ} as an operator transforms as

$$\begin{aligned} P^{\sigma} &\to U^{\dagger} P^{\sigma} U \\ &= \left(\mathbb{1} + \frac{\mathrm{i}}{2}\omega_{\mu\nu} M^{\mu\nu}\right) P^{\sigma} \left(\mathbb{1} - \frac{\mathrm{i}}{2}\omega_{\mu\nu} M^{\mu\nu}\right) \\ &= P^{\sigma} - \frac{\mathrm{i}}{2}\omega_{\mu\nu}\left(P^{\sigma} M^{\mu\nu} - M^{\mu\nu} P^{\sigma}\right) \end{aligned} \tag{2.1.18}$$

Comparing both expressions we find the commutator $[P^{\sigma}, M^{\mu\upsilon}]$:

$$\left[P^{\sigma}, M^{\mu\nu}\right] = -\mathrm{i}\left(P^{\mu}\eta^{\nu\sigma} - P^{\nu}\eta^{\mu\sigma}\right) \tag{2.1.19}$$

A similar argument can be used for the commutators of $M^{\mu\upsilon}$.

Therefore the generators of the Poincaré group are $M^{\mu\upsilon}$ and P^{σ} with algebra:

Poincaré algebra

$$\left[P^{\mu}\ ,\ P^{\nu}\right] = 0 \tag{2.1.20}$$

$$\left[M^{\mu\nu}\ ,\ P^{\sigma}\right] = \mathrm{i}\left(P^{\mu}\,\eta^{\nu\sigma} - P^{\nu}\,\eta^{\mu\sigma}\right) \tag{2.1.21}$$

$$\left[M^{\mu\nu}\ ,\ M^{\rho\sigma}\right] = \mathrm{i}\left(M^{\mu\sigma}\,\eta^{\nu\rho} + M^{\nu\rho}\,\eta^{\mu\sigma} - M^{\mu\rho}\,\eta^{\nu\sigma} - M^{\nu\sigma}\,\eta^{\mu\rho}\right) \tag{2.1.22}$$

As an example, a 4-dimensional matrix representation for the $M^{\mu\upsilon}$ is

$$(M^{\rho\sigma})^{\mu}{}_{\nu} = -\mathrm{i}\left(\eta^{\mu\sigma}\,\delta^{\rho}{}_{\nu} - \eta^{\rho\mu}\,\delta^{\sigma}{}_{\nu}\right)\,. \tag{2.1.23}$$

Also, the definition of the operators

$$(M^{\rho\sigma})^{\mu}{}_{\nu} = \mathrm{i}\,(x^{\mu}\,\partial_{\nu} - x_{\nu}\,\partial^{\mu}) \tag{2.1.24}$$

is a representation of the Lorentz generators acting on the space of functions.

Similarly, the operator

$$P^{\mu} = \mathrm{i}\,\partial^{\mu} \tag{2.1.25}$$

is the generator of translations in the representation defined by (2.1.24).

It is easy to verify that these operators satisfy the Poincaré algebra.

2.1.2 Properties of the Poincaré group

Let us summarize the basic properties of the Poincaré group.

Conservation laws

Recall that $P^0 = H$ corresponds to the Hamiltonian and thus

$$[P^0, P^\mu] = 0 \tag{2.1.26}$$

implies *conservation of energy and momentum*, whereas

$$[P^0, M^{ij}] = 0 \quad i, j = 1, 2, 3 \tag{2.1.27}$$

amounts to *conservation of angular momentum*.

There is no conservation law associated to the M^{0i} generators since they do not commute with P^0.

Correspondence of $\mathrm{SO}(3, 1)$ **and** $\mathrm{SU}(2) \otimes \mathrm{SU}(2)$

There is a correspondence between the algebras of $\mathrm{SO}(3, 1)$ and $\mathrm{SU}(2) \otimes \mathrm{SU}(2)$, namely

$$\mathrm{SO}(3, 1) \;\leftrightarrow\; \mathrm{SU}(2) \otimes \mathrm{SU}(2)\,. \tag{2.1.28}$$

This means that the representations of the $\mathrm{SO}(3, 1)$ algebra can be determined by those of $\mathrm{SU}(2) \oplus \mathrm{SU}(2)$.

This works as follows.

The generators J_i of rotations and K_i of Lorentz boosts can be defined as

$$J_i = \frac{1}{2}\,\epsilon_{ijk}\,M_{jk}\ , \qquad K_i = M_{0i}\ , \tag{2.1.29}$$

therefore, using the Poincaré algebra we can easily derive the commutation relations

$$\left[J_i, J_j\right] = \mathrm{i}\epsilon_{ijk}J_k\,, \quad \left[J_i, K_j\right] = \mathrm{i}\epsilon_{ijk}K_k\,, \quad \left[K_i, K_j\right] = -\mathrm{i}\epsilon_{ijk}J_k\,. \tag{2.1.30}$$

In order to identify the different representations of the Lorentz group, it is instructive to consider the linear combinations

$$A_i = \frac{1}{2}\left(J_i + \mathrm{i}K_i\right)\,, \qquad B_i = \frac{1}{2}\left(J_i - \mathrm{i}K_i\right) \tag{2.1.31}$$

which are neither Hermitian nor anti-Hermitian if J_i and K_i are Hermitian.

They satisfy SU(2) commutation relations,

$$\left[A_i\ ,\ A_j\right] = \mathrm{i}\epsilon_{ijk}\,A_k\ , \qquad \left[B_i\ ,\ B_j\right] = \mathrm{i}\epsilon_{ijk}\,B_k\ , \qquad \left[A_i\ ,\ B_j\right] = 0\,. \tag{2.1.32}$$

These are two independent copies of the SU(2) algebra, but keeping in mind that the operators A_i and B_i are not Hermitian.

In contrast, the combination **J** = **A** + **B** is Hermitian and corresponds to the physical spin.

We can then interpret **J** as the physical spin which itself generates an SU(2) group.

Recall that irreducible representations $\bar{R}_{\Lambda_j}$ of SU(2) are labelled by j = 0, 1/2 , . . . with j determined from the eigenvalues of the *quadratic Casimir Operator* which is nothing but the "total angular momentum" $J^2 = J^2_1 + J^2_2 + J^2_3$.

It commutes with the three generators J_i and satisfies

$$J^2 |j\rangle = j(j+1) |j\rangle \qquad j = 0, \frac{1}{2}, 1, \cdots . \tag{2.1.33}$$

Hence, we can use those eigenvalues to label the irreducible representations of SU(2).

In the case of SO(3, 1), we denote representations of SU(2) ⊗ SU(2) as

$$(A, B)\,, \qquad \text{with } A, B = 0, \frac{1}{2}, 1, \ldots \tag{2.1.34}$$

Since **J** = **A** + **B**, we can see using the standard addition of angular momenta that the representation (A, B) corresponds to spins $j = |A - B| \oplus (|A - B| + 1)\cdots \oplus (A + B)$.

Under parity P with $x^0 \mapsto x^0$ and $\mathbf{x} \to -\mathbf{x}$, we have

$$J_i \mapsto J_i\,, \qquad K_i \mapsto -K_i \quad \Longrightarrow \quad A_i \leftrightarrow B_i\,. \tag{2.1.35}$$

Therefore, A and B are interchanged under parity transformation

$$(A, B) \overset{P}{\leftrightarrow} (B, A)\,. \tag{2.1.36}$$

We then call A the *left-handed* and B the *right-handed* component of (A, B).

Below, we will use this notion to label representations of SO(3,1) and to define their handedness (or chirality).

Universal cover of SO(3, 1)

There is a homomorphism (not an isomorphism)

$$\mathrm{SO}(3,1) \;\cong\; \mathrm{SL}(2,\mathbb{C})\,, \tag{2.1.37}$$

where SL(2, C) is the group of 2 x 2 complex matrices with unit determinant.

To see this, take a 4 vector X and a corresponding (2 x 2)-matrix $\tilde{x}$,

$$X = x_\mu\, e^\mu = (x_0, x_1, x_2, x_3)\,, \quad \tilde{x} = x_\mu\, \sigma^\mu = \begin{pmatrix} x_0 + x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix}\,, \tag{2.1.38}$$

where σ^μ is the 4 vector of Pauli matrices

$$\sigma^\mu = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}\,. \tag{2.1.39}$$

Transformations $X \mapsto \Lambda X$ under SO(3, 1) leave the modulus square

$$|X|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 \tag{2.1.40}$$

invariant, whereas the action of SL(2, C) mapping

$$\tilde{x} \mapsto N\tilde{x}N^\dagger \tag{2.1.41}$$

with $N \in SL(2, C)$ preserves the determinant

$$\det \tilde{x} = x_0^2 - x_1^2 - x_2^2 - x_3^2 \,. \tag{2.1.42}$$

This equals $|X|^2$.

The map between SL(2, C) and SO(3, 1) is 2-1, since $N = \pm\mathbb{1}_2$ both correspond to $\Lambda = 1_4$, but SL(2, C) has the advantage of being simply connected, that means that SL(2, C) is the *Universal Covering Group* of the Lorentz group.

This is important since it is the simply connected group manifold that is continuously connected to the identity operator.

Furthermore, since the map between SL(2, C) and SO(3, 1) is 2-1, a rotation by an angle θ in SO(3, 1) is mapped to the matrix diag($e^{i\theta/2}$, $e^{-i\theta/2}$) in SL(2, C).

This in turn implies that it is only rotations by $\theta = 4\pi$ (and not 2π) that give the identity in SL(2, C).

This will turn out to be a crucial observation in order to describe particles of half-integer spin as we will see later.

Let us briefly see why the manifold of SL(2, C) is simply connected.

By the polar decomposition of matrices, an SL(2, C) matrix N can be written as N= e^H U where e^H is a positive Hermitian matrix and U is unitary.

Since e^H has positive determinant then det N = 1 implies Tr H = 0 and det U = 1.

Therefore H and U can be written as

$$H = \begin{pmatrix} a & b + ic \\ b - ic & -a \end{pmatrix} \qquad U = \begin{pmatrix} x + iy & z + iw \\ -z + iw & x - iy \end{pmatrix} \tag{2.1.43}$$

with a, b, c real parameters and x, y, z, w constrained by $x^2 + y^2 + z^2 + w^2 = 1$ therefore the manifold of SL(2, C) is $\mathbb{R}^3$ x S^3 which are simply connected, whereas the manifold for SO(3, 1) is $\mathbb{R}^3 \simeq S^3/\mathbb{Z}_2$ which is doubly connected.

Being the covering group, it is SL(2, C) that is reached by exponentiating the algebra and therefore we are let to consider the representations of SL(2, C) that we discuss next.

2.2 Spinor representations of the Lorentz group

Above, we already established that SL(2, C) is the universal covering group of the Lorentz group SO(3, 1).

This should ring a bell: in quantum mechanics, we learned that SU(2) is the double cover of SO(3) and therefore it is representations of SU(2) that are the ones to be considered since SU(2) is simply connected and SO(3) is not.

We observe a similar phenomenon for the Lorentz group in the sense that representation theory of SL(2, C) is the relevant one to study.

Concentrating only on representations of SO(3, 1) we would miss the fundamental representations which are the spinor representations which we define next.

2.2.1 Representations and invariant tensors of SL(2,C)

To begin, we define the basic representations of SL(2, C). Let N ϵ SL(2, C), then we have

- The fundamental representation ψ_β transforming as
$$\psi'_\alpha = N_\alpha{}^\beta \psi_\beta \ , \qquad \alpha, \beta = 1, 2 \tag{2.2.1}$$
The elements of this representation ψ_α are called **left-handed Weyl spinors**.
- The conjugate fundamental representation $\overline{\chi}_{\dot\beta}$ transforming as
$$\bar\chi'_{\dot\alpha} = N^{*}_{\dot\alpha}{}^{\dot\beta} \bar\chi_{\dot\beta} \ , \qquad \dot\alpha, \dot\beta = 1, 2 \tag{2.2.2}$$
Here $\bar\chi_{\dot\beta}$ are called **right-handed Weyl spinors**.

- The contravariant representations ψ^β and $\bar{\chi}^{\dot\beta}$

$$\psi'^{\alpha} = \psi^{\beta}\,(N^{-1})_{\beta}{}^{\alpha}\ , \qquad \bar{\chi}'^{\dot\alpha} = \bar{\chi}^{\dot\beta}\,(N^{*-1})_{\dot\beta}{}^{\dot\alpha} \tag{2.2.3}$$

as the dual representations of the two above.

The fundamental and conjugate representations are the basic representations of SL(2, C) and the Lorentz group, giving then the importance to spinors as the basic objects of special relativity, a fact that could be missed by not realizing the connection of the Lorentz group and SL(2, C).

We will show now that the contravariant representations are however not independent by explicitly showing how indices can be raised (or lowered) using specific tensors.

To see this, we consider three different ways to raise and lower indices.

- The metric tensor $\eta^{\mu\nu} = (\eta_{\mu\nu})^{-1}$ is invariant under $\mathrm{SO}(3,1)$ and is therefore used to lower and raise spacetime indices.
- The analogue for $\mathrm{SL}(2,\mathbb{C})$ is

$$\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta} = -\epsilon_{\dot\alpha\dot\beta}\ , \tag{2.2.4}$$

since it is invariant under $\mathrm{SL}(2,\mathbb{C})$ transformations

$$\epsilon'^{\alpha\beta} = \epsilon^{\rho\sigma}\,\left(N^{-1}\right)_{\rho}{}^{\alpha}\,\left(N^{-1}\right)_{\sigma}{}^{\beta} = \epsilon^{\alpha\beta}\cdot\det N^{-1} = \epsilon^{\alpha\beta}\ . \tag{2.2.5}$$

That is why $\epsilon^{\rho\sigma}$ is used to raise and lower indices

$$\psi^{\alpha} = \epsilon^{\alpha\beta}\,\psi_{\beta}\ , \qquad \bar{\chi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\,\bar{\chi}_{\dot{\beta}}\ , \tag{2.2.6}$$

so contravariant representations are not independent.

- To handle mixed SO(3,1)- and SL(2,$\mathbb{C}$)-indices, recall that the transformed components x_{μ} should look the same, whether we transform the vector X via SO(3,1) or the matrix $\tilde{x} = x_{\mu}\sigma^{\mu}$

$$(x_{\mu}\,\sigma^{\mu})_{\alpha\dot{\alpha}} \ \ \mapsto \ \ N_{\alpha}{}^{\beta}\,(x_{\nu}\,\sigma^{\nu})_{\beta\dot{\gamma}}\,N^{*}_{\dot{\alpha}}{}^{\dot{\gamma}} = \left(\Lambda^{-1}\right)_{\mu}{}^{\nu}\,x_{\nu}\,\sigma^{\mu}\ , \tag{2.2.7}$$

so the right transformation rule is

$$(\sigma^{\mu})_{\alpha\dot{\alpha}} = N_{\alpha}{}^{\beta}\,(\sigma^{\nu})_{\beta\dot{\gamma}}\,(\Lambda)^{\mu}{}_{\nu}\,N^{*}_{\dot{\alpha}}{}^{\dot{\gamma}}\ . \tag{2.2.8}$$

This may be interpreted by saying that the Pauli matrices are invariant under a combined SO(3,1) transformation on its spacetime index with a SL(2,$\mathbb{C}$) on its matrix indices. Similar relations hold for the quantity

$$(\bar{\sigma}^{\mu})^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\,\epsilon^{\dot{\alpha}\dot{\beta}}\,(\sigma^{\mu})_{\beta\dot{\beta}} = (\mathbb{1},\ -\boldsymbol{\sigma})\ . \tag{2.2.9}$$

Note that this is the definition of $\bar{\sigma}$ and no other connection with σ such as complex conjugation should be assumed despite the notation. Note in particular the chosen location of the dotted and undotted indices which differ between σ^{μ} and $\bar{\sigma}^{\mu}$. The order is conventional and keeps track of how the corresponding quantity transforms under SL(2,$\mathbb{C}$). Both σ and $\bar{\sigma}$ will play an important role next. In fact, we can already deduce that the *Clifford algebra*

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2\eta^{\mu\nu} \mathbb{1}_2 \tag{2.2.10}$$

appears naturally in our analysis which will give rise to *Dirac spinors* further below.

2.2.2 Generators of SL(2, C) and Weyl spinors

Let us define tensors $\sigma^{\mu\upsilon}$, $\bar{\sigma}^{\mu\nu}$ as antisymmetrized products of σ matrices

$$\begin{aligned} (\sigma^{\mu\nu})_\alpha{}^\beta &:= \frac{\mathrm{i}}{4} \left(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu \right)_\alpha{}^\beta \\ (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} &:= \frac{\mathrm{i}}{4} \left(\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \right)^{\dot{\alpha}}{}_{\dot{\beta}} \end{aligned}$$

which satisfy the Lorentz algebra

$$\left[\sigma^{\mu\nu} , \sigma^{\lambda\rho} \right] = \mathrm{i} \left(\eta^{\mu\rho} \sigma^{\nu\lambda} + \eta^{\nu\lambda} \sigma^{\mu\rho} - \eta^{\mu\lambda} \sigma^{\nu\rho} - \eta^{\nu\rho} \sigma^{\mu\lambda} \right) \tag{2.2.11}$$

and similarly for $\bar{\sigma}^{\mu\nu}$.

Then $\sigma^{\mu\upsilon}$ and $\bar{\sigma}^{\mu\nu}$ correspond to the generators of the Lorentz algebra in the spinor representation.

Under a finite Lorentz transformation with parameters $\omega_{\mu\upsilon}$, Weyl spinors transform as follows

$$
\begin{aligned}
\psi_\alpha &\mapsto \exp\left(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha{}^\beta \psi_\beta && \text{(left-handed)} \\
\bar{\chi}^{\dot{\alpha}} &\mapsto \exp\left(-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\chi}^{\dot{\beta}} && \text{(right-handed)}
\end{aligned}
$$

Now consider the spins with respect to the SU(2)s spanned by the A_i and B_i :

$$
\begin{aligned}
J_i &= \frac{1}{2}\sigma_i \ , \qquad K_i = -\frac{\mathrm{i}}{2}\sigma_i \quad \Longrightarrow \quad \psi_\alpha : \ (A,\ B) = \left(\frac{1}{2},\ 0\right) \ , \quad \text{left-handed} \\
J_i &= \frac{1}{2}\sigma_i \ , \qquad K_i = +\frac{\mathrm{i}}{2}\sigma_i \quad \Longrightarrow \quad \bar{\chi}^{\dot{\alpha}} : \ (A,\ B) = \left(0,\ \frac{1}{2}\right) \ , \quad \text{right-handed} \, .
\end{aligned}
$$

Recall the the Pauli matrices correspond to generators of the rotation group in the j = 1/2 representation since

$$
\sum_i \left(\frac{\sigma_i}{2}\right)^2 = \frac{3}{4} = j(j+1) \qquad \text{for} \qquad j = \frac{1}{2} \, . \tag{2.2.12}
$$

The expressions above also justify the aforementioned term left- and right-handed components for A, B.

The difference and independence between the left- and right-handed representations of SL(2, C) indicates that there is no reason to assume that parity is a fundamental symmetry and will be the reason behind the fact that the Standard Model is chiral in the sense that left- and right-handed representations are not the same.

The belief that physicists before Yang and Lee had assuming parity should be an inherent symmetry of Nature is not justified and it is not surprising then that the laws of Nature are not invariant under parity as we will see later.

The concept of chirality is ubiquitous not only in the Standard Model, but also more generally in various areas of modern physics (and biology).

Some useful identities concerning the σ^μ and $\sigma^{\mu\upsilon}$ can be found in various sources.

For now, let us just mention the identities

$$\sigma^{\mu\nu} = \frac{1}{2\mathrm{i}}\,\epsilon^{\mu\nu\rho\sigma}\,\sigma_{\rho\sigma}\,, \qquad \bar{\sigma}^{\mu\nu} = -\frac{1}{2\mathrm{i}}\,\epsilon^{\mu\nu\rho\sigma}\,\bar{\sigma}_{\rho\sigma}\,, \tag{2.2.13}$$

known as *self duality* and *anti-self dualit*y respectively.

They are important because naively $\sigma^{\mu\upsilon}$ being antisymmetric seems to have 4x3/2 components, but the self duality conditions reduces this by half.

We then need the two sets of generators $\sigma^{\mu\upsilon}$ and $\bar{\sigma}^{\mu\nu}$ to complete the 6 independent generators of SL(2, C).

There are reference books illustrating many of the calculations for 2-component spinors.

Products of Weyl spinors

We define the product of two Weyl spinors as

$$\chi\psi = \chi^\alpha \psi_\alpha = -\chi_\alpha \psi^\alpha , \quad \bar{\chi}\bar{\psi} = \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} = -\bar{\chi}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}} , \tag{2.2.14}$$

particularly,

$$\psi\psi = \psi^\alpha \psi_\alpha = \epsilon^{\alpha\beta} \psi_\beta \psi_\alpha = \psi_2 \psi_1 - \psi_1 \psi_2 . \tag{2.2.15}$$

Choose the ψ_α to be *anticommuting Grassmann numbers*, $\psi_1 \psi_2 = -\psi_2 \psi_1$, so $\psi\psi = 2\psi_2 \psi_1$.

From the definitions

$$\psi^\dagger_\alpha = \bar{\psi}_{\dot{\alpha}} , \quad \bar{\psi}^{\dot{\alpha}} = \psi^*_\beta (\sigma^0)^{\beta\dot{\alpha}} \tag{2.2.16}$$

it follows that

$$(\chi\psi)^\dagger = \bar{\chi}\bar{\psi} , \qquad (\psi \sigma^\mu \bar{\chi})^\dagger = \chi \sigma^\mu \bar{\psi} \tag{2.2.17}$$

which justifies the $\nearrow$ contraction of dotted indices in contrast to the $\searrow$ contraction of undotted ones.

In general we can generate all higher dimensional representations of the Lorentz group by products of the fundamental representation (1/2 , 0) and its conjugate (0,1/2).

The computation of tensor products

$$\left(\frac{r}{2}, \frac{s}{2}\right) = \left(\frac{1}{2}, 0\right)^{\otimes r} \otimes \left(0, \frac{1}{2}\right)^{\otimes s} \tag{2.2.18}$$

can be reduced to successive application of the elementary SU(2) rule (for $j \neq 0$)

$$\left(\frac{j}{2}\right) \otimes \left(\frac{1}{2}\right) = \left(\frac{j-1}{2}\right) \oplus \left(\frac{j+1}{2}\right) . \tag{2.2.19}$$

Let us give two examples for tensoring Lorentz representations:

- $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) \ = \ (\frac{1}{2}, \frac{1}{2})$

  Bispinors with different chiralities can be expanded in terms of the $\sigma^{\mu}_{\alpha\dot{\alpha}}$. Actually, the σ matrices form a complete orthonormal set of 2×2 matrices with respect to the trace $\mathrm{Tr}\{\sigma^{\mu}\bar{\sigma}^{\nu}\} = 2\eta^{\mu\nu}$:

  $$\psi_{\alpha}\,\bar{\chi}_{\dot{\alpha}} = \frac{1}{2}\ (\psi\,\sigma_{\mu}\,\bar{\chi})\ \sigma^{\mu}_{\alpha\dot{\alpha}} \tag{2.2.20}$$

  Hence, two spinor degrees of freedom with opposite chirality give rise to a Lorentz vector $\psi\sigma_{\mu}\bar{\chi}$.

- $(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) \ = \ (0, 0) \oplus (1, 0)$

  Alike bispinors require a different set of matrices to expand, $\epsilon_{\alpha\beta}$ and $(\sigma^{\mu\nu})_{\alpha}{}^{\gamma}\epsilon_{\gamma\beta} =: (\sigma^{\mu\nu}\epsilon^{T})_{\alpha\beta}$. The former represents the unique antisymmetric 2×2 matrix, the latter provides the symmetric ones. Note that the (anti-)self duality reduces the number of linearly independent $\sigma^{\mu\nu}$'s (over $\mathbb{C}$) from 6 to 3:

$$\psi_\alpha \, \chi_\beta = \frac{1}{2} \, \epsilon_{\alpha\beta} \, (\psi\chi) \; + \; \frac{1}{2} \, \left(\sigma^{\mu\nu} \, \epsilon^T\right)_{\alpha\beta} \, (\psi \, \sigma_{\mu\nu} \, \chi) \tag{2.2.21}$$

The product of spinors with alike chiralities decomposes into two Lorentz irreducibles, a scalar $\psi\chi$ and a self-dual antisymmetric rank two tensor $\psi \, \sigma_{\mu\nu} \, \chi$. The counting of independent components of $\sigma^{\mu\nu}$ from its self-duality property precisely provides the right number of three components for the $(1,0)$ representation. Similarly, there is an anti-self dual tensor $\bar{\chi}\bar{\sigma}^{\mu\nu}\bar{\psi}$ in $(0,1)$.

2.2.3 Dirac and Majorana spinors

Here, we give the dictionary connecting the ideas of Weyl spinors with the more standard Dirac theory in D = 4 dimensions.

Dirac spinor

A Dirac spinor Ψ_{D} is defined to be the direct sum of two Weyl spinors $\psi, \bar{\chi}$ of opposite chirality. It therefore falls into a reducible representation of the Lorentz group,

$$\Psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \left(0, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 0\right) \; . \tag{2.2.22}$$

The Dirac analogue of the Weyl spinors' sigma matrices are the 4 x 4 gamma matrices γ^μ subject to the *Clifford algebra*

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} , \qquad \left\{\gamma^\mu , \quad \gamma^\nu\right\} = 2\,\eta^{\mu\nu}\,\mathbb{1} \ . \tag{2.2.23}$$

Due to the reducibility, the generators of the Lorentz group take block diagonal form

$$\Sigma^{\mu\nu} = \frac{\mathrm{i}}{4}\,\gamma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix} \tag{2.2.24}$$

and naturally obey the same algebra like the irreducible blocks $\sigma^{\mu\upsilon}$, $\bar{\sigma}^{\mu\nu}$

$$\left[\Sigma^{\mu\nu} \ , \ \Sigma^{\lambda\rho}\right] = \mathrm{i}\,\left(\eta^{\mu\rho}\,\Sigma^{\nu\lambda} + \eta^{\nu\lambda}\,\Sigma^{\mu\rho} - \eta^{\mu\lambda}\,\Sigma^{\nu\rho} - \eta^{\nu\rho}\,\Sigma^{\mu\lambda}\right) \ . \tag{2.2.25}$$

To disentangle the two inequivalent Weyl representations, one defines the chiral matrix γ^5 as

$$\gamma^5 = \mathrm{i}\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} , \tag{2.2.26}$$

such that the $\psi(\chi)$ components of a Dirac spinors have eigenvalues (chirality) -1 (+1) under γ^5

$$\gamma^5\,\Psi_D = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} -\psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \ . \tag{2.2.27}$$

Hence, one can define projection operators P_L, P_R,

$$P_L = \frac{1}{2}\left(\mathbb{1} - \gamma^5\right), \qquad P_R = \frac{1}{2}\left(\mathbb{1} + \gamma^5\right), \tag{2.2.28}$$

eliminating one part of definite chirality, i.e.,

$$P_L\,\Psi_D = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}, \qquad P_R\,\Psi_D = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{2.2.29}$$

thThe fact that Lorentz generators preserve chirality can also be seen from $\{\gamma^5, \gamma^\mu\} = 0$ implying $[\gamma^5, \Sigma^{\mu\upsilon}] = 0$.

Finally, define the *Dirac conjugate* $\overline{\Psi}_D$ and *charge conjugate spinor* $\Psi_D{}^C$

$$\overline{\Psi}_D = (\chi^\alpha,\ \bar{\psi}_{\dot{\alpha}}) = \Psi_D^\dagger\,\gamma^0, \quad \Psi_D{}^C = C\,\overline{\Psi}_D^T = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \tag{2.2.307}$$

where C denotes the *charge conjugation matrix*

$$C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \tag{2.2.31}$$

There is a third basic type of spinors called *Majorana spinors* Ψ_M which have the property $\psi_\alpha = \chi_\alpha$,

$$
\Psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} = {\Psi_M}^C \ , \tag{2.2.32}
$$

which are neutral under charge conjugation.

A general Dirac spinor (and its charge conjugate) can be decomposed in terms of Majorana spinors as

$$
\Psi_D = \Psi_{M1} \ + \ \mathrm{i}\Psi_{M2} \ , \qquad {\Psi_D}^C = \Psi_{M1} \ - \ \mathrm{i}\Psi_{M2} \ . \tag{2.2.33}
$$

Note that there can be no spinors in 4 dimensions which are both Majorana and Weyl.

This is a dimension dependent property.

It can be shown that in dimensions 2 mod 8 it is possible to have spinors which are both Majorana and Weyl.

2.3 Unitary Representations of the Poincaré group

We now will combine the 2 fundamental theories of special relativity and quantum mechanics to find the unitary representations of the Poincaré group on quantum states.

As usually unitarity is required in order to have invariant observables (such as matrix elements).

Being non-compact, the Poincaré group does not have finite dimensional unitary representations.

Recap: the rotation group in Quantum Mechanics

Before we consider the Poincaré group, let us reiterate some facts about unitary representations of the rotation group SU(2) in quantum mechanics.

Recall that the rotation group SU(2) has generators $\{J_i : i = 1, 2, 3\}$ satisfying the algebra

$$\left[J_i\ ,\ J_j\right] = \mathrm{i}\epsilon_{ijk}\, J_k\ . \tag{2.3.1}$$

Next, we define the Casimir operator

$$J^2 = \sum_{i=1}^{3} J_i^2\ . \tag{2.3.2}$$

In general, for a given group G, the Casimir operators are operators that commute with all the generators.

They are important because Schur's Lemma guarantees that they are proportional to the identity within a given representation and therefore their eigenvalues can be used to label the representations.

For compact Lie groups, the number of Casimir operators equals the rank of the group.

In the present case of SU(2), the Casimir operator J^2 commutes indeed with all the J_i,

$$[J^2, J_i] = 0 \quad \forall i = 1, 2, 3\,, \tag{2.3.3}$$

and labels irreducible representations by eigenvalues $j(j + 1)$ of J^2, that is,

$$J^2 \, |j;\lambda\rangle = j(j+1) \, |j;\lambda\rangle \ . \tag{2.3.4}$$

Within these representations, λ parametrizes the degeneracy of states in the same representation obtained by acting with the ladder operators on the highest weight state.

Thus, we can make a choice and diagonalize the states with respect to J_3 with eigenvalues $\lambda = j_3$ so that

$$J_3 \, |j;j_3\rangle = j_3 \, |j;j_3\rangle \ , \quad j_3 = -j, -j+1, ..., j-1, j \ . \tag{2.3.5}$$

Hence, the corresponding states are labelled like $|j; j3>$.

These are in fact unitary representations and, since SU(2) is compact, they are finite-dimensional.

The latter will cease to be true for the Poincaré group.

The Poincaré group

Now, let us consider the Poincaré group.

The takeaway message from our recap about labelling irreducible representations of SU(2) is that we simply need to find the corresponding Casimir operators.

In this case there are two Casimir operators.

The first one corresponds to the square of the momenta: $C_1 = P^\mu P_\mu$ which can be easily checked that it commutes with all the generators P^μ, $M^{\mu\upsilon}$.

The second one involves the *Pauli-Ljubanski vector* $W\mu$,

$$W_{\mu} := \frac{1}{2}\, \epsilon_{\mu\nu\rho\sigma}\, P^{\nu}\, M^{\rho\sigma} \tag{2.3.6}$$

where $\epsilon_{0123} = -\epsilon^{0123} = -1$.

This operator satisfies the following commutation relations

$$[W_{\mu}, P_{\nu}] = 0\,, \tag{2.3.7}$$

$$[W_{\mu}, M_{\rho\sigma}] = \mathrm{i}\left(\eta_{\mu\rho}W_{\sigma} - \eta_{\mu\sigma}W_{\rho}\right)\,, \tag{2.3.8}$$

$$[W_{\mu}, W_{\nu}] = -\mathrm{i}\varepsilon_{\mu\nu\rho\sigma}W^{\rho}P^{\sigma}\,. \tag{2.3.9}$$

From these commutations relations we can check that a second Casimir corresponds to $C_2 = W^{\mu}W_{\mu}$.

Notice that at this level the Pauli-Ljubanski vector only provides a short way to express the second Casimir.

Even though W_{μ} has standard commutation relations with the generators of the Poincaré group $M_{\mu\upsilon}$, P_{μ} stating that it transforms as a vector under Lorentz transformations and commutes with P_{μ} (invariant under translations), the commutator $[W_{\mu}, W_{\upsilon}] \sim \epsilon_{\mu\upsilon\rho\sigma} W^{\rho}P^{\sigma}$ implies that the W_{μ}'s by themselves are not generators of any algebra since the right hand side is quadratic and not linear in the corresponding operators.

Summarizing, one can show that the Casimir operators for the Poincaré group are given by

Casimir operators of the Poincaré group

$$C_1 = P^\mu P_\mu\,, \quad C_2 = W^\mu W_\mu\,. \tag{2.3.10}$$

It is easy to verify that

$$[C_{1,2}, P^\mu] = [C_{1,2}, M^{\mu\nu}] = 0\,. \tag{2.3.11}$$

Poincaré multiplets are therefore labelled $| m, \omega ; \lambda_i >$ so that

$$C_1 \left|m, \omega; \lambda_i\right\rangle = m^2 \left|m, \omega; \lambda_i\right\rangle\,, \quad C_2 \left|m, \omega; \lambda_i\right\rangle = f(m, \omega) \left|m, \omega; \lambda_i\right\rangle\,, \tag{2.3.12}$$

that is, m^2 are the eigenvalues of C_1 and $f(m, \omega)$ the ones of C_2.

We have to work a little harder to determine the labels ω and the exact expression for the $f(m, \omega)$.

As above, states within those irreducible representations carry extra labels λ_i corresponding to all operators that can be diagonalized simultaneously (such as J_3 for SU(2)).

One of the λ_i corresponds to the eigenvalue p^μ of the generator P^μ as a label.

To find more labels, take the eigenvalue p^μ of P^μ as given and look for all elements of the Lorentz group that commute with P^μ.

This defines the *Little* or *Stability group* which we denote as $L(p_\mu)$.

Note that within a multiplet, at fixed momentum, the operator P^μ can be replaced by its eigenvalue p^μ and then the Pauli-Ljubanski vector can be seen as the combination of the generators of the Lorentz group that commutes with the momentum operator and its commutation relations determine the algebra of the Little group (now the right hand side of $[W_\mu, W_\upsilon]$ is a linear combination of the W_ρ's since the $P^\mu = p^\mu$'s are just numbers within the multiplet).

Our ultimate goal is to obtain unitary irreducible representations of the Poincaré group.

This can be achieved using the arguments above that can be summarized as the following theorem.

Theorem 2.3.1 Let p^μ be some fixed 4-vector.

a) On the orbit (all 4-vectors q^μ for which there exists $\Lambda \in SO(3,1)$ so that $\Lambda^\mu{}_\nu q^\nu = p^\mu$.) $O(p\mu)$, the independent components of W^μ form a Lie algebra $L(L(p^\mu))$ of the Little group $L(p^\mu)$.

b) For every *unitary* irreducible representation of $L(p^\mu)$, there exists an induced representation of the Poincaré group $P(3, 1)$.

c) The unitary irreducible representations of $P(3, 1)$ are characterized by eigenvalues of the Casimirs P^2 and W^2.

We will now consider the different representations determined by fixing the momenta for different values and signs of $P^\mu P_\mu$.

- $P^\mu P_\mu = m^2 > 0$ (**Massive particles**).

Valid choices of eigenvectors include $p^\mu = (m, 0, 0, 0)$ which have rotations as their little group since $p^i = 0, i = 1, 2, 3$, i.e., $L(p^\mu) = SO(3)$.

Due to the completely antisymmetric tensor $\epsilon_{\mu\upsilon\rho\sigma}$ in the definition of W_μ, it follow

$$W_0 = 0 \ , \qquad W_i = -m\, J_i \quad \Longrightarrow \quad C_2 = m^2 J^2 \ . \tag{2.3.13}$$

Thus, we have

$$C_1 \left|m, \omega; \lambda_i\right\rangle = m^2 \left|m, \omega; \lambda_i\right\rangle \ , \quad C_2 \left|m, \omega; \lambda_i\right\rangle = m^2 j(j+1) \left|m, \omega; \lambda_i\right\rangle \ , \tag{2.3.14}$$

This identifies ω with j, while the remaining labels are specified as $\lambda_i \in \{p^\mu, j_3\}$.

Note that once the p^μ are fixed within a representation the components of the Pauli-Ljubanski vector do satisfy an algebra (since within one representation we can replace P^μ by p^μ and in this case the algebra is essentially the same as the rotation group since $W_i \propto J_i$).

This algebra defines the Little group that has the well known finite dimensional representations.

Hence, every particle with nonzero mass is an irreducible representation of the Poincaré group with labels $| m, j \,; p^\mu, j_3 \rangle$.

This defines a one-particle state and, in particular, an elementary particle of mass m and spin j.

It is important to emphasize that the existence of these quantum states corresponding to elementary particles is a general consequence of the two basic theories, quantum mechanics and special relativity.

We may then define elementary particles as unitary irreducible representations of the Poincaré group.

This is a remarkable result since it is a way to mathematically define the basic building blocks of nature.

- $P^\mu P_\mu = 0$ (**Massless particles**).

The simplest realization is selecting the origin $p^\mu = (0, 0, 0, 0)$ which is Lorentz invariant.

Even though this seems like a trivial case, it corresponds to a state with no particles, the vacuum state $|0>$.

We will see the importance of this state later on.

In order to have non-trivial representations corresponding to particle states, we can take the momentum of the form $p^\mu = (E, 0, 0, E)$ which implies

$$(W_0,\ W_1,\ W_2,\ W_3) = E\left(J_3,\ -J_1 + K_2,\ -J_2 - K_1,\ -J_3\right) \tag{2.3.15}$$

$$\left[W_1\ ,\ W_2\right] = 0\,,\quad \left[W_3\ ,\ W_1\right] = -\mathrm{i}E\,W_2\,,\quad \left[W_3\ ,\ W_2\right] = \mathrm{i}E\,W_1\ . \tag{2.3.16}$$

These commutation relations are those for the Euclidean group in two dimensions (translations generated by W_1, W_2 and rotations generated by W_3, acting on an abstract two dimensional space) and again define the Little group for massless particles.

This group, contrary to the massive case, has infinite dimensional unitary representations known as *continuous spin representations*.

A simple way to see this is to realize that W_1 and W_2 commute with each other and can be simultaneously diagonalized with eigenvalues w_1, w_2.

If $w_1, w_2 \neq 0$, then

$$W^\mu W_\mu = -(w_1^2 + w_2^2) := -\rho^2 \quad \Rightarrow \quad w_1 = \rho\cos\theta\,, \quad w_2 = \rho\sin\theta\,. \tag{2.3.17}$$

Hence, the representation can be labelled as $|\, 0, \rho\,;\, p^\mu\,, \theta >$.

Therefore, the existence of these representations would imply particles with an extra continuous label on top of the momenta p^μ.

Since particles with these extra continuous labels have not been seen in nature, in order to proceed we concentrate only on the finite dimensional representations.

This is not entirely satisfactory since, contrary to the massive case in which we extracted the most general implications of special relativity and quantum mechanics without any further assumptions, here we have to make an ad-hoc restriction to concentrate only on finite dimensional representations.

Restricting to finite dimensional representations, SO(2) is the relevant subgroup of the Little group generated by W_3 as w_1, w_2 vanish.

In that case, $W^\mu = \lambda P^\mu$ and states are labelled as $|0,0, p^\mu, \lambda> := |p^\mu, \lambda>$, where λ is called helicity and corresponds to the component of angular momentum in the direction of motion of the particle.

Since we have seen that it is only rotations by 4π and not 2π that leave the physics invariant, we should expect

$$e^{2\pi i\lambda}\,|p^\mu,\lambda\rangle = \pm|p^\mu,\lambda\rangle \tag{2.3.18}$$

which requires λ to be integer or half integer $\lambda = 0, 1/2\ , 1, \ldots.$.

Notice that contrary to the massive case in which the integer or half-integer nature of spin was dictated by group theory, i.e., the representations of SU(2), for the massless case we need to use a topological argument related to the simply connected nature of SL(2, C).

We will see that essentially all the particles of the Standard Model will come from these massless representations of the Poincaré group, e.g., $\lambda = 0$ (Higgs), $\lambda = 1/2$ (quarks, leptons), $\lambda = 1$ (γ, $W^\pm$, Z^0, g) and $\lambda = 2$ (graviton).

We will see that parity transforms states of helicity λ, $|p^\mu, \lambda>$ to $|p^\mu, -\lambda>$ and therefore, if parity is conserved, states such as the photon and graviton have two degrees of freedom corresponding to $\lambda = \pm 1$, $\lambda = \pm 2$ respectively.

- $P^{\mu}P_{\mu} = -m^2 < 0$ (**tachyons**).

tA typical momentum can be $p^{\mu} = (0, m, 0, 0)$.

This would correspond to a particle moving in a space-like trajectory (moving faster than light).

In particular, it would contradict causality.

In some cases, these particles appear in physical theories when instead of expanding around a minimum of the energy we expand around a maximum and their presence would only indicate that we are expanding around the 'wrong' vacuum.

Once a minimum is identified and the expansion is done around the minimum of the energy the particle would correspond to a normal massive particle as described above.

We encounter such a situation further below when discussing the electroweak phase transition in Part 6.

All in all, we deduce that the states for massless and massive particles are finite dimensional representations of SO(3) (massive) or SO(2) (massless).

It is important to emphasize that the existence of the aforementioned quantum states corresponding to elementary particles is a general consequence of the two basic theories, quantum mechanics and special relativity.

We may then define elementary particles as unitary irreducible representations of the Poincaré group.

This is a remarkable result since it is a way to mathematically define the basic building blocks of nature.

Next we use these results to introduce quantum fields in an attempt to build up an off-shell framework to describe particle interactions.

But before we get there, we have to briefly discuss the effects of discrete spacetime transformations which, as we will see throughout these lectures, will also have important consequences for the Standard Model.

2.4 Discrete spacetime transformations

So far we have only considered the representations of the proper orthochronous Lorentz group.

Let us now consider the disconnected components of the Lorentz group and consider the action of parity and time reversal that, as we saw before, together with the identity and their product define the Klein group.

In the following, we denote operators representing a general Poincaré transformation $\{\Lambda|a\} \in \mathcal{P}(3,1)$ as $U(\Lambda, a)$.

The transformation matrices for parity P and time reversal T can be written

$$\Lambda_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \Lambda_T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.4.1}$$

As operators acting on the Hilbert space of quantum states, we denote them as

$$\mathcal{P} = U(\Lambda_P, 0)\,, \quad \mathcal{T} = U(\Lambda_T, 0)\,. \tag{2.4.1}$$

For a general Poincaré transformation $\{\Lambda|a\} \in \mathcal{P}(3,1)$, we have

$$\mathcal{P}^{-1} U \mathcal{P} = U(\Lambda_P^{-1}\Lambda\Lambda_P, \Lambda_P a)\,, \quad \mathcal{T}^{-1} U \mathcal{T} = U(\Lambda_T^{-1}\Lambda\Lambda_T, \Lambda_T a)\,. \tag{2.4.1}$$

Expanding as before around the identity using

$$\begin{aligned} &\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu\,, \quad a^\mu = \epsilon^\mu\,, \quad \omega^\mu{}_\nu\,, \quad \epsilon^\mu \ll 1 \\ &U(\Lambda, a) = \mathbb{1} - \frac{\mathrm{i}}{2}\omega_{\mu\nu} M^{\mu\nu} + \mathrm{i}\epsilon_\mu P^\mu\,, \end{aligned} \tag{2.4.4}$$

and recalling that $M^{\mu\nu}$ transforms as a tensor under Lorentz transformations, i.e.,

$$\frac{\mathrm{i}\omega_{\rho\sigma}}{2}\mathcal{P}^{-1} M^{\rho\sigma}\mathcal{P} = \frac{\mathrm{i}\omega_{\rho\sigma}}{2}\Lambda_P^\rho{}_\mu \Lambda_P^\sigma{}_\nu M^{\mu\nu}\,, \tag{2.4.5}$$

we compute

$$\mathcal{P}^{-1} J_i \mathcal{P} = J_i\,, \quad \mathcal{P}^{-1} K_i \mathcal{P} = -K_i\,, \tag{2.4.6}$$

$$\mathcal{P}^{-1} P_i \mathcal{P} = -P_i\,, \quad \mathcal{P}^{-1} P^0 \mathcal{P} = P^0\,. \tag{2.4.7}$$

This is as expected since under parity we expect that the 0-th component of the vector P^μ ($P^0 = E$) is invariant whereas the spatial components change sign.

Also the angular momentum J_i should be invariant (being an axial vector as in classical mechanics $\mathbf{J} = \mathbf{r} \times \mathbf{p}$).

It is straightforward to see that the parity operator is indeed unitary.

However, if we follow the same procedure to obtain the transformations under time reversal we encounter a problem.

If the time reversal operator is also unitary it would map $\mathcal{T}^{-1}P^0\mathcal{T} = -P^0$.

That means it would change positive energies to negative energies that seems unphysical.

To interpret this result, we have the following theorem due to Wigner:

Theorem 2.4.1 (Wigner). Transformations on a Hilbert space preserving probabilities are either

- unitary and linear, i.e.,

$$\langle U\Phi|U\Psi\rangle = \langle\Phi\Psi\rangle\ ,\ U(\alpha\,|\Phi\rangle + \beta\,|\Psi\rangle) = \alpha\,U\,|\Phi\rangle + \beta U\,|\Psi\rangle \tag{2.4.8}$$

- or anti-unitary and anti-linear, that is,

$$\langle U\Phi|U\Psi\rangle = \langle\Phi|\Psi\rangle^*\ ,\ U(\alpha\,|\Phi\rangle + \beta\,|\Psi\rangle) = \alpha^*\,U\,|\Phi\rangle + U\,\beta^*\,|\Psi\rangle \tag{2.4.9}$$

Now, in order to preserve positive energies, $\mathcal{T}$ must be an anti-unitary and anti-linear operator

$$\mathcal{T}(\mathrm{i}\,|\Phi\rangle) = -\mathrm{i}\mathcal{T}\,|\Phi\rangle\ , \tag{2.4.10}$$

so that

$$\mathcal{T}^{-1}J_i\mathcal{T} = -J_i\,,\quad \mathcal{T}^{-1}K_i\mathcal{T} = K_i\,, \tag{2.4.11}$$

$$\mathcal{T}^{-1}P_i\mathcal{T} = -P_i\,,\quad \mathcal{T}^{-1}P^0\mathcal{T} = P^0\,. \tag{2.4.12}$$

Then, the time reversal operator $\mathcal{T}$ maps positive energies to positive energies.

Let us come back to our one-particle states defined in section 2.3 and understand their transformation behavior under parity and time reversal.

One can show that:

- For massive particles, we have the transformation properties

$$|m,j;p^{\mu},j_3\rangle \begin{cases} \xrightarrow{\mathcal{P}} & \eta_P \ |m,j;p'^{\mu},j_3\rangle \ , \\ \xrightarrow{\mathcal{T}} & \eta_T \ (-1)^{j-j_3} \ |m,j;p'^{\mu},-j_3\rangle \end{cases} \tag{2.4.13}$$

  with $|\eta_P| = |\eta_T| = 1$ and p'^{μ} is the result of the corresponding transformation acting on p^{μ} (under parity the spatial components change sign, under time reversal only the time component change sign, etc.).

- For massless particles, one finds

$$|p^{\mu},\lambda\rangle \begin{cases} \xrightarrow{\mathcal{P}} & \eta'_P \ \mathrm{e}^{\mp \mathrm{i}\pi\lambda} \ |p'^{\mu},-\lambda\rangle \ , \\ \xrightarrow{\mathcal{T}} & \eta'_T \ \mathrm{e}^{\pm \mathrm{i}\pi\lambda} \ |p'^{\mu},\lambda\rangle \ . \end{cases} \tag{2.4.14}$$

  Note that for $\lambda \neq 0$ there has to exist the opposite helicity states.

  In particular,

– the photon and the graviton have $\lambda = \pm 1$, $\lambda = \pm 2$ respectively which means that for each of them the $\pm\lambda$ represent two states of the same particle, since both gravitation and electromagnetism are invariant under parity (both graviton and photon are their own antiparticle).

– if the neutrino were massless, then $\lambda = \pm 1/2$ may have a different interpretation, for instance $\lambda = +1/2$ could be identified with the neutrino and $\lambda = -1/2$ with the antineutrino, since the weak interactions are not invariant under parity.

Even though, as we will see, the neutrinos are expected to have a mass, it is still an open question if the neutrinos are or are not their own anti-particles.

We conclude this section with some comments that will be very relevant in the next chapters:

- Massive particles of spin $j = 1$ have $2j + 1 = 3$ polarisation states, namely $j_3 = -1, 0, 1$. In contrast, massless particles of helicity $\lambda = 1$ have only 2 polarisation states with $\lambda = \pm 1$.
- Massive particles of spin $j = 2$ have $2j + 1 = 5$ polarisation states, namely $j_3 = -2, -1, 0, 1, 2$. In contrast, massless particles of helicity $\lambda = 2$ have **still** only 2 polarisation states with $\lambda = \pm 2$.

3. FROM PARTICLES TO FIELDS

Particles vs Fields

If it turned out that some physical system could not be described by a quantum field theory, it would be a sensation; if it turned out that the system did not obey the rules of quantum mechanics and relativity, it would be a cataclysm.

Steven Weinberg

In the previous part we stressed that only assuming two fundamental theories of nature, special relativity and quantum mechanics, the fundamental physical entities are the elementary particles labeled by the quantum number specified by the representations of the Poincaré group $| m, j \,;\, p^\mu, j_3 >$ and $| p^\mu \,, \lambda >$ describing massive and massless particles.

This part is devoted to the study of interactions among these particles.

The general requirements of Poincaré invariance, locality and unitarity will let us to introduce *fields* as “functions” of spacetime which are operators made out of creation and annihilation operators that create and destroy the corresponding particles.

We emphasize that fields are only a tool to describe interactions among the particles.

Their introduction includes more conditions than just the assumptions of special relativity and quantum mechanics.

However, fields are the key objects to describe interactions among particles and their use goes beyond the study of interactions among particles.

They are key ingredients in any interacting theory which requires local interactions such as condensed matter systems.

So their use is across different disciplines.

They are often presented as the basic objects of high energy physics with the particles appearing as their excitations.

The two descriptions are somehow manifestations of the wave-particle duality of quantum mechanics.

Our emphasis on particles rather than fields as the fundamental objects resides on the fact that it may be possible that some of the ingredients assumed in the introduction of fields may be overcome in future descriptions of nature beyond the Standard Model.

The spirit of this class is not only to introduce the basic tools to describe the Standard Model, but also to identify the key ingredients and assumptions that may help in shaping formulations beyond the Standard Model.

However, for the rest of the class, we will use the powerful tool of field theory.

3.1 Particle interactions and fields

In the preceding section, we learned about the concept of one-particle states.

Here, our aim is to describe interactions among many particles.

Putting together Poincaré invariance with the extra assumptions of unitarity and locality will lead us to superpositions of the aforementioned one-particle states corresponding to fields.

These objects allow us to develop a formalism known as Quantum Field Theory (QFT) that is suitable to describe local interactions among particle states.

3.1.1 Many particle states

Let us begin by trying to understand how we can describe relativistic processes in a quantum mechanical theory.

Clearly, $E = mc^2$ tells us that mass and energy are on equal footing which further implies that particles can be annihilated into energy.

To describe such processes, we aim at combining, as we said several times before, Lorentz invariance of special relativity with the notion of quantum mechanics and add further conditions such as locality to describe interactions.

To this end, we initially need to introduce the space of multi-particle states.

The Hilbert space H of all particle states can be decomposed as

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots \tag{3.1.1}$$

where

- $\mathcal{H}_0$ encodes 0-particle states, i.e., the vacuum $|0\rangle$
- $\mathcal{H}_1$ includes 1-particle states, e.g.,those generated by the creation operator $a^\dagger(p,\lambda)$ from the vacuum state,
$$|p^\mu, \lambda\rangle = a^\dagger(p,\lambda)\,|0\rangle \;. \tag{3.1.2}$$
- $\mathcal{H}_2$ includes 2-particle states, e.g.,
$$|p_1^\mu, \lambda_1; p_2^\mu, \lambda_2\rangle = a^\dagger(p_2,\lambda_2)\,|p_1^\mu, \lambda_1\rangle = \pm\,|p_2^\mu, \lambda_2; p_1^\mu, \lambda_1\rangle \;. \tag{3.1.3}$$
Here, the + sign refers to integer spin/helicity states (**bosons**), whereas the − sign to half-integer spin/helicity states (**fermions**).
- ...

Note that this is one of the most important - signs in science since it is the origin of the Pauli exclusion principle that implies that nuclei, atoms, molecules and therefore all matter, have a non-trivial structure.

As usual the creation and annihilation operators satisfy

$$\begin{aligned} [a(p,\lambda), a(p',\lambda')]_\pm = [a^\dagger(p,\lambda), a^\dagger(p',\lambda')]_\pm = 0\,, \\ [a(p,\lambda), a^\dagger(p',\lambda')]_\pm \propto \delta(p-p')\delta_{\lambda\lambda'} \end{aligned} \tag{3.1.4}$$

for bosons $[\cdot,\cdot]_- = [\cdot,\cdot]$ which are the standard commutators, whereas for fermions we change commutators for anti-commutators which may be written as $[\cdot,\cdot]_+ = \{\cdot, \cdot\}$.

In this way, we construct the full Hilbert space of particle states that we would like to describe with a dedicated quantum theory.

3.1.2 Interactions and Fields

Interactions among many particle states are determined by computing the S-matrix.

Typically, we can think of scattering processes as starting from an initial state $|\, \alpha^{in} >$ at $t = -\infty$ and ending up with an out-state $|\, \beta^{out} >$ at $t = +\infty$, see Fig. 3.1.

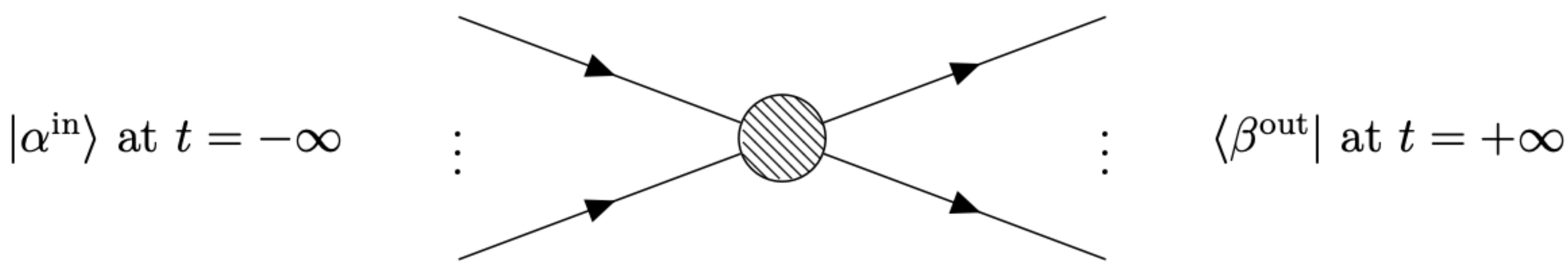

Figure 3.1: A cartoon representation of a scattering process.

In between, particles interact in a complicated way by, e.g., colliding with each other or splitting up in a bunch of new particles.

We define the S-matrix as

$$S_{\beta\alpha} = \langle\beta^{\mathrm{out}}|\alpha^{\mathrm{in}}\rangle = \delta_{\beta\alpha} + (2\pi)^4\delta(p_\beta - p_\alpha)\mathcal{M}_{\beta\alpha}\,. \tag{3.1.5}$$

The first term stands from the trivial event of no interactions at all.

So the interesting physics is encoded in the Matrix Elements $\mathcal{M}_{\beta\alpha}$.

The standard questions we can ask for particle interactions are:

a) **Decay Rates**. The probability of decay of one particle to several particles.

This is the simplest case in which the $|\,\alpha^{in}>$ state is one single particle.

The S-matrix reduces to the probability of decay of the original particle to its daughter states, see Fig. 3.2.

It is usually represented as $\Gamma(\alpha^{in} \rightarrow \beta^{out})$.

The decay rate per unit of phase space volume of the final states can be explicitly computed via

$$d\Gamma = \frac{d\Pi_{\mathrm{LIPS}}}{2E_\alpha}\,|\mathcal{M}_{\beta\alpha}|^2 \tag{3.1.6}$$

where $d\Pi_{\mathrm{LIPS}}$ stands for Lorentz invariant phase space volume

$$d\Pi_{\text{LIPS}} \equiv (2\pi)^4\delta^4(p^\mu_\alpha - \sum p) \tag{3.1.7}$$

The important point for us is that Γ is determined by $|\mathcal{M}_{\beta\alpha}|^2$ integrated and summed over all final momentum and spin states, see Appendix A for details.

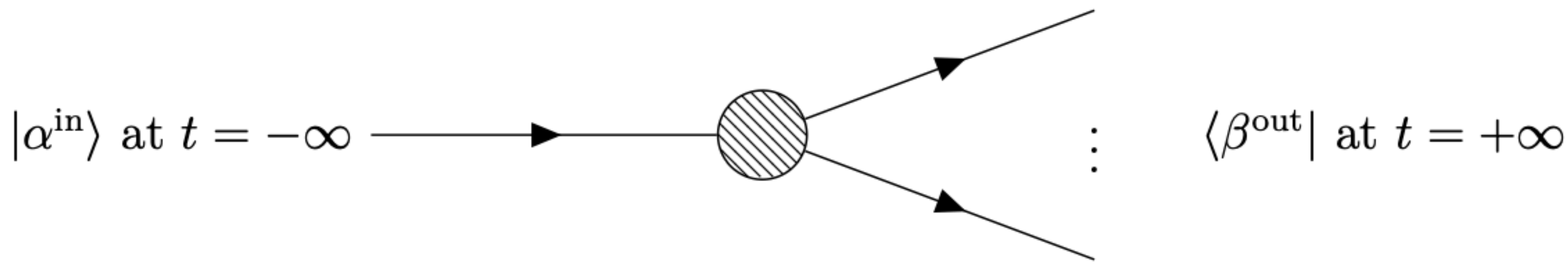

Figure 3.2: A cartoon representation of a decay process.

b) **Cross Sections.** As we know from Classical Mechanics, cross sections are the quantities that measure how often a scattering process between several particles happens.

Cross sections are usually labelled as σ and are also proportional to $|\mathcal{M}_{\beta\alpha}|^2$, see Appendix A for details.

Therefore our goal is to find a way to determine $|\mathcal{M}_{\beta\alpha}|^2$ given a configuration of initial and final states interacting in a particular way.

In order to determine the matrix elements $|\mathcal{M}_{\beta\alpha}|^2$, in general, we require the following conditions on interactions:

a) **Poincaré invariance of amplitudes.** The S-matrix is invariant under Poincaré transformations.

b) **Unitarity:** Probabilities add up to 1 and are preserved under time evolution by $U = e^{-iHt}$.

For the S-matrix, unitarity implies

$$\int \mathrm{d}\beta\, S^{*}_{\beta\gamma} S_{\beta\alpha} = \int \mathrm{d}\beta\, \langle \gamma^{\rm in} | \beta^{\rm out} \rangle \langle \beta^{\rm out} | \alpha^{\rm in} \rangle = \langle \gamma^{\rm in} | \alpha^{\rm in} \rangle = \delta(\alpha - \gamma)\,. \qquad (3.1.8)$$

or as an operator equation

$$S^{\dagger} S = 1\,. \qquad (3.1.9)$$

c) **Stability:** This is the requirement that the energy should be bounded from below and there is a state

of minimum energy, the vacuum $| 0 >$.

d) **Locality** (cluster decomposition): Observables $\mathcal{O}_1$, $\mathcal{O}_2$ commute at space-like distances, i.e.,

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0, \qquad (3.1.10)$$

which is usually referred to as *microcausality*.

As a consequence of the last requirement of locality, we deduce that the *Hamiltonian* H is determined by a local function, the *Hamiltonian density* $\mathcal{H}(\mathbf{x}, t)$, which is defined at each space point.

More precisely, we define the Hamiltonian as

$$H = \int \mathcal{H}(\mathbf{x}, t)\, \mathrm{d}^3 x \tag{3.1.11}$$

corresponding to the sum of energies at each point in space.

Similarly, the *Lagrangian* L is obtained from the *Lagrangian density* $\boldsymbol{\mathcal{L}}(\mathbf{x}, \mathrm{t})$ via

$$L = \int \mathcal{L}(\mathbf{x}, t)\, \mathrm{d}^3 x \tag{3.1.12}$$

with associated action

$$S = \int \mathcal{L}(\mathbf{x}, t)\, \mathrm{d}^4 x\,. \tag{3.1.13}$$

This locality requirement is crucial to determine interactions and is known as *cluster decomposition* which means that experiments performed at large enough distances do not affect each other.

Now, we arrive at a conundrum: $\mathscr{H}$ and $\boldsymbol{\mathcal{L}}$ are operators in position space, but particle states as derived above are defined in momentum space.

The way out is pretty obvious: we need to apply Fourier transformations to describe the corresponding states in terms of "functions" of spacetime coordinates – objects that we call **fields**.

For any particle of given momentum and spin/helicity, we define a field Φ_α as

$$\Phi_\alpha(x^\mu) = A_\alpha(x^\mu) + B^*_\alpha(x^\mu) \tag{3.1.14}$$

in terms of

$$A_\alpha(x^\mu) = \sum_\lambda \int \mathrm{d}p\, u_\alpha(p,\lambda)\, a(p,\lambda)\, \mathrm{e}^{\mathrm{i}p_\mu x^\mu}\,, \tag{3.1.15}$$

$$B^*_\alpha(x^\mu) = \sum_\lambda \int \mathrm{d}p\, v_\alpha(p,\lambda)\, b^\dagger(p,\lambda)\, \mathrm{e}^{-\mathrm{i}p_\mu x^\mu}\,. \tag{3.1.16}$$

We use a multi-index notation where $\alpha = \mu_1 \ldots \mu_n$ are spacetime indices of the corresponding representation under SO(3, 1).

Here, the operators $a(p, \lambda)$ and $b^\dagger(p, \lambda)$ are raising and lowering operators as defined in the previous section with the commutation relations (3.1.4).

The object A is the field annihilating the corresponding particle, whereas B is the field creating the anti-particle.

Stated otherwise, fields are always of the form

Quantum Fields

$$\Phi_\alpha(x^\mu) = \sum_\lambda \int \mathrm{d}p\, \left(u_\alpha(p,\lambda)\, a(p,\lambda)\, \mathrm{e}^{\mathrm{i}p_\mu x^\mu} + v_\alpha(p,\lambda)\, b^\dagger(p,\lambda)\, \mathrm{e}^{-\mathrm{i}p_\mu x^\mu}\right). \tag{3.1.17}$$

For integer spin or helicity n, the object on the left transforms as some rank n tensor under $SO(1,3)_\uparrow$, that is, it transforms under the (n/2, n/2) representation of the Lorentz algebra.

The integral and sum on the right is a superposition of all unitary irreducible representations of one-particle states as classified above via the little groups.

The parameter λ labels spins for massive particles and helicity for massless particles, while the integration over momenta is performed using the invariant measure

$$\int \mathrm{d}p = \int \mathrm{d}^4 p\, \delta(p^2 - m^2) = \int \mathrm{d}^3 p \frac{1}{2E_p (2\pi)^3}\,, \quad E_p = p_0\,, \quad E_p^2 = \mathbf{p}^2 + m^2\,. \tag{3.1.18}$$

The wave functions $u_\alpha(p, \lambda)$ and $v_\alpha(p, \lambda)$ in momentum space describe the dynamics of fields in spacetime given that they carry Lorentz indices α.

As we will in the next part, the functions $u_\alpha(p, \lambda)$ and $v_\alpha(p, \lambda)$ need to satisfy certain constraints in order to write down (off-shell) actions in terms of the fields.

The relation between the two sides in (3.1.17) is determined by the *coefficient functions* $e^{ip_\mu x^\mu}\, u_\alpha(p, \lambda)$ and $e^{-ip_\mu x^\mu}\, v_\alpha(p, \lambda)$ which carry both Lorentz indices α and x as well as Poincaré representation labels p and λ.

One import remark concern the form (3.1.14) that we started with which ensures the existence of anti-particles.

Let us explain this in more detail.

First, we note that writing the field $\Phi_\alpha(x^\mu)$ in terms of $A_\alpha(x^\mu)$ and $B_\alpha(x^\mu)$ as in (3.1.14) is essentially required by causality.

Above, we stated that all operators should commute at spacelike separations, cf. Eq. (3.1.10).

That is, for a fixed time and two different locations, we demand that

$$\left[\Phi_\alpha(x^i, t), \Phi^*_\alpha(y^i, t)\right] = 0\,. \tag{3.1.19}$$

However, it is impossible for both $A_\alpha(x^\mu)$ and $B_\alpha(x^\mu)$ to satisfy this condition by themselves because

$$\left[A_\alpha(x^i, t), A^*_\alpha(y^i, t)\right] \neq 0\,. \tag{3.1.20}$$

These commutators can be explicitly derived from those for the creation and annihilation operators $a(p, \lambda)$ and $a^\dagger(p, \lambda)$ as defined in (3.1.4).

Thus, both objects are needed to build fields $\Phi_\alpha(x^\mu)$ satisfying (3.1.19).

This is the requirement for the existence of anti-particles.

If $a(p, \lambda) = b(p, \lambda)$, the particle is simply its own anti-particle.

3.1.3 Field theories and their actions

So far, we have seen that putting special relativity and quantum mechanics together lead us to classifying one-particle states in terms of their masses and spins.

Interactions lead us to the concept of locality and to use fields rather than particles to describe our theory.

In finding all unitary irreducible representations of the Poincaré group, we defined states for fixed p^μ over which we have to integrate to get a suitable superposition of eigenstates, i.e., we found fields of the form

$$\Phi_\alpha(x^\mu) = \sum_\lambda \int \mathrm{d}p \left(u_\alpha(p,\lambda)\, a(p,\lambda)\, \mathrm{e}^{\mathrm{i}p_\mu x^\mu} + v_\alpha(p,\lambda)\, b^\dagger(p,\lambda)\, \mathrm{e}^{-\mathrm{i}p_\mu x^\mu} \right) . \tag{3.1.21}$$

The action S becomes a function of these fields Φ_α and their derivatives, i.e.,

$$S[\Phi_\alpha, \partial\Phi_\alpha] = \int \mathrm{d}^4x\, \mathcal{L}[\Phi_\alpha, \partial\Phi_\alpha] \tag{3.1.22}$$

where $\mathcal{L}[\Phi_\alpha,\ \partial\Phi_\alpha]$ is the Lagrange density or simply Lagrangian of the theory.

Here, translation invariance forbids the explicit dependence of $\mathcal{L}$ on the coordinates x^μ, i.e., $\mathcal{L} \neq \mathcal{L}[\Phi_\alpha,\ \partial\Phi_\alpha, x^\mu]$.

The Lagrangian is typically written as a sum of individual terms of the form

$$\mathcal{L}[\Phi_\alpha, \partial\Phi_\alpha] = \sum_i c_i\, \mathcal{O}_i(\Phi_\alpha, \partial\Phi_\alpha) . \tag{3.1.23}$$

Here, c_i are some “constant” coefficients and $\mathcal{O}_i$ are referred to as operators since they are functions of the fields Φ_α which are themselves operators (as it can be seen from their dependence on the creation and annihilation operators).

The equations of motion for Φ_α are obtained as usual from the Euler-Lagrange equations, i.e.,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \Phi_\alpha} - \frac{\partial \mathcal{L}}{\partial \Phi_\alpha} = 0\,. \tag{3.1.24}$$

When the field configuration satisfies these classical equations, we say that the field is on its mass shell or on-shell.

Otherwise we say it is off-shell.

Quantization of field theories proceeds most easily through the path integral approach where e.g. the partition function can be written as

$$\mathcal{Z} = \int \mathcal{D}\Phi_\alpha\, \mathrm{e}^{-S[\Phi_\alpha, \partial\Phi_\alpha]}\,. \tag{3.1.25}$$

Only in the classical limit the on-shell condition is satisfied. Similarly, correlations functions and amplitudes can be straightforwardly computed within this formalism through perturbation theory.

Note that there is an infinite-to-one mapping from actions to on-shell scattering amplitudes: infinitely many actions can give rise to the same on-shell amplitude due to field redefinitions $\Phi \to \Phi'(\Phi)$.

This begs the question: what is the point of introducing fields in the first place?

First and foremost, they provide us with an organizing principle for interactions among particles governed by symmetries.

What is more, non-perturbative effects, running couplings as well as off-shell correlation functions can be systematically studied.

As Weinberg himself always stresses, quantum fields are “the only way of satisfying the principles of Lorentz invariance plus quantum mechanics plus cluster decomposition”.

Recently, though, there has been much effort towards computing amplitudes directly without the use of Lagrangians.

No details here.

Let us now provide examples of free field theories focussing on spin/helicity states less than one for which the massless and massive states have the same number of degrees of freedom:

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