Part III

Spin One

54 Maxwell's Equations

The most common (and important) spin-one particle is the photon.

Emission and absorption of photons by matter is an important phenomenon in many areas of physics, and so that is the context in which most physicists first encounter a serious treatment of photons.

We will use a brief review of this subject (in this section and the next) as our entry point into the theory of quantum electrodynamics.

Let us begin with classical electrodynamics.

Maxwell's equations are

$$\nabla\cdot\mathbf{E} = \rho \,, \tag{54.1}$$

$$\nabla\times\mathbf{B} - \dot{\mathbf{E}} = \mathbf{J} \,, \tag{54.2}$$

$$\nabla\times\mathbf{E} + \dot{\mathbf{B}} = 0 \,, \tag{54.3}$$

$$\nabla\cdot\mathbf{B} = 0 \,, \tag{54.4}$$

where **E** is the electric field, **B** is the magnetic field, ρ is the charge density, and **J** is the current density.

We have written Maxwell's equations in *Heaviside-Lorentz units*, and also set $c = 1$.

In these units, the magnitude of the force between two charges of magnitude Q is $Q^2/4\pi r^2$.

Maxwell's equations must be supplemented by formulae that give us the dynamics of the charges and currents (such as the Lorentz force law for point particles).

For now, however, we will treat the charges and currents as specified sources, and focus on the dynamics of the electromagnetic fields.

The last two of Maxwell's equations, the ones with no sources on the right-hand side, can be solved by writing the **E** and **B** fields in terms of a scalar potential φ and a vector potential **A**,

$$\mathbf{E} = -\nabla\varphi - \dot{\mathbf{A}}\,, \tag{54.5}$$

$$\mathbf{B} = \nabla \times \mathbf{A}\,. \tag{54.6}$$

The potentials uniquely determine the fields, but the fields do not uniquely determine the potentials. Given a particular φ and **A** that result in a particular **E** and **B**, we will get the same **E** and **B** from any other potentials φ' and $\mathbf{A}'$ that are related by

$$\varphi' = \varphi + \dot{\Gamma}\,, \tag{54.7}$$

$$\mathbf{A}' = \mathbf{A} - \nabla\Gamma\,, \tag{54.8}$$

where Γ is an arbitrary function of spacetime.

A change of potentials that does not change the fields is called a *gauge transformation*.

The E and B fields are *gauge invariant*.

All this becomes more compact and elegant in a relativistic notation.

We define the four-vector potential or *gauge field*

$$A^{\mu} \equiv (\varphi, \mathbf{A}) \ . \tag{54.9}$$

We also define the *field strength*

$$F^{\mu\nu} \equiv \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} \ . \tag{54.10}$$

Obviously, $F^{\mu\nu}$ is antisymmetric: $F^{\mu\nu} = -F^{\nu\mu}$.

Comparing eqs. (54.5) and (54.6) with eqs. (54.9) and (54.10), we see that

$$F^{0i} = E^{i} \ , \tag{54.11}$$

$$F^{ij} = \varepsilon^{ijk} B_{k} \ . \tag{54.12}$$

The first two of Maxwell's equations can now be written as

$$\partial_{\nu} F^{\mu\nu} = J^{\mu} \ , \tag{54.13}$$

where

$$J^{\mu} \equiv (\rho, \mathbf{J}) \tag{54.14}$$

is the charge-current density four-vector.

If we take the four-divergence of eq. (54.13), we get $\partial_\mu \partial_\nu F^{\mu\nu} = \partial_\mu J^\mu$.

The left-hand side of this equation vanishes, because $\partial_\mu \partial_\nu$ is symmetric on exchange of μ and ν, while $F^{\mu\nu}$ is antisymmetric.

We conclude that we must have

$$\partial_\mu J^\mu = 0 \,, \tag{54.15}$$

or equivalently

$$\dot{\rho} + \nabla \cdot \mathbf{J} = 0 \,; \tag{54.16}$$

that is, the electromagnetic current must be conserved.

The last two of Maxwell's equations can be written as

$$\varepsilon_{\mu\nu\rho\sigma} \partial^\rho F^{\mu\nu} = 0 \,, \tag{54.17}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric Levi-Civita tensor; see section 34.

Plugging in eq. (54.10), we see that eq. (54.17) is automatically satisfied, since the antisymmetric combination of two derivatives vanishes.

Eqs. (54.7) and (54.8) can be combined into

$$A'^\mu = A^\mu - \partial^\mu \Gamma \,. \tag{54.18}$$

Setting $F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu$ and using eq. (54.18), we get

$$F'^{\mu\nu} = F^{\mu\nu} - (\partial^\mu\partial^\nu - \partial^\nu\partial^\mu)\Gamma \; . \tag{54.19}$$

The last term vanishes because derivatives commute; thus the field strength is gauge invariant,

$$F'^{\mu\nu} = F^{\mu\nu} \; . \tag{54.20}$$

Next we will find an action that results in Maxwell's equations as the equations of motion.

We will treat the current as an external source.

The action we seek should be Lorentz invariant, gauge invariant, parity and time-reversal invariant, and no more than second order in derivatives.

The only candidate is $S = \int d^4x\, \mathcal{L}$, where

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu \; . \tag{54.21}$$

The first term is obviously gauge invariant, because $F^{\mu\nu}$ is.

After a gauge transformation, eq. (54.18), the second term becomes $J^\mu A'_\mu$, and the difference is

$$\begin{aligned} J^\mu(A'_\mu - A_\mu) &= -J^\mu\partial_\mu\Gamma \\ &= -(\partial_\mu J^\mu)\Gamma - \partial_\mu(J^\mu\Gamma) \; . \end{aligned} \tag{54.22}$$

The first term in eq. (54.22) vanishes because the current is conserved.

The second term is a total divergence, and its integral over d^4x vanishes (assuming suitable boundary conditions at infinity).

Thus the action specified by eq. (54.21) is gauge invariant.

Setting $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and multiplying out the terms, eq. (54.21) becomes

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu + \tfrac{1}{2}\partial^\mu A^\nu \partial_\nu A_\mu + J^\mu A_\mu \tag{54.23}$$

$$= +\tfrac{1}{2} A_\mu (g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu) A_\nu + J^\mu A_\mu - \partial^\mu K_\mu \ , \tag{54.24}$$

where $K_\mu = 12\ A^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu)$.

The last term is a total divergence, and can be dropped.

From eq. (54.24), we can see that varying A^μ while requiring S to be unchanged yields the equation of motion

$$(g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu) A_\nu + J^\mu = 0 \ . \tag{54.25}$$

Noting that $\partial_\nu F^{\mu\nu} = \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = (\partial^\mu \partial^\nu - g^{\mu\nu}\partial^2)A_\nu$, we see that eq. (54.25) is equivalent to eq. (54.13), and hence to Maxwell's equations.

55 Electrodynamics in Coulomb Gauge

Next we would like to construct the hamiltonian, and quantize the electromagnetic field.

There is an immediate difficulty, caused by the gauge invariance: we have too many degrees of freedom.

This problem manifests itself in several ways.

For example, the lagrangian

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^{\mu}A_{\mu} \tag{55.1}$$

$$= -\tfrac{1}{2}\partial^{\mu}A^{\nu}\partial_{\mu}A_{\nu} + \tfrac{1}{2}\partial^{\mu}A^{\nu}\partial_{\nu}A_{\mu} + J^{\mu}A_{\mu} \tag{55.2}$$

does not contain the time derivative of A^0.

Thus, this field has no canonically conjugate momentum and no dynamics.

To deal with this problem, we must eliminate the gauge freedom.

We do this by *choosing a gauge*.

We choose a gauge by imposing a *gauge condition*.

This is a condition that we require $A^{\mu}(x)$ to satisfy.

The idea is that there should be only one $A^{\mu}(x)$ that results in a given $F^{\mu\nu}(x)$ and that also satisfies the gauge condition.

One possible class of gauge conditions is $n^\mu A_\mu(x) = 0$, where $n\mu$ is a constant four-vector.

If n is spacelike ($n^2 > 0$), then we have chosen *axial gauge*; if n is lightlike, ($n^2 = 0$), it is *lightcone gauge*; and if n is timelike, ($n^2 < 0$), it is *temporal gauge*.

Another gauge is *Lorenz gauge*, where the condition is $\partial^\mu A_\mu = 0$.

We will meet a family of closely related gauges in section 62.

In this section, we will work in *Coulomb gauge*, also known as *radiation gauge* or *transverse gauge*.

The condition for Coulomb gauge is

$$\nabla \cdot \mathbf{A}(x) = 0\ . \tag{55.3}$$

We can impose eq. (55.3) by acting on $A_i(x)$ with a projection operator,

$$A_i(x) \to \left(\delta_{ij} - \frac{\nabla_i \nabla_j}{\nabla^2}\right) A_j(x)\ . \tag{55.4}$$

We construct the right-hand side of eq. (55.4) by Fourier-transforming $A_i(x)$ to $A_i(k)$, multiplying $A_i(k)$ by the matrix $\delta_{ij} - k_i k_j/k^2$, and then Fourier-transforming back to position space.

From now on, whenever we write A_i, we will implicitly mean the right-hand side of eq. (55.4).

Now let us write out the lagrangian in terms of the scalar and vector potentials, $\varphi = A^0$ and A_i, with A_i obeying the Coulomb gauge condition.

Starting from eq. (55.2), we get

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{2}\dot{A}_i\dot{A}_i - \tfrac{1}{2}\nabla_j A_i \nabla_j A_i + J_i A_i \\ & + \tfrac{1}{2}\nabla_i A_j \nabla_j A_i + \dot{A}_i \nabla_i \varphi \\ & + \tfrac{1}{2}\nabla_i\varphi\nabla_i\varphi - \rho\varphi \,.\end{aligned} \tag{55.5}$$

In the second line of eq. (55.5), the ∇_i in each term can be integrated by parts; in the first term, we will then get a factor of $\nabla_j(\nabla_i A_i)$, and in the second term, we will get a factor of $\nabla_i \dot{A}_i$. .

Both of these vanish by virtue of the gauge condition $\nabla_i A_i = 0$, and so both of these terms can simply be dropped.

If we now vary φ (and require $S = \int d^4x\, \mathcal{L}$ to be stationary), we find that φ obeys Poisson's equation,

$$-\nabla^2\varphi = \rho \,. \tag{55.6}$$

The solution is

$$\varphi(\mathbf{x}, t) = \int d^3y \, \frac{\rho(\mathbf{y}, t)}{4\pi|\mathbf{x}-\mathbf{y}|} \,. \tag{55.7}$$

This solution is unique if we impose the boundary conditions that φ and ρ both vanish at spatial infinity.

Eq. (55.7) tells us that φ(x,t) is given entirely in terms of the charge density at the same time, and so has no dynamics of its own.

It is therefore legitimate to plug eq. (55.7) back into the lagrangian.

After an integration by parts to turn $\nabla_i\varphi\nabla_i\varphi$ into $-\varphi\nabla^2\varphi=\varphi\rho$, the result is

$$\mathcal{L} = \tfrac{1}{2}\dot{A}_i\dot{A}_i - \tfrac{1}{2}\nabla_j A_i\nabla_j A_i + J_i A_i + \mathcal{L}_{\rm coul}\ , \tag{55.8}$$

where

$$\mathcal{L}_{\rm coul} = -\frac{1}{2}\int d^3y\ \frac{\rho(\mathbf{x},t)\rho(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|}\ . \tag{55.9}$$

We can now vary A_i; keeping proper track of the implicit projection operator in eq. (55.4), we find that A_i obeys the massless Klein-Gordon equation with the projected current as a source,

$$-\partial^2 A_i(x) = \left(\delta_{ij} - \frac{\nabla_i\nabla_j}{\nabla^2}\right)J_j(x)\ . \tag{55.10}$$

For a free field ($J_i = 0$), the general solution is

$$\mathbf{A}(x) = \sum_{\lambda=\pm}\int \widetilde{dk}\left[\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k})a_\lambda(\mathbf{k})e^{ikx} + \boldsymbol{\varepsilon}_\lambda(\mathbf{k})a^\dagger_\lambda(\mathbf{k})e^{-ikx}\right], \tag{55.11}$$

where $k^0 = \omega = |\mathbf{k}|$, $\widetilde{dk} = d^3k/(2\pi)^3 2\omega$, and $\boldsymbol{\varepsilon}_+(\mathbf{k})$ and $\boldsymbol{\varepsilon}_-(\mathbf{k})$ are polarization vectors.

In order to satisfy the Coulomb gauge condition, the polarization vectors must be orthogonal to the wave vector **k**.

We will choose them to correspond to right- and left-handed circular polarizations; for **k** = (0,0,k), we then have

$$
\begin{aligned}
\boldsymbol{\varepsilon}_+(\mathbf{k}) &= \tfrac{1}{\sqrt{2}}(1,-i,0)\ , \\
\boldsymbol{\varepsilon}_-(\mathbf{k}) &= \tfrac{1}{\sqrt{2}}(1,+i,0)\ .
\end{aligned}
\tag{55.12}
$$

More generally, the two polarization vectors along with the unit vector in the **k** direction form an orthonormal and complete set,

$$\mathbf{k}\cdot\boldsymbol{\varepsilon}_\lambda(\mathbf{k}) = 0\ , \tag{55.13}$$

$$\boldsymbol{\varepsilon}_{\lambda'}(\mathbf{k})\cdot\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k}) = \delta_{\lambda'\lambda}\ , \tag{55.14}$$

$$\sum_{\lambda=\pm} \varepsilon^*_{i\lambda}(\mathbf{k})\varepsilon_{j\lambda}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\ . \tag{55.15}$$

The coefficients a_λ(**k**) and $a^\dagger_\lambda$(**k**) will become operators after quantization, which is why we have used the dagger symbol for conjugation.

In complete analogy with the procedure used for a scalar field in section 3, we can invert eq. (55.11) and its time derivative to get

$$a_\lambda(\mathbf{k}) = +i\,\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\int d^3x\ e^{-ikx}\,\overleftrightarrow{\partial_0}\,\mathbf{A}(x)\ , \tag{55.16}$$

$$a^\dagger_\lambda(\mathbf{k}) = -i\,\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k})\cdot\int d^3x\ e^{+ikx}\,\overleftrightarrow{\partial_0}\,\mathbf{A}(x)\ , \tag{55.17}$$

where $f\overleftrightarrow{\partial_\mu}g = f(\partial_\mu g) - (\partial_\mu f)g$.

Now we can proceed to the hamiltonian formalism. First, we compute the canonically conjugate momentum to A_i,

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \dot{A}_i \ . \tag{55.18}$$

Note that $\nabla_i A_i = 0$ implies $\nabla_i \Pi_i = 0$.

The hamiltonian density is then

$$\begin{aligned} \mathcal{H} &= \Pi_i \dot{A}_i - \mathcal{L} \\ &= \tfrac{1}{2}\Pi_i\Pi_i + \tfrac{1}{2}\nabla_j A_i \nabla_j A_i - J_i A_i + \mathcal{H}_{\rm coul} \ , \end{aligned} \tag{55.19}$$

the where $\mathcal{H}_{\rm coul} = -\mathcal{L}_{\rm coul}$.

To quantize the field, we impose the canonical commutation relations.

Keeping proper track of the implicit projection operator in eq. (55.4), we have

$$\begin{aligned} [A_i(\mathbf{x},t), \Pi_j(\mathbf{y},t)] &= i\left(\delta_{ij} - \frac{\nabla_i\nabla_j}{\nabla^2}\right)\delta^3(\mathbf{x}-\mathbf{y}) \\ &= i\int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right). \end{aligned} \tag{55.20}$$

The commutation relations of the $a_\lambda(\mathbf{k})$ and $a^\dagger_\lambda(\mathbf{k})$ operators follow from eq. (55.20) and $[A_i, A_j] = [\Pi_i, \Pi_j] = 0$ (at equal times).

The result is

$$[a_\lambda(\mathbf{k}), a_{\lambda'}(\mathbf{k}')] = 0\,, \tag{55.21}$$

$$[a^\dagger_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}')] = 0\,, \tag{55.22}$$

$$[a_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{k}')] = (2\pi)^3 2\omega\,\delta^3(\mathbf{k}'-\mathbf{k})\delta_{\lambda\lambda'}\,. \tag{55.23}$$

We interpret $a^\dagger_\lambda(\mathbf{k})$ and $a_\lambda(\mathbf{k})$ as creation and annihilation operators for photons of definite helicity, with helicity +1 corresponding to right-circular polarization and helicity −1 to left-circular polarization.

It is now straightfoward to write the hamiltonian explicitly in terms of these operators.

We find

$$H = \sum_{\lambda=\pm} \int \widetilde{dk}\; \omega\; a^\dagger_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) + 2\mathcal{E}_0 V - \int d^3x\; \mathbf{J}(x)\cdot\mathbf{A}(x) + H_{\text{coul}}\,, \tag{55.24}$$

where $\mathcal{E}_0 = \frac{1}{2}(2\pi)^{-3}\int d^3k\;\omega$ is the zero-point energy per unit volume that we found for a real scalar field in section 3, V is the volume of space, the Coulomb hamiltonian is

$$H_{\text{coul}} = \frac{1}{2}\int d^3x\, d^3y\; \frac{\rho(\mathbf{x},t)\rho(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|}\,, \tag{55.25}$$

and we use eq. (55.11) to express $A_i(x)$ in terms of $a_\lambda(\mathbf{k})$ and $a^\dagger_\lambda(\mathbf{k})$ at any one particular time (say, $t = 0$).

This is sufficient, because H itself is time independent.

This form of the hamiltonian of electrodynamics is often used as the starting point for calculations of atomic transition rates, with the charges and currents treated via the nonrelativistic Schrodinger equation.

The Coulomb interaction appears explicitly, and the $\mathbf{J} \cdot \mathbf{A}$ term allows for the creation and annihilation of photons of definite polarization.

56 LSZ Reduction for Photons

In section 55, we found that the creation and annihilation operators for free photons could be written as

$$a^\dagger_\lambda(\mathbf{k}) = -i\,\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k})\cdot\int d^3x\; e^{+ikx}\,\overleftrightarrow{\partial_0}\,\mathbf{A}(x)\;, \tag{56.1}$$

$$a_\lambda(\mathbf{k}) = +i\,\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\int d^3x\; e^{-ikx}\,\overleftrightarrow{\partial_0}\,\mathbf{A}(x)\;, \tag{56.2}$$

where $\varepsilon_\lambda(\mathbf{k})$ is a polarization vector.

From here, we can follow the analysis of section 5 line by line to deduce the LSZ reduction formula for photons.

The result is that the creation operator for each incoming photon should be replaced by

$$a_\lambda^\dagger(\mathbf{k})_{\rm in} \to i\,\varepsilon_\lambda^{\mu*}(\mathbf{k}) \int d^4x\; e^{+ikx}(-\partial^2)A_\mu(x)\;, \tag{56.3}$$

and the destruction operator for each outgoing photon should be replaced by

$$a_\lambda(\mathbf{k})_{\rm out} \to i\,\varepsilon_\lambda^{\mu}(\mathbf{k}) \int d^4x\; e^{-ikx}(-\partial^2)A_\mu(x)\;, \tag{56.4}$$

and then we should take the vacuum expectation value of the time-ordered product.

Note that, in writing eqs. (56.3) and (56.4), we have made them look nicer by introducing $\varepsilon^0{}_\lambda(\mathbf{k}) \equiv 0$, and then using four-vector dot products rather than three-vector dot products.

The LSZ formula is valid provided the field is normalized according to the free-field formulae

$$\langle 0|A^i(x)|0\rangle = 0\;, \tag{56.5}$$

$$\langle k,\lambda|A^i(x)|0\rangle = \varepsilon_\lambda^i(\mathbf{k})e^{ikx}\;, \tag{56.6}$$

where $|k,\lambda\rangle$ is a single photon state, normalized according to

$$\langle k',\lambda'|k,\lambda\rangle = (2\pi)^3 2\omega\,\delta^3(\mathbf{k}'-\mathbf{k})\delta_{\lambda\lambda'}\;. \tag{56.7}$$

The zero on the right-hand side of eq. (56.5) is required by rotation invariance, and only the overall scale of the right-hand side of eq. (56.6) might be different in an interacting theory.

The renormalization of A_i necessitates including appropriate Z factors in the lagrangian,

$$\mathcal{L} = -\tfrac{1}{4} Z_3 F^{\mu\nu} F_{\mu\nu} + Z_1 J^\mu A_\mu \ . \tag{56.8}$$

Here Z_3 and Z_1 are the traditional names; we will meet Z_2 in section 62.

We must choose Z_3 so that eq. (56.6) holds. We will fix Z_1 by requiring the corresponding vertex function to take on a certain value for a particular set of external momenta.

Next we must compute the correlation functions $\langle 0|\mathrm{T}A_i(x)\ldots|0\rangle$.

As usual, we begin by working with free field theory.

The analysis is again almost identical to the case of a scalar field.

We find that, in free field theory,

$$\langle 0|\mathrm{T}A^i(x)A^j(y)|0\rangle = \tfrac{1}{i}\Delta^{ij}(x-y) \ , \tag{56.9}$$

where the propagator is

$$\Delta^{ij}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - i\epsilon} \textstyle\sum_{\lambda=\pm} \varepsilon^{i*}_\lambda(\mathbf{k})\varepsilon^j_\lambda(\mathbf{k}) \ . \tag{56.10}$$

As with a free scalar field, correlations of an odd number of fields vanish, and correlations of an even number of fields are given in terms of the propagator by Wick's theorem; see section 8.

We would now like to evaluate the path integral for the free electromagnetic field

$$Z_0(J) \equiv \langle 0|0\rangle_J = \int \mathcal{D}A\; e^{i\int d^4x\,[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+J^\mu A_\mu]}\;. \tag{56.11}$$

Here we treat the current $J^\mu(x)$ as an external source.

We will evaluate $Z_0(J)$ in Coulomb gauge.

This means that we will integrate over only those field configurations that satisfy $\nabla\cdot\mathbf{A} = 0$.

We begin by integrating over A^0.

Because the action is quadratic in A^μ, this is equivalent to solving the variational equation for A^0, and then substituting the solution back into the lagrangian.

The result is that we have the Coulomb term in the action,

$$S_{\rm coul} = -\frac{1}{2}\int d^4x\; d^4y\; \delta(x^0{-}y^0)\;\frac{J^0(x)J^0(y)}{4\pi|\mathbf{x}{-}\mathbf{y}|}\;. \tag{56.12}$$

Since this term does not depend on the vector potential, we simply get a factor of $\exp(iS_{\rm coul})$ in front of the remaining path integral over A_i.

We will perform this integral formally (as we did for fermion fields in section 43) by requiring it to yield the correct results for the correlation functions of A_i when we take functional derivatives with respect to J_i.

In this way we find that

$$Z_0(J) = \exp\left[iS_{\text{coul}} + \frac{i}{2}\int d^4x\, d^4y\, J_i(x)\Delta^{ij}(x-y)J_j(y)\right] . \tag{56.13}$$

We can make $Z_0(J)$ look prettier by writing it as

$$Z_0(J) = \exp\left[\frac{i}{2}\int d^4x\, d^4y\, J_\mu(x)\Delta^{\mu\nu}(x-y)J_\nu(y)\right] , \tag{56.14}$$

where we have defined

$$\Delta^{\mu\nu}(x-y) \equiv \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, \tilde{\Delta}^{\mu\nu}(k) , \tag{56.15}$$

$$\tilde{\Delta}^{\mu\nu}(k) \equiv -\frac{1}{\mathbf{k}^2}\,\delta^{\mu 0}\delta^{\nu 0} + \frac{1}{k^2 - i\epsilon}\sum\nolimits_{\lambda=\pm}\varepsilon_\lambda^{\mu *}(\mathbf{k})\varepsilon_\lambda^{\nu}(\mathbf{k}) . \tag{56.16}$$

The first term on the right-hand side of eq. (56.16) reproduces the Coulomb term in eq. (56.13) by virtue of the facts that

$$\int_{-\infty}^{+\infty} \frac{dk^0}{2\pi}\, e^{-ik^0(x^0-y^0)} = \delta(x^0 - y^0) , \tag{56.17}$$

$$\int \frac{d^3k}{(2\pi)^3}\, \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\mathbf{k}^2} = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} . \tag{56.18}$$

The second term on the right-hand side of eq. (56.16) reproduces the second term in eq. (56.13) by virtue of the fact that $\varepsilon^0_\lambda(\mathbf{k}) = 0$.

Next we will simplify eq. (56.16). We begin by introducing a unit vector in the time direction,

$$\hat{t}^\mu = (1, \mathbf{0}) \ . \tag{56.19}$$

Next we need a unit vector in the **k** direction, which we will call $\hat{z}^\mu$. .

We first note that $\hat{t}\cdot k = -k^0$, and so we can write

$$(0, \mathbf{k}) = k^\mu + (\hat{t}\cdot k)\hat{t}^\mu \ . \tag{56.20}$$

The square of this four-vector is

$$\mathbf{k}^2 = k^2 + (\hat{t}\cdot k)^2 \ , \tag{56.21}$$

where we have used $\hat{t}^2 = -1$.

Thus the unit vector that we want is

$$\hat{z}^\mu = \frac{k^\mu + (\hat{t}\cdot k)\hat{t}^\mu}{[k^2 + (\hat{t}\cdot k)^2]^{1/2}} \ . \tag{56.22}$$

Now we recall from section 55 that

$$\sum_{\lambda=\pm} \varepsilon_\lambda^{i*}(\mathbf{k})\varepsilon_\lambda^{j}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \ . \tag{56.23}$$

This can be extended to i $\rightarrow \mu$ and j $\rightarrow \nu$ by writing

$$\sum_{\lambda=\pm} \varepsilon_\lambda^{\mu *}(\mathbf{k})\varepsilon_\lambda^{\nu}(\mathbf{k}) = g^{\mu\nu} + \hat{t}^\mu \hat{t}^\nu - \hat{z}^\mu \hat{z}^\nu \ . \tag{56.24}$$

It is not hard to check that the right-hand side of eq. (56.24) vanishes if $\mu = 0$ or $\nu = 0$, and agrees with eq. (56.23) for $\mu = i$ and $\nu = j$.

Putting all this together, we can now write eq. (56.16) as

$$\tilde{\Delta}^{\mu\nu}(k) = -\frac{\hat{t}^\mu \hat{t}^\nu}{k^2 + (\hat{t}\cdot k)^2} + \frac{g^{\mu\nu} + \hat{t}^\mu \hat{t}^\nu - \hat{z}^\mu \hat{z}^\nu}{k^2 - i\epsilon} \ . \tag{56.25}$$

The next step is to consider the terms in this expression that contain factors of k^μ or k^ν; from eq. (56.22), we see that these will arise from the $\hat{z}^\mu \hat{z}^\nu$ term.

In eq. (56.15), a factor of k^μ can be written as a derivative with respect to x^μ acting on $e^{ik(x-y)}$.

This derivative can then be integrated by parts in eq. (56.14) to give a factor of $\partial^\mu J_\mu(x)$.

But $\partial^\mu J_\mu(x)$ vanishes, because the current must be conserved.

Similarly, a factor of k^ν can be turned into $\partial^\nu J_\nu(y)$, and also leads to a vanishing contribution.

Therefore, we can *ignore any terms* in $\tilde{\Delta}^{\mu\nu}(k)$ *that contain factors* of k^μ or k^ν.

From eq. (56.22), we see that this means we can make the substitution

$$\hat{z}^\mu \to \frac{(\hat{t}\cdot k)\hat{t}^\mu}{[k^2+(\hat{t}\cdot k)^2]^{1/2}} \,. \tag{56.26}$$

Then eq. (56.25) becomes

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{1}{k^2-i\epsilon}\left[g^{\mu\nu} + \left(-\frac{k^2}{k^2+(\hat{t}\cdot k)^2} + 1 - \frac{(\hat{t}\cdot k)^2}{k^2+(\hat{t}\cdot k)^2}\right)\hat{t}^\mu\hat{t}^\nu\right], \tag{56.27}$$

where the three coefficients of $\hat{t}^\mu\hat{t}^\nu$ come from the Coulomb term, the $\hat{t}^\mu\hat{t}^\nu$ term in the polarization sum, an $\hat{z}^\mu\hat{z}^\nu$ term, respectively.

A bit of algebra now reveals that the net coefficient of $\hat{t}^\mu\hat{t}^\nu$ vanishes, leaving us with the elegant expression

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu}}{k^2-i\epsilon} \,. \tag{56.28}$$

Written in this way, the photon propagator is said to be in Feynman gauge.

(It would still be in Coulomb gauge if we had retained the k^μ and k^ν terms that we previously dropped.)

In the next section, we will rederive eq. (56.28) from a more explicit path-integral point of view.

57 The Path Integral for Photons

In this section, in order to get a better understanding of the photon path integral, we will evaluate it directly, using the methods of section 8.

We begin with

$$Z_0(J) = \int \mathcal{D}A \, e^{iS_0} \, , \tag{57.1}$$

$$S_0 = \int d^4x \left[-\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu \right] \, . \tag{57.2}$$

Following section 8, we Fourier-transform to momentum space, where we find

$$\begin{aligned} S_0 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \Big[& -\widetilde{A}_\mu(k) \Big(k^2 g^{\mu\nu} - k^\mu k^\nu \Big) \widetilde{A}_\nu(-k) \\ & + \widetilde{J}^\mu(k) \widetilde{A}_\mu(-k) + \widetilde{J}^\mu(-k) \widetilde{A}_\mu(k) \Big] \end{aligned} \tag{57.3}$$

The next step is to shift the integration variable $\widetilde{A}$ so as to "complete the square".

This involves inverting the 4×4 matrix $k^2 g^{\mu\nu} - k^\mu k^\nu$.

However, this matrix has a zero eigenvalue, and cannot be inverted.

To see this, let us write

$$k^2 g^{\mu\nu} - k^\mu k^\nu = k^2 P^{\mu\nu}(k) \, , \tag{57.4}$$

where we have defined

$$P^{\mu\nu}(k) \equiv g^{\mu\nu} - k^\mu k^\nu / k^2 \ . \tag{57.5}$$

This is a projection matrix because, as is easily checked,

$$P^{\mu\nu}(k) P_\nu{}^\lambda(k) = P^{\mu\lambda}(k) \ . \tag{57.6}$$

Thus the only allowed eigenvalues of P are zero and one.

There is at least one zero eigenvalue, because

$$P^{\mu\nu}(k) k_\nu = 0 \ . \tag{57.7}$$

On the other hand, the sum of the eigenvalues is given by the trace

$$g_{\mu\nu} P^{\mu\nu}(k) = 3 \ . \tag{57.8}$$

Thus the remaining three eigenvalues must all be one.

Now let us imagine carrying out the path integral of eq. (57.1), with S_0 given by eq. (57.3).

Let us decompose the field $\widetilde{A}_\mu(k)$ into components aligned along a set of linearly independent four-vectors, one of which is k^μ.

(It will not matter whether or not this basis set is orthonormal.)

Because the term quadratic in $\widetilde{A}_\mu$ involves the matrix $k^2P^{\mu\nu}(k)$, and $P^{\mu\nu}(k)k_\nu = 0$, the component of $\widetilde{A}_\mu(k)$ that lies along k_μ does not contribute to this either, because $\partial^\mu J_\mu(x) = 0$ implies $k^\mu J_\mu(k) = 0$.

Thus this component does not appear in the path integral at all!

It then makes no sense to integrate over it.

We therefore define $\int \mathcal{D}A$ to mean integration over only those components that are spanned by the remaining three basis vectors, and therefore satisfy $k^\mu \widetilde{A}_\mu(k) = 0$.

This is equivalent to imposing Lorenz gauge, $\partial^\mu A_\mu(x) = 0$.

The matrix $P^{\mu\nu}(k)$ is simply the matrix that projects a four-vector into the subspace orthogonal to k_μ.

Within the subspace, $P^{\mu\nu}(k)$ is equivalent to the identity matrix.

Therefore, within the subspace, the inverse of $k^2P^{\mu\nu}(k)$ is $(1/k^2)P^{\mu\nu}(k)$.

Employing the ϵ trick to pick out vacuum boundary conditions replaces k^2 with $k^2 - i\epsilon$.

We can now continue following the procedure of section 8, with the result that

$$\begin{aligned} Z_0(J) &= \exp\left[\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\, \widetilde{J}_\mu(k)\, \frac{P^{\mu\nu}(k)}{k^2 - i\epsilon}\, \widetilde{J}_\nu(-k)\right] \\ &= \exp\left[\frac{i}{2}\int d^4x\, d^4y\, J_\mu(x)\Delta^{\mu\nu}(x-y)J_\nu(y)\right], \end{aligned} \tag{57.9}$$

where

$$\Delta^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, \frac{P^{\mu\nu}(k)}{k^2 - i\epsilon} \tag{57.10}$$

is the photon propagator in *Lorenz gauge* (also known as *Landau gauge*).

Of course, because the current is conserved, the $k^\mu k^\nu$ term in $P^{\mu\nu}(k)$ does not contribute, and so the result is equivalent to that of Feynman gauge, where $P^{\mu\nu}(k)$ is replaced by $g^{\mu\nu}$.

58 Spinor Electrodynamics

In the section, we will study spinor electrodynamics: the theory of photons interacting with the electrons and positrons of a Dirac field.

(We will use the term quantum electrodynamics to denote any theory of photons, irrespective of the kinds of particles with which they interact.)

We construct spinor electrodynamics by taking the electromagnetic current $j^\mu(x)$ to be proportional to the Noether current corresponding to the U(1) symmetry of a Dirac field; see section 36.

Specifically,

$$j^{\mu}(x) = e\overline{\Psi}(x)\gamma^{\mu}\Psi(x)\ . \tag{58.1}$$

Here e = –0.302822 is the charge of the electron in Heaviside-Lorentz units, with $\hbar$ = c = 1.

(We will rely on context to distinguish this e from the base of natural logarithms.)

In these units, the fine-structure constant is $\alpha = e^2/4\pi = 1/137.036$.

With the normalization of eq. (58.1), $Q = \int d^3x\, j^0(x)$ is the electric charge operator.

Of course, when we specify a number in quantum field theory, we must always have a renormalization scheme in mind; e= –0.302822 corresponds to a specific version of on-shell renormalization that we will explore in sections 62 and 63.

The value of e is different in other renormalization schemes, such as $\overline{MS}$, as we will see in section 66.

The complete lagrangian of our theory is thus

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi + e\overline{\Psi}\gamma^{\mu}\Psi A_{\mu}\ . \tag{58.2}$$

In this section, we will be concerned with tree-level processes only, and so we omit renormalizing Z factors.

We have a problem, though.

A Noether current is conserved only when the fields obey the equations of motion, or, equivalently, only at points in field space where the action is stationary.

On the other hand, in our development of photon path integrals in sections 56 and 57, we assumed that the current was always conserved.

This issue is resolved by enlarging the definition of a gauge transformation to include a transformation on the Dirac field as well as the electromagnetic field.

Specifically, we define a gauge transformation to consist of

$$A^\mu(x) \to A^\mu(x) - \partial^\mu \Gamma(x) \; , \tag{58.3}$$

$$\Psi(x) \to \exp[-ie\Gamma(x)]\Psi(x) \; , \tag{58.4}$$

$$\overline{\Psi}(x) \to \exp[+ie\Gamma(x)]\overline{\Psi}(x) \; . \tag{58.5}$$

It is not hard to check that $\mathcal{L}$(x) is *invariant* under this transformation, whether or not the fields obey their equations of motion.

To perform this check most easily, we first rewrite $\mathcal{L}$ as

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\Psi}\not{D}\Psi - m\overline{\Psi}\Psi \; , \tag{58.6}$$

where we have defined the *gauge covariant derivative* (or just *covariant derivative* for short)

$$D_\mu \equiv \partial_\mu - ieA_\mu \; . \tag{58.7}$$

In the last section, we found that $F^{\mu\nu}$ is invariant under eq. (58.3), and so the FF term in $\mathcal{L}$ is obviously invariant as well.

It is also obvious that the m $\overline{\Psi}\Psi$ term in L is invariant under eqs. (58.4) and (58.5).

This leaves the $\overline{\Psi}\not{D}\Psi$ [1] term.

This term will also be invariant if, under the gauge transformation, the covariant derivative of Ψ transforms as

$$D_\mu \Psi(x) \to \exp[-ie\Gamma(x)] D_\mu \Psi(x) \ . \qquad (58.8)$$

To see if this is true, we note that

$$\begin{aligned} D_\mu \Psi &\to \Big(\partial_\mu - ie[A_\mu - \partial_\mu \Gamma]\Big)\Big(\exp[-ie\Gamma]\Psi\Big) \\ &= \exp[-ie\Gamma]\Big(\partial_\mu \Psi - ie(\partial_\mu \Gamma)\Psi - ie[A_\mu - \partial_\mu \Gamma]\Psi\Big) \\ &= \exp[-ie\Gamma]\Big(\partial_\mu - ieA_\mu\Big)\Psi \qquad (58.9) \\ &= \exp[-ie\Gamma] D_\mu \Psi \ . \end{aligned}$$

So eq. (58.8) holds, and $\overline{\Psi}\not{D}\Psi$ [1] is gauge invariant.

We can also write the transformation rule for D_μ a little more abstractly as

$$D_\mu \to e^{-ie\Gamma} D_\mu \, e^{+ie\Gamma} \; , \tag{58.10}$$

where the ordinary derivative in D_μ is defined to act on anything to its right, including any fields that are left unwritten in eq. (58.10).

Thus we have

$$\begin{aligned} D_\mu \Psi &\to \left(e^{-ie\Gamma} D_\mu e^{+ie\Gamma}\right)\left(e^{-ie\Gamma}\Psi\right) \\ &= e^{-ie\Gamma} D_\mu \Psi \; , \end{aligned} \tag{58.11}$$

which is, of course, the same as eq. (58.9).

We can also express the field strength in terms of the covariant derivative by noting that

$$[D^\mu, D^\nu]\Psi(x) = -ieF^{\mu\nu}(x)\Psi(x) \; . \tag{58.12}$$

We can write this more abstractly as

$$F^{\mu\nu} = \frac{i}{e}[D^\mu, D^\nu] \; , \tag{58.13}$$

where, again, the ordinary derivative in each covariant derivative acts on anything to its right.

From eqs. (58.10) and (58.13), we see that, under a gauge transformation,

$$
\begin{aligned}
F^{\mu\nu} &\to \frac{i}{e}\Big[e^{-ie\Gamma}D^{\mu}\,e^{+ie\Gamma}, e^{-ie\Gamma}D^{\nu}\,e^{+ie\Gamma}\Big] \\
&= e^{-ie\Gamma}\Big(\frac{i}{e}[D^{\mu}, D^{\nu}]\Big)e^{+ie\Gamma} \\
&= e^{-ie\Gamma}F^{\mu\nu}e^{+ie\Gamma} \qquad (58.14) \\
&= F^{\mu\nu}\ .
\end{aligned}
$$

In the last line, we are able to cancel the $e^{\pm ie\Gamma}$ factors against each other because no derivatives act on them.

Eq. (58.14) shows us that (as we already knew) $F^{\mu\nu}$ is gauge invariant.

It is interesting to note that the gauge transformation on the fermion fields, eqs. (58.4–58.5), is a generalization of the U(1) transformation

$$\Psi \to e^{-i\alpha}\Psi\ , \qquad (58.15)$$

$$\overline{\Psi} \to e^{+i\alpha}\overline{\Psi}\ , \qquad (58.16)$$

that is a symmetry of the free Dirac lagrangian.

The difference is that, in the gauge transformation, the phase factor is allowed to be a function of spacetime, rather than a constant that is the same everywhere.

Thus, the gauge transformation is also called a local U(1) transformation, while eqs. (58.15–58.16) correspond to a *global* U(1) transformation.

We say that, in a gauge theory, the global U(1) symmetry is promoted to a local U(1) symmetry, or that we have *gauged* the U(1) symmetry.

In section 57, we argued that the path integral over A_μ should be restricted to those components of $\widetilde{A}_\mu(k)$ that are orthogonal to k_μ, because the component parallel to k_μ did not appear in the integrand.

Now we must make a slightly more subtle argument.

We argue that the path integral over the parallel component is redundant, because the fermionic path integral over Ψ and $\overline{\Psi}$ already includes all possible values of $\Gamma(x)$.

Therefore, as in section 57, we should not integrate over the parallel component. (We will make a more precise and careful version of this argument when we discuss the quantization of nonabelian gauge theories in section 71.)

By the standard procedure, this leads us to the following form of the path integral for spinor electrodynamics:

$$Z(\overline{\eta},\eta,J) \propto \exp\left[ie\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)\left(i\frac{\delta}{\delta\eta_\alpha(x)}\right)(\gamma^\mu)_{\alpha\beta}\left(\frac{1}{i}\frac{\delta}{\delta\overline{\eta}_\beta(x)}\right)\right] \times Z_0(\overline{\eta},\eta,J)\ , \tag{58.17}$$

where

$$Z_0(\overline{\eta},\eta,J) = \exp\left[i\int d^4x\, d^4y\, \overline{\eta}(x)S(x-y)\eta(y)\right]$$
$$\times \exp\left[\frac{i}{2}\int d^4x\, d^4y\, J^\mu(x)\Delta_{\mu\nu}(x-y)J^\nu(y)\right], \tag{58.18}$$

and

$$S(x-y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{(-\not{p}+m)}{p^2+m^2-i\epsilon}\, e^{ip(x-y)}\ , \tag{58.19}$$

$$\Delta_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4}\, \frac{g_{\mu\nu}}{k^2-i\epsilon}\, e^{ik(x-y)} \tag{58.20}$$

are the appropriate Feynman propagators for the corresponding free fields, with the photon propagator in Feynman gauge.

We impose the normalization Z(0,0,0) = 1, and write

$$Z(\overline{\eta},\eta,J) = \exp[iW(\overline{\eta},\eta,J)]\ . \tag{58.21}$$

Then iW $(\overline{\eta},\eta,J)$ can be expressed as a series of connected Feynman diagrams with sources.

The rules for internal and external Dirac fermions were worked out in the context of Yukawa theory in section 45, and they follow here with no change.

For external photons, the LSZ analysis of section 56 implies that each external photon line carries a factor of the polarization vector $\varepsilon^{\mu}(\mathbf{k})$.

Putting everything together, we get the following set of Feynman rules for tree-level processes in spinor electrodynamics.

1. For each *incoming electron*, draw a solid line with an arrow pointed *towards* the vertex, and label it with the electron's four-momentum, p_i.

2. For each *outgoing electron*, draw a solid line with an arrow pointed *away* from the vertex, and label it with the electron's four-momentum, p_i'.

3. For each *incoming positron*, draw a solid line with an arrow pointed *away* from the vertex, and label it with *minus* the positron's four-momentum, $-p_i$.

4. For each *outgoing positron*, draw a solid line with an arrow pointed *towards* the vertex, and label it with *minus* the positron's four-momentum, $-p_i'$.

5. For each *incoming photon*, draw a wavy line with an arrow pointed *towards* the vertex, and label it with the photon's four-momentum, k_i. (Wavy lines for photons is a standard convention.)

6. For each *outgoing photon*, draw a wavy line with an arrow pointed *away* from the vertex, and label it with the photon's four-momentum, k'_i.

7. The only allowed vertex joins two solid lines, one with an arrow pointing towards it and one with an arrow pointing away from it, and one wavy line (whose arrow can point in either direction). Using this vertex, join up all the external lines, including extra internal lines as needed. In this way, draw all possible diagrams that are *topologically inequivalent.*

8. Assign each internal line its own four-momentum. Think of the four-momenta as flowing along the arrows, and conserve four-momentum at each vertex. For a tree diagram, this fixes the momenta on all the internal lines.

9. The value of a diagram consists of the following factors:

   for each incoming photon, $\varepsilon^{\mu *}_{\lambda_i}(\mathbf{k}_i)$;

   for each outgoing photon, $\varepsilon^{\mu}_{\lambda'_i}(\mathbf{k}'_i)$;

   for each incoming electron, $u_{s_i}(\mathbf{p}_i)$;

   for each outgoing electron, $\overline{u}_{s'_i}(\mathbf{p}'_i)$;

   for each incoming positron, $\overline{v}_{s_i}(\mathbf{p}_i)$;

   for each outgoing positron, $v_{s'_i}(\mathbf{p}'_i)$;

   for each vertex, $ie\gamma^{\mu}$;

   for each internal photon, $-ig^{\mu\nu}/(k^2 - i\epsilon)$;

   for each internal fermion, $-i(-\not{p} + m)/(p^2 + m^2 - i\epsilon)$.

10. Spinor indices are contracted by starting at one end of a fermion line: specifically, the end that has the arrow pointing away from the vertex. The factor associated with the external line is either $\overline{u}$ or $\overline{v}$.

    Go along the complete fermion line, following the arrows backwards, and write down (in order from left to right) the factors associated with the vertices and propagators that you encounter. The last factor is either a u or v. Repeat this procedure for the other fermion lines, if any. The vector index on each vertex is contracted with the vector index on either the photon propagator (if the attached photon line is internal) or the photon polarization vector (if the attached photon line is external).

11. The overall sign of a tree diagram is determined by drawing all contributing diagrams in a standard form: all fermion lines horizontal, with their arrows pointing from left to right, and with the left endpoints labeled in the same fixed order (from top to bottom); if the ordering of the labels on the right endpoints of the fermion lines in a given diagram is an even (odd) permutation of an arbitrarily chosen fixed ordering, then the sign of that diagram is positive (negative).

12. The value of $i\mathcal{T}$ (at tree level) is given by a sum over the values of all the contributing diagrams.

In the next section, we will do a sample calculation.

59 Scattering in Spinor Electrodynamics

In the last section, we wrote down the Feynman rules for spinor electrodynamics.

In this section, we will compute the scattering amplitude (and its spin-averaged square) at tree level for the process of electron-positron annihilation into a pair of photons, $e^+ e^- \to \gamma\gamma$.

The contributing diagrams are shown in fig. (59.1), and the associated expression for the scattering amplitude is

$$\mathcal{T} = e^2\, \varepsilon^{\mu}_{1'} \varepsilon^{\nu}_{2'}\, \overline{v}_2 \left[\gamma_\nu \left(\frac{-\not{p}_1 + \not{k}'_1 + m}{-t + m^2} \right) \gamma_\mu + \gamma_\mu \left(\frac{-\not{p}_1 + \not{k}'_2 + m}{-u + m^2} \right) \gamma_\nu \right] u_1 \,, \tag{59.1}$$

where $\varepsilon^{\mu}{}_{1'}$ is shorthand for $\varepsilon^{\mu}_{\lambda'_1}(\mathbf{k}'_1)$, $\overline{v}_2$ is shorthand for $\overline{v}_{s_2}(\mathbf{p}_2)$, and so on.

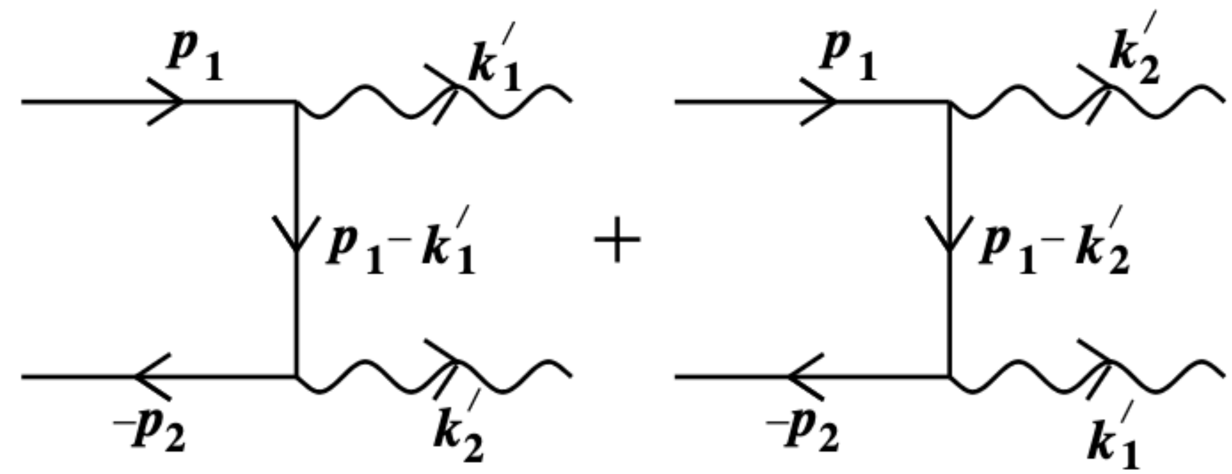


Figure 59.1: Diagrams for $e^+ e^- \to \gamma\gamma$, corresponding to eq. (59.1).

The Mandelstam variables are

$$\begin{aligned} s &= -(p_1 + p_2)^2 = -(k'_1 + k'_2)^2 \,, \\ t &= -(p_1 - k'_1)^2 = -(p_2 - k'_2)^2 \,, \\ u &= -(p_1 - k'_2)^2 = -(p_2 - k'_1)^2 \,, \end{aligned} \tag{59.2}$$

and they obey $s + t + u = 2m^2$

Following the procedure of section 46, we write eq. (59.1) as

$$\mathcal{T} = \varepsilon^{\mu}_{1'}\varepsilon^{\nu}_{2'}\,\overline{v}_2 A_{\mu\nu} u_1 \ , \tag{59.3}$$

where

$$A_{\mu\nu} \equiv e^2\Big[\gamma_\nu\Big(\frac{-\not{p}_1+\not{k}'_1+m}{-t+m^2}\Big)\gamma_\mu + \gamma_\mu\Big(\frac{-\not{p}_1+\not{k}'_2+m}{-u+m^2}\Big)\gamma_\nu\Big] . \tag{59.4}$$

We also have

$$\mathcal{T}^* = \overline{\mathcal{T}} = \varepsilon^{\rho*}_{1'}\,\varepsilon^{\sigma*}_{2'}\,\overline{u}_1\,\overline{A_{\rho\sigma}}\,v_2 \ . \tag{59.5}$$

Using $\overline{\not{a}\not{b}\ldots} = \ldots\not{b}\not{a}$, we see from eq. (59.4) that

$$\overline{A_{\rho\sigma}} = A_{\sigma\rho} \ . \tag{59.6}$$

Thus we have

$$|\mathcal{T}|^2 = \varepsilon^{\mu}_{1'}\varepsilon^{\nu}_{2'}\,\varepsilon^{\rho*}_{1'}\,\varepsilon^{\sigma*}_{2'}\,(\overline{v}_2 A_{\mu\nu} u_1)(\overline{u}_1 A_{\sigma\rho} v_2) \ . \tag{59.7}$$

Next, we will average over the initial electron and positron spins, using the technology of section 46; the result is

$$\tfrac{1}{4}\sum_{s_1,s_2}|\mathcal{T}|^2 = \tfrac{1}{4}\varepsilon^{\mu}_{1'}\varepsilon^{\nu}_{2'}\varepsilon^{\rho*}_{1'}\,\varepsilon^{\sigma*}_{2'}\,\mathrm{Tr}\Big[A_{\mu\nu}(-\not{p}_1{+}m)A_{\sigma\rho}(-\not{p}_2{-}m)\Big] \ . \tag{59.8}$$

We would also like to sum over the final photon polarizations.

From eq. (59.8), we see that we must evaluate

$$\sum_{\lambda=\pm} \varepsilon^{\mu}_{\lambda}(\mathbf{k})\varepsilon^{\rho*}_{\lambda}(\mathbf{k}) \ . \tag{59.9}$$

We did this polarization sum in Coulomb gauge in section 56, with the result that

$$\sum_{\lambda=\pm} \varepsilon^{\mu}_{\lambda}(\mathbf{k})\varepsilon^{\rho*}_{\lambda}(\mathbf{k}) = g^{\mu\rho} + \hat{t}^{\mu}\hat{t}^{\rho} - \hat{z}^{\mu}\hat{z}^{\rho} \ , \tag{59.10}$$

where $\hat{t}^{\mu}$ is a unit vector in the time direction, and $\hat{z}^{\mu}$ is a unit vector in the **k** direction that can be expressed as

$$\hat{z}^{\mu} = \frac{k^{\mu} + (\hat{t}\cdot k)\hat{t}^{\mu}}{[k^2 + (\hat{t}\cdot k)^2]^{1/2}} \ . \tag{59.11}$$

It is tempting to drop the k^{μ} and k^{ρ} terms in eq. (59.10), on the grounds that the photons couple to a conserved current, and so these terms should not contribute. (We indeed used this argument to drop the analogous terms in the photon propagator.)

This also follows from the notion that the scattering amplitude should be invariant under a gauge transformation, as represented by a transformation of the external polarization vectors of the form

$$\varepsilon^{\mu}_{\lambda}(\mathbf{k}) \rightarrow \varepsilon^{\mu}_{\lambda}(\mathbf{k}) - i\tilde{\Gamma}(k)k^{\mu} \ . \tag{59.12}$$

Thus, if we write a scattering amplitude $\mathcal{T}$ for a process that includes a particular outgoing photon with four-momentum k^μ as

$$\mathcal{T} = \varepsilon^{\mu}_{\lambda}(\mathbf{k})\mathcal{M}_\mu \ , \tag{59.13}$$

or a particular incoming photon with four-momentum k^μ as

$$\mathcal{T} = \varepsilon^{\mu *}_{\lambda}(\mathbf{k})\mathcal{M}_\mu \ , \tag{59.14}$$

then in either case we should have

$$k^\mu \mathcal{M}_\mu = 0 \ . \tag{59.15}$$

Eq. (59.15) is in fact valid; we will give a proof of it, based on the Ward identity for the electromagnetic current, in section 67.

For now, we will take eq. (59.15) as given, and so drop the k^μ and k^ρ terms in eq. (59.10).

This leaves us with

$$\sum_{\lambda=\pm} \varepsilon^{\mu}_{\lambda}(\mathbf{k})\varepsilon^{\rho *}_{\lambda}(\mathbf{k}) \to g^{\mu\rho} + \hat{t}^\mu \hat{t}^\rho - \frac{(\hat{t}\cdot k)^2}{k^2 + (\hat{t}\cdot k)^2}\,\hat{t}^\mu \hat{t}^\rho \ . \tag{59.16}$$

But, for an external photon, $k^2 = 0$.

the the $(\overline{\eta}, \eta, J)$ $\overline{v}$

Thus the second and third terms in eq. (59.16) cancel, leaving us with the beautifully simple substitution rule

$$\sum_{\lambda=\pm} \varepsilon^{\mu}_{\lambda}(\mathbf{k})\varepsilon^{\rho*}_{\lambda}(\mathbf{k}) \to g^{\mu\rho} \; . \tag{59.17}$$

Using eq. (59.17), we can sum $|\mathcal{T}|^2$ over the polarizations of the outgoing photons, in addition to averaging over the spins of the incoming fermions; the result is

$$\begin{aligned}
\langle|\mathcal{T}|^2\rangle &\equiv \tfrac{1}{4}\sum_{\lambda_1',\lambda_2'}\sum_{s_1,s_2}|\mathcal{T}|^2 \\
&= \tfrac{1}{4}\mathrm{Tr}\Big[A_{\mu\nu}(-\not{p}_1+m)A^{\nu\mu}(-\not{p}_2-m)\Big] \\
&= e^4\left[\frac{\langle\Phi_{tt}\rangle}{(m^2-t)^2}+\frac{\langle\Phi_{tu}\rangle+\langle\Phi_{ut}\rangle}{(m^2-t)(m^2-u)}+\frac{\langle\Phi_{uu}\rangle}{(m^2-u)^2}\right],
\end{aligned} \tag{59.18}$$

where

$$\begin{aligned}
\langle\Phi_{tt}\rangle &= \tfrac{1}{4}\mathrm{Tr}\Big[\gamma_\nu(-\not{p}_1+\not{k}_1'+m)\gamma_\mu(-\not{p}_1+m)\gamma^\mu(-\not{p}_1+\not{k}_1'+m)\gamma^\nu(-\not{p}_2-m)\Big] \; , \\
\langle\Phi_{uu}\rangle &= \tfrac{1}{4}\mathrm{Tr}\Big[\gamma_\mu(-\not{p}_1+\not{k}_2'+m)\gamma_\nu(-\not{p}_1+m)\gamma^\nu(-\not{p}_1+\not{k}_2'+m)\gamma^\mu(-\not{p}_2-m)\Big] \; , \\
\langle\Phi_{tu}\rangle &= \tfrac{1}{4}\mathrm{Tr}\Big[\gamma_\nu(-\not{p}_1+\not{k}_1'+m)\gamma_\mu(-\not{p}_1+m)\gamma^\nu(-\not{p}_1+\not{k}_2'+m)\gamma^\mu(-\not{p}_2-m)\Big] \; , \\
\langle\Phi_{ut}\rangle &= \tfrac{1}{4}\mathrm{Tr}\Big[\gamma_\mu(-\not{p}_1+\not{k}_2'+m)\gamma_\nu(-\not{p}_1+m)\gamma^\mu(-\not{p}_1+\not{k}_1'+m)\gamma^\nu(-\not{p}_2-m)\Big] \; .
\end{aligned} \tag{59.19}$$

Examinging $\langle\Phi_{tt}\rangle$ and $\langle\Phi_{uu}\rangle$, we see that they are transformed into each other by $k'1 \leftrightarrow k'2$, which is equivalent to $t \leftrightarrow u$.

The same is true of $\langle\Phi_{tu}\rangle$ and $\langle\Phi ut\rangle$.

Thus we need only compute $\langle\Phi_{tt}\rangle$ and $\langle\Phi_{tu}\rangle$, and then take $t \leftrightarrow u$ to get $\langle\Phi_{uu}\rangle$ and $\langle\Phi_{ut}\rangle$.

Now we can apply the gamma-matrix technology of section 47.

In particular, we will need the $d = 4$ relations

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= -4 \,, \\
\gamma^\mu \not{a} \gamma_\mu &= 2\not{a} \,, \\
\gamma^\mu \not{a}\not{b} \gamma_\mu &= 4(ab) \,, \\
\gamma^\mu \not{a}\not{b}\not{c} \gamma_\mu &= 2\not{c}\not{b}\not{a} \,,
\end{aligned} \tag{59.20}$$

in addition to the trace formulae.

We also need

$$\begin{aligned}
p_1p_2 &= -\tfrac{1}{2}(s-2m^2) \,, \\
k_1'k_2' &= -\tfrac{1}{2}s \,, \\
p_1k_1' = p_2k_2' &= +\tfrac{1}{2}(t-m^2) \,, \\
p_1k_2' = p_2k_1' &= +\tfrac{1}{2}(u-m^2) \,.
\end{aligned} \tag{59.21}$$

which follow from eq. (59.2) plus the mass-shell conditions $p^2_1 = p^2_2 = -m^2$ and $k'^2_1 = k'^2_2 = 0$.

After a lengthy and tedious calculation, we find

$$\langle\Phi_{tt}\rangle = 2[tu - m^2(3t+u) - m^4]\ , \tag{59.22}$$

$$\langle\Phi_{tu}\rangle = 2m^2(s - 4m^2)\ , \tag{59.23}$$

which then implies

$$\langle\Phi_{uu}\rangle = 2[tu - m^2(3u+t) - m^4]\ , \tag{59.24}$$

$$\langle\Phi_{ut}\rangle = 2m^2(s - 4m^2)\ . \tag{59.25}$$

This completes our calculation.

Other tree-level scattering processes in spinor electrodynamics pose no new calculational diffculties.

In the high-energy limit, where the electron can be treated as massless, we can reduce our labor with the method of spinor helicity,

60 Spinor Helicity for Spinor Electrodynamics

In section 50, we introduced a special notation for u and v spinors of definite helicity for massless electrons and positrons.

This notation greatly simplifies calculations in the high-energy limit (s, |t|, and |u| all much greater than m^2).

We define the twistors

$$\begin{aligned} |p] &\equiv u_-(p) = v_+(p)\ , \\ |p\rangle &\equiv u_+(p) = v_-(p)\ , \\ [p| &\equiv \overline{u}_+(p) = \overline{v}_-(p)\ , \\ \langle p| &\equiv \overline{u}_-(p) = \overline{v}_+(p)\ . \end{aligned} \tag{60.1}$$

We then have

$$\begin{aligned} [k|\ |p] &= [k\,p]\ , \\ \langle k|\ |p\rangle &= \langle k\,p\rangle\ , \\ [k|\ |p\rangle &= 0\ , \\ \langle k|\ |p] &= 0\ , \end{aligned} \tag{60.2}$$

where the *twistor products* [kp] and ⟨kp⟩ are antisymmetric,

$$\begin{aligned} [k\,p] &= -[p\,k]\ , \\ \langle k\,p\rangle &= -\langle p\,k\rangle\ , \end{aligned} \tag{60.3}$$

and related by complex conjugation, $\langle pk\rangle^* = [kp]$.

They can be expressed explicitly in terms of the components of the massless four-momenta k and p.

However, more useful are the relations

$$\begin{aligned} \langle k\,p\rangle\,[p\,k] &= \mathrm{Tr}\,\tfrac{1}{2}(1-\gamma_5)\not{k}\not{p} \\ &= -2k\cdot p \\ &= -(k+p)^2 \end{aligned} \tag{60.4}$$

and

$$\begin{aligned} \langle p\,q\rangle\,[q\,r]\,\langle r\,s\rangle\,[s\,p] &= \mathrm{Tr}\,\tfrac{1}{2}(1-\gamma_5)\not{p}\not{q}\not{r}\not{s} \\ &= 2[(p\cdot q)(r\cdot s) - (p\cdot r)(q\cdot s) + (p\cdot s)(q\cdot r) \\ &\qquad + i\varepsilon^{\mu\nu\rho\sigma}p_\mu q_\nu r_\rho s_\sigma]\ . \end{aligned} \tag{60.5}$$

Finally, for any massless four-momentum p we can write

$$-\not{p} = |p\rangle[p| + |p]\langle p|\ . \tag{60.6}$$

We will quote other results from section 50 as we need them.

To apply this formalism to spinor electrodynamics, we need to write photon polarization vectors in terms of twistors.

The formulae we need are

$$\varepsilon^\mu_+(k) = -\frac{\langle q|\gamma^\mu|k]}{\sqrt{2}\,\langle q\,k\rangle}\ , \tag{60.7}$$

$$\varepsilon^\mu_-(k) = -\frac{[q|\gamma^\mu|k\rangle}{\sqrt{2}\,[q\,k]}\ , \tag{60.8}$$

where q is an arbitrary massless *reference momentum*.

We will verify eqs. (60.7) and (60.8) for a specific choice of k, and then rely on the Lorentz transformation properties of twistors to conclude that the result must hold in any frame (and therefore for any massless four-momentum k).

We will choose $k^\mu = (\omega, \omega\hat{\mathbf{z}}) = \omega(1,0,0,1)$.

Then, the most general form of $\varepsilon^\mu{}_+(k)$ is

$$\varepsilon^\mu_+(k) = e^{i\phi}\tfrac{1}{\sqrt{2}}(0,1,-i,0) + Ck^\mu \ . \tag{60.9}$$

Here $e^{i\phi}$ is an arbitrary phase factor, and C is an arbitrary complex number; the freedom to add a multiple of k comes from the underlying gauge invariance.

To verify that eq. (60.7) reproduces eq. (60.9), we need the explicit form of the twistors $|k]$ and $|k\rangle$ when the three-momentum is in the z direction.

Using results in section 50 we find

$$|k] = \sqrt{2\omega}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad |k\rangle = \sqrt{2\omega}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}. \tag{60.10}$$

For any value of q, the twistor $\langle q|$ takes the form

$$\langle q| = (0,\ 0,\ \alpha,\ \beta)\ , \tag{60.11}$$

where α and β are complex numbers.

Plugging eqs. (60.10) and (60.11) into eq. (60.7), and using

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{60.12}$$

along with $\sigma^\mu = (I, \vec{\sigma})$ and $\bar{\sigma}^\mu = (I, -\vec{\sigma})$, we find that we reproduce eq. (60.9) with $e^{i\phi} = 1$ and $C = -\beta/(\sqrt{2}\,\alpha\omega)$.

There is now no need to check eq. (60.8), because $\varepsilon^\mu_-(k) = -[\varepsilon^\mu_+(k)]^*$, as can be seen by using $\langle qk\rangle^* = -[qk]$ along with another result from section 50, $\langle q|\gamma^\mu|k]^* = \langle k|\gamma^\mu|q]$.

In spinor electodynamics, the vector index on a photon polarization vector is always contracted with the vector index on a gamma matrix.

We can get a convenient formula for $\not{\varepsilon}_\pm(k)$ by using the Fierz identities

$$-\tfrac{1}{2}\gamma^\mu\langle q|\gamma_\mu|k] = |k]\langle q| + |q\rangle[k| \;, \tag{60.13}$$

$$-\tfrac{1}{2}\gamma^\mu[q|\gamma_\mu|k\rangle = |k\rangle[q| + |q]\langle k| \;. \tag{60.14}$$

We then have

$$\not{\varepsilon}_+(k;q) = \frac{\sqrt{2}}{\langle q\,k\rangle}\Big(|k]\langle q| + |q\rangle[k|\Big), \tag{60.15}$$

$$\not{\varepsilon}_-(k;q) = \frac{\sqrt{2}}{[q\,k]}\Big(|k\rangle[q| + |q]\langle k|\Big), \tag{60.16}$$

where we have added the reference momentum as an explicit argument on the left-hand sides.

Now we have all the tools we need for doing calculations.

However, we can simplify things even further by making maximal use of crossing symmetry.

Note from eq. (60.1) that u_- (which is the factor associated with an incoming electron) and v_+ (an outgoing positron) are both represented by the twistor $|p]$, while $\overline{u}_+$ (an outgoing electron) and $\overline{v}_-$(an incoming positron) are both represented by $[p|$.

Thus the square-bracket twistors correspond to outgoing fermions with positive helicity, and incoming fermions with negative helicity.

Similarly, the angle-bracket twistors correspond to outgoing fermions with negative helicity, and incoming fermions with positive helicity.

Let us adopt a convention in which all particles are assigned four-momenta that are treated as outgoing.

A particle that has an assigned four-momentum p then has physical four-momentum $\epsilon_p p$, where $\epsilon_p = \text{sign}(p^0) = +1$ if the particle is physically outgoing, and $\epsilon_p = \text{sign}(p^0) = -1$ if the particle is physically incoming.

Since the physical three-momentum of an incoming particle is opposite to its assigned three-momentum, a particle with negative helicity relative to its physical three-momentum has positive helicity relative to its assigned three-momentum.

From now on, we will refer to the helicity of a particle relative to its assigned momentum.

Thus a particle that we say has “positive helicity” actually has negative physical helicity if it is incoming, and positive physical helicity if it is outgoing.

With this convention, the square-bracket twistors $|p]$ and $[p|$ represent positive-helicity fermions, and the angle-bracket twistors $|p\rangle$ and $\langle p|$ represent negative-helicity fermions.

When $\epsilon_p = \mathrm{sign}(p^0) = -1$, we analytically continue the twistors by replacing each $\omega^{1/2}$ in eq. (60.10) with $i|\omega|^{1/2}$.

Then all of our formulae for twistors and polarizations hold without change, with the exception of the rule for complex conjugation of a twistor product, which becomes

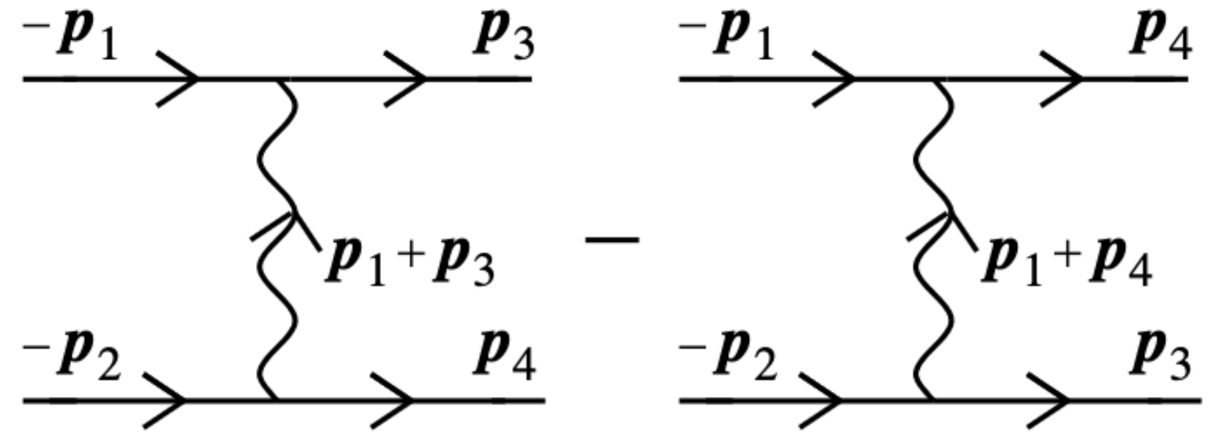
$$\langle p\,k\rangle^* = \epsilon_p \epsilon_k [k\,p]\ . \tag{60.17}$$

Now we are ready to calculate some amplitudes.

Consider first the process of fermion-fermion scattering.

The contributing tree-level diagrams are shown in fig. (60.1).

Figure 60.1: Diagrams for fermion-fermion scattering, with all momenta treated as outgoing.



The first thing to notice is that a diagram is zero if two external fermion lines that meet at a vertex have the same helicity.

This is because (as shown in section 50) we get zero if we sandwich the product of an odd number of gamma matrices between two twistors of the same helicity.

In particular, we have $\langle p|\gamma^\mu|k\rangle = 0$ and $[p|\gamma^\mu|k] = 0$.

Thus, we will get a nonzero result for the tree-level amplitude only if two of the helicities are positive, and two are negative.

This means that, of the $2^4 = 16$ possible combinations of helicities, only six give a nonzero tree-level amplitude: $\mathcal{T}_{++--}$, $\mathcal{T}_{+-+-}$, $\mathcal{T}_{+--+}$, $\mathcal{T}_{--++}$, $\mathcal{T}_{-+-+}$, and $\mathcal{T}_{-++-}$, where the notation is $\mathcal{T}_{s1s2s3s4}$.

Furthermore, the last three of these are related to the first three by complex conjugation, so we only have three amplitudes to compute.

Let us begin with $\mathcal{T}_{+--+}$. Only the first diagram of fig. (60.1) contributes, because the second has two postive-helicity lines meeting at a vertex.

To evaluate the first diagram, we note that the two vertices contribute a factor of $(ie)^2 = -e^2$, and the internal photon line contributes a factor of $ig_{\mu\nu}/s_{13}$, where we have defined the Mandelstam variable

$$s_{ij} \equiv -(p_i + p_j)^2 \ . \tag{60.18}$$

Following the charge arrows backwards on each fermion line, and dividing by i to get $\mathcal{T}$ (rather than i$\mathcal{T}$), we find

$$\begin{aligned} \mathcal{T}_{+--+} &= -e^2 \langle 3|\gamma^\mu|1]\, [4|\gamma_\mu|2\rangle \,/s_{13} \\ &= +2e^2\, [1\,4]\, \langle 2\,3\rangle \,/s_{13} \ , \end{aligned} \tag{60.19}$$

where $\langle 3|$ is short for $\langle p_3|$, etc, and we have used yet another form of the Fierz identity to get the second line.

The computation of $\mathcal{T}_{+-+-}$ is exactly analogous, except that now it is only the second diagram of fig. (60.1) that contributes.

According to the Feynman rules, this diagram comes with a relative minus sign, and so we have

$$\mathcal{T}_{+-+-} = -2e^2\, [1\,3]\, \langle 2\,4\rangle \,/s_{14} \ . \tag{60.20}$$

Finally, we turn to $\mathcal{T}_{++--}$.

Now both diagrams contribute, and we have

$$
\begin{aligned}
\mathcal{T}_{++--} &= -e^2 \left(\frac{\langle 3|\gamma^\mu|1]\,\langle 4|\gamma_\mu|2]}{s_{13}} - \frac{\langle 4|\gamma^\mu|1]\,\langle 3|\gamma_\mu|2]}{s_{14}} \right) \\
&= -2e^2\,[1\,2]\,\langle 3\,4\rangle \left(\frac{1}{s_{13}} + \frac{1}{s_{14}} \right) \\
&= +2e^2\,[1\,2]\,\langle 3\,4\rangle \left(\frac{s_{12}}{s_{13}s_{14}} \right) ,
\end{aligned} \tag{60.21}
$$

where we used the Mandelstam relation $s_{12} + s_{13} + s_{14} = 0$ to get the last line.

To get the cross section for a particular set of helicities, we must take the absolute squares of the amplitudes.

These follow from eqs. (60.4), (60.17), and (60.18) :

$$
|\langle 1\,2\rangle|^2 = |[1\,2]|^2 = \epsilon_1\epsilon_2 s_{12} = |s_{12}| \,. \tag{60.22}
$$

We can then compute the spin-averaged cross section by summing the absolute squares of eqs. (60.19–60.21), multiplying by two to account for the processes in which all helicities are opposite (and which have amplitudes that are related by complex conjugation), and then dividing by four to average over the initial helicities.

Making use of $s_{34} = s1_2$ and its permutations, we find

$$\begin{aligned}\langle|\mathcal{T}|^2\rangle &= 2e^4\left(\frac{s_{14}^2}{s_{13}^2}+\frac{s_{13}^2}{s_{14}^2}+\frac{s_{12}^4}{s_{13}^2 s_{14}^2}\right)\\ &= 2e^4\left(\frac{s_{12}^4+s_{13}^4+s_{14}^4}{s_{13}^2 s_{14}^2}\right).\end{aligned} \tag{60.23}$$

For the processes of $e^-e^- \to e^-e^-$ and $e^+e^+ \to e^+e^+$, we have $s_{12} = s$, $s_{13} = t$, and $s_{14} = u$, for $e^+e^- \to e^+e^-$, we have $s_{13} = s$, $s_{14} = t$, and $s_{12} = u$,

Now we turn to processes with two external fermions and two external photons, as shown in fig. (60.2).

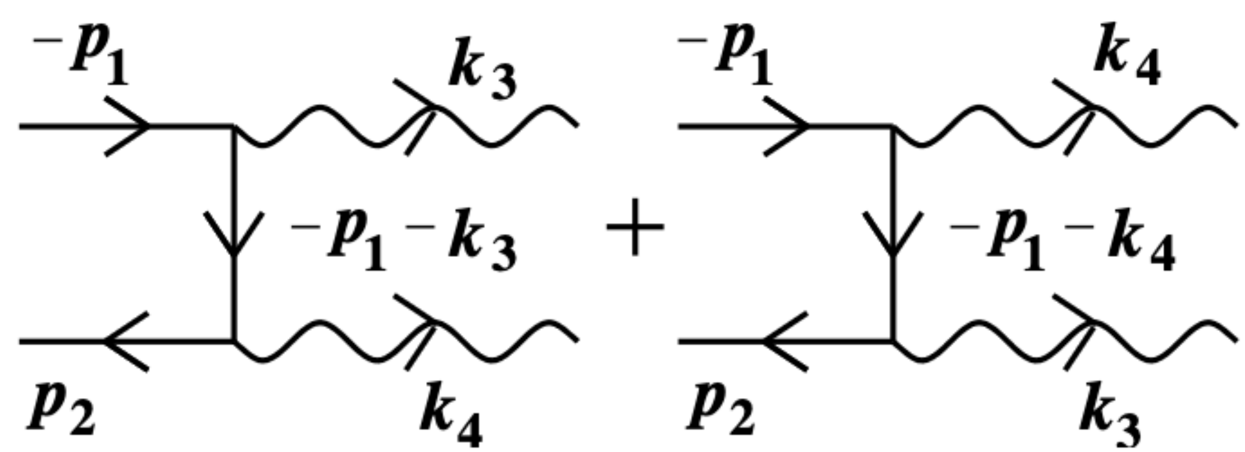


Figure 60.2: Diagrams for fermion-photon scattering, with all momenta treated as outgoing.

The first thing to notice is that a diagram is zero if the two external fermion lines have the same helicity.

This is because the corresponding twistors sandwich an odd number of gamma matrices: one from each vertex, and one from the massless fermion propagator .

Thus we need only compute $\mathcal{T}_{+-\lambda_3\lambda_4}$ since $\mathcal{T}_{-+\lambda_3\lambda_4}$ is related by complex conjugation.

Next we use eqs. (60.15–60.16) and (60.2–60.3) to get

$$\not{\varepsilon}_-(k;p)|p] = 0 \, , \tag{60.24}$$

$$[p|\not{\varepsilon}_-(k;p) = 0 \, . \tag{60.25}$$

$$\not{\varepsilon}_+(k;p)|p\rangle = 0 \, , \tag{60.26}$$

$$\langle p|\not{\varepsilon}_+(k;p) = 0 \, , \tag{60.27}$$

Thus we can get some amplitudes to vanish with appropriate choices of the reference momenta in the photon polarizations.

So, let us consider

$$\begin{aligned} \mathcal{T}_{+-\lambda_3\lambda_4} = & -e^2 \, \langle 2|\not{\varepsilon}_{\lambda_4}(k_4;q_4)(\not{p}_1 + \not{k}_3)\not{\varepsilon}_{\lambda_3}(k_3;q_3)|1] \, / s_{13} \\ & -e^2 \, \langle 2|\not{\varepsilon}_{\lambda_3}(k_3;q_3)(\not{p}_1 + \not{k}_4)\not{\varepsilon}_{\lambda_4}(k_4;q_4)|1] \, / s_{14} \, . \end{aligned} \tag{60.28}$$

If we take $\lambda_3 = \lambda_4 = -$, then we can get both terms in eq. (60.28) to vanish by choosing $q_3 = q_4 = p_1$, and using eq. (60.24).

If we take $\lambda_3 = \lambda_4 = +$, then we can get both terms in eq. (60.28) to vanish by choosing $q_3 = q_4 = p_2$, and using eq. (60.27).

Thus, we need only compute $\mathcal{T}_{+--+}$ and $\mathcal{T}_{+-+-}$.

For $\mathcal{T}_{+-+-}$, we can get the second term in eq. (60.28) to vanish by choosing $q_3 = p_2$, and using eq. (60.27).

Then we have

$$\begin{aligned} \mathcal{T}_{+-+-} &= -e^2 \langle 2| \not{\varepsilon}_-(k_4;q_4)(\not{p}_1 + \not{k}_3)\not{\varepsilon}_+(k_3;p_2)|1] \, /s_{13} \\ &= -e^2 \frac{\sqrt{2}}{[q_4 4]} \langle 2\, 4\rangle \, [q_4|(\not{p}_1 + \not{k}_3)|2\rangle \, [3\, 1] \, \frac{\sqrt{2}}{\langle 2\, 3\rangle} \frac{1}{s_{13}} \, . \end{aligned} \tag{60.29}$$

Next we note that $[p|\not{p} = 0$, and so it is useful to choose either $q_4 = p_1$ or $q_4 = k_3$.

There is no obvious advantage in one choice over the other, and they must give equivalent results, so let us take $q_4 = k_3$.

Then, using eq. (60.6) for $\not{k}_3$, we get

$$\mathcal{T}_{+-+-} = 2e^2 \frac{\langle 2\, 4\rangle \, [3\, 1] \, \langle 1\, 2\rangle \, [3\, 1]}{[3\, 4] \, \langle 2\, 3\rangle \, s_{13}} \tag{60.30}$$

Now we use $[3\ 1]\ \langle 1\ 2\rangle = -[3\ 4]\ \langle 4\ 2\rangle$ in the numerator, and set $s_{13} = \langle 1\ 3\rangle\ [3\ 1]$ in the denominator.

Canceling common factors and using antisymmetry of the twistor product then yields

$$\mathcal{T}_{+-+-} = 2e^2 \, \frac{\langle 2\,4\rangle^2}{\langle 1\,3\rangle\langle 2\,3\rangle} \; . \tag{60.31}$$

We can now get $\mathcal{T}_{+--+}$ simply by exchanging the labels 3 and 4,

$$\mathcal{T}_{+--+} = 2e^2 \, \frac{\langle 2\,3\rangle^2}{\langle 1\,4\rangle\langle 2\,4\rangle} \; . \tag{60.32}$$

We can compute the spin-averaged cross section by summing the absolute squares of eqs. (60.31) and (60.32), multiplying by two to account for the processes in which all helicities are opposite (and which have amplitudes that are related by complex conjugation), and then dividing by four to average over the initial helicities.

The result is

$$\langle |\mathcal{T}|^2 \rangle = 2e^4 \left(\left| \frac{s_{13}}{s_{14}} \right| + \left| \frac{s_{14}}{s_{13}} \right| \right) . \tag{60.33}$$

For the processes of $e^-\gamma \rightarrow e^-\gamma$ and $e^+\gamma \rightarrow e^+\gamma$, we have $s_{13} = s$, $s_{12} = t$, and $s_{14} = u$; for $e^+e^- \rightarrow \gamma\gamma$ and $\gamma\gamma \rightarrow e^+e^-$ we have $s_{12} = s$, $s_{13} = t$, and $s_{14} = u$.

61 Scalar Electrodynamics

In this section, we will consider how charged spin-zero particles interact with photons.

We begin with the lagrangian for a complex scalar field with a quartic interaction,

$$\mathcal{L} = -\partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 \varphi^\dagger \varphi - \tfrac{1}{4}\lambda(\varphi^\dagger \varphi)^2 \ . \tag{61.1}$$

This lagrangian is obviously invariant under the global U(1) symmetry

$$\begin{aligned} \varphi(x) &\to e^{-i\alpha}\varphi(x) \ , \\ \varphi^\dagger(x) &\to e^{+i\alpha}\varphi^\dagger(x) \ . \end{aligned} \tag{61.2}$$

We would like to promote this global symmetry to a local symmetry,

$$\varphi(x) \to \exp[-ie\Gamma(x)]\varphi(x) \ , \tag{61.3}$$

$$\varphi^\dagger(x) \to \exp[+ie\Gamma(x)]\varphi^\dagger(x) \ . \tag{61.4}$$

To do so, we must replace each ordinary derivative in eq. (61.1) with a covariant derivative

$$D_\mu \equiv \partial_\mu - ieA_\mu \ , \tag{61.5}$$

where A_μ transforms as

$$A^\mu(x) \to A^\mu(x) - \partial^\mu\Gamma(x) \ , \tag{61.6}$$

which implies that D_μ transforms as

$$D_\mu \to \exp[-ie\Gamma(x)] D_\mu \exp[+ie\Gamma(x)] \ . \tag{61.7}$$

Our complete lagrangian for scalar electrodynamics is then

$$\mathcal{L} = -(D^\mu\varphi)^\dagger D_\mu\varphi - m^2\varphi^\dagger\varphi - \tfrac{1}{4}\lambda(\varphi^\dagger\varphi)^2 - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \ . \tag{61.8}$$

We have added the usual gauge-invariant kinetic term for the gauge field.

The quartic interaction term has a dimensionless coefficient, and so is necessary for renormalizability.

For now, we omit the renormalizing Z factors.

Of course, eq. (61.8) is invariant under a global U(1) transformation as well as a local U(1) transformation: we simply set $\Gamma(x)$ to a constant.

Then we can find the conserved Noether current corresponding to this symmetry, following the procedure of section 22.

In the case of spinor electrodynamics, this current is same as it is for a free Dirac field, $j^\mu = \overline{\Psi}\gamma^\mu\Psi$.

In the case of a complex scalar field, we find

$$j^{\mu} = -i[\varphi^{\dagger} D^{\mu} \varphi - (D^{\mu} \varphi)^{\dagger} \varphi] \ . \tag{61.9}$$

With a factor of e, this current should be identified as the electromagnetic current.

Because the covariant derivative appears in eq. (61.9), the electromagnetic current depends explicitly on the gauge field.

We had not previously contemplated this possibility, but in scalar electrodynamics it arises naturally, and is essential for gauge invariance.

It also poses no special problem in the quantum theory.

We will make the same assumption that we did for spinor electrodynamics: namely, that the correct procedure is to omit integration over the component of $\tilde{A}_{\mu}(k)$ that is parallel to k_{μ}, on the grounds that this integration is redundant.

This leads to the same Feynman rules for internal and external photons as in section 58.

We will call the spin-zero particle with electric charge +e a scalar electron or selectron (recall that our convention is that e is negative), and the spin-zero particle with electric charge –e a scalar positron or s positron.

Scalar lines (traditionally drawn as dashed in scalar electrodynamics) carry a charge arrow whose direction must be preserved when lines are joined by vertices.

To determine the kinds of vertices we have, we first write out the interaction terms in the lagrangian of eq. (61.8):

$$\mathcal{L}_1 = ieA^\mu[(\partial_\mu\varphi^\dagger)\varphi - \varphi^\dagger\partial_\mu\varphi] - e^2A^\mu A_\mu\varphi^\dagger\varphi - \tfrac{1}{4}\lambda(\varphi^\dagger\varphi)^2 \; . \tag{61.10}$$

This leads to the vertices shown in fig. (61.1).

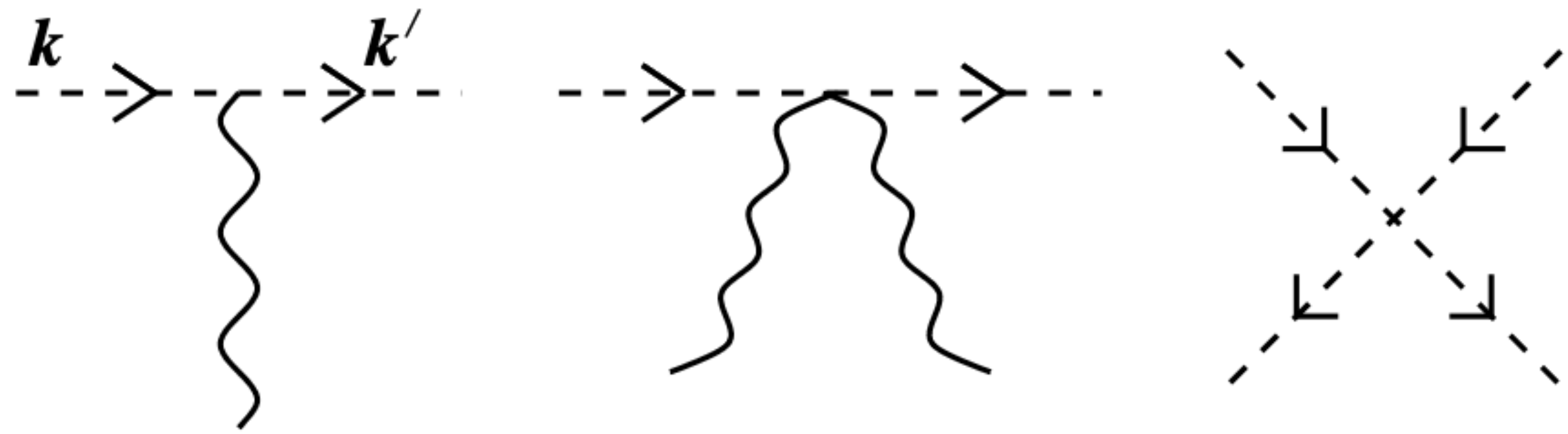


Figure 61.1: The three vertices of scalar electrodynamics; the corresponding vertex factors are $ie(k+k')_\mu$, $-2ie^2g_{\mu\nu}$, and $-i\lambda$.

The vertex factors associated with the last two terms are $-2ie^2g_{\mu\nu}$ and $-i\lambda$.

To get the vertex factor for the first term, we note that if $|k\rangle$ is an incoming selectron state, then $\langle 0|\varphi(x)|k\rangle = e^{ikx}$ and $\langle 0|\varphi^\dagger(x)|k\rangle = 0$; and if $\langle k'|$ is an outgoing selectron state, then $\langle k'|\varphi^\dagger(x)|0\rangle = e^{-ik'x}$ and $\langle 0|\varphi(x)|k\rangle = 0$.

Therefore, in free field theory,

$$\langle k'|(\partial_\mu\varphi^\dagger)\varphi|k\rangle = -ik'_\mu e^{-i(k'-k)x} \; , \tag{61.11}$$

$$\langle k'|\varphi^\dagger\partial_\mu\varphi|k\rangle = +ik_\mu e^{-i(k'-k)x} \; . \tag{61.12}$$

This implies that the vertex factor for the first term in eq. (61.10) is given by $i(ie)[(-ik'_\mu) - (ik_\mu)] = ie(k + k')_\mu$.

Putting everything together, we get the following set of Feynman rules for tree-level processes in scalar electrodynamics.

1. For each *incoming selectron*, draw a dashed line with an arrow pointed *towards* the vertex, and label it with the selectron's four-momentum, k_i.

2. For each *outgoing selectron*, draw a dashed line with an arrow pointed *away* from the vertex, and label it with the selectron's four-momentum, k'_i.

3. For each *incoming spositron*, draw a dashed line with an arrow pointed *away* from the vertex, and label it with *minus* the spositron's four-momentum, $-k_i$.

4. For each *outgoing spositron*, draw a dashed line with an arrow pointed *towards* the vertex, and label it with *minus* the spositron's four-momentum, $-k'_i$.

5. For each *incoming photon*, draw a wavy line with an arrow pointed *towards* the vertex, and label it with the photon's four-momentum, k_i.

6. For each *outgoing photon*, draw a wavy line with an arrow pointed *away* from the vertex, and label it with the photon's four-momentum, k'_i.

7. There are three allowed vertices, shown in fig. (61.1). Using these vertices, join up all the external lines, including extra internal lines as needed. In this way, draw all possible diagrams that are *topologically inequivalent.*

8. Assign each internal line its own four-momentum. Think of the four-momenta as flowing along the arrows, and conserve four-momentum at each vertex. For a tree diagram, this fixes the momenta on all the internal lines.

9. The value of a diagram consists of the following factors:

   for each incoming photon, $\varepsilon^{\mu *}_{\lambda_i}(\mathbf{k}_i)$;

   for each outgoing photon, $\varepsilon^{\mu}_{\lambda_i}(\mathbf{k}_i)$;

   for each incoming or outgoing selectron or spositron, 1;

   for each scalar-scalar-photon vertex, $ie(k + k')_\mu$;

   for each scalar-scalar-photon-photon vertex, $-2ie^2 g_{\mu\nu}$;

   for each four-scalar vertex, $-i\lambda$;

   for each internal photon, $-ig^{\mu\nu}/(k^2 - i\epsilon)$;

   for each internal scalar, $-i/(k^2 + m^2 - i\epsilon)$.

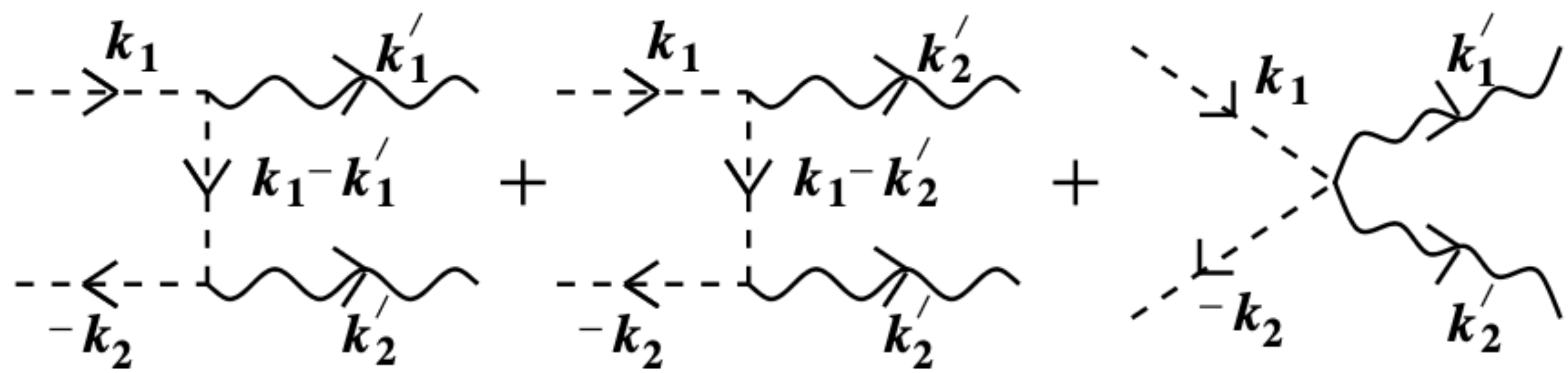
10. The vector index on each vertex is contracted with the vector index on either the photon propagator (if the attached photon line is internal) or the photon polarization vector (if the attached photon line is external).

11. The value of $i\mathcal{T}$ (at tree level) is given by a sum over the values of all the contributing diagrams.

Let us compute the scattering amplitude for a particular process, $\tilde{e}^+\tilde{e}^- \to \gamma\gamma$, where e–denotes a selectron.

We have the diagrams of fig. (61.2)

Figure 61.2: Diagrams for $e_+e_- \to \gamma\gamma$.



The amplitude is

$$i\mathcal{T} = (ie)^2 \frac{1}{i} \frac{(2k_1 - k_1')_\mu \varepsilon^\mu_{1'} (k_1 - k_1' - k_2)_\nu \varepsilon^\nu_{2'}}{m^2 - t} + (1' \leftrightarrow 2') \tag{61.13}$$
$$- 2ie^2 g_{\mu\nu} \varepsilon^\mu_{1'} \varepsilon^\nu_{2'} \,,$$

where $t = -(k_1 - k'_1)^2$ and $u = -(k_1 - k'_2)^2$.

This expression can be simplified by noting that $k_1 - k'_1 - k_2 = k'_2 - 2\, k_2$, and that $k'_i \cdot \varepsilon'_i = 0$.

Then we have

$$\mathcal{T} = -e^2 \left[\frac{4(k_1 \cdot \varepsilon_{1'})(k_2 \cdot \varepsilon_{2'})}{m^2 - t} + \frac{4(k_1 \cdot \varepsilon_{2'})(k_2 \cdot \varepsilon_{1'})}{m^2 - u} + 2(\varepsilon_{1'} \cdot \varepsilon_{2'}) \right] . \qquad (61.14)$$

To get the polarization-summed cross section, we take the absolute square of eq. (61.14), and use the substitution rule

$$\sum_{\lambda=\pm} \varepsilon_\lambda^\mu(\mathbf{k}) \varepsilon_\lambda^{\rho *}(\mathbf{k}) \to g^{\mu\rho} \, . \qquad (61.15)$$

This is now a straightforward calculation.

62 Loop Corrections in Spinor Electrodynamics

In this section we will compute the one-loop corrections in spinor electrodynamics.

First let us note that the general discussion of sections 18 and 29 leads us to expect that we will need to add to the free lagrangian

$$\mathcal{L}_0 = i\overline{\Psi} \partial\!\!\!/ \Psi - m\overline{\Psi}\Psi - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} \qquad (62.1)$$

all possible terms whose coefficients have positive or zero mass dimension, and that respect the symmetries of the original lagrangian.

These include Lorentz symmetry, the U(1) gauge symmetry, and the discrete symmetries of parity, time reversal, and charge conjugation.

The mass dimensions of the fields (in four spacetime dimensions) are $[A^\mu] = 1$ and $[\Psi] = 3/2$.

Gauge invariance requires that A^μ appear only in the form of a covariant derivative D^μ.

(Recall that the field strength $F^{\mu\nu}$ can be expressed as the commutator of two covariant derivatives.)

Thus, the only possible term we could add to $\mathcal{L}_0$ that does not involve the Ψ field, and that has mass dimension four or less, is $\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$.

This term, however, is odd under parity and time reversal.

Similarly, there are no terms meeting all the requirements that involve Ψ: the only candidates contain either γ_5 (e.g., $i\,\overline{\Psi}\gamma_5\Psi$) and are forbidden by parity, or C (e.g, $\overline{\Psi}^{T}C\Psi$) and are forbidden by the U(1) symmetry.

Therefore, the theory we will consider is specified by $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$, where $\mathcal{L}_0$ is given by eq. (62.1), and

$$\mathcal{L}_1 = Z_1 e\overline{\Psi}\not{A}\Psi + \mathcal{L}_{\rm ct} \ , \tag{62.2}$$

$$\mathcal{L}_{\rm ct} = i(Z_2{-}1)\overline{\Psi}\not{\partial}\Psi - (Z_m{-}1)m\overline{\Psi}\Psi - \tfrac{1}{4}(Z_3{-}1)F^{\mu\nu}F_{\mu\nu} \ . \tag{62.3}$$

We will use an on-shell renormalization scheme.

We can write the exact photon propagator (in momentum space) as a geometric series of the form

$$\tilde{\mathbf{\Delta}}_{\mu\nu}(k) = \tilde{\Delta}_{\mu\nu}(k) + \tilde{\Delta}_{\mu\rho}(k)\Pi^{\rho\sigma}(k)\tilde{\Delta}_{\sigma\nu}(k) + \dots \,, \tag{62.4}$$

where $i\Pi^{\mu\nu}(k)$ is given by a sum of one-particle irreducible (1PI for short; see section 14) diagrams with two external photon lines (and the external propagators removed), and $\tilde{\Delta}_{\mu\nu}(k)$ is the free photon propagator,

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{1}{k^2 - i\epsilon}\left(g_{\mu\nu} - (1{-}\xi)\frac{k_\mu k_\nu}{k^2}\right). \tag{62.5}$$

Here we have used the freedom to add k_μ or k_ν terms to put the propagator into *generalized Feynman gauge* or R_ξ *gauge*.

(The name R_ξ *gauge* has historically been used only in the context of spontaneous symmetry breaking—see section 85—but we will use it here as well; R stands for *renormalizable* and ξ stands for ξ.)

Setting $\xi = 1$ gives Feynman gauge, and setting $\xi = 0$ gives Lorenz gauge (also known as Landau gauge).

Observable squared amplitudes should not depend on the value of ξ.

This suggests that $\Pi^{\mu\nu}(k)$ should be *transverse*,

$$k_\mu \Pi^{\mu\nu}(k) = k_\nu \Pi^{\mu\nu}(k) = 0 \,, \tag{62.6}$$

so that the ξ dependent term in $\tilde{\Delta}_{\mu\nu}(k)$ vanishes when an internal photon line is attached to $\Pi^{\mu\nu}(k)$.

Eq. (62.6) is in fact valid; the proof is based on the Ward identity for the electromagnetic current.

Without the proof, we will take eq. (62.6) as given.

This implies that we can write

$$\Pi^{\mu\nu}(k) = \Pi(k^2)\Big(k^2 g^{\mu\nu} - k^\mu k^\nu\Big) \tag{62.7}$$

$$= k^2\,\Pi(k^2)P^{\mu\nu}(k)\;, \tag{62.8}$$

where $\Pi(k^2)$ is a scalar function, and $P^{\mu\nu}(k) = g^{\mu\nu} - k^\mu k^\nu/k^2$ is the projection matrix introduced in section 57.

Note that we can also write

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{1}{k^2 - i\epsilon}\left(P_{\mu\nu}(k) + \xi\,\frac{k_\mu k_\nu}{k^2}\right). \tag{62.9}$$

Then, using eqs. (62.8) and (62.9) in eq. (62.4), and summing the geometric series, we find

$$\tilde{\mathbf{\Delta}}_{\mu\nu}(k) = \frac{P_{\mu\nu}(k)}{k^2[1 - \Pi(k^2)] - i\epsilon} + \xi\,\frac{k_\mu k_\nu/k^2}{k^2 - i\epsilon}\;. \tag{62.10}$$

The ξ dependent term should be physically irrelevant (and can be set to zero by the gauge choice $\xi = 0$, corresponding to Lorenz gauge).

The remaining term has a pole at $k^2 = 0$ with residue $P_{\mu\nu}(k)/[1 - \Pi(0)]$.

In our on-shell renormalization scheme, we should have

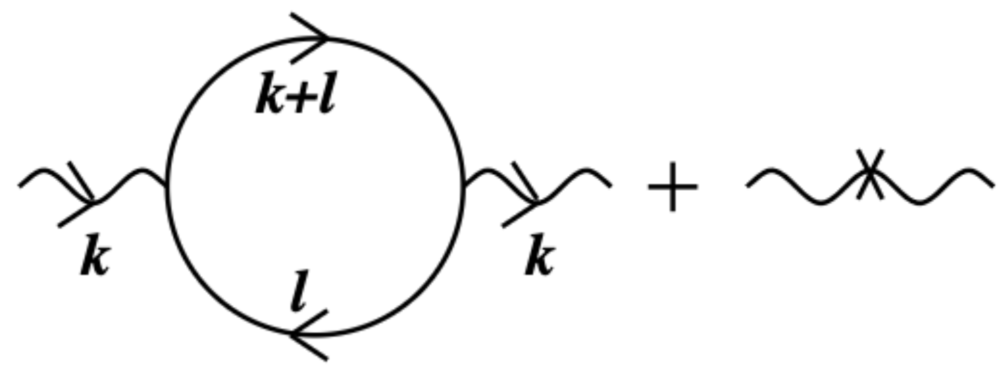
$$\Pi(0) = 0\ . \tag{62.11}$$

This corresponds to the field normalization that is needed for validity of the LSZ formula.

Let us now turn to the calculation of $\Pi^{\mu\nu}(k)$.

The one-loop and counterterm contributions are shown in fig. (62.1).

Figure 62.1: The one-loop and counterterm corrections to the photon propagator in spinor electrodynamics.



We have

$$\begin{aligned} i\Pi^{\mu\nu}(k) &= (-1)(iZ_1e)^2\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\Big[\tilde S(\not\ell+\not k)\gamma^\mu \tilde S(\not\ell)\gamma^\nu\Big] \\ &\quad - i(Z_3-1)(k^2g^{\mu\nu} - k^\mu k^\nu) + O(e^4)\ , \end{aligned} \tag{62.12}$$

where the factor of minus one is for the closed fermion loop, and $\tilde S(\not p) = (-\not p+m)/(p^2+m^2-i\epsilon)$ is the free fermion propagator in momentum space.

Anticipating that $Z_1 = 1 + O(e^2)$, we set $Z_1 = 1$ in the first term.

We can write

$$\mathrm{Tr}\Big[\tilde{S}(\ell\!\!\!/+k\!\!\!/)\gamma^\mu \tilde{S}(\ell\!\!\!/)\gamma^\nu\Big] = \int_0^1 dx\, \frac{4N^{\mu\nu}}{(q^2+D)^2}\,, \tag{62.13}$$

where we have combined denominators in the usual way: $q = \ell + xk$ and

$$D = x(1-x)k^2 + m^2 - i\epsilon\,. \tag{62.14}$$

The numerator is

$$4N^{\mu\nu} = \mathrm{Tr}\Big[(-\ell\!\!\!/-k\!\!\!/+m)\gamma^\mu(-\ell\!\!\!/+m)\gamma^\nu\Big] \tag{62.15}$$

Completing the trace, we get

$$N^{\mu\nu} = (\ell+k)^\mu \ell^\nu + \ell^\mu(\ell+k)^\nu - [\ell(\ell+k) + m^2]g^{\mu\nu}\,. \tag{62.16}$$

Setting $\ell = q - xk$ and and dropping terms linear in q (because they integrate to zero), we find

$$N^{\mu\nu} \to 2q^\mu q^\nu - 2x(1-x)k^\mu k^\nu - [q^2 - x(1-x)k^2 + m^2]g^{\mu\nu}\,. \tag{62.17}$$

The integrals diverge, and so we analytically continue to $d = 4 - \varepsilon$ dimensions, and replace e with $e\tilde{\mu}^{\varepsilon/2}$ (so that e remains dimensionless for any d).

Next we use a standard mathematical result

$$\int d^d q\; q^\mu q^\nu f(q^2) = \frac{1}{d}\, g^{\mu\nu} \int d^d q\; q^2 f(q^2)\;. \tag{62.18}$$

This allows the replacement

$$N^{\mu\nu} \to -2x(1-x)k^\mu k^\nu + \Big[\Big(\tfrac{2}{d}-1\Big)q^2 + x(1-x)k^2 - m^2\Big]g^{\mu\nu}\;. \tag{62.19}$$

Using the results of section 14, along with a little manipulation of gamma functions, we can show that

$$\Big(\tfrac{2}{d}-1\Big)\int \frac{d^d q}{(2\pi)^d}\,\frac{q^2}{(q^2+D)^2} = D\int \frac{d^d q}{(2\pi)^d}\,\frac{1}{(q^2+D)^2}\;. \tag{62.20}$$

Thus we can make the replacement $(2/d-1)q^2 \to D$ in eq. (62.19), and we find

$$N^{\mu\nu} \to 2x(1-x)(k^2 g^{\mu\nu} - k^\mu k^\nu)\;. \tag{62.21}$$

This guarantees that the one-loop contribution to $\Pi^{\mu\nu}(k)$ is transverse (as we expected) in any number of spacetime dimensions.

Now we evaluate the integral over q, using

$$\begin{aligned} \tilde{\mu}^\varepsilon \int \frac{d^d q}{(2\pi)^d}\,\frac{1}{(q^2+D)^2} &= \frac{i}{16\pi^2}\,\Gamma(\tfrac{\varepsilon}{2})\Big(4\pi\tilde{\mu}^2/D\Big)^{\varepsilon/2} \\ &= \frac{i}{8\pi^2}\left[\frac{1}{\varepsilon} - \tfrac{1}{2}\ln(D/\mu^2)\right]\;, \end{aligned} \tag{62.22}$$

where $\mu^2 = 4\pi e^{-\gamma}\tilde{\mu}^2$, and we have dropped terms of order ε in the last line.

Combining eqs. (62.7), (62.12), (62.13), (62.21), and (62.22), we get

$$\Pi(k^2) = -\frac{e^2}{\pi^2}\int_0^1 dx\, x(1-x)\left[\frac{1}{\varepsilon} - \tfrac{1}{2}\ln(D/\mu^2)\right] - (Z_3-1) + O(e^4)\ . \tag{62.23}$$

Imposing $\Pi(0) = 0$ fixes

$$Z_3 = 1 - \frac{e^2}{6\pi^2}\left[\frac{1}{\varepsilon} - \ln(m/\mu)\right] + O(e^4) \tag{62.24}$$

and

$$\Pi(k^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln(D/m^2) + O(e^4)\ . \tag{62.25}$$

Next we turn to the fermion propagator.

The exact propagator can be written in Lehmann-Kallen form as

$$\tilde{\mathbf{S}}(\not{p}) = \frac{1}{\not{p} + m - i\epsilon} + \int_{m^2_{\rm th}}^{\infty} ds\, \frac{\rho_\Psi(s)}{\not{p} + \sqrt{s} - i\epsilon}\ . \tag{62.26}$$

We see that the first term has a pole at $\not{p} = -m$ with residue one.

This residue corresponds to the field normalization that is needed for the validity of the LSZ formula.

There is a problem, however: in quantum electrodynamics, the threshold mass m_{th} is m, corresponding to the contribution of a fermion and a zero-energy photon.

Thus the second term has a branch point at $\not{p} = -m$.

The pole in the first term is therefore not isolated, and its residue is ill defined.

This is a reflection of an underlying infrared divergence, associated with the massless photon.

To deal with it, we must impose an infrared cutoff that moves the branch point away from the pole.

The most direct method is to change the denominator of the photon propagator from k^2 to $k^2 + m^2_\gamma$, where m_γ is a fictitious photon mass.

Ultimately, as in section 26, we must deal with this issue by computing cross-sections that take into account detector inefficiencies.

In quantum electrodynamics, we must specify the lowest photon energy ωmin that can be detected.

Only after computing cross sections with extra undedectable photons, and then summing over them, is it safe to take the limit $m_\gamma \to 0$.

It turns out that it is not also necessary to abandon the on-shell shell renormalization scheme (as we were forced to do in massless φ3 theory in section 27), as long as the electron is massive.

An alternative is to use dimensional regularization for the infrared divergences as well as the ultraviolet ones.

As discussed in section 25, there are no soft-particle infrared divergences for $d > 4$ (and no colinear divergences at all in quantum electrodynamics with massive charged particles).

In practice, infrared-divergent integrals are finite away from even-integer dimensions, just like ultraviolet-divergent integrals.

Thus we simply keep $d = 4-\varepsilon$ all the way through to the very end, taking the $\varepsilon \to 0$ limit only after summing over cross sections with extra undetectable photons, all computed in $4-\varepsilon$ dimensions.

This method is calculationally the simplest, but requires careful bookkeeping to segregate the infrared and ultraviolet singularities.

For that reason, we will not pursue it further.

We can write the exact fermion propagator in the form

$$\tilde{\mathbf{S}}(\not{p})^{-1} = \not{p} + m - i\epsilon - \Sigma(\not{p}) \,, \tag{62.27}$$

where $i\Sigma(\not{p})$ is given by the sum of 1PI diagrams with two external fermion lines (and the external propagators removed).

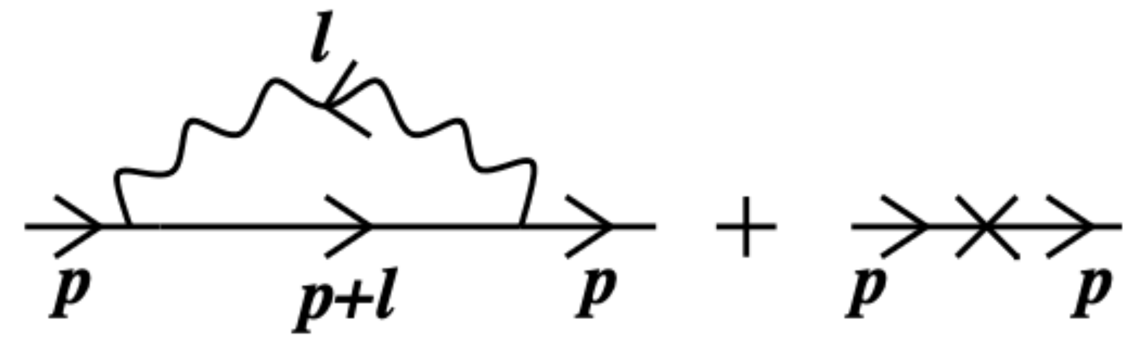
The fact that $\tilde{\mathbf{S}}(\not{p})$ has a pole at $\not{p} = -m$ with residue one implies that $\Sigma(-m) = 0$ and $\Sigma'(-m) = 0$; this fixes the coefficients Z_2 and Z_m.

As we will see, we must have an infrared cutoff in place in order to have a finite value for $\Sigma'(-m)$.

Let us now turn to the calculation of $\Sigma(\not{p})$.

The one-loop and counterterm contributions are shown in fig. (62.2).

Figure 62.2: The one-loop and counterterm corrections to the fermion propagator in spinor electrodynamics.



We have

$$i\Sigma(\not{p}) = (iZ_1 e)^2 \left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \left[\gamma^\nu \tilde{S}(\not{p}+\not{\ell})\gamma^\mu\right]\tilde{\Delta}_{\mu\nu}(\ell)$$
$$- i(Z_2{-}1)\not{p} - i(Z_m{-}1)m + O(e^4)\ . \qquad (62.28)$$

It is simplest to work in Feynman gauge, where we take

$$\tilde{\Delta}_{\mu\nu}(\ell) = \frac{g_{\mu\nu}}{\ell^2 + m_\gamma^2 - i\epsilon}\ ; \qquad (62.29)$$

here we have included the fictitious photon mass m_γ as an infrared cutoff.

We now apply the usual bag of tricks to get

$$i\Sigma(\not{p}) = e^2\tilde{\mu}^\varepsilon \int_0^1 dx \int \frac{d^dq}{(2\pi)^d}\,\frac{N}{(q^2+D)^2}$$
$$- i(Z_2{-}1)\not{p} - i(Z_m{-}1)m + O(e^4)\ , \qquad (62.30)$$

where q = ℓ + xp and

$$D = x(1-x)p^2 + xm^2 + (1-x)m_\gamma^2 \ , \tag{62.31}$$

$$\begin{aligned} N &= \gamma_\mu(-\not{p} - \not{\ell} + m)\gamma^\mu \\ &= -(d-2)(\not{p} + \not{\ell}) - dm \\ &= -(d-2)[\not{q} + (1-x)\not{p}] - dm \ , \end{aligned} \tag{62.32}$$

where we have used (from section 47) $\gamma_\mu\gamma^\mu = -d$ and $\gamma_\mu\not{p}\gamma^\mu = (d-2)\not{p}$.

The term linear in q integrates to zero, and then, using eq. (62.22), we get

$$\begin{aligned} \Sigma(\not{p}) = &- \frac{e^2}{8\pi^2}\int_0^1 dx\Big((2-\varepsilon)(1-x)\not{p} + (4-\varepsilon)m\Big)\left[\frac{1}{\varepsilon} - \tfrac{1}{2}\ln(D/\mu^2)\right] \\ &- (Z_2-1)\not{p} - (Z_m-1)m + O(e^4) \ . \end{aligned} \tag{62.33}$$

We see that finiteness of $\Sigma(\not{p})$ requires

$$Z_2 = 1 - \frac{e^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) + O(e^4) \ , \tag{62.34}$$

$$Z_m = 1 - \frac{e^2}{2\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) + O(e^4) \ . \tag{62.35}$$

We can impose $\Sigma(-m) = 0$ by writing

$$\Sigma(\not{p}) = \frac{e^2}{8\pi^2}\left[\int_0^1 dx\,\Big((1-x)\not{p} + 2m\Big)\ln(D/D_0) + \kappa_2(\not{p} + m)\right] + O(e^4)\ , \tag{62.36}$$

where D_0 is D evaluated at $p^2 = -m^2$.

$$D_0 = x^2m^2 + (1-x)m_\gamma^2\ , \tag{62.37}$$

and κ^2 is a constant to be determined.

We fix κ^2 by imposing $\Sigma'(-m) = 0$.

In differentiating with respect to $\not{p}$, we take the p^2 in D, eq. (62.31), to be $-\not{p}^2$; we find

$$\begin{aligned}\kappa_2 &= -2\int_0^1 dx\, x(1-x^2)m^2/D_0 \\ &= -2\ln(m/m_\gamma) + 1\ ,\end{aligned} \tag{62.38}$$

where we have dropped terms that go to zero with the infrared cutoff m_γ.

Next we turn to the loop correction to the vertex.

We define the vertex function $i\mathbf{V}^\mu(p',p)$ as the sum of one-particle irreducible diagrams with one incoming fermion with momentum p, one outgoing fermion with momentum p′, and one incoming photon with momentum $k = p'-p$.

The original vertex $iZ_1e\gamma^\mu$ is the first term in this sum, and the diagram of fig. (62.3) is the second.

Thus we have

$$i\mathbf{V}^\mu(p',p) = iZ_1 e\gamma^\mu + i\mathbf{V}^\mu_{1\,\text{loop}}(p',p) + O(e^5)\;, \tag{62.39}$$

where

$$i\mathbf{V}^\mu_{1\,\text{loop}}(p',p) = (ie)^3\left(\tfrac{1}{i}\right)^3\int\frac{d^4\ell}{(2\pi)^4}\left[\gamma^\rho\tilde{S}(\not{p}'+\not{\ell})\gamma^\mu\tilde{S}(\not{p}+\not{\ell})\gamma^\nu\right]\tilde{\Delta}_{\nu\rho}(\ell)\;. \tag{62.40}$$

We again use eq. (62.29) for the photon propagator, and combine denominators in the usual way.

We then get

$$i\mathbf{V}^\mu_{1\,\text{loop}}(p',p) = e^3\int dF_3\int\frac{d^4q}{(2\pi)^4}\,\frac{N^\mu}{(q^2+D)^3}\;, \tag{62.41}$$

where the integral over Feynman parameters is

$$\int dF_3 \equiv 2\int_0^1 dx_1 dx_2 dx_3\,\delta(x_1{+}x_2{+}x_3{-}1)\;, \tag{62.42}$$

and

$$q = \ell + x_1 p + x_2 p'\;, \tag{62.43}$$

$$\begin{aligned} D = {}& x_1(1{-}x_1)p^2 + x_2(1{-}x_2)p'^2 - 2x_1x_2\,p{\cdot}p' \\ & + (x_1{+}x_2)m^2 + x_3 m_\gamma^2\;, \end{aligned} \tag{62.44}$$

$$
\begin{aligned}
N^\mu &= \gamma_\nu(-\not{p}' - \not{\ell} + m)\gamma^\mu(-\not{p} - \not{\ell} + m)\gamma^\nu \\
&= \gamma_\nu[-\not{q} + x_1\not{p} - (1-x_2)\not{p}' + m]\gamma^\mu[-\not{q} - (1-x_1)\not{p} + x_2\not{p}' + m]\gamma^\nu \\
&= \gamma_\nu\not{q}\gamma^\mu\not{q}\gamma^\nu + \widetilde{N}^\mu + (\text{linear in } q)\ ,
\end{aligned}
\tag{62.45}
$$

where

$$
\widetilde{N}^\mu = \gamma_\nu[x_1\not{p} - (1-x_2)\not{p}' + m]\gamma^\mu[-(1-x_1)\not{p} + x_2\not{p}' + m]\gamma^\nu\ . \tag{62.46}
$$

The terms linear in q in eq. (62.45) integrate to zero, and only the first term is divergent.

After continuing to d dimensions, we can use eq. (62.18) to make the replacement

$$
\gamma_\nu\not{q}\gamma^\mu\not{q}\gamma^\nu \to \frac{1}{d}\, q^2\, \gamma_\nu\gamma_\rho\gamma^\mu\gamma^\rho\gamma^\nu\ . \tag{62.47}
$$

then we use $\gamma_\rho\gamma^\mu\gamma_\rho = (d-2)\gamma^\mu$ twice to get

$$
\gamma_\nu\not{q}\gamma^\mu\not{q}\gamma^\nu \to \frac{(d-2)^2}{d}\, q^2\, \gamma^\mu\ . \tag{62.48}
$$

Performing the usual manipulations, we find

$$
\mathbf{V}^\mu_{1\,\text{loop}}(p',p) = \frac{e^3}{8\pi^2}\left[\left(\frac{1}{\varepsilon} - 1 - \tfrac{1}{2}\int dF_3\, \ln(D/\mu^2)\right)\gamma^\mu + \tfrac{1}{4}\int dF_3\, \frac{\widetilde{N}^\mu}{D}\right]. \tag{62.49}
$$

From eq. (62.39), we see that finiteness of $\mathbf{V}^\mu(p', p)$ requires

$$Z_1 = 1 - \frac{e^2}{8\pi^2}\left(\frac{1}{\varepsilon} + \text{finite}\right) + O(e^4)\ .$$

To completely fix $\mathbf{V}^\mu(p', p)$, we need a suitable condition to impose on it.

We take this up in the next section.

63 The Vertex Function in Spinor Electrodynamics

In the last section, we computed the one-loop contribution to the vertex function $\mathbf{V}^\mu(p', p)$ in spinor electrodynamics, where p is the four-momentum of an incoming electron (or outgoing positron), and p′ is the four-momentum of an outgoing electron (or incoming positron).

We left open the issue of the renormalization condition we wish to impose on $\mathbf{V}^\mu(p', p)$.

For the theories we have studied previously, we have usually made the mathematically convenient (but physically obscure) choice to define the coupling constant as the value of the vertex function when all external four-momenta are set to zero.

However, in the case of spinor electrodynamics, the masslessness of the photon gives us the opportunity to do something more physically meaningful: we can define the coupling constant as the value of the vertex function when all three particles are on shell: $p^2 = p'^2 = -m^2$, and $q^2 = 0$, where $q \equiv p' - p$ is the photon four-momentum.

Because the photon is massless, these three on-shell conditions are compatible with momentum conservation.

To be more precise, let us sandwich $\mathbf{V}^\mu(p', p)$ between the spinor factors that are appropriate for an incoming electron with momentum p and an outgoing electron with momentum p', impose the on-shell conditions, and define the electron charge e via

$$\overline{u}_{s'}(\mathbf{p}')\mathbf{V}^\mu(p',p)u_s(\mathbf{p})\Big|_{\substack{p^2=p'^2=-m^2\\(p'-p)^2=0}} = e\,\overline{u}_{s'}(\mathbf{p}')\gamma^\mu u_s(\mathbf{p})\Big|_{\substack{p^2=p'^2=-m^2\\(p'-p)^2=0}}\;. \qquad (63.1)$$

This definition is in accord with the usual one provided by Coulomb's law.

To see why, consider the process of electron-electron scattering.

According to the general discussion in section 19, we compute the exact amplitude for this process by using tree diagrams with exact internal propagators and vertices, as shown in fig. (63.1).

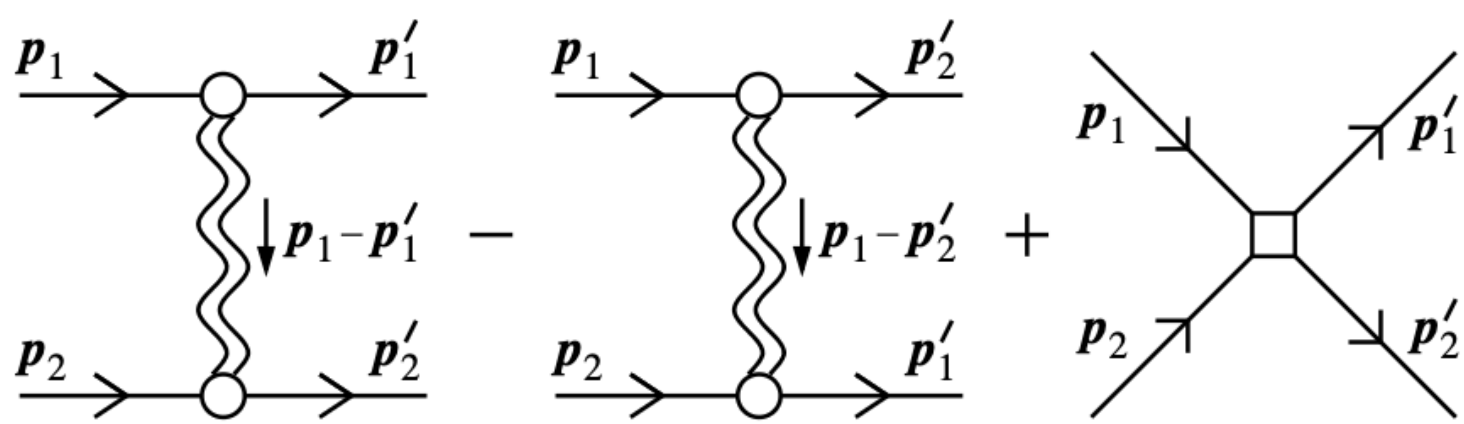


Figure 63.1: Diagrams for the exact electron-electron scattering amplitude.The vertices and photon propagator are exact; external lines stand for the usual u and spinor factors, times the unit residue of the po $\overline{u}$ at $p^2 = -m^2$.

In the last section, we renormalized the photon propagator so that it approaches its tree-level value $\widetilde{\Delta}_{\mu\nu}(q)$ when $q^2 \to 0$.

And we have just chosen to renormalize the electron-photon vertex function by requiring it to approach its tree-level value $e\gamma^\mu$ when $q^2 \to 0$, and when sandwiched between external spinors for on-shell incoming and outgoing electrons.

Therefore, as $q^2 \to 0$, the first two diagrams in fig. (63.1) approach the tree-level scattering amplitude, with the electron charge equal to e.

Furthermore, the third diagram does not have a pole at $q^2 = 0$, and so can be neglected in this limit.

Physically, $q^2 \to 0$ means that the electron's momentum changes very little during the scattering.

Measuring a slight deflection in the trajectory of one charged particle (due to the presence of another) is how we measure the coefficient in Coulomb's law.

Thus, eq. (63.1) corresponds to this traditional definition of the charge of the electron.

We can simplify eq. (63.1) by noting that the on-shell conditions actually enforce $p' = p$.

So we can rewrite eq. (63.1) as

$$\begin{aligned} \overline{u}_s(\mathbf{p})\mathbf{V}^\mu(p,p)u_s(\mathbf{p}) &= e\,\overline{u}_s(\mathbf{p})\gamma^\mu u_s(\mathbf{p}) \\ &= 2ep^\mu \ , \end{aligned} \tag{63.2}$$

where $p^2 = -m^2$ is implicit.

We have taken $s' = s$, because otherwise the right-hand side vanishes (and hence does not specify a value for e).

Now we can use eq. (63.2) to completely determine $\mathbf{V}^\mu(p',p)$.

Using the freedom to choose the finite part of Z_1, we write

$$\mathbf{V}^\mu(p',p) = e\gamma^\mu - \frac{e^3}{16\pi^2}\int dF_3\left[\Big(\ln(D/D_0)+2\kappa_1\Big)\gamma^\mu - \frac{N^\mu}{2D}\right] + O(e^5)\ , \tag{63.3}$$

where

$$\begin{aligned} D = {} & x_1(1-x_1)p^2 + x_2(1-x_2)p'^2 - 2x_1x_2p\cdot p' \\ & + (x_1+x_2)m^2 + x_3m_\gamma^2\ , \end{aligned} \tag{63.4}$$

D_0 is D evaluated at $p' = p$ and $p^2 = -m^2$,

$$\begin{aligned} D_0 &= (x_1+x_2)^2m^2 + x_3m_\gamma^2 \\ &= (1-x_3)^2m^2 + x_3m_\gamma^2\ , \end{aligned} \tag{63.5}$$

and

$$N^\mu = \gamma_\nu[x_1\not{p} - (1-x_2)\not{p}' + m]\gamma^\mu[-(1-x_1)\not{p} + x_2\not{p}'m]\gamma^\nu\ ; \tag{63.6}$$

N^μ was called $\widetilde{N}^\mu$ in section 62, but we have dropped the tilde for notational convenience.

We fix the constant κ_1 in eq. (63.3) by imposing eq. (63.2).

This yields

$$4\kappa_1 p^\mu = \int dF_3 \, \frac{\overline{u}_s(\mathbf{p}) N_0^\mu u_s(\mathbf{p})}{2D_0} \, , \tag{63.7}$$

where $N^\mu{}_0$ is N^μ with $p' = p$ and $p^2 = p'^2 = -m^2$.

So now we must evaluate $\overline{u}$ $N^\mu{}_0$ u.

To do so, we first write

$$N^\mu = \gamma_\nu (\not{a}_1 {+} m) \gamma^\mu (\not{a}_2 {+} m) \gamma^\nu \, , \tag{63.8}$$

where

$$\begin{aligned} a_1 &= x_1 p - (1{-}x_2) p' \, , \\ a_2 &= x_2 p' - (1{-}x_1) p \, . \end{aligned} \tag{63.9}$$

Now we use the gamma matrix contraction identities to get

$$N^\mu = 2\not{a}_2 \gamma^\mu \not{a}_1 + 4m(a_1{+}a_2)^\mu + 2m^2 \gamma^\mu \, . \tag{63.10}$$

Here we have set d = 4, because we have already removed the divergence and taken the limit $\varepsilon \to 0$.

Setting $p' = p$, and using $\overline{u}$ u = −mu and u $\overline{u}$ = −mu, along with $\overline{u}\,\gamma^\mu u = 2p^\mu$ and $\overline{u}\,u = 2m$, and recalling that $x_1{+}x_2{+}x_3 = 1$, we find

$$\overline{u}N_0^\mu u = 4(1-4x_3+x_3^2)m^2p^\mu \; . \tag{63.11}$$

Using eqs. (63.5), (63.7), and (63.11), we get

$$\begin{aligned}
\kappa_1 &= \frac{1}{2}\int dF_3 \, \frac{1-4x_3+x_3^2}{(1-x_3)^2 + x_3 m_\gamma^2/m^2} \\
&= \int_0^1 dx_3 \, (1-x_3) \, \frac{1-4x_3+x_3^2}{(1-x_3)^2 + x_3 m_\gamma^2/m^2} \\
&= -2\ln(m/m_\gamma) + \tfrac{5}{2}
\end{aligned} \tag{63.12}$$

in the limit of $m_\gamma \to 0$.

We see that an infrared regulator is necessary for the vertex function as well as the fermion propagator.

Now that we have $\mathbf{V}^\mu(p',p)$, we can extract some physics from it.

Consider again the process of electron-electron scattering, shown in fig. (63.1).

In order to compute the contributions of these diagrams, we must evaluate $\overline{u}_{s'}(p')\, \mathbf{V}^\mu(p',p)\, u_s(p)$ with $p^2 = -p'^2 = -m^2$, but with $q^2 = (p'- p)^2$ arbitrary.

To evaluate $\overline{u}'N^\mu u$, we start with eq. (63.10), and use the anticommutation relations of the gamma matrices to move all the $\not{p}$'s in N^μ to the far right (where we can use $\not{p}u = -mu$) and all the $\not{p}'$'s to the far left (where we can use $\overline{u}'\not{p}' = -m\overline{u}'$).

This results in

$$N^\mu \to [4(1-x_1-x_2+x_1x_2)p\cdot p' + 2(2x_1-x_1^2+2x_2-x_2^2)m^2]\gamma^\mu \\ + 4m(x_1^2-x_2+x_1x_2)p^\mu + 4m(x_2^2-x_1+x_1x_2)p'^\mu \ . \qquad (63.13)$$

Next, replace $p\cdot p'$ with $-1/2\ q^2 - m^2$, group the p^μ and p'^μ terms into $p'+p$ and $p'-p$ combinations, and make use of $x_1+x_2+x_3 = 1$ to simplify some coefficients.

The result is

$$N^\mu \to 2[(1-2x_3-x_3^2)m^2 - (x_3+x_1x_2)q^2]\gamma^\mu \\ - 2m(x_3-x_3^2)(p'+p)^\mu \\ - 2m[(x_1+x_1^2) - (x_2+x_2^2)](p'-p)^\mu \ . \qquad (63.14)$$

In the denominator, set $p^2 = p'^2 = -m^2$ and $p\cdot p' = -1/2\ q^2 - m^2$ to get

$$D \to x_1x_2q^2 + (1-x_3)^2m^2 + x_3m_\gamma^2 \ . \qquad (63.15)$$

Note that the right-hand side of eq. (63.15) is symmetric under $x_1 \leftrightarrow x_2$.

Thus the last line of eq. (63.14) will vanish when we integrate $\overline{u}'N^\mu u/D$ over the Feynman parameters.

Finally, we use the Gordon identity from section 38,

$$\overline{u}'(p'+p)^\mu u = \overline{u}'[2m\gamma^\mu + 2iS^{\mu\nu}q_\nu]u \;, \tag{63.16}$$

where $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, to get

$$\begin{aligned} N^\mu \;\rightarrow\; & 2[(1{-}4x_3{+}x_3^2)m^2 - (x_3{+}x_1x_2)q^2]\gamma^\mu \\ & - 4im(x_3{-}x_3^2)S^{\mu\nu}q_\nu \;. \end{aligned} \tag{63.17}$$

So now we have

$$\overline{u}_{s'}(\mathbf{p}')\mathbf{V}^\mu(p',p)u_s(\mathbf{p}) = e\overline{u}'\Big[F_1(q^2)\gamma^\mu - \tfrac{i}{m}F_2(q^2)S^{\mu\nu}q_\nu\Big]u \;, \tag{63.18}$$

where we have defined the *form factors*

$$\begin{aligned} F_1(q^2) \;=\; 1 - \frac{e^2}{16\pi^2}\int dF_3\,\Bigg[&\ln\!\left(1 + \frac{x_1x_2q^2/m^2}{(1{-}x_3)^2}\right) + \frac{1{-}4x_3{+}x_3^2}{(1{-}x_3)^2 + x_3m_\gamma^2/m^2} \\ &+ \frac{(x_3{+}x_1x_2)q^2/m^2 - (1{-}4x_3{+}x_3^2)}{x_1x_2q^2/m^2 + (1{-}x_3)^2 + x_3m_\gamma^2/m^2}\Bigg] + O(e^4) \;, \end{aligned} \tag{63.19}$$

$$F_2(q^2) \;=\; \frac{e^2}{8\pi^2}\int dF_3\;\frac{x_3{-}x_3^2}{x_1x_2q^2/m^2 + (1{-}x_3)^2} + O(e^4) \;. \tag{63.20}$$

We have set $m_\gamma = 0$ in eq. (63.20), and in the logarithm term in eq. (63.19), because these terms do not suffer from infrared divergences.

We can simplify $F_2(q^2)$ by using the delta function in dF_3 to do the integral over x_2 (which replaces x_2 with $1 - x_3 - x_1$), making the change of variable $x_1 = (1 - x_3)y$, and performing the integral over x_3 from zero to one; the result is

$$F_2(q^2) = \frac{e^2}{8\pi^2} \int_0^1 \frac{dy}{1 - y(1-y)q^2/m^2} + O(e^4) \ . \tag{63.21}$$

This last integral can also be done in closed form, but we will be mostly interested in its value at $q^2 = 0$, corresponding to an on-shell photon:

$$F_2(0) = \frac{\alpha}{2\pi} + O(\alpha^2) \ , \tag{63.22}$$

where $\alpha = e^2/4\pi = 1/137.036$ is the fine-structure constant.

We will explore the physical consequences of eq. (63.22) in the next section.

64 The Magnetic Moment of the Electron

In the last section, we computed the one-loop contribution to the vertex function $V\mu(p',p)$ in spinor electrodynamics, where p is the four-momentum of an incoming electron, and p′ is the four-momentum of an outgoing electron.

We found

$$\overline{u}_{s'}(\mathbf{p}')\mathbf{V}^{\mu}(p',p)u_s(\mathbf{p}) = e\overline{u}'\Big[F_1(q^2)\gamma^{\mu} - \tfrac{i}{m}F_2(q^2)S^{\mu\nu}q_{\nu}\Big]u \ , \tag{64.1}$$

where $q = p' - p$ is the four-momentum of the photon (treated as incoming), and with complicated expressions for the *form factors* $F_1(q^2)$ and $F_2(q^2)$.

For our purposes in this section, all we will need to know is that

$$\begin{aligned} F_1(0) &= 1 \quad \text{exactly}, \\ F_2(0) &= \frac{\alpha}{2\pi} + O(\alpha^2)\ . \end{aligned} \tag{64.2}$$

Eq. (64.1) follows from a quantum action of the form

$$\Gamma = \int d^4x \left[eF_1(0)\overline{\Psi}\not{A}\Psi + \frac{e}{2m}F_2(0)F_{\mu\nu}\overline{\Psi}S^{\mu\nu}\Psi + \ldots \right], \tag{64.3}$$

where the ellipses stand for terms with more derivatives.

The displayed terms yield the vertex factor of eq. (64.1) with $q^2 = 0$.

To see this, recall that an incoming photon translates into a factor of $A_\mu \sim \varepsilon^*{}_\mu e^{iqx}$, and therefore of $F_{\mu\nu} \sim i(q_\mu\varepsilon^*{}_\nu - q_\nu\varepsilon^*{}_\mu)e^{iqx}$; the two terms in $F_{\mu\nu}$ cancel the extra factor of one half in the second term in eq. (64.3).

Now we will see what eq. (64.3) predicts for the *magnetic moment of the electron*.

We define the magnetic moment by the following procedure.

We take the photon field A^μ to be a classical field that corresponds to a constant magnetic field in the z direction: $A^0 = 0$ and $\mathbf{A} = (0,Bx,0)$.

This yields $F_{12} = -F_{21} = B$, with all other components of $F_{\mu\nu}$ vanishing.

Then we define a normalized state of an electron at rest, with spin up along the z axis:

$$|e\rangle \equiv \int \widetilde{dp}\, f(\mathbf{p}) b^\dagger_+(\mathbf{p})|0\rangle \ , \tag{64.4}$$

where the wave packet is rotationally invariant (so that there is no orbital angular momentum) and sharply peaked at $\mathbf{p} = 0$, something like

$$f(\mathbf{p}) \sim \exp(-a^2\mathbf{p}^2/2) \tag{64.5}$$

with $a \ll 1/m$.

We normalize the wave packet by $\int \widetilde{dp}\, |f(\mathbf{p})|^2 = 1$; then we have $\langle e|e\rangle = 1$.

Now we define the interaction hamiltonian as what we get from the two displayed terms in eq. (64.3), using our specified field A^μ, and with the form-factor values of eq. (64.2):

$$H_1 \equiv -eB \int d^3x\, \overline{\Psi} \left[x\gamma^2 + \frac{\alpha}{2\pi m} S^{12} \right] \Psi \ . \tag{64.6}$$

Then the electron's magnetic moment μ is specified by

$$\mu B \equiv -\langle e|H_1|e\rangle \ . \tag{64.7}$$

In quantum mechanics in general, if we identify H_1 as the piece of the hamiltonian that is linear in the external magnetic field, then eq. (64.7) defines the magnetic moment of a normalized quantum state with definite angular momentum in the **B** direction.

Now we turn to the computation.

We need to evaluate $\langle e|\overline{\Psi}_\alpha(x)\Psi_\beta(x)|e\rangle$.

Using the usual plane-wave expansions, we have

$$\langle 0|b_+(\mathbf{p}')\overline{\Psi}_\alpha(x)\Psi_\beta(x)b_+^\dagger(\mathbf{p})|0\rangle = \overline{u}_+(\mathbf{p}')_\alpha u_+(\mathbf{p})_\beta \, e^{i(p-p')x} \ . \tag{64.8}$$

Thus we get

$$\begin{aligned} \langle e|H_1|e\rangle = -eB \int \widetilde{dp}\,\widetilde{dp}'\, d^3x\, e^{i(p-p')x} \\ \times f^*(\mathbf{p}')\overline{u}_+(\mathbf{p}')\Big[x\gamma^2 + \frac{\alpha}{2\pi m}S^{12}\Big]u_+(\mathbf{p})f(\mathbf{p}) \ . \end{aligned} \tag{64.9}$$

We can write the factor of x as $-i\partial_{p1}$ acting on $e^{i(p-p')x}$, and integrate by parts to put this derivative onto $u_+(\mathbf{p})f(\mathbf{p})$; the wave packets kill any surface terms.

Then we can complete the integral over d^3x to get a factor of $(2\pi)^3\delta^3(\mathbf{p}' - \mathbf{p})$, and do the integral over $\widetilde{dp}'$.

The result is

$$\langle e|H_1|e\rangle = -eB \int \frac{\widetilde{dp}}{2\omega}\, f^*(\mathbf{p})\overline{u}_+(\mathbf{p})\Big[\, i\gamma^2\partial_{p_1} + \frac{\alpha}{2\pi m}S^{12}\Big]u_+(\mathbf{p})f(\mathbf{p})\;. \tag{64.11}$$

where $K^j = S^{j0} = i/2\ \gamma^j\gamma^0$ is the boost matrix, $\hat{\mathbf{p}}$ is a unit vector in the $\mathbf{p}$ direction, and $\eta = \sinh^{-1}(|\mathbf{p}|/m)$ is the rapidity.

Since the wave packet is sharply peaked at $\mathbf{p} = 0$, we can expand eq. (64.11) to linear order in $\mathbf{p}$, take the derivative with respect to p_1, and then set $\mathbf{p} = 0$; the result is

$$\begin{aligned} \partial_{p_1} u_+(\mathbf{p})\Big|_{\mathbf{p}=0} &= \frac{i}{m}K^1 u_+(\mathbf{0}) \\ &= -\frac{1}{2m}\gamma^1\gamma^0 u_+(\mathbf{0}) \\ &= -\frac{1}{2m}\gamma^1 u_+(\mathbf{0})\;, \end{aligned} \tag{64.12}$$

where we used $\gamma^0 u_s(0) = u_s(0)$ to get the last line.

Then we have

$$\begin{aligned} \overline{u}_+(\mathbf{p})i\gamma^2\partial_{p_1} u_+(\mathbf{p})\Big|_{\mathbf{p}=0} &= \overline{u}_+(\mathbf{0})\frac{-i}{2m}\gamma^2\gamma^1 u_+(\mathbf{0}) \\ &= \frac{1}{m}\,\overline{u}_+(\mathbf{0})S^{12}u_+(\mathbf{0}) \end{aligned} \tag{64.13}$$

Plugging this into eq. (64.10) yields

$$\begin{aligned}\langle e|H_1|e\rangle &= -eB\int \widetilde{dp}\,\frac{1}{2\omega}\,|f(\mathbf{p})|^2\left(1+\tfrac{\alpha}{2\pi}\right)\tfrac{1}{m}\,\overline{u}_+(\mathbf{0})S^{12}u_+(\mathbf{0}) \\ &= -\frac{eB}{2m^2}\left(1+\tfrac{\alpha}{2\pi}\right)\overline{u}_+(\mathbf{0})S^{12}u_+(\mathbf{0})\ .\end{aligned} \tag{64.14}$$

Next we use $S^{12}u_\pm(\mathbf{0}) = \pm 1/2\ u_\pm(\mathbf{0})$ and $\overline{u}_\pm(\mathbf{0})u_\pm(\mathbf{0}) = 2m$ to get

$$\langle e|H_1|e\rangle = -\frac{eB}{2m}\left(1+\tfrac{\alpha}{2\pi}\right)\ . \tag{64.15}$$

Comparing with eq. (64.7), we see that the magnetic moment of the electron is

$$\mu = g\,\frac{1}{2}\,\frac{e}{2m}\ , \tag{64.16}$$

where e/2m is the *Bohr magneton*, the extra factor of one-half is for the electron's spin (a classical spinning ball of charge would have a magnetic moment equal to the Bohr magneton times its angular momentum), and g is the *Lande g factor*, given by

$$g = 2\Big(1+\tfrac{\alpha}{2\pi}+O(\alpha^2)\Big)\ . \tag{64.17}$$

Since g can be measured to high precision, calculations of μ provide a stringent test of spinor electrodynamics.

Corrections up through the α^4 term have been computed; the result is currently in good agreement with experiment.

65 Loop Corrections in Scalar Electrodynamics

In this section we will compute the one-loop corrections in scalar electrodynamics.

We will concentrate on the divergent parts of the diagrams, enabling us to compute the renormalizing Z factors in the $\overline{MS}$ scheme, and hence the beta functions.

This gives us the most important qualitative information about the theory: whether it becomes strongly coupled at high or low energies.

Our lagrangian for scalar electrodynamics in section 61 already includes all possible terms whose coefficients have positive or zero mass dimension, and that respect Lorentz symmetry, the U(1) gauge symmetry, parity, time reversal, and charge conjugation.

Therefore, the theory we will consider is

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \ , \tag{65.1}$$

$$\mathcal{L}_0 = -\partial^\mu\varphi^\dagger\partial_\mu\varphi - m^2\varphi^\dagger\varphi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \ , \tag{65.2}$$

$$\begin{aligned}\mathcal{L}_1 = {} & iZ_1 e[\varphi^\dagger\partial^\mu\varphi - (\partial^\mu\varphi^\dagger)\varphi]A_\mu - Z_4 e^2\varphi^\dagger\varphi A^\mu A_\mu \\ & - \tfrac{1}{4}Z_\lambda\lambda(\varphi^\dagger\varphi)^2 + \mathcal{L}_{\rm ct} \ ,\end{aligned} \tag{65.3}$$

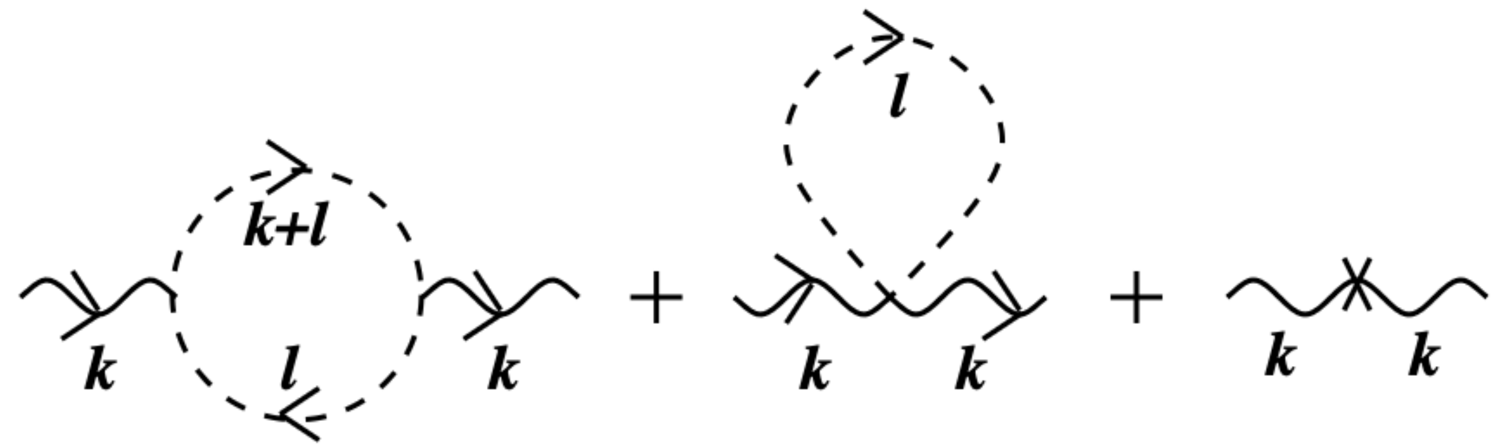
$$\mathcal{L}_{\rm ct} = -(Z_2{-}1)\partial^\mu\varphi^\dagger\partial_\mu\varphi - (Z_m{-}1)m^2\varphi^\dagger\varphi - \tfrac{1}{4}(Z_3{-}1)F^{\mu\nu}F_{\mu\nu} \ . \tag{65.4}$$

We will use the $\overline{MS}$ renormalization scheme to fix the values of the Z's.

We begin with the photon self-energy, $\Pi^{\mu\nu}(k)$.

The one-loop and counterterm contributions are shown in fig. (65.1).

Figure 65.1: The one-loop and counterterm corrections to the photon propagator in scalar electrodynamics.



We have

$$i\Pi^{\mu\nu}(k) = (iZ_1 e)^2\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\,\frac{(2\ell+k)^\mu(2\ell+k)^\nu}{((\ell{+}k)^2+m^2)(\ell^2+m^2)}$$
$$+ (-2iZ_4)e^2 g^{\mu\nu}\int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell^2+m^2} \qquad (65.5)$$
$$- i(Z_3{-}1)(k^2 g^{\mu\nu} - k^\mu k^\mu) + \ldots\ ,$$

where the ellipses stand for higher-order (in e^2 and/or λ) terms.

We can set $Z_i = 1 + O(e^2\ , \lambda)$ in the first two terms.

It will prove convenient to combine these first two terms into

$$
\begin{aligned}
i\Pi^{\mu\nu}(k) = e^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{N^{\mu\nu}}{((\ell{+}k)^2 + m^2)(\ell^2 + m^2)} \\
- i(Z_3{-}1)(k^2 g^{\mu\nu} - k^\mu k^\mu) + \ldots\ ,
\end{aligned}
\tag{65.6}
$$

where

$$
N^{\mu\nu} = (2\ell + k)^\mu (2\ell + k)^\nu - 2[(\ell{+}k)^2 + m^2]g^{\mu\nu}\ . \tag{65.7}
$$

Then we continue to d dimensions, replace e with $e\tilde{\mu}^{\varepsilon/2}$, and combine the denominators with Feynman's formula; the result is

$$
\begin{aligned}
i\Pi^{\mu\nu}(k) = e^2 \tilde{\mu}^{\varepsilon} \int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{N^{\mu\nu}}{(q^2 + D)^2} \\
- i(Z_3{-}1)(k^2 g^{\mu\nu} - k^\mu k^\mu) + \ldots\ ,
\end{aligned}
\tag{65.8}
$$

where $q = \ell + xk$ and $D = x(1-x)k^2 + m^2$.

The numerator is

$$
\begin{aligned}
N^{\mu\nu} &= (2q + (1{-}2x)k)^\mu (2q + (1{-}2x)k)^\nu - 2[(q + (1{-}x)k)^2 + m^2]g^{\mu\nu} \\
&= 4q^\mu q^\nu + (1{-}2x)^2 k^\mu k^\nu - 2[q^2 + (1{-}x)^2 k^2 + m^2]g^{\mu\nu} \\
&\quad + (\text{linear in } q) \\
&\to 4d^{-1} g^{\mu\nu} q^2 + (1{-}2x)^2 k^\mu k^\nu - 2[q^2 + (1{-}x)^2 k^2 + m^2]g^{\mu\nu}\ ,
\end{aligned}
\tag{65.9}
$$

where we used the symmetric-integration identity from problem 14.3 to get the last line.

We can rearrange eq. (65.9) into

$$N^{\mu\nu} = 2\Big(\tfrac{2}{d} - 1\Big) g^{\mu\nu} q^2 + (1-2x)^2 k^\mu k^\nu - 2[(1-x)^2 k^2 + m^2] g^{\mu\nu} \; . \tag{65.10}$$

Now we recall from section 62 that, when q^2 is integrated against $(q^2+D)^{-2}$, we can make the replacement $(2/d - 1)q^2 \to D$; thus we have

$$\begin{aligned} N^{\mu\nu} &\to 2D g^{\mu\nu} + (1-2x)^2 k^\mu k^\nu - 2[(1-x)^2 k^2 + m^2] g^{\mu\nu} \\ &= (1-2x)^2 k^\mu k^\nu - 2(1-2x)(1-x) k^2 g^{\mu\nu} \; . \end{aligned} \tag{65.11}$$

Next we note that if we make the change of variable $x = y + 1/2$, then we have $D = (1-1/4\, y^2)k^2 + m^2$, and y is integrated from $-1/2$ to $+ 1/2$.

Therefore, any term in $N^{\mu\nu}$ that is even in y will integrate to zero.

We then get

$$\begin{aligned} N^{\mu\nu} &= 4y^2 k^\mu k^\nu - 2(2y^2 - y) k^2 g^{\mu\nu} \\ &\to -4y^2 (k^2 g^{\mu\nu} - k^\mu k^\nu) \; . \end{aligned} \tag{65.12}$$

Thus we see that $\Pi^{\mu\nu}(k)$ is transverse, as expected.

Performing the integral over q in eq. (65.8), and focusing on the divergent part, we get

$$\tilde{\mu}^{\varepsilon} \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2+D)^2} = \frac{i}{8\pi^2} \frac{1}{\varepsilon} + O(\varepsilon^0) \ . \tag{65.13}$$

Then performing the integral over y yields

$$\int_{-1/2}^{1/2} dy\, N^{\mu\nu} = -\tfrac{1}{3}(k^2 g^{\mu\nu} - k^{\mu}k^{\nu}) \ . \tag{65.14}$$

Combining eqs. (65.8), (65.13), and (65.14), we get

$$\Pi^{\mu\nu}(k) = \Pi(k^2)(k^2 g^{\mu\nu} - k^{\mu}k^{\nu}) \ , \tag{65.15}$$

where

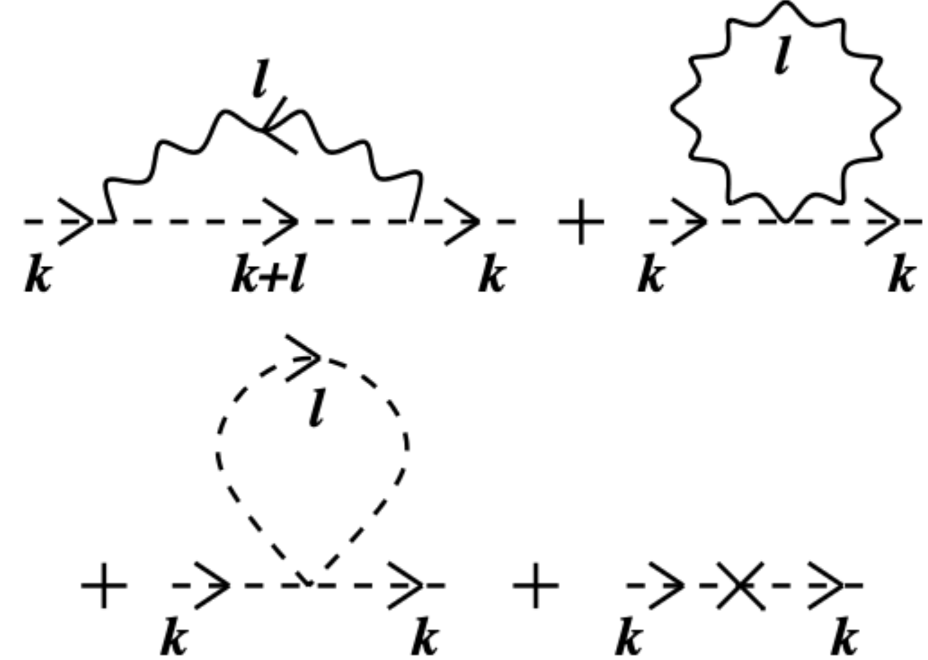
$$\Pi(k^2) = \ - \frac{e^2}{24\pi^2} \frac{1}{\varepsilon} + \text{finite} - (Z_3 - 1) + \ldots \ . \tag{65.16}$$

Thus we find, in the $\overline{MS}$ scheme,

$$Z_3 = 1 - \frac{e^2}{24\pi^2} \frac{1}{\varepsilon} + \ldots \ . \tag{65.17}$$

Now we turn to the one-loop corrections to the scalar propagator, shown in fig. (65.2).

Figure 65.2: The one-loop and counterterm corrections to the scalar propagator in scalar electrodynamics.



It will prove very convenient to work in Lorenz gauge, where the photon propagator is

$$\tilde{\Delta}_{\mu\nu}(\ell) = \frac{P_{\mu\nu}(\ell)}{\ell^2 - i\epsilon} \,, \tag{65.18}$$

with $P_{\mu\nu}(\ell) = g_{\mu\nu} - \ell_\mu \ell_\nu / \ell^2$.

The diagrams in fig. (65.2) then yield

$$\begin{aligned} i\Pi_\varphi(k^2) = {} & (iZ_1 e)^2 \left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \, \frac{P_{\mu\nu}(\ell)(\ell+2k)^\mu(\ell+2k)^\nu}{\ell^2((\ell+k)^2+m^2)} \\ & + (-2iZ_4 e^2 g^{\mu\nu})\left(\tfrac{1}{i}\right) \int \frac{d^4\ell}{(2\pi)^4} \, \frac{P_{\mu\nu}(\ell)}{\ell^2+m_\gamma^2} \\ & + (-i\lambda)\left(\tfrac{1}{i}\right) \int \frac{d^4\ell}{(2\pi)^4} \, \frac{1}{\ell^2+m^2} \\ & - i(Z_2-1)k^2 - i(Z_m-1)m^2 + \ldots \,. \end{aligned} \tag{65.19}$$

In the second line, m_γ is a fictitious photon mass; it appears here as an infrared regulator.

We can set $Z_i = 1 + O(e^2, \lambda)$ in the first three lines.

Continuing to d dimensions, making the replacements $e \to e\tilde{\mu}^{\varepsilon/2}$ and $\lambda \to \lambda\tilde{\mu}^{\varepsilon}$, , and using the relations $\ell^\mu P_{\mu\nu}(\ell) = \ell^\nu P_{\mu\nu}(\ell) = 0$ and $g^{\mu\nu}P_{\mu\nu}(\ell) = d-1$, we get

$$
\begin{aligned}
i\Pi_\varphi(k^2) = 4e^2\tilde{\mu}^\varepsilon \int \frac{d^d\ell}{(2\pi)^d} \frac{P_{\mu\nu}(\ell)k^\mu k^\nu}{\ell^2((\ell+k)^2+m^2)} \\
- 2(d-1)e^2\tilde{\mu}^\varepsilon \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\ell^2+m_\gamma^2} \\
- \lambda\tilde{\mu}^\varepsilon \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2+m^2} \\
- i(Z_2-1)k^2 - i(Z_m-1)m^2 + \ldots \ .
\end{aligned}
\tag{65.20}
$$

We evaluate the second and third lines via

$$
\tilde{\mu}^\varepsilon \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{\ell^2+m^2} = -\frac{i}{8\pi^2}\frac{1}{\varepsilon}m^2 + O(\varepsilon^0) \ . \tag{65.21}
$$

Then, taking the limit $m^2 \to 0$ (with ε fixed) in eq. (65.21) shows that the second line of eq. (65.20) vanishes when the infrared regulator is removed.

To evaluate the first line of eq. (65.20), we multiply the numerator and denominator by ℓ^2 and use Feynman's formula to get

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2 P_{\mu\nu}(\ell)k^\mu k^\nu}{\ell^2\ell^2((\ell+k)^2+m^2)} = \int dF_3 \int \frac{d^dq}{(2\pi)^d} \frac{N}{(q^2+D)^3} \,, \tag{65.22}$$

where $q = \ell + x_3 k$, $D = x_3(1-x_3)k^2 + x_3 m^2$, and

$$\begin{aligned} N &= \ell^2 k^2 - (\ell\cdot k)^2 \\ &= (q - x_3 k)^2 k^2 - (q\cdot k - x_3 k^2)^2 \\ &= q^2 k^2 - (q\cdot k)^2 + (\text{linear in } q) \\ &\to q^2 k^2 - d^{-1} q^2 k^2 \,. \end{aligned} \tag{65.23}$$

Now we use

$$\tilde{\mu}^\varepsilon \int \frac{d^dq}{(2\pi)^d} \frac{q^2}{(q^2+D)^3} = \frac{i}{8\pi^2}\frac{1}{\varepsilon} + O(\varepsilon^0) \,. \tag{65.24}$$

Combining eqs. (65.20–65.24), and requiring $\Pi_\varphi(k^2)$ to be finite, we find

$$Z_2 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\varepsilon} + \ldots \,, \tag{65.25}$$

$$Z_m = 1 + \frac{\lambda}{8\pi^2}\frac{1}{\varepsilon} + \ldots \tag{65.26}$$

in the $\overline{MS}$ scheme.

Now we turn to the one-loop corrections to the three-point (scalar– scalar–photon) vertex, shown in fig. (65.3).

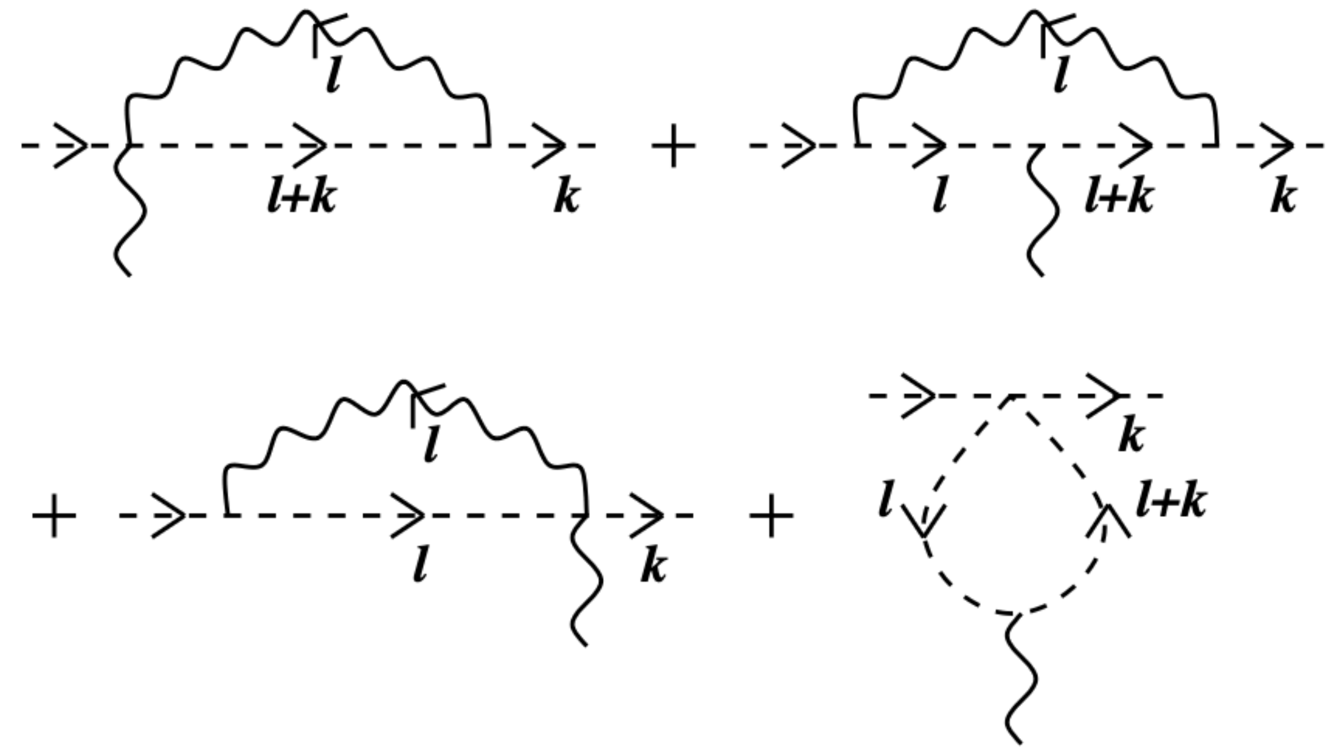


Figure 65.3: The one-loop corrections to the three-point vertex in scalar electrodynamics.

In order to simplify the calculation of the divergent terms as much as possible, we have chosen a special set of external momenta.

(If we wanted the complete vertex function, including the finite terms, we would need to use a general set of external momenta.)

We take the incoming scalar to have zero four-momentum, and the photon (treated as incoming) to have four-momentum k; then, by momentum conservation, the outgoing scalar also has four-momentum k.

We take the internal photon to have four-momentum ℓ.

Now comes the magic of Lorenz gauge: in the second and third diagrams of fig. (65.3), the vertex factor for the leftmost vertex is $ie\ell^\mu$, and this is zero when contracted with the $P_{\mu\nu}(\ell)$ of the internal photon propagator.

Thus the second and third diagrams vanish.

Alas, we will have to do some more work to evaluate the first and fourth diagrams.

We have

$$
\begin{aligned}
i\mathbf{V}_3^{\mu}(k,0) &= ieZ_1 k^{\mu} \\
&\quad + (iZ_1 e)(-2iZ_4 e^2 g^{\mu\nu})\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\,\frac{P_{\nu\rho}(\ell)(\ell+2k)^{\rho}}{\ell^2((\ell{+}k)^2+m^2)} \\
&\quad + (-iZ_\lambda \lambda)(iZ_1 e)\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\,\frac{(2\ell+k)^{\mu}}{(\ell^2+m^2)((\ell{+}k)^2+m^2)} \\
&\quad + \ldots\ .
\end{aligned}
\tag{65.27}
$$

We can set $Z_i = 1 + O(e^2, \lambda)$ in the second and third lines.

We then do the usual manipulations; the integral in the third line becomes

$$
\int_0^1 dx \int \frac{d^dq}{(2\pi)^d}\,\frac{2q^{\mu}+(1{-}2x)k^{\mu}}{(q^2+D)^2}\ , \tag{65.28}
$$

where $q = \ell + xk$ and $D = x(1{-}x)k^2 + m^2$.

The term linear in q vanishes upon integration over q, and the term linear in k vanishes upon integration over x.

Thus the third line of eq. (65.27) evaluates to zero.

To evaluate the second line of eq. (65.27), we note that, since $P_{\nu\rho}(\ell)\ell^\rho = 0$, it already has an overall factor of k.

We can then treat k as infinitesimal, and set k = 0 in the denominator.

We can then use the symmetric-integration identity to make the replacement $\ell_\nu\ell_\rho/\ell 2 \rightarrow d{-}1 g_{\nu\rho}$ in the numerator.

Putting all of this together, and using eq. (65.13), we find

$$\mathbf{V}_3^\mu(k,0)/e = Z_1 k^\mu - \frac{3e^2}{8\pi^2}\frac{1}{\varepsilon}k^\mu + O(\varepsilon^0) + \ldots\ , \tag{65.29}$$

and so

$$Z_1 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\varepsilon} + \ldots\ , \tag{65.30}$$

in the $\overline{MS}$ scheme.

Next up is the four-point, scalar–scalar–photon–photon vertex.

Because the tree-level vertex factor,$-2iZ_4e^2g^{\mu\nu}$, does not depend on the external four-momenta, we can simply set them all to zero.

Then, whenever an internal photon line attaches to an external scalar with a three-point vertex, the diagram is zero, for the same reason that the second and third diagrams of fig. (65.3) were zero.

This kills a lot of diagrams; the survivors are shown in fig. (65.4).

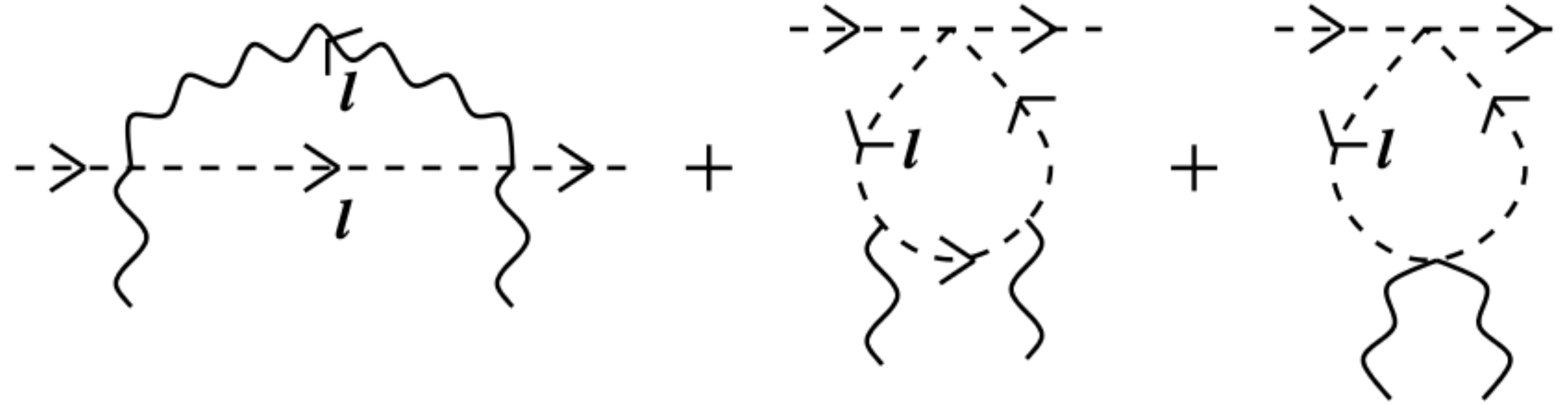


Figure 65.4: The nonvanishing one-loop corrections to the scalar–scalar–photon–photon vertex in scalar electrodynamics (in Lorenz gauge with vanishing external momenta).

We have

$$
\begin{aligned}
i\mathbf{V}_4^{\mu\nu}(0,0,0) = {} & -2iZ_4 e^2 g^{\mu\nu} \\
& + (-2iZ_4 e^2)^2 \left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{g^{\mu\rho} P_{\rho\sigma}(\ell) g^{\sigma\nu}}{\ell^2(\ell^2+m^2)} + (\mu\!\leftrightarrow\!\nu) \\
& + (iZ_1 e)^2 (-iZ_\lambda \lambda) \left(\tfrac{1}{i}\right)^3 \int \frac{d^4\ell}{(2\pi)^4} \frac{(2\ell)^\mu (2\ell)^\nu}{(\ell^2+m^2)^3} + (\mu\!\leftrightarrow\!\nu) \\
& + (-iZ_\lambda \lambda)(-2iZ_4 e^2 g^{\mu\nu}) \left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2+m^2)^2} \\
& + \ldots\,.
\end{aligned}
\tag{65.31}
$$

The notation $+(\mu\leftrightarrow\nu)$ in the second and third lines means that we must add the same expression with these indices swapped; this is because the original and swapped versions of each diagram are topologically distinct, and contribute separately to the vertex function.

As usual, we set $Z_i = 1 + O(e^2, \lambda)$ in the second through fourth lines.

After using the symmetric-integration identity, along with eqs. (65.13) and (65.24), we can see that the divergent parts of the third and fourth lines cancel each other.

The first line is easily evaluated with symmetric integration and eq. (65.13).

Then we have

$$\mathbf{V}_4^{\mu\nu}(0,0,0)/e^2 = -2Z_4 g^{\mu\nu} + \frac{3e^2}{4\pi^2}\frac{1}{\varepsilon}\, g^{\mu\nu} + O(\varepsilon^0) + \ldots\ , \qquad (65.32)$$

and so

$$Z_4 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\varepsilon} + \ldots \qquad (65.33)$$

in the $\overline{MS}$ scheme.

Finally, we have the one-loop corrections to the four-scalar vertex.

Once again, because the tree-level vertex factor, $-iZ_\lambda\lambda$, does not depend on the external four-momenta, we can set them all to zero.

Then, whenever an internal photon line attaches to an external scalar with a three-point vertex, the diagram is zero.

The remaining diagrams are shown in fig. (65.5).

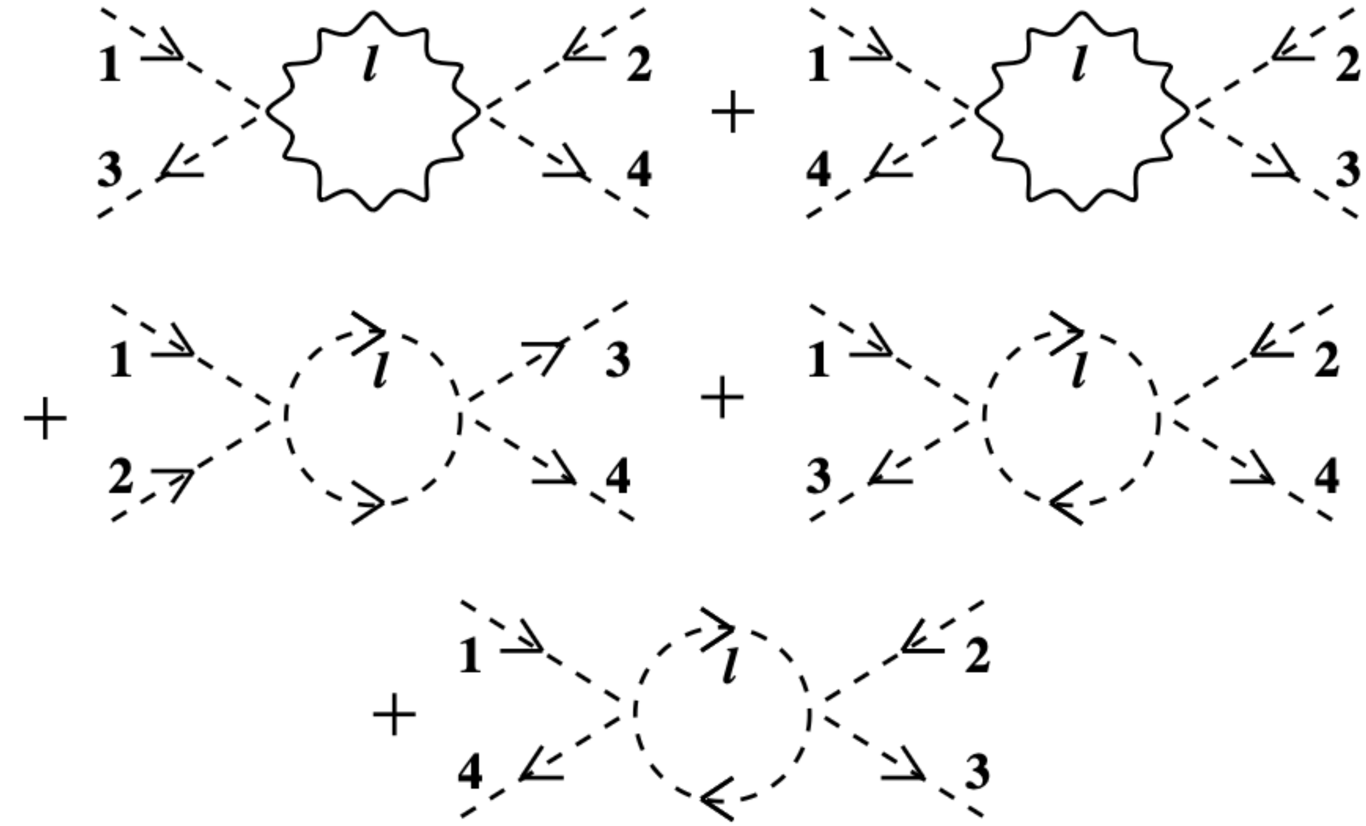


Figure 65.5: The nonvanishing one-loop corrections to the four-scalar vertex in scalar electrodynamics (in Lorenz gauge with vanishing external momenta).

Even though we have set the external momenta to zero, we still have to keep track of which particle is which, in order to count the diagrams correctly; thus the external lines are labelled 1 through 4.

Lines 1 and 2 have arrows pointing towards their vertices, and 3 and 4 have arrows pointing away from their vertices.

The symmetry factor for each of the first three diagrams is S = 2; for each of the last two, it is S = 1.

The difference arises because the last two diagrams have the charge arrows pointing in opposite directions on the two internal propagators, and so these propagators cannot be exchanged.

It is clear that the first two diagrams will yield identical contributions to the vertex function (when the external momenta are all zero).

Similarly, except for symmetry factors, the contributions of the last three diagrams are also identical.

Thus we have

$$\begin{aligned} i\mathbf{V}_{4\varphi}(0,0,0) \;=\; & -iZ_\lambda\lambda \\ & + \left(\tfrac{1}{2}+\tfrac{1}{2}\right)(-2iZ_4e^2)^2\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\,\frac{g^{\mu\nu}P_{\nu\rho}(\ell)g^{\rho\sigma}P_{\rho\mu}(\ell)}{(\ell^2+m_\gamma^2)^2} \\ & + \left(\tfrac{1}{2}+1+1\right)(-iZ_\lambda\lambda)^2\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{(\ell^2+m^2)^2} \\ & + \ldots\,. \end{aligned} \tag{65.34}$$

Using the familiar techniques, we find

$$\mathbf{V}_{4\varphi}(0,0,0) = -iZ_\lambda\lambda + \frac{3e^4}{2\pi^2}\,\frac{1}{\varepsilon} + \frac{5\lambda^2}{16\pi^2}\,\frac{1}{\varepsilon} + O(\varepsilon^0) + \ldots\,, \tag{65.35}$$

and so

$$Z_\lambda = 1 + \left(\frac{3e^4}{2\pi^2\lambda} + \frac{5\lambda}{16\pi^2}\right)\frac{1}{\varepsilon} + \ldots \tag{65.36}$$

in the $\overline{MS}$ scheme.

66 Beta Functions in Quantum Electrodynamics

In this section we will compute the beta function for the electromagnetic coupling e in spinor electrodynamics and scalar electrodynamics.

We will also compute the beta function for the φ^4 coupling λ in scalar electrodynamics.

In spinor electrodynamics, the relation between the bare and renormalized couplings is

$$e_0 = Z_3^{-1/2} Z_2^{-1} Z_1 \, \tilde{\mu}^{\varepsilon/2} e \; . \tag{66.1}$$

It is convenient to recast this formula in terms of the fine-structure constant $\alpha = e^2/4\pi$ and its bare counterpart $\alpha_0 = e^2_0/4\pi$,

$$\alpha_0 = Z_3^{-1} Z_2^{-2} Z_1^2 \, \tilde{\mu}^{\varepsilon} \alpha \; . \tag{66.2}$$

From section 62, we have

$$Z_1 = 1 - \frac{\alpha}{2\pi}\frac{1}{\varepsilon} + O(\alpha^2) \; , \tag{66.3}$$

$$Z_2 = 1 - \frac{\alpha}{2\pi}\frac{1}{\varepsilon} + O(\alpha^2) \; , \tag{66.4}$$

$$Z_3 = 1 - \frac{2\alpha}{3\pi}\frac{1}{\varepsilon} + O(\alpha^2) \; , \tag{66.5}$$

in the $\overline{MS}$ scheme.

Let us write

$$\ln\left(Z_3^{-1} Z_2^{-2} Z_1^2\right) = \sum_{n=1}^{\infty} \frac{E_n(\alpha)}{\varepsilon^n} . \tag{66.6}$$

Then we have

$$\ln \alpha_0 = \sum^{\infty} \frac{E_n(\alpha)}{\varepsilon^n} + \ln \alpha + \varepsilon \ln \tilde{\mu} . \tag{66.7}$$

From eqs. (66.3–66.5), we get

$$E_1(\alpha) = \frac{2\alpha}{3\pi} + O(\alpha^2) . \tag{66.8}$$

Then, the general analysis of section 28 yields

$$\beta(\alpha) = \alpha^2 E_1'(\alpha) , \tag{66.9}$$

where the prime denotes differentiation with respect to α.

Thus we find

$$\beta(\alpha) = \frac{2\alpha^2}{3\pi} + O(\alpha^3) \tag{66.10}$$

in spinor electrodynamics.

We can, if we like, restate this in terms of e as

$$\beta(e) = \frac{e^3}{12\pi^2} + O(e^5) . \tag{66.11}$$

To go from eq. (66.10) to eq. (66.11), we use $\alpha = e^2/4\pi$ and $\dot{\alpha} = e\dot{e}/2\pi$, where the dot denotes $d/d\ln\mu$.

The most important feature of either eq. (66.10) or eq. (66.11) is that the beta function is positive: the electromagnetic coupling in spinor electrodynamics gets stronger at high energies, and weaker at low energies.

It is easy to generalize eqs. (66.10) and (66.11) to the case of N Dirac fields with electric charges $Q_i e$.

There is now a factor of Z_{2i} for each field, and of Z_{1i} for each interaction.

These are found by replacing α in eqs. (66.3) and (66.4) with $Q^2_i\alpha$.

Then we find $Z_{1i}/Z_{2i} = 1 + O(\alpha^2)$, so that this ratio is universal, at least through $O(\alpha)$.

In fact, as we will see in section 67, Z_{1i}/Z_{2i} is always exactly equal to one, and so it always cancels in eq. (66.6).

As for Z_3, now each Dirac field contributes separately to the fermion loop in the photon self-energy, and so we should replace α in eq. (66.5) with $iQ^2_i\alpha$.

Thus we find that the generalization of eq. (66.11) is

$$\beta(e) = \frac{\sum_{i=1}^{N} Q_i^2}{12\pi^2}\, e^3 + O(e^5)\ . \qquad (66.12)$$

Now we turn to scalar electrodynamics.

The relations between the bare and renormalized couplings are

$$e_0 = Z_3^{-1/2} Z_\varphi^{-1} Z_1 \, \tilde{\mu}^{\varepsilon/2} e \,. \tag{66.13}$$

$$e_0^2 = Z_3^{-2} Z_\varphi^{-2} Z_4 \, \tilde{\mu}^{\varepsilon} e^2 \,. \tag{66.14}$$

$$\lambda_0 = Z_\varphi^{-2} Z_\lambda \tilde{\mu}^{\varepsilon} \lambda \,. \tag{66.15}$$

We have two different relations between e and e_0, coming from the two types of vertices.

We can guess (and will demonstrate in section 67) that these two renormalizations must work out to give the same answer. I

ndeed, from section 65, we have

$$Z_1 = 1 + \frac{3e^2}{8\pi^2} \frac{1}{\varepsilon} + \ldots \,, \tag{66.16}$$

$$Z_2 = 1 + \frac{3e^2}{8\pi^2} \frac{1}{\varepsilon} + \ldots \,, \tag{66.17}$$

$$Z_3 = 1 - \frac{e^2}{24\pi^2} \frac{1}{\varepsilon} + \ldots \,, \tag{66.18}$$

$$Z_4 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\varepsilon} + \ldots\ , \tag{66.19}$$

$$Z_\lambda = 1 + \left(\frac{3e^4}{2\pi^2\lambda} + \frac{5\lambda}{16\pi^2}\right)\frac{1}{\varepsilon} + \ldots\ , \tag{66.20}$$

in Lorenz gauge in the $\overline{MS}$ scheme; the ellipses stand for higher powers of e^2 and/or λ.

We see that $Z_1 = Z_2 = Z_4$, at least through $O(e^2, \lambda)$.

The correct guess is that this is true exactly.

Thus eqs. (66.13) and (66.14) both collapse to $e_0 = Z_3^{-1/2}\, e$, just as in spinor electrodynamics.

Thus we can write

$$\ln\left(Z_3^{-1/2}\right) = \sum_{n=1}^{\infty} \frac{E_n(e,\lambda)}{\varepsilon^n}\ , \tag{66.21}$$

$$\ln\left(Z_2^{-2} Z_\lambda\right) = \sum_{n=1}^{\infty} \frac{L_n(e,\lambda)}{\varepsilon^n}\ . \tag{66.22}$$

Then we have

$$\ln e_0 = \sum_{n=1}^{\infty} \frac{E_n(e,\lambda)}{\varepsilon^n} + \ln e + \tfrac{1}{2} e \ln \tilde{\mu}\ , \tag{66.23}$$

$$\ln \lambda_0 = \sum_{n=1}^{\infty} \frac{L_n(e,\lambda)}{\varepsilon^n} + \ln \lambda + e \ln \tilde{\mu}\ . \tag{66.24}$$

Using eqs. (66.17), (66.18) and (66.20), we have

$$E_1(e,\lambda) = \frac{e^2}{24\pi^2} + \ldots\ , \tag{66.25}$$

$$L_1(e,\lambda) = \frac{1}{16\pi^2}\Big(5\lambda + 24e^4/\lambda - 12e^2\Big) + \ldots\ , \tag{66.26}$$

Now applying the general analysis of section 52 yields

$$\beta_e(e,\lambda) = \frac{e^3}{48\pi^2} + \ldots\ , \tag{66.27}$$

$$\beta_\lambda(e,\lambda) = \frac{1}{16\pi^2}\Big(5\lambda^2 - 6\lambda e^2 + 24e^4\Big) + \ldots\ . \tag{66.28}$$

Both right-hand sides are strictly positive, and so both e and λ become large at high energies, and small at low energies.

Generalizing eq. (66.25) to the case of several complex scalar fields with charges $Q_i e$ works in the same way as it does in spinor electrodynamics.

67 Ward Identities in Quantum Electrodynamics I

In section 59, we assumed that scattering amplitudes would be gauge invariant, in the sense that they would be unchanged if we replaced any photon polarization vector ε^μ with $^\mu$ + ck^μ, where k^μ is the photon's four-momentum and c is an arbitrary constant.

Thus, if we write a scattering amplitude $\mathcal{T}$ for a process that includes an external photon with four-momentum k^μ as

$$\mathcal{T} = \varepsilon^\mu \mathcal{M}_\mu \ , \tag{67.1}$$

then we should have

$$k^\mu \mathcal{M}_\mu = 0 \ . \tag{67.2}$$

In this section, we will use the Ward identity for the electromagnetic current to prove eq. (67.2).

We begin by recalling the LSZ formula for scalar fields,

$$\langle f|i\rangle = i\int d^4x_1\, e^{-ik_1x_1}(-\partial_1^2 + m^2)\ldots\langle 0|\mathrm{T}\varphi(x_1)\ldots|0\rangle \ . \tag{67.3}$$

Ve have treated all external particles as outgoing; an incoming particle has $k_i^0 < 0$.

can rewrite eq. (67.3) as

$$\langle f|i\rangle = \lim_{k_i^2\to -m^2}(k_1^2 + m^2)\ldots\langle 0|\mathrm{T}\tilde{\varphi}(k_1)\ldots|0\rangle \ . \tag{67.4}$$

Here $\tilde{\varphi}(k) = i\int d^4x\, e^{-ikx}\varphi(x)$ is the field in momentum space (with an extra factor of i), and we do not fix $k^2 = -m^2$.

We know that the right-hand side of eq. (67.4) must include an overall energy-momentum delta function, so let us write

$$\langle 0|\mathrm{T}\tilde{\varphi}(k_1)\ldots|0\rangle = (2\pi)^4\delta^4(\textstyle\sum_i k_i)\mathcal{F}(k_i^2, k_i\cdot k_j)\ , \tag{67.5}$$

where $\mathcal{F}(k_i^2, k_i\cdot k_j)$ is a function of the Lorentz scalars k_i^2 and $k_i\cdot k_j$.

Then, since

$$\langle f|i\rangle = i(2\pi)^4\delta^4(\textstyle\sum_i k_i)\mathcal{T}\ , \tag{67.6}$$

eq. (67.4) tells us that $\mathcal{F}$ should have a multivariable pole as each k_i^2 approaches $-m^2$, and that $i\,\mathcal{T}$ is the residue of this pole.

That is, near $k_i^2 = -m^2$, $\mathcal{F}$ takes the form

$$\mathcal{F}(k_i^2, k_i\cdot k_j) = \frac{i\mathcal{T}}{(k_1^2+m^2)\ldots(k_n^2+m^2)} + \text{nonsingular}\ . \tag{67.7}$$

The key point is this: contributions to $\mathcal{F}$ that do not have this multivariable pole do not contribute to $\mathcal{T}$.

We have framed this discussion in terms of scalar fields in order to keep the notation as simple as possible, but the general point holds for fields of any spin.

In section 22, we analyzed how various classical field equations apply to quantum correlation functions.

For example, we derived the *Schwinger-Dyson equations*

$$\begin{aligned} &\langle 0|\mathrm{T}\frac{\delta S}{\delta\phi_a(x)}\,\phi_{a_1}(x_1)\ldots\phi_{a_n}(x_n)|0\rangle \\ &\qquad = i\sum_{j=1}^{n}\langle 0|\mathrm{T}\phi_{a_1}(x_1)\ldots\delta_{aa_j}\delta^4(x-x_j)\ldots\phi_{a_n}(x_n)|0\rangle\ . \end{aligned} \tag{67.8}$$

Here we have used $\varphi_a(x)$ to denote any kind of field, not necessarily a scalar field, carrying any kind of index or indices.

The classical equation of motion for the field $\varphi_a(x)$ is $\delta S/\delta\varphi_a(x) = 0$.

Thus, eq. (67.8) tells us that the classical equation of motion holds for a field inside a quantum correlation function, as long as its spacetime argument and indices do not match up exactly with those of any other field in the correlation function.

These matches, which constitute the right-hand side of eq. (67.8), are called contact terms.

Suppose we have a correlation function that, for whatever reason, includes a contact term with a factor of, say, $\delta^4(x_1-x_2)$.

φ $'$ $\mathcal{T}$ $\mathbb{R}$ ρ $\mathcal{L}$ $\int$ $\bar{q}$ $\tilde{q}$ $\widetilde{\Delta}$ $\tilde{\mu}$ $\tilde{\varphi}$ $\widetilde{dk}$ $\overline{MS}$ $\overline{\eta}(x)$ $\overline{\eta}$ $\overline{u}$ $\overline{\Psi}$ the the $\mathcal{F}$ the the the $(\overline{\eta}, \eta, J)$ $\overline{v}$

After Fourier-transforming to momentum space, this contact term is a function of $k_1 + k_2$, but is independent of $k_1 - k_2$; hence it cannot take the form of the singular term in eq. (67.7).

Therefore, contact terms in a correlation function $\mathcal{F}$ *do not contribute to the scattering amplitude* $\mathcal{T}$.

Now let us consider a scattering process in quantum electrodyamics that involves an external photon with four-momentum k.

In Lorenz gauge (the simplest for this analysis), the LSZ formula reads

$$\langle f|i\rangle = i\varepsilon^{\mu}\int d^4x\, e^{-ikx}(-\partial^2)\ldots\langle 0|\mathrm{T}A_{\mu}(x)\ldots|0\rangle\ , \tag{67.9}$$

and the classical equation of motion for A^{μ} is

$$-Z_3\,\partial^2 A_{\mu} = \frac{\partial\mathcal{L}}{\partial A^{\mu}}\ . \tag{67.10}$$

In spinor electrodynamics, the right-hand side of eq. (67.10) is $Z_1 j^{\mu}$, where j^{μ} is the electromagnetic current.

(This is also true for scalar electrodynamics if $Z_4 = Z_1^2/Z_2$; we saw in problem 65.2 that this condition is necessary for gauge invariance.)

We therefore have

$$\langle f|i\rangle = iZ_3^{-1}Z_1\varepsilon^{\mu}\int d^4x\, e^{-ikx}\ldots\Big[\langle 0|\mathrm{T}j_{\mu}(x)\ldots|0\rangle + \text{contact terms}\Big]. \tag{67.11}$$

The contact terms arise because, as we saw in eq. (67.8), the classical equations of motion hold inside quantum correlation functions only up to contact terms.

However, the contact terms cannot generate singularities in the k^2's of the other particles, and so they do not contribute to the left-hand side.

(Remember that, for each of the other particles, there is still an appropriate wave operator, such as the Klein-Gordon wave operator for a scalar, acting on the correlation function. These wave operators kill any term that does not have an appropriate singularity.)

Now let us try replacing ε^μ in eq. (67.11) with k^μ.

We are attempting to prove that the result is zero, and we are almost there.

We can write the factor of ik^μ as $-\partial^\mu$ acting on the e^{-ikx}, and then we can integrate by parts to get ∂^μ acting on the correlation function.

(Strictly speaking, we need a wave packet for the external photon to kill surface terms.)

Then we have $\partial^\mu\langle 0|Tj_\mu(x)...|0\rangle$ on the right-hand side.

Now we use another result from section 22, namely that a Noether current for an exact symmetry obeys $\partial^\mu j_\mu = 0$ classically, and

$$\partial^\mu \langle 0|\mathrm{T} j_\mu(x) \ldots |0\rangle = \text{contact terms} \tag{67.12}$$

quantum mechanically; this is the *Ward (or Ward-Takahashi) identity*.

But once again, the contact terms do not have the right singularities to contribute to $\langle f \mid i \rangle$.

Thus we conclude that $\langle f \mid i \rangle$ vanishes if we replace an external photon's polarization vector ε^μ with its four-momentum k^μ, quo erat demonstratum

68 Ward Identities in Quantum Electrodynamics II

In this section, we will show that $Z_1 = Z_2$ in spinor electrodynamics, and that $Z_1 = Z_2 = Z_4$ in scalar electrodyanmics (in the OS and $\overline{MS}$ renormalization schemes).

Let us specialize to the case of spinor electrodynamics with a single Dirac field, and consider the correlation function

$$C^\mu_{\alpha\beta}(k,p',p) \equiv iZ_1 \int d^4x\, d^4y\, d^4z\, e^{ikx-ip'y+ipz}\, \langle 0|\mathrm{T} j^\mu(x)\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0\rangle\ , \qquad (68.1)$$

where $j^\mu = e\overline{\Psi}\gamma^\mu\Psi$ is the electromagnetic current.

As we saw in section 67, including $Z_1\, j^\mu(x)$ inside a correlation function adds a vertex for an external photon; the factor of Z_1 provides the necessary renormalization of this vertex.

The explicit fermion fields on the right-hand side of eq. (68.1) combine with the fermion fields in the current to generate propagators.

Thus we have

$$C^\mu_{\alpha\beta}(k,p',p) = (2\pi)^4\delta^4(k+p-p')\Big[\tfrac{1}{i}\tilde{\mathbf{S}}(p')i\mathbf{V}^\mu(p',p)\tfrac{1}{i}\tilde{\mathbf{S}}(p)\Big]_{\alpha\beta}\ , \qquad (68.2)$$

where is the exact fermion propagator, and $V\mu(p',p)$ is the exact 1PI photon–fermion–fermion vertex function.

Now let us consider $k_\mu C^\mu{}_{\alpha\beta}(k, p', p)$.

Using eq. (68.1), we can write thefactor of ik_μ on the right-hand side as ∂_μ acting on e^{ikx}, and then integrate by parts to get $-\partial_\mu$ acting on $j^\mu(x)$.

(Strictly speaking, we need a wave packet for the external photon to kill surface terms.)

Thus we have

$$k_\mu C^\mu_{\alpha\beta}(k,p',p) = -\int d^4x\, d^4y\, d^4z\, e^{ikx-ip'y+ipz}\, \partial_\mu \langle 0|\mathrm{T} j^\mu(x)\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0\rangle\ . \tag{68.3}$$

Now we use the Ward identity from section 22, which in general reads

$$\begin{aligned} &-\partial_\mu \langle 0|\mathrm{T} J^\mu(x)\phi_{a_1}(x_1)\ldots\phi_{a_n}(x_n)|0\rangle \\ &\quad = i\sum_{j=1}^{n}\langle 0|\mathrm{T}\phi_{a_1}(x_1)\ldots\delta\phi_{a_j}(x)\delta^4(x-x_j)\ldots\phi_{a_n}(x_n)|0\rangle\ . \end{aligned} \tag{68.4}$$

Here $\delta\varphi_a(x)$ is the change in a field $\varphi_a(x)$ under an infinitesimal transformation that leaves the action invariant, and

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a \tag{68.5}$$

is the corresponding Noether current.

In the case of spinor electrodynamics,

$$\begin{aligned}\delta\Psi(x) &= -ie\Psi(x)\,, \\ \delta\overline{\Psi}(x) &= +ie\overline{\Psi}(x)\,,\end{aligned} \tag{68.6}$$

where we have dropped an infinitesimal parameter on the right-hand sides, but included a factor of the electron charge e.

Using $\partial\mathcal{L}/\partial(\partial_\mu\Psi) = iZ_2\overline{\Psi}\gamma^\mu$ and $\partial\mathcal{L}/\partial(\partial_\mu\overline{\Psi}) = 0$ we find that, with these conventions, the Noether current is $J^\mu = Z_2 e\overline{\Psi}\gamma^\mu\Psi = Z_2 j^\mu$.

Thus the Ward identity becomes

$$\begin{aligned}-Z_2\partial_\mu\langle 0|\mathrm{T}j^\mu(x)\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0\rangle &= +e\,\delta^4(x-y)\langle 0|\mathrm{T}\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0\rangle \\ &\quad -e\,\delta^4(x-z)\langle 0|\mathrm{T}\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0\rangle\,.\end{aligned} \tag{68.7}$$

Recall that

$$\langle 0|\mathrm{T}\Psi_\alpha(y)\overline{\Psi}_\beta(z)|0\rangle = \frac{1}{i}\int\frac{d^4q}{(2\pi)^4}\,e^{iq(y-z)}\,\tilde{\mathbf{S}}(q)_{\alpha\beta}\,. \tag{68.8}$$

Using eqs. (68.7) and (68.8) in eq. (68.3), and carrying out the coordinate integrals, we get

$$k_\mu C^\mu_{\alpha\beta}(k,p',p) = -iZ_2^{-1}Z_1(2\pi)^4\delta^4(k{+}p{-}p')\Big[e\tilde{\mathbf{S}}(p) - e\tilde{\mathbf{S}}(p')\Big]_{\alpha\beta} . \tag{68.9}$$

On the other hand, from eq. (68.2) we have

$$k_\mu C^\mu_{\alpha\beta}(k,p',p) = -i(2\pi)^4\delta^4(k{+}p{-}p')\Big[\tilde{\mathbf{S}}(p')k_\mu \mathbf{V}^\mu(p',p)\tilde{\mathbf{S}}(p)\Big]_{\alpha\beta} . \tag{68.10}$$

Comparing eqs. (68.9) and (68.10) shows that

$$(p'{-}p)_\mu \tilde{\mathbf{S}}(p')\mathbf{V}^\mu(p',p)\tilde{\mathbf{S}}(p) = Z_2^{-1}Z_1 e\Big[\tilde{\mathbf{S}}(p) - \tilde{\mathbf{S}}(p')\Big] , \tag{68.11}$$

where we have dropped the spin indices.

We can simplify eq. (68.11) by multiplying on the left by $\tilde{\mathbf{S}}(p')^{-1}$, and on the right by $\tilde{\mathbf{S}}(p)^{-1}$, to get

$$(p'{-}p)_\mu \mathbf{V}^\mu(p',p) = Z_2^{-1}Z_1 e\Big[\tilde{\mathbf{S}}(p')^{-1} - \tilde{\mathbf{S}}(p)^{-1}\Big] . \tag{68.12}$$

Thus we find a relation between the exact photon–fermion–fermion vertex function $\mathbf{V}^\mu(p', p)$ and the exact fermion propagator $\tilde{\mathbf{S}}(p)$.

Since both $\tilde{\mathbf{S}}(p)$ and $\mathbf{V}^\mu(p', p)$ are finite, eq. (68.12) implies that Z_1/Z_2 must be finite as well.

In the $\overline{MS}$ scheme (where all corrections to $Z_i = 1$ are divergent), this immediately implies that

$$Z_1 = Z_2 . \tag{68.13}$$

In the OS scheme, we recall that near the on-shell point $p^2 = p'^2 = -m^2$ and $(p'- p)^2 = 0$ we have $\tilde{\mathbf{S}}(p) = \not{p} + m$ and $\mathbf{V}^\mu(p', p) = e\gamma^\mu$.

Plugging these expressions into eq. (68.12) then yields $Z_1 = Z_2$ for the OS scheme.

To better understand this result, we note that when $Z_1 = Z_2$, we can combine the fermion kinetic term $iZ_2\overline{\Psi}\not{\partial}\Psi$ and the interaction term $Z_1 e\overline{\Psi}\not{A}\Psi$ into $iZ_2\overline{\Psi}\not{D}\Psi,$, where $D^\mu = \partial^\mu - ieA^\mu$ is the covariant derivative.

Recall that it is D^μ that has a simple gauge transformation, and so we might expect the lagrangian, written in terms of renormalized fields, to include ∂^μ and A^μ only in the combination D^μ.

It is still necessary to go through the analysis that led to eq. (68.12), however, because quantization requires fixing a gauge, and this renders suspect any naive arguments based on gauge invariance.

Still, in this case, those arguments yield the correct result.

A similar analysis is made in scalar electrodynamics.

φ $'$ $\mathcal{T}$ $\mathbb{R}$ ρ $\mathcal{L}$ $\int$ $\bar{q}$ $\tilde{q}$ $\widetilde{\Delta}$ $\tilde{\mu}$ $\tilde{\varphi}$ $\widetilde{dk}$ $\overline{MS}$ $\overline{\eta}(x)$ $\overline{\eta}$ $\overline{u}$ $\overline{\Psi}$ the the the the the the $(\overline{\eta}, \eta, J)$ $\mathcal{F}$ $\overline{\upsilon}$

69 Nonabelian Gauge Theory

Consider a lagrangian with N scalar or spinor fields $\phi_i(x)$ that is invariant under a continuous SU(N) or SO(N) symmetry,

$$\phi_i(x) \to U_{ij}\phi_j(x) \ , \tag{69.1}$$

where U_{ij} is an $N \times N$ special unitary matrix in the case of SU(N), or an $N \times N$ special orthogonal matrix in the case of SO(N).

(Special means that the determinant of U is one.)

Eq. (69.1) is called a global symmetry transformation, because the matrix U does not depend on the spacetime label x.

In section 58, we saw that quantum electrodynamics could be understood as having a *local* U(1) symmetry,

$$\phi(x) \to U(x)\phi(x) \ , \tag{69.2}$$

where $U(x) = \exp[-ie\Gamma(x)]$ can be thought of as a 1×1 unitary matrix that *does* depend on the spacetime label x.

Eq. (69.2) can be a symmetry of the lagrangian only if we include a U(1) *gauge field* $A_\mu(x)$, and promote ordinary derivatives ∂_μ of $\phi(x)$ to covariant derivatives $D_\mu = \partial_\mu - ieA_\mu$.

Under the transformation of eq. (69.2), we have

$$D_\mu \to U(x) D_\mu U^\dagger(x) \; . \tag{69.3}$$

With this transformation rule, a scalar kinetic term like $-(D_\mu\phi)^\dagger D_\mu\phi$, or a fermion kinetic term like $i\overline{\Psi}\not{D}\Psi$, is invariant, as are mass terms like $m^2\phi^\dagger\phi$ and m $\overline{\Psi}\,\Psi$.

We call eq. (69.3) a *gauge transformation*, and say that the lagrangian is *gauge invariant*.

Eq. (69.3) implies that the gauge field transforms as

$$A_\mu(x) \to U(x) A_\mu(x) U^\dagger(x) + \tfrac{i}{e} U(x) \partial_\mu U^\dagger(x) \; . \tag{69.4}$$

If we use $U(x) = \exp[-ie\Gamma(x)]$, then eq. (69.4) simplifies to

$$A_\mu(x) \to A_\mu(x) - \partial_\mu \Gamma(x) \; , \tag{69.5}$$

which is what we originally had in section 54.

We can now easily generalize this construction of U(1) gauge theory to SU(N) or SO(N).

(We will consider other possibilities later.)

To be concrete, let us consider SU(N).

Recall from section 24 that we can write an infinitesimal SU(N) transformation as

$$U_{jk}(x) = \delta_{jk} - ig\theta^a(x)(T^a)_{jk} + O(\theta^2)\ , \tag{69.6}$$

where we have inserted a coupling constant g for later convenience.

The indices j and k run from 1 to N, the index a runs from 1 to $N^2 - 1$ (and is implicitly summed), and the *generator matrices* T^a are hermitian and traceless.

(These properties of T^a follow immediately from the special unitarity of U.)

The generator matrices obey commutation relations of the form

$$[T^a, T^b] = if^{abc}T^c\ , \tag{69.7}$$

where the real numerical factors f_{abc} are called the structure coefficients of the group.

If f_{abc} does not vanish, the group is *nonabelian*.

We can choose the generator matrices so that they obey the normalization condition

$$\mathrm{Tr}(T^aT^b) = \tfrac{1}{2}\delta^{ab}\ ; \tag{69.8}$$

then eqs. (69.7) and (69.8) can be used to show that f_{abc} is completely antisymmetric.

For SU(2), we have $T^a = 1/2\ \sigma^a$, where σ^a is a Pauli matrix, and $f_{abc} = \varepsilon_{abc}$, where ε_{abc} is the completely antisymmetric Levi-Civita symbol.

Now we define an SU(N) gauge field $A_\mu(x)$ as a traceless hermitian $N \times N$ matrix of fields with the gauge transformation property

$$A_\mu(x) \to U(x)A_\mu(x)U^\dagger(x) + \tfrac{i}{g}U(x)\partial_\mu U^\dagger(x) \ . \tag{69.9}$$

Note that this is identical to eq. (69.4), except that now U(x) is a special unitary matrix (rather than a phase factor), and $A_\mu(x)$ is a traceless hermitian matrix (rather than a real number).

(Also, the electromagnetic coupling e has been replaced by g.)

We can write U(x) in terms of the generator matrices as

$$U(x) = \exp[-ig\Gamma^a(x)T^a] \ , \tag{69.10}$$

where the real parameters $\Gamma^a(x)$ are no longer infinitesimal.

The covariant derivative is

$$D_\mu = \partial_\mu - igA_\mu(x) \ , \tag{69.11}$$

where there is an understood $N \times N$ identity matrix multiplying ∂_μ.

Acting on the set of N fields $\phi_i(x)$ that transform according to eq. (69.2), the covariant derivative can be written more explicitly as

$$(D_\mu\phi)_j(x) = \partial_\mu\phi_j(x) - igA_\mu(x)_{jk}\phi_k(x) \;, \tag{69.12}$$

with an understood sum over k.

The covariant derivative transforms according to eq. (69.3).

Replacing all ordinary derivatives in $\mathcal{L}$ with covariant derivatives renders $\mathcal{L}$ gauge invariant (assuming, of course, that $\mathcal{L}$ originally had a global SU(N) symmetry).

We still need a kinetic term for $A_\mu(x)$.

Let us define the *field strength*

$$F_{\mu\nu}(x) \equiv \frac{i}{g}[D_\mu, D_\nu] \tag{69.13}$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \;. \tag{69.14}$$

Because A_μ is a matrix, the final term in eq. (69.14) does not vanish, as it does in U(1) gauge theory.

Eqs. (69.3) and (69.13) imply that, under a gauge transformation,

$$F_{\mu\nu}(x) \to U(x)F_{\mu\nu}(x)U^\dagger(x) \;. \tag{69.15}$$

Therefore,

$$\mathcal{L}_{\rm kin} = -\tfrac{1}{2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) \tag{69.16}$$

is gauge invariant, and can serve as a kinetic term for the SU(N) gauge field.

(Note, however, that the field strength itself is not gauge invariant, in contrast to the situation in U(1) gauge theory.)

Since we have taken $A_\mu(x)$ to be hermitian and traceless, we can expand it in terms of the generator matrices

$$A_\mu(x) = A^a_\mu(x)T^a \ . \tag{69.17}$$

Then we can use eq. (69.8) to invert eq. (69.17):

$$A^a_\mu(x) = 2\,\mathrm{Tr}\,A_\mu(x)T^a \ . \tag{69.18}$$

Similarly, we have

$$F_{\mu\nu}(x) = F^a_{\mu\nu}T^a \ , \tag{69.19}$$

$$F^a_{\mu\nu}(x) = 2\,\mathrm{Tr}\,F_{\mu\nu}T^a \ . \tag{69.20}$$

Using eq. (69.19) in eq. (69.14), we get

$$\begin{aligned} F^c_{\mu\nu}T^c &= (\partial_\mu A^c_\nu - \partial_\nu A^c_\mu)T^c - igA^a_\mu A^b_\nu[T^a,T^b] \\ &= (\partial_\mu A^c_\nu - \partial_\nu A^c_\mu + gf^{abc}A^a_\mu A^b_\nu)T^c \ . \end{aligned} \tag{69.21}$$

Then using eqs. (69.20) and (69.8) yields

$$F^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g f^{abc} A^a_\mu A^b_\nu \ . \tag{69.22}$$

Also, using eqs. (69.19) and (69.8) in eq. (69.16), we get

$$\mathcal{L}_{\text{kin}} = -\tfrac{1}{4} F^{c\mu\nu} F^c_{\mu\nu} \ . \tag{69.23}$$

From eq. (69.22), we see that $\mathcal{L}_{\text{kin}}$ includes interactions among the gauge fields.

A theory of this type, with nonzero fabc, is called *nonabelian gauge theory* or *Yang–Mills theory*.

Everything we have just said about SU(N) also goes through for SO(N), with unitary replaced by orthogonal, and traceless replaced by antisymmetric.

There is also another class of compact nonabelian groups called Sp(2N), and five exceptional compact groups: G(2), F(4), E(6), E(7), and E(8).

Compact means that $\mathrm{Tr}(T^a T^b)$ is a positive definite matrix.

Non-abelian gauge theory must be based on a compact group, because otherwise some of the terms in $\mathcal{L}_{\text{kin}}$ would have the wrong sign, leading to a hamiltonian that is unbounded below.

As a specific example, let us consider *quantum chromodynamics*, or QCD, which is based on the gauge group SU(3).

There are several Dirac fields corresponding to *quarks*.

Each quark comes in three *colors*; these are the values of the SU(3) index.

(These colors have nothing to do with ordinary color.)

There are also six *flavors*: up, down, strange, charm, bottom (or beauty) and top (or truth).

Thus we consider the Dirac field $\Psi_{iI}(x)$, where i is the color index and I is the flavor index.

The lagrangian is

$$\mathcal{L} = i\overline{\Psi}_{iI} \not{D}_{ij} \Psi_{jI} - m_I \overline{\Psi}_I \Psi_I - \tfrac{1}{2}\mathrm{Tr}(F^{\mu\nu} F_{\mu\nu}) \ , \qquad (69.24)$$

where all indices are summed.

The different quark flavors have different masses, ranging from a few MeV for the up and down quarks to 178 GeV for the top quark.

(The quarks also have electric charges: +2/3 |e| for the u, c, and t quarks, and −1/3 |e| for the d, s, and b quarks. For now, however, we omit the appropriate coupling to the electromagnetic field.)

The covariant derivative in eq. (69.24) is

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - igA^a_\mu T^a_{ij} \ . \qquad (69.25)$$

The index a on A^a_μ runs from 1 to 8, and the corresponding massless spin-one particles are the eight *gluons*.

In a nonabelian gauge theory in general, we can consider scalar or spinor fields in different *representations* of the group.

A representation of a compact nonabelian group is a set of finite-dimensional hermitian matrices T^a_R (the R is part of the name, not an index) that obey that same commutation relations as the original generator matrices T^a.

Given such a set of $D(R) \times D(R)$ matrices (where $D(R)$ is the *dimension* of the representation), and a field $\phi(x)$ with $D(R)$ components, we can write its covariant derivative as $D_\mu = \partial_\mu - igA^a_\mu T^a_R$, with an understood $D(R) \times D(R)$ identity matrix multiplying ∂_μ.

Under a gauge transformation, $\phi(x) \to U_R(x)\phi(x)$, where $U_R(x)$ is given by eq. (69.10) with T^a replaced by T^a_R .

The theory will be gauge invariant provided that the transformation rule for A^a_μ is independent of the representation used in eq. (69.9); it is.

We will not need to know a lot of representation theory, but we collect some useful facts in the next section.

70 Group Representations

Given the structure coefficients f_{abc} of a compact nonabelian group, a *representation* of that group is specified by a set of D(R) × D(R) traceless hermitian matrices T^a_R (the R is part of the name, not an index) that obey that same commutation relations as the original generators matrices T^a, namely

$$[T^a_{\rm R}, T^b_{\rm R}] = i f^{abc} T^c_{\rm R} \,. \tag{70.1}$$

The number D(R) is the *dimension* of the representation.

The original T^a's correspond to the *fundamental* or *defining representation*.

Consider taking the complex conjugate of the commutation relations, eq. (70.1).

Since the structure coefficients are real, we see that the matrices $-(T^a_R)^*$ also obey these commutation relations.

If $-(T^a_R)^* = T^a_R$ or if we can find a unitary transformation $T^a_R \to U^{-1} T^a_R U$ that makes $-(T^a_R)^* = T^a_R$ for every a, then the representation R is *real*.

If such a unitary transformation does not exist, but we can find a unitary matrix $V \neq I$ such that $-(T^a_R)^* = V^{-1} T^a_R V$ for every a, then the representation R is *pseudoreal*.

If such a unitary matrix also does not exist, then the representation R is *complex*.

In this case, the *complex conjugate representation* R is specified by

$$T^a_{\overline{\mathrm{R}}} = -(T^a_{\mathrm{R}})^* \ . \tag{70.2}$$

One way to prove that a representation is complex is to show that at least one generator matrix T^a_R (or a real linear combination of them) has eigenvalues that do not come in plus-minus pairs.

This is the case for the fundamental representation of SU(N) with $N \geq 3$.

For SU(2), the fundamental representation is pseudoreal, because $-(1/2\ \sigma^a)^* \neq 1/2\ \sigma_a$, but $-(1/2\ \sigma_a)^* = V^{-1}(1/2\ \sigma_a)V$ with $V = \sigma_2$.

For SO(N), the fundamental representation is real, because the generator matrices are antisymmetric, and every antisymmetric hermitian matrix is equal to minus its complex conjugate.

An important representation for any compact nonabelian group is the *adjoint representation* A.

This is given by

$$(T^a_{\mathrm{A}})^{bc} = -if^{abc} \ . \tag{70.3}$$

Because f_{abc} is real and completely antisymmetric, T^a_A is manifestly hermitian, and also satisfies eq. (70.2); thus the adjoint representation is real.

The dimension of the adjoint representation D(A) is equal to the number of generators of the group; this number is also called the dimension of the group.

To see that the $T^a{}_A$'s satisfy the commutation relations, we use the Jacobi identity

$$f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0\,, \tag{70.4}$$

which holds for the structure coefficients of any group.

To prove the Jacobi identity, we note that

$$\mathrm{Tr}\, T^e \Big([[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] \Big) = 0\,, \tag{70.5}$$

where the T^a's are the original generator matrices.

That the left-hand side of eq. (70.5) vanishes can be seen by writing out all the commutators as matrix products, and noting that they cancel in pairs.

Employing the commutation relations twice in each term, followed by

$$\mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\delta^{ab}\,, \tag{70.6}$$

ultimately yields eq. (70.4).

Then, using the antisymmetry of the structure coefficients, inserting some judicious factors of i, and moving the last term of eq. (70.4) to the right-hand side, we can rewrite it as

$$(-if^{abd})(-if^{cde}) - (-if^{cbd})(-if^{ade}) = if^{acd}(-if^{dbe})\,. \tag{70.7}$$

Now we use eq. (70.3) in eq. (70.7) to get

$$(T^a_{\text{A}})^{bd}(T^c_{\text{A}})^{de} - (T^c_{\text{A}})^{bd}(T^a_{\text{A}})^{de} = if^{acd}(T^d_{\text{A}})^{be} \ , \tag{70.8}$$

or equivalently $[T^a_A, T^c_A] = if_{acd}T^d_A$.

Thus the T^a_A's satisfy the appropriate commutation relations.

Two related numbers usefully characterize a representation: the *index* T(R) and the *quadratic Casimir* C(R).

The index is defined via

$$\text{Tr}(T^a_{\text{R}}T^b_{\text{R}}) = T(\text{R})\delta^{ab} \ . \tag{70.9}$$

Now it can be shown that the matrix $T^a_R T^a_R$ commutes with every generator, and so must be a number times the identity matrix; this number is the quadratic Casimir C(R).

It is easy to show that

$$T(\text{R})D(\text{A}) = C(\text{R})D(\text{R}) \ . \tag{70.10}$$

With the standard normalization conventions for the generators, we have T(N) = 1/2 for the fundamental representation of SU(N) and T(N) = 2 for the fundamental representation of SO(N).

It can be shown that T(A) = N for the adjoint representation of SU(N), and also shown that T(A) = 2N– 4 for the adjoint representation of SO(N).

A representation R is reducible if there is a unitary transformation $T^a_R \rightarrow U^{-1} T^a_R U$ that puts all the nonzero entries into the same diagonal blocks for each a; otherwise it is irreducible.

Consider a reducible representation R whose generators can be put into (for example) two blocks, with the blocks forming the generators of the irreducible representations R_1 and R_2.

Then R is the direct sum representation $R = R_1 \oplus R_2$, and we have

$$D(\mathrm{R}_1 \oplus \mathrm{R}_2) = D(\mathrm{R}_1) + D(\mathrm{R}_2) \ , \tag{70.11}$$

$$T(\mathrm{R}_1 \oplus \mathrm{R}_2) = T(\mathrm{R}_1) + T(\mathrm{R}_2) \ . \tag{70.12}$$

Suppose we have a field $\varphi_{iI}(x)$ that carries two group indices, one for the representation R_1 and one for the representation R_2, denoted by i and I respectively.

This field is in the direct product representation $R_1 \oplus R_2$. The corresponding generator matrix is

$$(T^a_{\mathrm{R}_1 \otimes \mathrm{R}_2})_{iI,jJ} = (T^a_{\mathrm{R}_1})_{ij}\delta_{IJ} + \delta_{ij}(T^a_{\mathrm{R}_2})_{IJ} \ , \tag{70.13}$$

where i and I together constitute the row index, and j and J together constitute the column index.

We then have

$$D(\mathrm{R}_1 \otimes \mathrm{R}_2) = D(\mathrm{R}_1) D(\mathrm{R}_2) \ , \tag{70.14}$$

$$T(\mathrm{R}_1 \otimes \mathrm{R}_2) = T(\mathrm{R}_1) D(\mathrm{R}_2) + D(\mathrm{R}_1) T(\mathrm{R}_2) \ . \tag{70.15}$$

To get eq. (70.15), we need to use the fact that the generator matrices are traceless, $(T^a_R)_{ii} = 0$.

At this point it is helpful to introduce a slightly more refined notation for the indices of a complex representation.

Consider a field φ in the complex representation R.

We will adopt the convention that such a field carries a "down" index: φ_i, where $i = 1, 2, \ldots, D(R)$.

Hermitian conjugation changes the representation from R to $\overline{R}$, and we will adopt the convention that this also raises the index on the field,

$$(\varphi_i)^\dagger = \varphi^{\dagger\, i} \,. \tag{70.16}$$

Thus a down index corresponds to the representation R, and an up index to $\overline{R}$.

Indices can be contracted only if one is up and one is down.

Generator matrices for R are then written with the first index down and the second index up: $(T^a_R)_i{}^j$.

An infinitesimal group transformation of φ_i takes the form

$$\begin{aligned} \varphi_i &\to (1 - i\theta^a T^a_{\mathrm{R}})_i{}^j \varphi_j \\ &= \varphi_i - i\theta^a (T^a_{\mathrm{R}})_i{}^j \varphi_j \,. \end{aligned} \tag{70.17}$$

The generator matrices for $\overline{\mathrm{R}}$ are then given by

$$(T^a_{\overline{\mathrm{R}}})^i{}_j = -(T^a_{\mathrm{R}})_j{}^i \ , \tag{70.18}$$

where we have used the hermiticity to trade complex conjugation for transposition of the indices.

An infinitesimal group transformation of $\varphi^{\dagger i}$ takes the form

$$\begin{aligned} \varphi^{\dagger i} &\to (1 - i\theta^a T^a_{\overline{\mathrm{R}}})^i{}_j \varphi^{\dagger j} \\ &= \varphi^{\dagger i} - i\theta^a (T^a_{\overline{\mathrm{R}}})^i{}_j \varphi^{\dagger j} \\ &= \varphi^{\dagger i} + i\theta^a (T^a_{\mathrm{R}})_j{}^i \varphi^{\dagger j} \ , \end{aligned} \tag{70.19}$$

where we used eq. (70.18) to get the last line.

Note that eqs. (70.17) and (70.19) together imply that $\varphi^{\dagger i} \varphi_i$ is invariant, as expected.

Consider the Kronecker delta symbol with one index down and one up: $\delta_i{}^j$.

Under a group transformation, we have

$$\begin{aligned} \delta_i{}^j &\to (1 + i\theta^a T^a_{\mathrm{R}})_i{}^k (1 + i\theta^a T^a_{\overline{\mathrm{R}}})^j{}_l \delta_k{}^l \\ &= (1 + i\theta^a T^a_{\mathrm{R}})_i{}^k \delta_k{}^l (1 - i\theta^a T^a_{\mathrm{R}})_l{}^j \\ &= \delta_i{}^j + O(\theta^2) \ . \end{aligned} \tag{70.20}$$

Eq. (70.20) shows that $\delta_i{}^j$ is an *invariant symbol* of the group.

This existence of this invariant symbol, which carries one index for R and one for $\overline{\mathrm{R}}$, tells us that the product of the representations R and $\overline{\mathrm{R}}$ must contain the *singlet* representation 1, specified by $T_1^a = 0$.

We therefore can write

$$\mathrm{R} \otimes \overline{\mathrm{R}} = 1 \oplus \ldots \,. \tag{70.21}$$

The generator matrix $(T^a_{\mathrm{R}})_i{}^j$, which carries one index for R, one for $\overline{\mathrm{R}}$, and one for the adjoint representation A, is also an invariant symbol.

To see this, we make a simultaneous infinitesimal group transformation on each of these indices,

$$\begin{aligned}(T^b_{\mathrm{R}})_i{}^j &\to (1 - i\theta^a T^a_{\mathrm{R}})_i{}^k (1 - i\theta^a T^a_{\overline{\mathrm{R}}})^j{}_l (1 - i\theta^a T^a_{\mathrm{A}})^{bc} (T^c_{\mathrm{R}})_k{}^l \\ &= (T^b_{\mathrm{R}})_i{}^j - i\theta^a [(T^a_{\mathrm{R}})_i{}^k (T^b_{\mathrm{R}})_k{}^j + (T^a_{\overline{\mathrm{R}}})^j{}_l (T^b_{\mathrm{R}})_i{}^l + (T^a_{\mathrm{A}})^{bc} (T^c_{\mathrm{R}})_i{}^j] \\ &\quad + O(\theta^2)\,.\end{aligned} \tag{70.22}$$

The factor in square brackets should vanish if (as we claim) the generator matrix is an invariant symbol.

Using eqs. (70.3) and (70.18), we have

$$\begin{aligned}[\ldots] &= (T^a_{\mathrm{R}})_i{}^k (T^b_{\mathrm{R}})_k{}^j - (T^a_{\mathrm{R}})_l{}^j (T^b_{\mathrm{R}})_i{}^l - i f^{abc} (T^c_{\mathrm{R}})_i{}^j \\ &= (T^a_{\mathrm{R}} T^b_{\mathrm{R}})_i{}^j - (T^b_{\mathrm{R}} T^a_{\mathrm{R}})_i{}^j - i f^{abc} (T^c_{\mathrm{R}})_i{}^j \\ &= 0\,,\end{aligned} \tag{70.23}$$

where the last line follows from the commutation relations.

The fact that $(T^a{}_R)_i{}^j$ is an invariant symbol implies that

$$R \otimes \overline{R} \otimes A = 1 \oplus \ldots . \tag{70.24}$$

If we now multiply both sides of eq. (70.24) by A, and use $A \otimes A = 1 \oplus \ldots$ [which follows from eq. (70.21) and the reality of A], we find $R \otimes \overline{R} = A \oplus \ldots$.

Combining this with eq. (70.21), we have

$$R \otimes \overline{R} = 1 \oplus A \oplus \ldots . \tag{70.25}$$

That is, the product of a representation with its complex conjugate is always reducible into a sum that includes (at least) the singlet and adjoint representations.

For the fundamental representation N of SU(N), we have

$$N \otimes \overline{N} = 1 \oplus A\ , \tag{70.26}$$

with no other representations on the right-hand side.

To see this, recall that $D(1) = 1$, $D(N) = D(\overline{N}) = N$, and, as shown in section 24, $D(A) = N^2-1$.

From eq. (70.14), we see that there is no room for anything else on the right-hand side of eq. (70.26).

Consider now a real representation R.

From eq. (70.25), with $\overline{\mathrm{R}} = \mathrm{R}$, we have

$$\mathrm{R} \otimes \mathrm{R} = 1 \oplus \mathrm{A} \oplus \ldots . \tag{70.27}$$

The singlet on the right-hand side implies the existence of an invariant symbol with two R indices; this symbol is the Kronecker delta δ_{ij}.

It is invariant because

$$\begin{aligned} \delta_{ij} &\to (1 - i\theta^a T^a_{\mathrm{R}})_i{}^k (1 - i\theta^a T^a_{\mathrm{R}})_j{}^l \delta_{kl} \\ &= \delta_{ij} - i\theta^a [(T^a_{\mathrm{R}})_{ij} + (T^a_{\mathrm{R}})_{ji}] + O(\theta^2) . \end{aligned} \tag{70.28}$$

The term in square brackets vanishes by hermiticity and eq. (70.18).

The fact that $\delta_{ij} = \delta_{ji}$ implies that the singlet on the right-hand side of eq. (70.28) appears in the symmetric part of this product of two identical representations.

The fundamental representation N of SO(N) is real, and we have

$$\mathrm{N} \otimes \mathrm{N} = 1_{\mathrm{S}} \oplus \mathrm{A}_{\mathrm{A}} \oplus \mathrm{S}_{\mathrm{S}} . \tag{70.29}$$

The subscripts tell whether the representation appears in the symmetric or antisymmetric part of the product.

The representation S corresponds to a field with a symmetric traceless pair of fundamental indices: $\phi_{ij} = \phi_{ji}$, $\phi_{ii} = 0$, where the repeated index is summed.

We have D(1) = 1, D(N) = N, and, as shown in section 24, D(A) = 1/2 N(N−1).

Also, a traceless symmetric tensor has D(S) = 1/2 N(N+1) −1 independent components; thus eq. (70.14) is fulfilled.

Consider now a pseudoreal representation R.

Since R is equivalent to its complex conjugate, up to a change of basis, eq. (70.27) still holds.

However, we cannot identify δ_{ij} as the corresponding invariant symbol, because then eq. (70.28) shows that R would have to be real, rather than pseudoreal.

From the perspective of the direct product, the only alternative is to have the singlet appear in the antisymmetric part of the product, rather than the symmetric part.

The corresponding invariant symbol must then be antisymmetric on exchange of its two R indices.

An example (the only one that will be of interest to us) is the fundamental representation of SU(2).

For SU(N) in general, another invariant symbol is the Levi-Civita tensor $\varepsilon_{i_1 \ldots i_N}$, which carries N fundamental indices and is completely antisymmetric.

It is invariant because, under an SU(N) transformation,

$$\begin{aligned} \varepsilon_{i_1 \ldots i_N} &\to U_{i_1}{}^{j_1} \ldots U_{i_N}{}^{j_N} \varepsilon_{j_1 \ldots j_N} \\ &= (\det U) \varepsilon_{i_1 \ldots i_N} \ . \end{aligned} \tag{70.30}$$

Since det U = 1 for SU(N), we see that the Levi-Civita symbol is invariant.

We can similarly consider $\varepsilon^{i_1 \ldots i_N}$, which carries N completely antisymmetric antifundamental indices.

For SU(2), the Levi-Civita symbol is $\varepsilon_{ij} = -\varepsilon_{ji}$; this is the two-index invariant symbol that corresponds to the singlet in the product

$$2 \otimes 2 = 1_\mathrm{A} \oplus 3_\mathrm{S} \ , \tag{70.31}$$

where 3 is the adjoint representation.

We can use ε_{ij} and ε_{ij} to raise and lower SU(2) indices.

This is another way to see that there is no distinction between the fundamental representation 2 and its complex conjugate $\overline{2}$.

That is, if we have a field φ_i in the representation 2, we can get a field in the representation $\overline{2}$ by raising the index: $\varphi^i = \varepsilon^{ij}\varphi_j$.

The structure constants f^{abc} are another invariant symbol.

This follows from $(T_A{}^a)^{bc} = -if^{abc}$, since we have seen that generator matrices (in any representation) are invariant.

Alternatively, given the generator matrices in a representation R, we can write

$$T(\mathrm{R})f^{abc} = -i\,\mathrm{Tr}(T^a_{\mathrm{R}}[T^b_{\mathrm{R}}, T^c_{\mathrm{R}}])\ . \tag{70.32}$$

Since the right-hand side is invariant, the left-hand side must be as well.

If we use an anticommutator in place of the commutator in eq. (70.32), we get another invariant symbol,

$$A(\mathrm{R})d^{abc} \equiv \tfrac{1}{2}\mathrm{Tr}(T^a_{\mathrm{R}}\{T^b_{\mathrm{R}}, T^c_{\mathrm{R}}\})\ , \tag{70.33}$$

where A(R) is the *anomaly coefficient* of the representation.

The cyclic property of the trace implies that $A(R)d^{abc}$ is symmetric on exchange of any pair of indices.

Using eq. (70.18), we can see that

$$A(\overline{\mathrm{R}}) = -A(\mathrm{R})\ . \tag{70.34}$$

Thus, if R is real or pseudoreal, A(R) = 0.

We also have

$$A(\mathrm{R}_1 \oplus \mathrm{R}_2) = A(\mathrm{R}_1) + A(\mathrm{R}_2) \, , \tag{70.35}$$

$$A(\mathrm{R}_1 \otimes \mathrm{R}_2) = A(\mathrm{R}_1) D(\mathrm{R}_2) + D(\mathrm{R}_1) A(\mathrm{R}_2) \, . \tag{70.36}$$

We normalize the anomaly coefficient so that it equals one for the smallest complex representation.

In particular, for SU(N) with $N \geq 3$, the smallest complex representation is the fundamental, and $A(N) = 1$.

For SU(2), all representations are real or pseudoreal, and $A(R) = 0$ for all of them.

71 The Path Integral for Nonabelian Gauge Theory

We wish to evaluate the path integral for nonabelian gauge theory (also known as Yang–Mills theory),

$$Z(J) \propto \int \mathcal{D}A \; e^{iS_{\mathrm{YM}}(A,J)} \, , \tag{71.1}$$

$$S_{\mathrm{YM}}(A, J) = \int d^4x \left[-\tfrac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + J^{a\mu} A^a_\mu \right] \, . \tag{71.2}$$

In section 57, we evaluated the path integral for U(1) gauge theory by arguing that, in momentum space, the component of the U(1) gauge field parallel to the four-momentum k^μ did not appear in the action, and hence should not be integrated over.

This argument relied on the form of the U(1) gauge transformation,

$$A_\mu(x) \to A_\mu(x) - \partial_\mu \Gamma(x) \; . \tag{71.3}$$

In the nonabelian case, however, the gauge transformation is nonlinear,

$$A_\mu(x) \to U(x) A_\mu(x) U^\dagger(x) + \tfrac{i}{g} U(x) \partial_\mu U^\dagger(x) \; , \tag{71.4}$$

where $A_\mu(x) = A^a{}_\mu(x) T^a$.

For an infinitesimal transformation,

$$\begin{aligned} U(x) &= I - ig\theta(x) + O(\theta^2) \\ &= I - ig\theta^a(x) T^a + O(\theta^2) \; , \end{aligned} \tag{71.5}$$

we have

$$A_\mu(x) \to A_\mu(x) + ig[A_\mu(x), \theta(x)] - \partial_\mu \theta(x) \; , \tag{71.6}$$

or equivalently

$$\begin{aligned} A^a_\mu(x) \to \; & A^a_\mu(x) - g f^{abc} A^b_\mu(x) \theta^c(x) - \partial_\mu \theta^a(x) \\ = \; & A^a_\mu(x) - [\delta^{ac} \partial_\mu + g f^{abc} A^b_\mu(x)] \theta^c(x) \\ = \; & A^a_\mu(x) - [\delta^{ac} \partial_\mu - ig A^b_\mu (-i f^{bac})] \theta^c(x) \\ = \; & A^a_\mu(x) - [\delta^{ac} \partial_\mu - ig A^b_\mu (T^b_{\mathrm{A}})^{ac}] \theta^c(x) \\ = \; & A^a_\mu(x) - D^{ac}_\mu \theta^c(x) \; , \end{aligned} \tag{71.7}$$

where $D^{ac}{}_{\mu}$ is the covariant derivative in the adjoint representation.

We see the similarity with the abelian case, eq. (71.3).

However, the fact that it is D_μ that appears in eq. (71.7), rather than ∂_μ, means that we cannot account for gauge redundancy in the path integral by simply excluding the components of $A^a{}_\mu$ that are parallel to k_μ.

We will have to do something more clever.

Consider an ordinary integral of the form

$$Z \propto \int dx\, dy\, e^{iS(x)} \ , \qquad (71.8)$$

where both x and y are integrated from minus to plus infinity.

Because y does not appear in S(x), the integral over y is redundant.

We can then define Z by simply dropping the integral over y,

$$Z \equiv \int dx\, e^{iS(x)} \ . \qquad (71.9)$$

This is how we dealt with gauge redundancy in the abelian case.

We could get the same answer by inserting a delta function, rather than by dropping the y integral:

$$Z = \int dx\, dy\, \delta(y)\, e^{iS(x)}\ . \tag{71.10}$$

Furthermore, the argument of the delta function can be shifted by an arbitrary function of x, without changing the result:

$$Z = \int dx\, dy\, \delta(y - f(x))\, e^{iS(x)}\ . \tag{71.11}$$

Suppose we are not given $f(x)$ explicitly, but rather are told that $y = f(x)$ is the unique solution, for fixed x, of $G(x,y) = 0$.

Then we can write

$$\delta(G(x,y)) = \frac{\delta(y - f(x))}{|\partial G/\partial y|}\ , \tag{71.12}$$

where we have used a standard rule for delta functions.

We can drop the absolute-value signs if we assume that $\partial G/\partial y$ is positive when evaluated at $y = f(x)$.

Then we have

$$Z = \int dx\, dy\, \frac{\partial G}{\partial y}\, \delta(G)\, e^{iS}\ . \tag{71.13}$$

Now let us generalize this result to an integral over $d^n x\, d^n y$.

We will need n functions $G_i(x,y)$ to fix all n components of y.

The generalization of eq. (71.13) is

$$Z = \int d^n x \, d^n y \, \det\left(\frac{\partial G_i}{\partial y_j}\right) \prod_i \delta(G_i) \, e^{iS} \ . \tag{71.14}$$

Now we are ready to translate these results to path integrals over non-abelian gauge fields.

The role of the redundant integration variable y is played by the set of all gauge transformations $\theta^a(x)$.

The role of the integration variables x and y together is played by the gauge field $A^a{}_\mu(x)$.

The role of G is played by a *gauge-fixing function*.

We will use the gauge-fixing function appropriate for R_ξ gauge, which is

$$G^a(x) \equiv \partial^\mu A^a_\mu(x) - \omega^a(x) \ , \tag{71.15}$$

where $\omega^a(x)$ is a fixed, arbitrarily chosen function of x.

We will see how the parameter ξ enters later.

In eq. (71.15), the spacetime argument x and the index a play the role of the index i in eq. (71.14).

Our path integral becomes

$$Z(J) \propto \int \mathcal{D}A \, \det\left(\frac{\delta G}{\delta \theta}\right) \prod_{x,a} \delta(G) \, e^{iS_{\text{YM}}} \ , \tag{71.16}$$

where S_{YM} is given by eq. (71.2).

Now we have to evaluate the functional derivative $\delta G^a(x)/\delta\theta^b(y)$, and then its functional determinant.

From eqs. (71.7) and (71.15), we find that, under an infinitesimal gauge transformation,

$$G^a(x) \to G^a(x) - \partial^\mu D_\mu^{ab}\theta^b(x) \; . \tag{71.17}$$

Thus we have

$$\frac{\delta G^a(x)}{\delta\theta^b(y)} = -\partial^\mu D_\mu^{ab}\delta^4(x-y) \; , \tag{71.18}$$

where the derivatives are with respect to x.

Now we need to compute the functional determinant of eq. (71.18).

Luckily, we learned how to do this in section 53.

A functional determinant can be written as a path integral over complex Grassmann variables.

So let us introduce the complex Grassmann field $c^a(x)$, and its hermitian conjugate $\bar{c}^a(x)$.

We use a bar rather than a dagger to keep the notation a little less cluttered.

These fields are called F*addeev-Popov ghosts*.

Then we can write

$$\det \frac{\delta G^a(x)}{\delta \theta^b(y)} \propto \int \mathcal{D}c\,\mathcal{D}\bar{c}\; e^{iS_{\rm gh}}\ , \tag{71.19}$$

where the ghost action is $S_{\rm gh} = \int d^4x\, \mathcal{L}_{\rm gh},$ and the ghost lagrangian is

$$\begin{aligned} \mathcal{L}_{\rm gh} &= \bar{c}^a \partial^\mu D^{ab}_\mu c^b \\ &= -\partial^\mu \bar{c}^a D^{ab}_\mu c^b \\ &= -\partial^\mu \bar{c}^a \partial_\mu c^a + ig\partial^\mu \bar{c}^a A^c_\mu (T^c_{\rm A})^{ab} c^b \\ &= -\partial^\mu \bar{c}^a \partial_\mu c^a + gf^{abc} A^c_\mu \partial^\mu \bar{c}^a c^b\ . \end{aligned} \tag{71.20}$$

We dropped a total divergence in the second line.

We see that $c^a(x)$ has the standard kinetic term for a complex scalar field.

We need the factor of i in front of S_{gh} in eq. (71.19) for this to work out; this factor affects only the overall phase of Z(J), and so we can choose it at will.

The ghost field is also a Grassmann field, and so a closed loop of ghost lines in a Feynman diagram carries an extra factor of minus one.

We see from eq. (71.20) that the ghost field interacts with the gauge field, and so we will have such loops.

Since the particles associated with the ghost field do not in fact exist (and would violate the spin-statistics theorem if they did), it must be that the amplitude to produce them in any scattering process is zero.

This is indeed the case, as we will discuss in section 74.

We note that in abelian gauge theory, where $f^{abc} = 0$, there is no interaction term for the ghost field.

In that case, it is simply an extra free field, and we can absorb its path integral into the overall normalization.

We have one final trick to perform.

Our gauge-fixing function, $G^a(x)$ contains an arbitrary function ωa(x).

The path integral Z(J) is, however, independent of $\omega^a(x)$.

So, we can multiply Z(J) by an arbitrary functional of ω, and then perform a path integral over ω; the result can change only the overall normalization of Z(J).

In particular, let us multiply Z(J) by

$$\exp\left[-\frac{i}{2\xi}\int d^4x\,\omega^a\omega^a\right]. \tag{71.21}$$

Because of the delta-functional in eq. (71.16), it is easy to integrate over ω.

The final result for Z(J) is

$$Z(J) \propto \int \mathcal{D}A\,\mathcal{D}\bar{c}\,\mathcal{D}c\ \exp\Bigl(iS_{\rm YM} + iS_{\rm gh} + iS_{\rm gf}\Bigr)\ , \tag{71.22}$$

where $S_{\rm YM}$ is given by eq. (71.2), $S_{\rm gh}$ is given by the integral over d^4x of eq. (71.20), and S_{gf} (g*f* stands for *gauge fixing*) is given by the integral over d^4x of

$$\mathcal{L}_{\rm gf} = -\tfrac{1}{2}\xi^{-1}\partial^\mu A^a_\mu \partial^\nu A^a_\nu\ . \tag{71.23}$$

In the next section, we will derive the Feynman rules that follow from this path integral.

72 The Feynman Rules for Nonabelian Gauge Theory

Let us begin by considering nonabelian gauge theory without any scalar or spinor fields.

The lagrangian is

$$\begin{aligned}\mathcal{L}_{\rm YM} &= -\tfrac{1}{4}F^{e\mu\nu}F^e_{\mu\nu}\\ &= -\tfrac{1}{4}(\partial^\mu A^{e\nu} - \partial^\nu A^{e\mu} + gf^{abe}A^{a\mu}A^{b\nu})(\partial_\mu A^e_\nu - \partial_\nu A^e_\mu + gf^{cde}A^c_\mu A^d_\nu)\\ &= -\tfrac{1}{2}\partial^\mu A^{e\nu}\partial_\mu A^e_\nu + \tfrac{1}{2}\partial^\mu A^{e\nu}\partial_\nu A^e_\mu\\ &- gf^{abe}A^{a\mu}A^{b\nu}\partial_\mu A^e_\nu - \tfrac{1}{4}g^2 f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu\ .\end{aligned} \tag{72.1}$$

To this we should add the gauge-fixing term for R_ξ gauge,

$$\mathcal{L}_{\rm gf} = -\tfrac{1}{2}\xi^{-1}\partial^\mu A^e_\mu \partial^\nu A^e_\nu \,. \tag{72.2}$$

Adding eqs. (72.1) and (72.2), and doing some integrations-by-parts in the quadratic terms, we find

$$\begin{aligned}\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm gf} = {}& \tfrac{1}{2}A^{e\mu}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^{e\nu} + \tfrac{1}{2}\xi^{-1}A^{e\mu}\partial_\mu\partial_\nu A^{e\nu} \\ & - gf^{abc}A^{a\mu}A^{b\nu}\partial_\mu A^c_\nu - \tfrac{1}{4}g^2 f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu \,.\end{aligned} \tag{72.3}$$

The first line of eq. (72.3) yields the gluon propagator in R_ξ gauge,

$$\tilde{\Delta}^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2 - i\epsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \xi\frac{k_\mu k_\nu}{k^2}\right) \,. \tag{72.4}$$

The second line of eq. (72.3) yields three- and four-gluon vertices, shown in fig. (72.1).

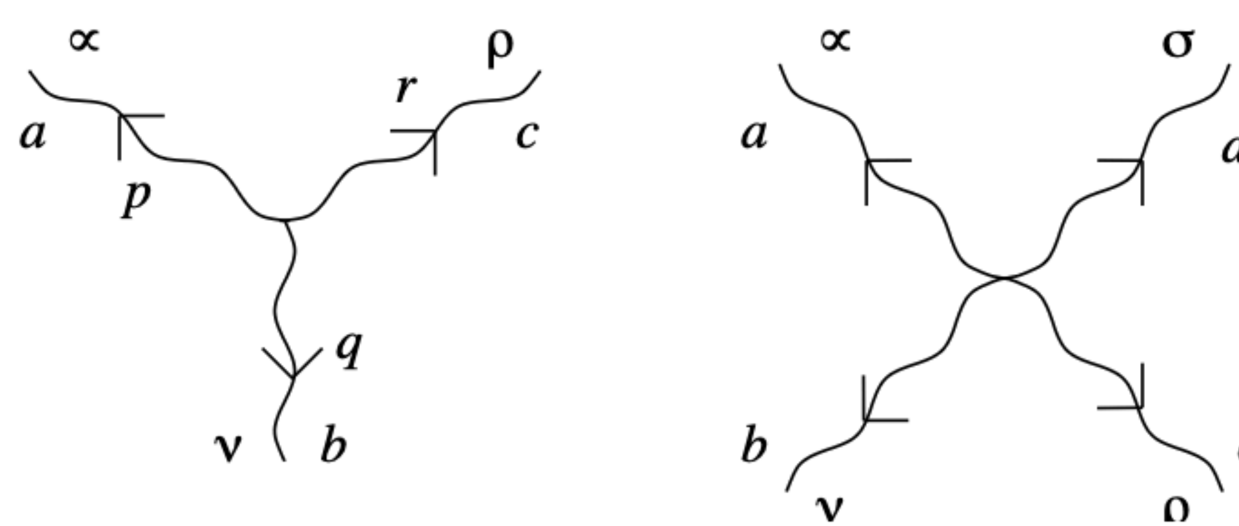


Figure 72.1: The three-gluon and four-gluon vertices in nonabelian gauge theory.

The three-gluon vertex factor is

$$
\begin{aligned}
i\mathbf{V}^{abc}_{\mu\nu\rho}(p,q,r) &= i(-gf^{abc})(-ir_\mu g_{\nu\rho}) \\
&\qquad + [\,5 \text{ permutations of } (a,\mu,p),(b,\nu,q),(c,\rho,r)\,] \\
&= gf^{abc}[(q-r)_\mu g_{\nu\rho} + (r-p)_\nu g_{\rho\mu} + (p-q)_\rho g_{\mu\nu}]\,.
\end{aligned}
\tag{72.5}
$$

The four-gluon vertex factor is

$$
\begin{aligned}
i\mathbf{V}^{abcd}_{\mu\nu\rho\sigma} &= -ig^2 f^{abe} f^{cde} g_{\mu\rho} g_{\nu\sigma} \\
&\qquad + [\,5 \text{ permutations of } (b,\nu),(c,\rho),(d,\sigma)\,] \\
&= -ig^2\,[\,f^{abe} f^{cde}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \\
&\qquad + f^{ace} f^{dbe}(g_{\mu\sigma}g_{\rho\nu} - g_{\mu\nu}g_{\rho\sigma}) \\
&\qquad + f^{ade} f^{bce}(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\rho}g_{\sigma\nu})\,]\,.
\end{aligned}
\tag{72.6}
$$

These vertex factors are quite a bit more complicated that the ones we are used to, and they lead to rather involved formulae for scattering cross sections.

For example, the tree-level gg → gg cross section (where g is a gluon), averaged over initial spins and colors and summed over final spins and colors, has 12,996 terms!

Of course, many are identical and the final result can be expressed much more simply, but this is no help to us at the initial stages of computation.

For this reason, we postpone any attempt at tree-level calculations until section 81, where we will make use of some techniques (color ordering and Gervais-Neveu gauge) that, combined with spinor-helicity methods, greatly reduce the necessary labor.

For loop calculations, we need to include the ghosts.

The ghost lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{gh}} &= -\partial^\mu \bar{c}^b D_\mu^{bc} c^c \\ &= -\partial^\mu \bar{c}^c \partial_\mu c^c + ig\partial^\mu \bar{c}^b A_\mu^a (T_{\text{A}}^a)^{bc} c^c \\ &= -\partial^\mu \bar{c}^c \partial_\mu c^c + gf^{abc} A_\mu^a \partial^\mu \bar{c}^b c^c \,.\end{aligned} \tag{72.7}$$

The ghost propagator is

$$\tilde{\Delta}^{ab}(k^2) = \frac{\delta^{ab}}{k^2 - i\epsilon} \,. \tag{72.8}$$

Because the ghosts are complex scalars, their propagators carry a charge arrow.

The ghost-ghost-gluon vertex shown in fig. (72.2); the associated vertex factor is

$$\begin{aligned}i\mathbf{V}_\mu^{abc}(q,r) &= i(gf^{abc})(-iq_\mu) \\ &= gf^{abc} q_\mu \,.\end{aligned} \tag{72.9}$$

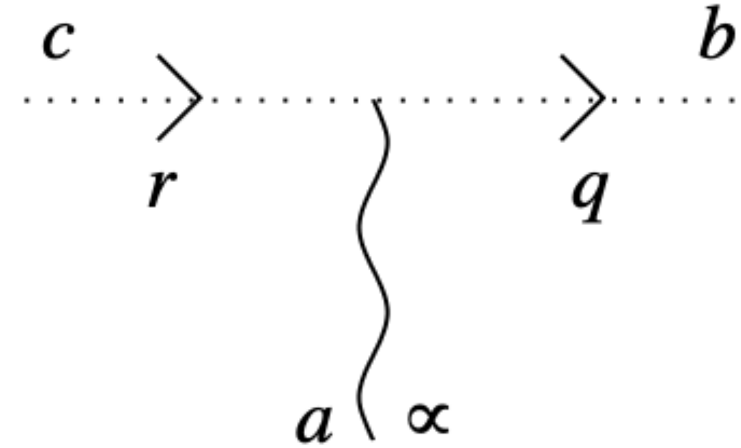


Figure 72.2: The ghost-ghost-gluon vertex in nonabelian gauge theory.

If we include a quark coupled to the gluons, we have the quark lagrangian

$$\begin{aligned}\mathcal{L}_{\rm q} &= i\overline{\Psi}_i \not{D}_{ij} \Psi_j - m\overline{\Psi}_i \Psi_i \\ &= i\overline{\Psi}_i \not{\partial} \Psi_i - m\overline{\Psi}_i \Psi_i + gA^a_\mu \overline{\Psi}_i \gamma^\mu T^a_{ij} \Psi_j \ . \end{aligned} \tag{72.10}$$

The quark propagator is

$$\tilde{S}_{ij}(p) = \frac{(-\not{p} + m)\delta_{ij}}{p^2 + m^2 - i\epsilon} \ . \tag{72.11}$$

The quark-quark-gluon vertex shown in fig. (72.3);

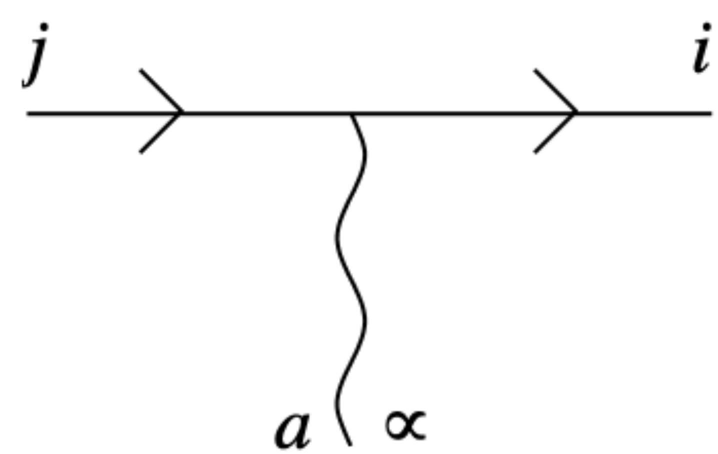


Figure 72.3: The quark-quark-gluon vertex in nonabelian gauge theory.

the associated vertex factor is

$$i\mathbf{V}^{\mu a}_{ij} = ig\gamma^\mu T^a_{ij} \ . \tag{72.12}$$

If the quark is in a representation R other than the fundamental, then $T^a{}_{ij}$ becomes $(T^a{}_R)_{ij}$.

73 The Beta Function in Nonabelian Gauge Theory

In this section, we will do enough loop calculations to compute the beta function for the Yang–Mills coupling g.

We can write the complete lagrangian, including Z factors, as

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{2}Z_3 A^{a\mu}(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^{a\nu} + \tfrac{1}{2}\xi^{-1}A^{a\mu}\partial_\mu\partial_\nu A^{a\nu} \\ & - Z_{3g}\, g f^{abc} A^{a\mu}A^{b\nu}\partial_\mu A^c_\nu - \tfrac{1}{4}Z_{4g}\, g^2 f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu \\ & - Z_{2'}\partial^\mu \bar{C}^a\partial_\mu C^a + Z_{1'}\, g f^{abc}A^c_\mu \partial^\mu \bar{C}^a C^b \\ & + iZ_2\overline{\Psi}_i \partial\!\!\!/ \Psi_i - Z_m m\overline{\Psi}_i\Psi_i + Z_1 g A^a_\mu \overline{\Psi}_i\gamma^\mu T^a_{ij}\Psi_j \ .\end{aligned} \tag{73.1}$$

Note that the gauge-fixing term in the first line does not need a Z factor; we saw in section 62 that the ξ-dependent term in the propagator is not renormalized.

We see that g appears in several places in $\mathcal{L}$, and gauge invariance leads us to expect that it will renormalize in the same way in each place.

If we rewrite $\mathcal{L}$ in terms of bare fields and parameters, and compare with eq. (73.1), we find that

$$g_0^2 = \frac{Z_1^2}{Z_2^2 Z_3} g^2 \tilde{\mu}^{\varepsilon} = \frac{Z_{1'}^2}{Z_{2'}^2 Z_3} g^2 \tilde{\mu}^{\varepsilon} = \frac{Z_{3g}^2}{Z_3^3} g^2 \tilde{\mu}^{\varepsilon} = \frac{Z_{4g}}{Z_3^2} g^2 \tilde{\mu}^{\varepsilon} \,, \tag{73.2}$$

where $d = 4 - \varepsilon$ is the number of spacetime dimensions.

To prove eq. (73.2), we have to derive the nonabelian analogs of the Ward identities, known as *Slavnov-Taylor identities*.

For now, we simply assume that eq. (73.2) holds; we will return to this issue in section 74.

The simplest computation to perform is the renormalization of the quark-quark-gluon vertex.

This is partly because much of the calculation is the same as it is in spinor electrodynamics, and so we can make use of our results in section 62.

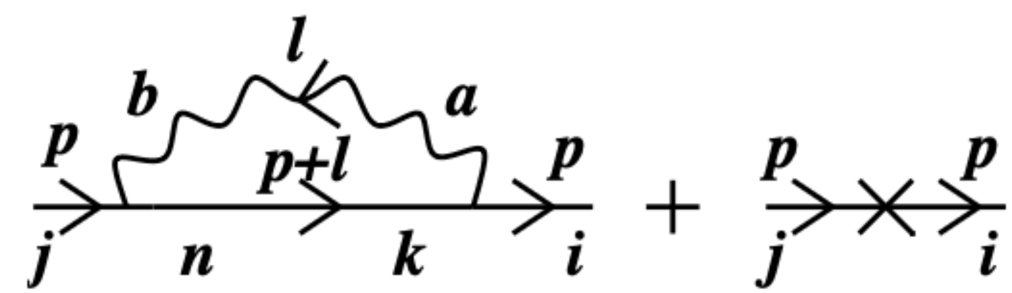
We then must compute Z_1, Z_2, and Z_3.

We will work in Feynman gauge, and use the $\overline{MS}$ renormalization scheme.

We begin with Z_2.

The $O(g^2)$ corrections to the fermion propagator are shown in fig. (73.1).

Figure 73.1: The one-loop and counterterm corrections to the quark propagator in quantum chromodynamics.



These diagrams are the same as in spinor electrodynamics, except for the factors related to the color indices.

The loop diagram has a color-factor of $(T^aT^a)_{ij} = C(R)\delta_{ij}$.

Here we have allowed the quark to be in an arbitrary representation R; for notational simplicity, we will continue to omit the label R on the generator matrices.

In section 62, we found that, in spinor electrodynamics, the divergent part of this diagram contributes $-(e^2/8\pi^2\varepsilon)\not{p}$ to the electron self-energy $\Sigma(\not{p})$.

Thus in Yang–Mills gauge theory, the divergent part of this diagram contribute $-(g^2/8\pi^2\varepsilon)C(\mathrm{R})\delta_{ij}\not{p}$ to the quark self-energy $\Sigma_{ij}(\not{p})$.

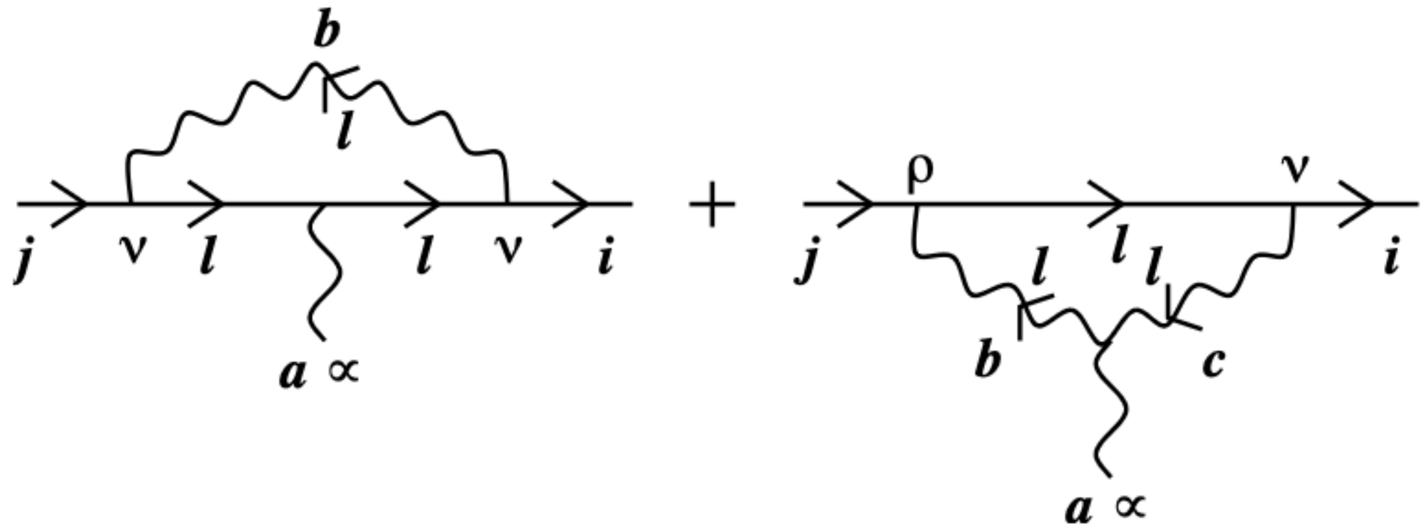
This divergent term must be cancelled by the counterterm contribution of $-(Z_2-1)\delta_{ij}\not{p}$.

Therefore, in Yang–Mills theory, with a quark in the representation R, using Feynman gauge and the $\overline{MS}$ renormalization scheme, we have

$$Z_2 = 1 - C(\mathrm{R})\,\frac{g^2}{8\pi^2}\,\frac{1}{\varepsilon} + O(g^4)\,. \tag{73.3}$$

Moving on to the quark-quark-gluon vertex, the contributing one- loop diagrams are shown in fig. (73.2).

Figure 73.2: The one-loop corrections to the quark–quark–gluon vertex in in quantum chromodynamics.



The first diagram is again the same as it is in spinor electrodynamics, except for the color factor of $(T^bT^aT^b)_{ij}$.

We can simplify this via

$$\begin{aligned} T^bT^aT^b &= T^b\Big(T^bT^a + if^{abc}T^c\Big) \\ &= C(\mathrm{R})T^a + \tfrac{1}{2}if^{abc}[T^b, T^c] \\ &= C(\mathrm{R})T^a + \tfrac{1}{2}(if^{abc})(if^{bcd})T^d \\ &= C(\mathrm{R})T^a - \tfrac{1}{2}(T^a_{\mathrm{A}})^{bc}(T^d_{\mathrm{A}})^{cb}T^d \\ &= \Big[C(\mathrm{R}) - \tfrac{1}{2}T(\mathrm{A})\Big]T^a\,. \end{aligned} \tag{73.4}$$

In the second line, we used the complete antisymmetry of f^{abc} to replace T^bT^c with 1/2 $[T^b, T^c]$.

To get the last line, we used $\mathrm{Tr}(T^a_A T^d_A) = T(A)\delta^{ad}$.

In section 62, we found that, in spinor electrodynamics, the divergent part of this diagram contributes $(e^2/8\pi^2\varepsilon)ie\gamma^\mu$ to the vertex function $i\mathbf{V}^\mu(p', p)$.

Thus in Yang–Mills theory, the divergent part of this diagram contributes

$$\left[C(\mathrm{R}) - \tfrac{1}{2}T(\mathrm{A})\right] \frac{g^2}{8\pi^2\varepsilon}\, igT^a_{ij}\gamma^\mu \tag{73.5}$$

to the quark-quark-gluon vertex function $i\mathbf{V}^{a\mu}{}_{ij}(p',p)$.

This divergent term, along with any divergent term from the second diagram of fig. (73.2), must be cancelled by the tree-level vertex $iZ_1 g\gamma^\mu T^a{}_{ij}$.

Now we must evaluate the second diagram of fig. (73.2).

The divergent part is independent of the external momenta, and so we can set them to zero.

Then we get a contribution to $i\mathbf{V}^{a\mu}{}_{ij}(0,0)$ of

$$\begin{aligned}(ig)^2 g f^{abc}(T^cT^b)_{ij}\left(\tfrac{1}{i}\right)^3 \int \frac{d^4\ell}{(2\pi)^4}\, \frac{\gamma_\rho(-\not{\ell}+m)\gamma_\nu}{\ell^2\ell^2(\ell^2+m^2)} \\ \times \left[(\ell-(-\ell))^\mu g^{\nu\rho} + (-\ell-0)^\nu g^{\rho\mu} + (0-\ell)^\rho g^{\mu\nu}\right].\end{aligned} \tag{73.6}$$

We can simplify the color factor with the manipulations of eq. (73.4),

$$\begin{aligned} f^{abc}T^cT^b &= \tfrac{1}{2}f^{abc}[T^c,T^b] \\ &= \tfrac{1}{2}if^{abc}f^{cbd}T^d \\ &= -\tfrac{1}{2}iT(\mathrm{A})T^a \ . \end{aligned} \tag{73.7}$$

The numerator in eq. (73.6) is

$$N^\mu = \gamma_\rho(-\gamma_\sigma \ell^\sigma + m)\gamma_\nu(2\ell^\mu g^{\nu\rho} - \ell^\nu g^{\rho\mu} - \ell^\rho g^{\mu\nu}) \ . \tag{73.8}$$

We can drop the terms linear in ℓ, and make the replacement $\ell^\sigma\ell^\mu \to d^{-1}\ell^2 g^{\sigma\mu}$.

Thus we have

$$\begin{aligned} N^\mu &\to -d^{-1}\ell^2\,(\gamma_\rho\gamma_\sigma\gamma_\nu)(2g^{\sigma\mu}g^{\nu\rho} - g^{\sigma\nu}g^{\rho\mu} - g^{\sigma\rho}g^{\mu\nu}) \\ &\to -d^{-1}\ell^2\,(2\gamma^\nu\gamma^\mu\gamma_\nu - \gamma^\mu\gamma^\nu\gamma_\nu - \gamma_\rho\gamma^\rho\gamma^\mu) \\ &\to -d^{-1}\ell^2\,(2(d-2) + d + d)\gamma^\mu \ . \end{aligned} \tag{73.9}$$

Because we are only keeping track of the divergent term, we are free to set $d = 4$, which yields

$$N^\mu \to -3\ell^2\gamma^\mu \ . \tag{73.10}$$

Using eqs. (73.7) and (73.10) in eq. (73.6), we get

$$\tfrac{3}{2}T(\mathrm{A})\, g^3\, T^a_{ij}\gamma^\mu \int \frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell^2(\ell^2{+}m^2)}\;. \tag{73.11}$$

After continuing to d dimensions, the integral becomes $i/8\pi^2\varepsilon + O(\varepsilon^0)$.

Combining eqs. (73.5) and (73.6), we find that the divergent part of the quark-quark-gluon vertex function is

$$\mathbf{V}^{a\mu}_{ij}(0,0)_{\mathrm{div}} = \left(Z_1 + \Big[C(\mathrm{R}){-}\tfrac{1}{2}T(\mathrm{A})\Big]\,\frac{g^2}{8\pi^2\varepsilon} + \tfrac{3}{2}T(\mathrm{A})\,\frac{g^2}{8\pi^2\varepsilon}\right) gT^a_{ij}\gamma^\mu\,. \tag{73.12}$$

Requiring $\mathbf{V}^{a\mu}{}_{ij}(0,0)$ to be finite yields

$$Z_1 = 1 - \Big[C(\mathrm{R}) + T(\mathrm{A})\Big]\,\frac{g^2}{8\pi^2}\,\frac{1}{\varepsilon} + O(g^4) \tag{73.13}$$

in Feynman gauge and the $\overline{MS}$ renormalization scheme.

Note that we have found that Z_1 does not equal Z_2.

In electrodynamics, we argued that gauge invariance requires all derivatives in the lagrangian to be covariant derivatives, and that both pieces of $D_\mu = \partial_\mu - ieA_\mu$ should therefore be renormalized by the same factor; this then implies that Z_1 must equal Z_2.

In Yang–Mills theory, however, this argument fails.

This failure is due to the introduction of the ordinary derivative in the gauge- fixing function for R_ξ gauge: once we have added $\mathcal{L}_{\rm gf}$ and $\mathcal{L}_{\rm gh}$ to $\mathcal{L}_{\rm YM}$, we find that both ordinary and covariant derivatives appear.

This is especially obvious for $\mathcal{L}_{\rm gh}$.

Therefore, to be certain of what gauge invariance does and does not imply, we must derive the appropriate Slavnov-Taylor identities, a subject we will take up in section 74.

Next we turn to the calculation of Z_3.

The $O(g^2)$ corrections to the gluon propagator are shown in fig. (73.3).

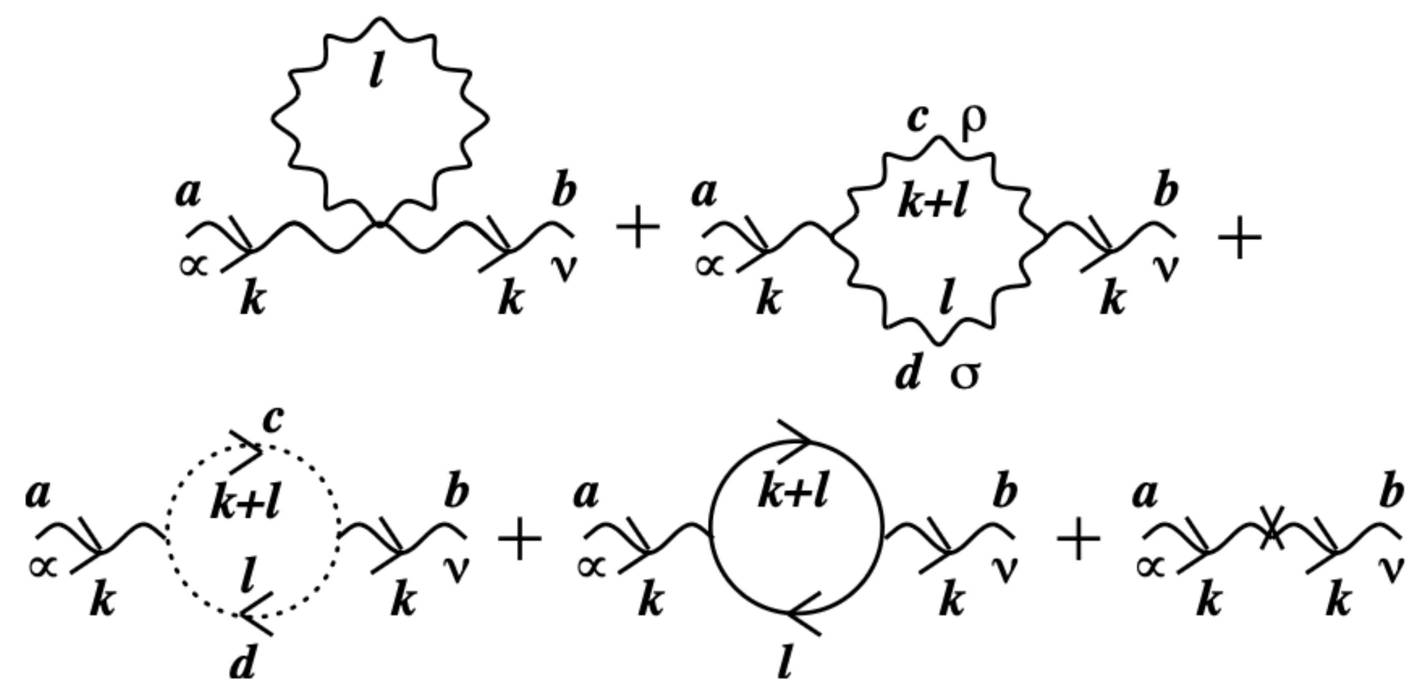


Figure 73.3: The one-loop and counterterm corrections to the gluon propagator in quantum chromodynamics.

The first diagram is proportional to $\int d^4\ell/\ell^2$; as we saw in section 65, this integral vanishes after dimensional regularization.

φ $'$ $\mathcal{T}$ $\mathbb{R}$ ρ $\mathcal{L}$ $\int$ $\bar{q}$ $\tilde{q}$ $\widetilde{\Delta}$ $\tilde{\mu}$ $\tilde{\varphi}$ $\widetilde{dk}$ $\overline{MS}$ $\overline{\eta}(x)$ $\overline{\eta}$ $\overline{u}$ $\overline{\Psi}$ the the the the the the $(\overline{\eta}, \eta, J)$ $\mathcal{F}$ $\overline{\upsilon}$

The second diagram yields a contribution to $i\Pi^{\mu\nu ab}(k)$ of

$$\tfrac{1}{2}g^2 f^{acd} f^{bcd}\left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4}\,\frac{N^{\mu\nu}}{\ell^2(\ell+k)^2}\;, \tag{73.14}$$

where the one-half is a symmetry factor, and

$$\begin{aligned} N^{\mu\nu} &= [(k+\ell)-(-\ell))^\mu g^{\rho\sigma} + (-\ell-(-k))^\rho g^{\sigma\mu} + ((-k)-(k+\ell))^\sigma g^{\mu\rho}] \\ &\quad\times[(-k-\ell)-\ell)^\nu g_{\rho\sigma} + (\ell-k)_\rho \delta_\sigma{}^\nu + (k-(-k-\ell))_\sigma \delta^\nu{}_\rho] \\ &= -[(2\ell+k)^\mu g^{\rho\sigma} - (\ell-k)^\rho g^{\sigma\mu} - (\ell+2k)^\sigma g^{\mu\rho}] \\ &\quad\times[(2\ell+k)^\nu g_{\rho\sigma} - (\ell-k)_\rho \delta_\sigma{}^\nu - (\ell+2k)_\sigma \delta^\nu{}_\rho] \end{aligned} \tag{73.15}$$

The color factor can be simplified via $f^{acd} f^{bcd} = T(A)\delta^{ab}$.

We combine denominators with Feynman's formula, and continue to $d = 4 - \varepsilon$ dimensions; we now have

$$-\tfrac{1}{2}g^2\, T(\mathrm{A})\delta^{ab}\,\tilde{\mu}^\varepsilon \int_0^1 dx \int \frac{d^dq}{(2\pi)^d}\,\frac{N^{\mu\nu}}{(q^2+D)^2}\;, \tag{73.16}$$

where $D = x(1-x)k^2$ and $q = \ell + xk$.

The numerator is

$$\begin{aligned} N^{\mu\nu} &= -[(2q+(1-2x)k)^\mu g^{\rho\sigma} - (q-(1+x)k)^\rho g^{\sigma\mu} - (q+(2-x)k)^\sigma g^{\mu\rho}] \\ &\quad\times[(2q+(1-2x)k)^\nu g_{\rho\sigma} - (q-(1+x)k)_\rho \delta_\sigma{}^\nu - (q+(2-x)k)_\sigma \delta^\nu{}_\rho]\,. \end{aligned} \tag{73.17}$$

Terms linear in q will integrate to zero, and so we have

$$\begin{aligned} N^{\mu\nu} \rightarrow\ & -2q^2 g^{\mu\nu} - (4d{-}6)q^\mu q^\nu \\ & - [(1{+}x)^2 + (2{-}x)^2]k^2 g^{\mu\nu} \\ & - [d(1{-}2x)^2 + 2(1{-}2x)(1{+}x) \\ & \quad - 2(2{-}x)(1{+}x) - 2(2{-}x)(1{-}2x)]k^\mu k^\nu \ . \end{aligned} \tag{73.18}$$

Since we are only interested in the divergent part, we can go ahead and set d = 4 in the numerator.

We can also make the replacement $q^\mu q^\nu \rightarrow 1/4\ q^2 g^{\mu\nu}$.

Then we find

$$N^{\mu\nu} \rightarrow -\tfrac{9}{2}q^2 g^{\mu\nu} - (5{-}2x{+}2x^2)k^2 g^{\mu\nu} + (2{+}10x{-}10x^2)k^\mu k^\nu \ . \tag{73.19}$$

We saw in section 62 that, when integrated against $(q^2 + D)^{-2}$, q^2 can be replaced with $(2/d-1)^{-1}D$; in our case this is $-2x(1-x)k^2$.

This yields

$$N^{\mu\nu} \rightarrow -(5{-}11x{+}11x^2)k^2 g^{\mu\nu} + (2{+}10x{-}10x^2)k^\mu k^\nu \ . \tag{73.20}$$

We now use

$$\tilde{\mu}^\varepsilon \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + D)^2} = \frac{i}{8\pi^2 \varepsilon} + O(\varepsilon^0) \tag{73.21}$$

in eq. (73.16) to get

$$-\frac{ig^2}{16\pi^2}T(\mathrm{A})\delta^{ab}\,\frac{1}{\varepsilon}\int_0^1 dx\,N^{\mu\nu}+O(\varepsilon^0)\;. \tag{73.22}$$

Performing the integral over x yields

$$-\frac{ig^2}{16\pi^2}T(\mathrm{A})\delta^{ab}\,\frac{1}{\varepsilon}\left(-\tfrac{19}{6}k^2g^{\mu\nu}+\tfrac{11}{3}k^\mu k^\nu\right) \tag{73.23}$$

as the divergent contribution of the second diagram to i $i\Pi^{\mu\nu ab}(k)$.

Next we have the third diagram of fig. (73.3), which makes a contribution to $i\Pi^{\mu\nu ab}(k)$ of

$$(-1)g^2f^{acd}f^{bdc}\left(\tfrac{1}{i}\right)^2\int\frac{d^4\ell}{(2\pi)^4}\,\frac{(\ell{+}k)^\mu\ell^\nu}{\ell^2(\ell{+}k)^2}\;, \tag{73.24}$$

where the factor of minus one is from the closed ghost loop.

The color factor is $f^{acd}f^{bcd}$ = -T(A)δ^{ab}.

After combining denominators, the numerator becomes

$$\begin{aligned}(\ell{+}k)^\mu\ell^\nu \;&=\; (q+(1{-}x)k)^\mu(q-xk)^\nu\\ &\to \tfrac{1}{4}q^2g^{\mu\nu}-x(1{-}x)k^\mu k^\nu\\ &\to -\tfrac{1}{2}x(1{-}x)k^2g^{\mu\nu}-x(1{-}x)k^\mu k^\nu\;.\end{aligned} \tag{73.25}$$

We then use eq. (73.21) in eq. (73.24); performing the integral over x yields

$$-\frac{ig^2}{8\pi^2}T(\mathrm{A})\delta^{ab}\frac{1}{\varepsilon}\left(-\tfrac{1}{12}k^2g^{\mu\nu}-\tfrac{1}{6}k^\mu k^\nu\right) \tag{73.26}$$

as the divergent contribution of the third diagram to $i\Pi^{\mu\nu ab}(k)$.

Finally, we have the fourth diagram.

This is the same as it is in spinor electrodynamics, except for the color factor of $\mathrm{Tr}(T^aT^b) = T(R)\delta^{ab}$.

If there is more than one flavor of quark, each contributes separately, leading to a factor of the number of flavors n_F.

Then, using our results in section 62, we find

$$-\frac{ig^2}{6\pi^2}n_{\mathrm{F}}T(\mathrm{R})\delta^{ab}\frac{1}{\varepsilon}\left(k^2g^{\mu\nu}-k^\mu k^\nu\right) \tag{73.27}$$

as the divergent contribution of the fourth diagram to $i\Pi^{\mu\nu ab}(k)$.

Adding up eqs. (73.23), (73.26), and (73.27), as well as the counterterm contribution, we find that the gluon self-energy is transverse,

$$\Pi^{\mu\nu ab}(k) = \Pi(k^2)(k^2g^{\mu\nu}-k^\mu k^\nu)\delta^{ab}\ , \tag{73.28}$$

and that

$$\Pi(k^2)_{\text{div}} = -(Z_3-1) + \left[\tfrac{5}{3}T(\text{A}) - \tfrac{4}{3}n_{\text{F}}T(\text{R})\right]\frac{g^2}{8\pi^2}\frac{1}{\varepsilon} + O(g^4)\;. \tag{73.29}$$

Thus we find

$$Z_3 = 1 + \left[\tfrac{5}{3}T(\text{A}) - \tfrac{4}{3}n_{\text{F}}T(\text{R})\right]\frac{g^2}{8\pi^2}\frac{1}{\varepsilon} + O(g^4) \tag{73.30}$$

in Feynman gauge and the $\overline{MS}$ renormalization scheme.

Let us collect our results:

$$Z_1 = 1 - \left[C(\text{R}) + T(\text{A})\right]\frac{g^2}{8\pi^2}\frac{1}{\varepsilon} + O(g^4)\;, \tag{73.31}$$

$$Z_2 = 1 - C(\text{R})\frac{g^2}{8\pi^2}\frac{1}{\varepsilon} + O(g^4)\;, \tag{73.32}$$

$$Z_3 = 1 + \left[\tfrac{5}{3}T(\text{A}) - \tfrac{4}{3}n_{\text{F}}T(\text{R})\right]\frac{g^2}{8\pi^2}\frac{1}{\varepsilon} + O(g^4)\;, \tag{73.33}$$

in Feynman gauge and the $\overline{MS}$ renormalization scheme.

We define

$$\alpha \equiv \frac{g^2}{4\pi}\;. \tag{73.34}$$

Then we have

$$\alpha_0 = \frac{Z_1^2}{Z_2^2 Z_3}\alpha\tilde{\mu}^{\varepsilon}\;. \tag{73.35}$$

Let us write

$$\ln\left(Z_3^{-1}Z_2^{-2}Z_1^2\right) = \sum_{n=1}^{\infty} \frac{G_n(\alpha)}{\varepsilon^n} \; . \tag{73.36}$$

Then we have

$$\ln \alpha_0 = \sum_{n=1}^{\infty} \frac{G_n(\alpha)}{\varepsilon^n} + \ln\alpha + \varepsilon \ln\tilde{\mu} \; . \tag{73.37}$$

From eqs. (73.31–73.33), we get

$$G_1(\alpha) = -\left[\tfrac{11}{3}T(\mathrm{A}) - \tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right]\frac{\alpha}{2\pi} + O(\alpha^2) \; . \tag{73.38}$$

Then, the general analysis of section 28 yields

$$\beta(\alpha) = \alpha^2 G_1'(\alpha) \; , \tag{73.39}$$

where the prime denotes differentiation with respect to α.

Thus we find

$$\beta(\alpha) = -\left[\tfrac{11}{3}T(\mathrm{A}) - \tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right]\frac{\alpha^2}{2\pi} + O(\alpha^3) \; . \tag{73.40}$$

in nonabelian gauge theory with n_F Dirac fermions in the representation R of the gauge group.

We can, if we like, restate eq. (73.40) in terms of g as

$$\beta(g) = -\left[\tfrac{11}{3}T(\mathrm{A}) - \tfrac{4}{3}n_{\mathrm{F}}T(\mathrm{R})\right]\frac{g^3}{16\pi^2} + O(g^5) \; . \tag{73.41}$$

To go from eq. (73.40) to eq. (73.41), we use $\alpha = g^2/4\pi$ and $\dot{\alpha} = g\dot{g}/2\pi$, where the dot denotes $d/d \ln \mu$.

In quantum chromodynamics, the gauge group is SU(3), and the quarks are in the fundamental representation.

Thus T(A) = 3 and T(R) = 1/2 , and the factor in square brackets in eq. (73.41) is $11 - 23\, n_F$.

So for $n_F \leq 16$, the beta function is negative: the gauge coupling in quantum chromodynamics gets weaker at high energies, and stronger at low energies.

This has dramatic physical consequences.

Perturbation theory cannot serve as a reliable guide to the low-energy physics.

And indeed, in nature we do not see isolated quarks or gluons.

Quarks, in particular, have fractional electric charges and would be easy to discover.

The appropriate conclusion is that color is confined: all finite-energy states are invariant under a global SU(3) transformation.

This has not yet been rigorously proven, but it is the only hypothesis that is consistent with all of the available theoretical and experimental information.

74 BRST Symmetry

In this section we will rederive the gauge-fixed path integral for nonabelian gauge theory from a different point of view.

We will discover that the complete gauge-fixed lagrangian, $\mathcal{L} \equiv \mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{gh}}$, still has a residual form of the gauge symmetry, known as *Becchi-Rouet-Stora-Tyutin symmetry*, or *BRST symmetry* for short.

BRST symmetry can be used to derive the Slavnov-Taylor identities that, among other useful things, show that the coupling constant is renormalized by the same factor at each of its appearances in $\mathcal{L}$.

Also, we can use BRST symmetry to show that gluons whose polarizations are not both spacelike and transverse (perpendicular to the four-momentum) decouple from physical scattering amplitudes (as do particles that are created by the ghost field).

Consider a nonabelian gauge theory with a gauge field $A^a{}_\mu(x)$, and a scalar or spinor field $\phi_i(x)$ in the representation R.

Then, under an infinitesimal gauge transformation parameterized by $\theta^a(x)$, we have

$$\delta A^a_\mu(x) = -D^{ab}\theta^b(x) \ , \tag{74.1}$$

$$\delta\phi_i(x) = -ig\theta^a(x)(T^a_{\mathrm{R}})_{ij}\phi_j(x) \ . \tag{74.2}$$

We now introduce a scalar Grassmann field $c^a(x)$ in the adjoint representation; this field will turn out to be the ghost field that we introduced in section 71.

We define an infinitesimal BRST transformation via

$$\delta_B A^a_\mu(x) \equiv D^{ab}_\mu c^b(x) \tag{74.3}$$

$$= \partial_\mu c^a(x) - g f^{abc} A^c_\mu(x) c^b(x) , \tag{74.4}$$

$$\delta_B \phi_i(x) \equiv igc^a(x)(T^a_R)_{ij}\phi_j(x) . \tag{74.5}$$

This is simply an infinitesimal gauge transformation, with the ghost field $c^a(x)$ in place of the infinitesimal parameter $-\theta^a(x)$.

Therefore, any combination of fields that is gauge invariant is also BRST invariant.

In particular, the Yang–Mills lagrangian $\mathcal{L}_{YM}$ (including the appropriate lagrangian for the scalar or spinor field ϕ_i) is BRST invariant,

$$\delta_B \mathcal{L}_{YM} = 0 . \tag{74.6}$$

We now place a further restriction on the BRST transformation: we require a BRST variation of a BRST variation to be zero.

This requirement will determine the BRST transformation of the ghost field.

Consider

$$\delta_B(\delta_B \phi_i) = ig(\delta_B c^a)(T^a_R)_{ij}\phi_j - igc^a(T^a_R)_{ij}\delta_B \phi_j . \tag{74.7}$$

There is a minus sign in front of the second term because δ_B acts as an anticommuting object, and it generates a minus sign when it passes through another anticommuting object, in this case c^a.

Using eq. (74.5), we have

$$\delta_{\rm B}(\delta_{\rm B}\phi_i) = ig(\delta_{\rm B}c^a)(T^a_{\rm R})_{ij}\phi_j - g^2 c^a c^b (T^a_{\rm R}T^b_{\rm R})_{ik}\phi_k \; . \tag{74.8}$$

We now use $c^b c^a = -c^a c^b$ in the second term to replace $T^a_R T^b_R$ with its anti-symmetric part, $1/2\,[T^a_R,T^b_R] = i/2\, f^{abc}T^c_R$.

Then, after relabeling some dummy indices, we have

$$\delta_{\rm B}(\delta_{\rm B}\phi_i) = ig(\delta_{\rm B}c^c + \tfrac{1}{2}gf^{abc}c^a c^b)(T^c_{\rm R})_{ij}\phi_j \; . \tag{74.9}$$

The right-hand side of eq. (74.9) will vanish for all $\phi_j(x)$ if and only if

$$\delta_{\rm B}c^c(x) = -\tfrac{1}{2}gf^{abc}c^a(x)c^b(x) \; . \tag{74.10}$$

We therefore adopt eq. (74.10) as the BRST variation of the ghost field.

Let us now check to see that the BRST variation of the BRST variation of the gauge field also vanishes.

From eq. (74.4) we have

$$\begin{aligned}
\delta_{\rm B}(\delta_{\rm B}A^a_\mu) &= (\delta^{ab}\partial_\mu - gf^{abc}A^c_\mu)(\delta_{\rm B}c^b) - gf^{abc}(\delta_{\rm B}A^c_\mu)c^b \\
&= D^{ab}_\mu(\delta_{\rm B}c^b) - gf^{abc}(D^{cd}_\mu c^d)c^b \\
&= D^{ab}_\mu(\delta_{\rm B}c^b) - gf^{abc}(\partial_\mu c^c)c^b + g^2 f^{abc}f^{cde}A^e_\mu c^d c^b \; .
\end{aligned} \tag{74.11}$$

We now use the antisymmetry of f^{abc} in the second term to replace $(\partial_\mu c^c)c^b$ with its antisymmetric part,

$$\begin{aligned}(\partial_\mu c^{[c})c^{b]} &\equiv \tfrac{1}{2}(\partial_\mu c^c)c^b - \tfrac{1}{2}(\partial_\mu c^b)c^c \\ &= \tfrac{1}{2}(\partial_\mu c^c)c^b + \tfrac{1}{2}c^c(\partial_\mu c^b) \\ &= \tfrac{1}{2}\partial_\mu(c^c c^b)\ .\end{aligned} \tag{74.12}$$

Similarly, we use the antisymmetry of $c^d c^b$ in the third term to replace $f^{abc}f^{cde}$ with its antisymmetric part,

$$\begin{aligned}\tfrac{1}{2}(f^{abc}f^{cde} - f^{adc}f^{cbe}) &= -\tfrac{1}{2}[(T^b_{\rm A})^{ac}(T^d_{\rm A})^{ce} - (T^d_{\rm A})^{ac}(T^b_{\rm A})^{ce}] \\ &= -\tfrac{1}{2}if^{bdh}(T^h_{\rm A})^{ae} \\ &= -\tfrac{1}{2}f^{bdh}f^{hae}\ ,\end{aligned} \tag{74.13}$$

which is just the Jacobi identity.

Now we have

$$\begin{aligned}\delta_{\rm B}(\delta_{\rm B}A^a_\mu) &= D^{ab}_\mu(\delta_{\rm B}c^b) - \tfrac{1}{2}gf^{abc}(\partial_\mu c^c c^b) - \tfrac{1}{2}g^2 f^{bdh}f^{hae}A^e_\mu c^d c^b \\ &= D^{ah}_\mu(\delta_{\rm B}c^h) - (\delta^{ah}\partial_\mu - gf^{ahe}A^e_\mu)\tfrac{1}{2}gf^{bch}c^c c^b \\ &= D^{ah}_\mu(\delta_{\rm B}c^h + \tfrac{1}{2}gf^{bch}c^b c^c)\ .\end{aligned} \tag{74.14}$$

We see that this vanishes if the BRST variation of the ghost field is given by eq. (74.10).

Now we introduce the *antighost* field $\bar{c}^a(x)$.

We take its BRST transformation to be

$$\delta_{\rm B}\bar{c}^a(x) = B^a(x) \ , \tag{74.15}$$

where $B^a(x)$ is a commuting (as opposed to Grassmann) scalar field, the *Lautrup-Nakanishi auxiliary field*.

Because $B^a(x)$ is itself a BRST variation, we have

$$\delta_{\rm B} B^a(x) = 0 \ . \tag{74.16}$$

Note that eq. (74.15) is in apparent contradiction with eq. (74.10).

However, there is actually no need to identify $\bar{c}^a(x)$ as the hermitian conjugate of $c^a(x)$.

The role of these fields (in producing the functional determinant that must accompany the gauge-fixing delta functional) is fulfilled as long as $c^a(x)$ and $\bar{c}^a(x)$ are treated as independent when we integrate them; whether or not they are hermitian conjugates of each other is irrelevant.

We identified them as hermitian conjugates in section 71 only for the sake of familiarity in deriving the associated Feynman rules.

Now, however, we must abandon this notion.

In fact, it will be most convenient to treat $c^a(x)$ and $\bar{c}^a(x)$ as two real Grassmann fields.

Now that we have introduced a collection of new fields — $c^a(x)$, $\bar{c}^a(x)$, and $B^a(x)$—what are we to do with them?

Consider adding to $\mathcal{L}_{YM}$ a new term that is the BRST variation of some object O,

$$\mathcal{L} = \mathcal{L}_{\mathrm{YM}} + \delta_{\mathrm{B}}\mathcal{O} \, . \tag{74.17}$$

Clearly $\mathcal{L}$ is BRST invariant, because $\mathcal{L}_{YM}$ is, and because $\delta_B(\delta_B O) = 0$.

We will see that adding $\delta_B O$ corresponds to fixing a gauge; which gauge we get depends on O.

We will choose

$$\mathcal{O}(x) = \bar{c}^a(x)\Big[\tfrac{1}{2}\xi B^a(x) - G^a(x)\Big] \, , \tag{74.18}$$

where $G^a(x)$ is a gauge-fixing function, and ξ is a parameter.

If we further choose

$$G^a(x) = \partial^\mu A^a_\mu(x) \, , \tag{74.19}$$

then we end up with R_ξ gauge.

Let us see how this works. We have

$$\delta_{\mathrm{B}}\mathcal{O} = (\delta_{\mathrm{B}}\bar{c}^a)\Big[\tfrac{1}{2}\xi B^a - \partial^\mu A^a_\mu\Big] - \bar{c}^a\Big[\tfrac{1}{2}\xi(\delta_{\mathrm{B}}B^a) - \partial^\mu(\delta_{\mathrm{B}}A^a_\mu)\Big] \, . \tag{74.20}$$

There is a minus sign in front of the second set of terms because δ_B acts as an anticommuting object, and so it generates a minus sign when it passes through another anticommuting object, in this case $\bar{c}^a(x)$.

Now using eqs. (74.3), (74.15), and (74.16), we get

$$\delta_B \mathcal{O} = \tfrac{1}{2}\xi B^a B^a - B^a \partial^\mu A^a_\mu + \bar{c}^a \partial^\mu D^{ab}_\mu c^b \; . \tag{74.21}$$

We see that the last term is the ghost lagrangian $\mathcal{L}_{gh}$ that we found in section 71.

If we like, we can integrate the ordinary derivative by parts, so that it acts on the antighost field,

$$\delta_B \mathcal{O} \to \tfrac{1}{2}\xi B^a B^a - B^a \partial^\mu A^a_\mu - \partial^\mu \bar{c}^a D^{ab}_\mu c^b \; . \tag{74.22}$$

Examining the first two terms in eq. (74.22), we see that no derivatives act on the auxiliary field $B^a(x)$.

Furthermore, it appears only quadratically and linearly in $\delta_B O$.

We can, therefore, perform the path integral over it; the result is equivalent to solving the classical equation of motion

$$\frac{\partial(\delta_B \mathcal{O})}{\partial B^a(x)} = \xi B^a(x) - \partial^\mu A^a_\mu(x) = 0 \; , \tag{74.23}$$

and substituting the result back into $\delta_B O$.

This yields

$$\delta_B \mathcal{O} \to -\tfrac{1}{2}\xi^{-1} \partial^\mu A^a_\mu \partial^\nu A^a_\nu - \partial^\mu \bar{c}^a D^{ab}_\mu c^b \; . \tag{74.24}$$

We see that the first term is the gauge-fixing lagrangian $\mathcal{L}_{gf}$ that we found in section 71.

We now take note of all the symmetries of the action $S = \int d^4x\, \mathcal{L}$, where $\mathcal{L} = \mathcal{L}_{YM} + \delta_B O$.

With our choice of O, they are: (1) Lorentz invariance; (2) the discrete symmetries of parity, time reversal, and charge conjugation; (3) global gauge invariance (that is, invariance under a gauge transformation with a spacetime-independent parameter θ^a); (4) BRST invariance; (5) ghost number conservation; and (6) antighost translation invariance.

Global gauge invariance simply requires every term in $\mathcal{L}$ to have all the group indices contracted in a group-invariant manner.

Ghost number conservation corresponds to assigning ghost number +1 to c^a, −1 to $\bar{c}^a$, and zero to all other fields, and requiring every term in $\mathcal{L}$ to have ghost number zero.

Antighost translation invariance corresponds to $\bar{c}^a(x) \to \bar{c}^a(x) + \chi$, where χ is a Grassmann constant.

This leaves L invariant because, in the form of eq. (74.22), $\mathcal{L}$ contains only a derivative of $\bar{c}^b(x)$.

We now claim that L already includes all terms consistent with these symmetries that have coefficients with positive or zero mass dimension.

This means that we will not encounter any divergences in perturbation theory that cannot be absorbed by including a Z factor for each term in $\mathcal{L}$.

Furthermore, loop corrections should respect the symmetries, and BRST symmetry requires that g renormalize in the same way at each of its appearances. (Filling in the mathematical details of these claims is a lengthy project that we will not undertake.)

We can regard a BRST transformation as infinitesimal, and hence construct the associated Noether current via the standard formula

$$j^{\mu}_{\mathrm{B}}(x) = \sum_{I} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\Phi_{I}(x))}\delta_{\mathrm{B}}\Phi_{I}(x)\ , \tag{74.25}$$

where $\Phi_I(x)$ stands for all the fields, including the matter (scalar and/or spinor), gauge, ghost, antighost, and auxiliary fields.

We can then define the BRST charge

$$Q_{\mathrm{B}} = \int d^3x\, j^0_{\mathrm{B}}(x)\ . \tag{74.26}$$

If we think of $c^a(x)$ and $\bar{c}^a(x)$ as independent hermitian fields, then Q_{B} is hermitian.

The BRST charge generates a BRST transformation,

$$i[Q_{\mathrm{B}}, A^a_{\mu}(x)] = D^{ab}_{\mu}c^b(x)\ , \tag{74.27}$$

$$i\{Q_{\mathrm{B}}, c^a(x)\} = -\tfrac{1}{2}gf^{abc}c^b(x)c^c(x)\ , \tag{74.28}$$

$$i\{Q_{\mathrm{B}}, \bar{c}^a(x)\} = B^a(x)\ , \tag{74.29}$$

$$i[Q_{\rm B}, B^a(x)] = 0 \,, \tag{74.30}$$

$$i[Q_{\rm B}, \phi_i(x)]_{\pm} = igc^a(x)(T^a_{\rm R})_{ij}\phi_j(x) \,. \tag{74.31}$$

where {A,B} = AB + BA is the anticommutator, and [,]$_{\pm}$ is the commutator if ϕ_i is a scalar field, and the anticommutator if ϕ_i is a spinor field.

Also, since the BRST transformation of a BRST transformation is zero, Q_B must be *nilpotent*,

$$Q_{\rm B}^2 = 0 \,. \tag{74.32}$$

Eq. (74.32) has far-reaching consequences.

In order for it to be satisfied, many states must be annihilated by Q_B; such states are said to be in the kernel of Q_B.

A state $|\psi\rangle$ which is annihilated by Q_B may take the form of Q_B acting on some other state; such states are said to be in the image of Q_B. There may be some states in the *kernel* of Q_B that are not in the image; such states are said to be in the *cohomology* of Q_B.

> *Cohomology is a mathematical concept in homology theory and algebraic topology that attaches algebraic invariants to topological spaces, helping to encode their properties in a computable way. It is often used to study the relationship between local and global properties of spaces.*

Two states in the *cohomology* of Q_B are identified if their difference is in the image; that is, if $Q_B\,|\psi\rangle = 0$ but $|\psi\rangle \neq Q_B\,|\chi\rangle$ for any state $|\chi\rangle$, and if $|\psi'\rangle = |\psi\rangle + Q_B\,|\zeta\rangle$ for some state $|\zeta\rangle$, then we identify $|\psi\rangle$ and $|\psi'\rangle$ as a single element of the cohomology of Q_B.

Note any state in the image of Q_B has zero norm, since if $|\psi\rangle = Q_B|\chi\rangle$, then $\langle\psi|\psi\rangle = \langle\psi|\, Q_B\, |\chi\rangle = 0$. (Here we have used the hermiticity of Q_B to conclude that $Q_B|\psi\rangle = 0$ implies $\langle\psi|Q_B = 0$.)

Now consider starting at some initial time with a normalized state $|\psi\rangle$ in the cohomology: $\langle\psi|\psi\rangle = 1$, $Q_B\,|\psi\rangle = 0$, $|\psi\rangle \neq Q_B\,|\chi\rangle$. (This last equation is actually redundant, because if $|\psi\rangle = Q_B\,|\chi\rangle$ for some state $|\chi\rangle$, then $|\psi\rangle$ has zero norm.)

Since $\mathcal{L}$ is BRST invariant, the hamiltonian that we derive from it must commute with the BRST charge: $[H, Q_B] = 0$.

Thus, an initial state $|\psi\rangle$ that is annihilated by Q_B must still be annihilated by it at later times, since $Q_B\, e^{-iHt}|\psi\rangle = e^{-iHt}\, Q_B\,|\psi\rangle = 0$.

Also, since unitary time evolution does not change the norm of a state, the time-evolved state must still be in the cohomology.

We now claim that the *physical states of the theory correspond to the cohomology of* Q_B.

We have already shown that if we start with a state in the cohomology, it remains in the cohomology under time evolution.

Consider, then, an initial state of widely separated wave packets of incoming particles.

According to our discussion in section 5, we can treat these states as being created by the appropriate Fourier modes of the fields, and ignore interactions.

We will suppress the group index (because it plays no essential role when interactions can be neglected) and write the mode expansions

$$A^\mu(x) = \sum_{\substack{\lambda=>,<,\\+,-}} \int \widetilde{dk}\left[\varepsilon_\lambda^{\mu*}(\mathbf{k})a_\lambda(\mathbf{k})e^{ikx} + \varepsilon_\lambda^{\mu}(\mathbf{k})a_\lambda^\dagger(\mathbf{k})e^{-ikx}\right] , \tag{74.33}$$

$$c(x) = \int \widetilde{dk}\left[c(\mathbf{k})e^{ikx} + c^\dagger(\mathbf{k})e^{-ikx}\right] , \tag{74.34}$$

$$\bar{c}(x) = \int \widetilde{dk}\left[b(\mathbf{k})e^{ikx} + b^\dagger(\mathbf{k})e^{-ikx}\right] , \tag{74.35}$$

$$\phi(x) = \int \widetilde{dk}\left[a_\phi(\mathbf{k})e^{ikx} + a_\phi^\dagger(\mathbf{k})e^{-ikx}\right] , \tag{74.36}$$

Here, for maximum simplicity, we have taken $\phi(x)$ to be a real scalar field. (This is possible if R is a real representation.)

In eq. (74.33), we have included four polarization vectors that span four-dimensional spacetime.

For $k^\mu = (\omega, \mathbf{k}) = \omega(1,0,0,1)$, we choose these four polarization vectors to be

$$\begin{aligned}
\varepsilon^\mu_>(\mathbf{k}) &= \tfrac{1}{\sqrt{2}}(1,0,0,1) \ , \\
\varepsilon^\mu_<(\mathbf{k}) &= \tfrac{1}{\sqrt{2}}(1,0,0,-1) \ , \\
\varepsilon^\mu_+(\mathbf{k}) &= \tfrac{1}{\sqrt{2}}(0,1,-i,0) \ , \\
\varepsilon^\mu_-(\mathbf{k}) &= \tfrac{1}{\sqrt{2}}(0,1,+i,0) \ .
\end{aligned} \tag{74.37}$$

The first two of these, $>$ and $<$, are lightlike vectors; $\varepsilon^\mu_>(\mathbf{k})$ is parallel to k^μ, and $\varepsilon^\mu_<(\mathbf{k})$ is spatially opposite.

The latter two, $+$ and $-$, are spacelike and transverse: they correspond to physical photon polarizations of definite helicity.

We set $g = 0$, plug eqs. (74.33–74.36) into eqs. (74.27–74.31), and use eq. (74.23) to eliminate the auxiliary field.

Matching coefficients of e^{-ikx}, we find

$$[Q_{\rm B}, a^\dagger_\lambda(\mathbf{k})] = \sqrt{2}\omega\, \delta_{\lambda>}\, c^\dagger(\mathbf{k})\ , \tag{74.38}$$

$$\{Q_{\rm B}, c^\dagger(\mathbf{k})\} = 0\ , \tag{74.39}$$

$$\{Q_{\rm B}, b^\dagger(\mathbf{k})\} = \xi^{-1}\sqrt{2}\omega\, a^\dagger_<(\mathbf{k})\ , \tag{74.40}$$

$$[Q_{\rm B}, a^\dagger_\phi(\mathbf{k})] = 0\ . \tag{74.41}$$

Consider a normalized state $|\psi\rangle$ in the cohomology: $\langle\psi|\psi\rangle = 1$, $Q_B|\psi\rangle = 0$.

Eq. (74.38) tells us that if we add a photon with the unphysical polarization $>$ by acting on $|\psi\rangle$ with $a^\dagger_>(k)$, then this state is not annihilated by Q_B; hence the state $a^\dagger_>(k)|\psi\rangle$ is not in the cohomology.

Eq. (74.40) tells us that the state $a^\dagger_<(\mathbf{k})|\psi\rangle$ is proportional to $Q_B b^\dagger(\mathbf{k})|\psi\rangle$; hence the state $a^\dagger_<(\mathbf{k})|\psi\rangle$ is also not in the cohomology.

On the other hand, the states $a^\dagger_+(\mathbf{k})|\psi\rangle$ and $a^\dagger_-(\mathbf{k})|\psi\rangle$ are annihilated by Q_B, but they cannot be written as QB acting on some other state; hence these states are in the cohomology.

Also, by similar reasoning, the state with one extra ϕ particle, $a^\dagger_\phi(\mathbf{k})|\psi\rangle$, is in the cohomology.

Eq. (74.38) tells us that if we add a ghost particle by acting on $|\psi\rangle$ with $c^\dagger(\mathbf{k})$, then this state is proportional to $Q_B a^\dagger_>(\mathbf{k})|\psi\rangle$; hence the state $c^\dagger(\mathbf{k})|\psi\rangle$ is not in the cohomology.

Eq. (74.40) tells us that if we add an antighost particle by acting on $|\psi\rangle$ with $b^\dagger(\mathbf{k})$, then this state is not annihilated by Q_B; hence the state $b^\dagger(\mathbf{k})|\psi\rangle$ is also not in the cohomology.

We conclude that the only particle creation operators that do not take a state out of the cohomology are $a^\dagger_\phi(\mathbf{k})$, $a^\dagger_+(\mathbf{k})$, and $a^\dagger_-(\mathbf{k})$.

Of course, it is precisely these operators that create the expected physical particles.

Finally, we note that the vacuum $|0\rangle$ must be in the cohomology, because it is the unique state with zero energy and positive norm.

Thus we can conclude that we can build an initial state of widely separated particles that is in the cohomology only if we do not include any ghost or antighost particles, or photons with polarizations other than + and−.

Since a state in the cohomology must evolve to another state in the cohomology, no ghosts, antighosts, or unphysically polarized photons can be produced in the scattering process.

75 Chiral Gauge Theories and Anomalies

So far, we have only discussed gauge theories with Dirac fermion fields.

Recall that a Dirac field Ψ can be written in terms of two left-handed Weyl fields χ and ξ as

$$\Psi = \begin{pmatrix} \chi \\ \xi^\dagger \end{pmatrix} . \tag{75.1}$$

If Ψ is in a representation R of the gauge group, then χ and $\xi\dagger$ must be as well.

Equivalently, χ must be in the representation R, and ξ must be in the complex conjugate representation R. (For an abelian theory, this means that if Ψ has charge +Q, then χ has charge +Q and ξ has charge -Q.)

Thus a Dirac field in a representation R is equivalent to two left-handed Weyl fields, one in R and one in $\overline{\mathrm{R}}$. .

If the representation R is real, then we can have a Majorana field

$$\Psi = \begin{pmatrix} \psi \\ \psi^\dagger \end{pmatrix} \tag{75.2}$$

instead of a Dirac field; the left-handed Weyl field ψ and and its hermitian conjugate $\psi^\dagger$ are both in the representation R.

Thus a Majorana field in a real representation R is equivalent to a single left-handed Weyl field in R.

Now suppose that we have a single left-handed Weyl field ψ in a *complex* representation R.

Such a gauge theory is automatically parity violating (because the right-handed hermitian conjugate of the left-handed Weyl field is in an inequivalent representation of the gauge group), and is said to be *chiral* .

The lagrangian is

$$\mathcal{L} = i\psi^\dagger \bar{\sigma}^\mu D_\mu \psi - \tfrac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} \ , \tag{75.3}$$

where $D_\mu = \partial_\mu \rightarrow igA^a_\mu T^a_R$.

Since T^a_R is a hermitian matrix (even when R is a complex representation), $i\psi^\dagger \bar{\sigma}^\mu D_\mu \psi$ is hermitian (up to a total divergence, as usual).

We cannot include a mass term for ψ, though, because $\psi\psi$ transforms as $R \otimes R$, and $R \otimes R$ does not contain a singlet if R is complex.

Thus, $\psi\psi$ is not gauge invariant.

But without a mass term, this lagrangian would appear to possess all the required properties: Lorentz invariance, gauge invariance, and no terms with coefficients with negative mass dimension.

However, it turns out that most chiral gauge theories do not exist as quantum field theories; they are *anomalous*.

The problem can ultimately be traced back to the functional measure for the fermion field; it turns out that this measure is, in general, not gauge invariant.

We will explore this surprising fact in section 77.

For now we will content ourselves with analyzing Feynman diagrams.

We will find an insuperable problem with gauge invariance at the one-loop level that afflicts most chiral gauge theories.

We will work with the simplest possible example: a U(1) theory with a single Weyl field ψ with charge +1.

The lagrangian is

$$\mathcal{L} = i\psi^\dagger\bar{\sigma}^\mu(\partial_\mu - igA_\mu)\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \ . \tag{75.4}$$

We can use the following trick to write this theory in terms of a Dirac field Ψ.

We note that

$$P_{\mathrm{L}}\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix} , \tag{75.5}$$

where $P_L = 1/2(1 - \gamma_5)$ is the left-handed projection matrix, does not involve the right-handed components of Ψ.

Then we can write eq. (75.4) as

$$\mathcal{L} = i\overline{\Psi}\gamma^\mu(\partial_\mu - igA_\mu)P_{\mathrm{L}}\Psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \ , \tag{75.6}$$

and treat Ψ as a Dirac field when we derive the Feynman rules.

To better understand the physical consequences of eq. (75.5), consider the case of a free field.

The mode expansion is

$$P_{\rm L}\Psi(x) = \sum_{s=\pm}\int \widetilde{dp}\,\left[b_s(\mathbf{p})P_{\rm L}u_s(\mathbf{p})e^{ipx} + d_s^\dagger(\mathbf{p})P_{\rm L}v_s(\mathbf{p})e^{-ipx}\right] . \qquad (75.7)$$

For a massless field, we learned in section 38 that $P_L u_+(\mathbf{p}) = 0$ and $P_L v_-(\mathbf{p}) = 0$.

Thus we can write eq. (75.7) as

$$P_{\rm L}\Psi(x) = \int \widetilde{dp}\,\left[b_-(\mathbf{p})u_-(\mathbf{p})e^{ipx} + d_+^\dagger(\mathbf{p})v_+(\mathbf{p})e^{-ipx}\right] . \qquad (75.8)$$

Eq. (75.8) shows us that there are only two kinds of particles associated with this field (as opposed to four with a Dirac field): $b^\dagger_-(p)$ creates a particle with charge +1 and helicity -1/2, and $d^\dagger_+(p)$ creates a particle with charge -1 and helicity +1/2.

In this theory, charge and spin are correlated.

We can easily read the Feynman rules off of eq. (75.6).

In particular, the fermion propagator in momentum space is $-P_{\rm L}\,\not{p}/p^2$, and the fermion–fermion–photon vertex is $ig\gamma^\mu P_L$.

When we go to evaluate loop diagrams, we need a method of regulating the divergent integrals.

However, our usual choice, dimensional regularization, is problematic, due to the close connection between γ_5 and four-dimensional spacetime.

In particular, in four dimensions we have

$$\mathrm{Tr}[\gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = -4i\varepsilon^{\mu\nu\rho\sigma} \,, \tag{75.9}$$

where $\varepsilon^{0123} = +1$. It is not obvious what should be done with this formula in d dimensions.

One possibility is to take $d > 4$ and define $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$.

Then eq. (75.9) holds, but with each of the four vector indices restricted to span 0, 1, 2, 3.

We also have $\{\gamma^\mu, \gamma_5\} = 0$ for $\mu = 0, 1, 2, 3$, but $[\gamma^\mu, \gamma 5] = 0$ for $\mu > 3$.

This approach is workable, but cumbersome in practice.

It is therefore tempting to abandon dimensional regularization in favor of, say, Pauli–Villars regularization, which involves the replacement

$$P_{\mathrm{L}} \frac{\not{p}}{p^2} \rightarrow P_{\mathrm{L}} \left(\frac{-\not{p}}{p^2} - \frac{-\not{p} + \Lambda}{p^2 + \Lambda^2} \right) . \tag{75.10}$$

Pauli–Villars regularization is equivalent to adding an extra fermion field with mass Λ, and a propagator with the wrong sign (corresponding to changing the signs of the kinetic and mass terms in the lagrangian).

But, a Dirac field with a chiral coupling to the gauge field cannot have a mass, since the mass term would not be gauge invariant.

So, in a chiral gauge theory, Pauli–Villars regularization violates gauge invariance, and hence is unacceptable.

Given the difficulty with regulating chiral gauge theories (which is a hint that they may not make sense), we will sidestep the issue for now, and see what we can deduce about loop diagrams without a regulator in place.

Consider the correction to the photon propagator, shown in fig. (75.1).

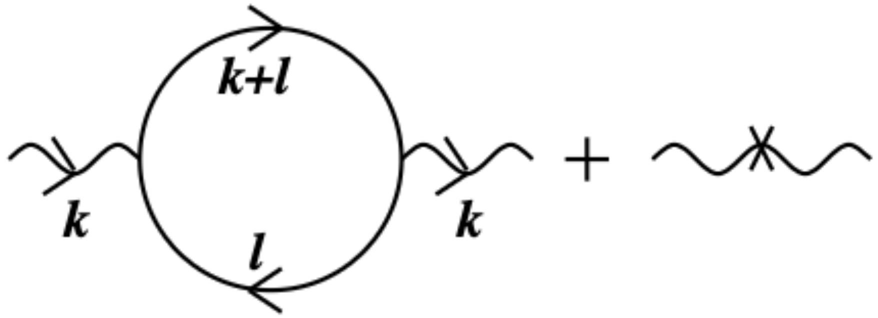


Figure 75.1: The one-loop and counterterm corrections to the photon propagator.

We have

$$
\begin{aligned}
i\Pi^{\mu\nu}(k) &= (-1)(ig)^2 \left(\tfrac{1}{i}\right)^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{N^{\mu\nu}}{(\ell+k)^2\ell^2} \\
&\quad - i(Z_3-1)(k^2 g^{\mu\nu} - k^\mu k^\nu) + O(g^4) \;,
\end{aligned}
\tag{75.11}
$$

where the numerator is

$$N^{\mu\nu} = \mathrm{Tr}[P_{\rm L}(\not\ell+\not k)\gamma^\mu P_{\rm L} P_{\rm L}\not\ell\gamma^\nu P_{\rm L}] \,. \tag{75.12}$$

We have $P^2_L = P_L$ and $P_L\gamma^\mu = \gamma^\mu P_R$ (and hence $P_L\gamma^\mu\gamma^\omega = \gamma^\mu\gamma^\upsilon P_L$), and so all the P_L's in eq. (75.12) can be collapsed into just one; this is generically true along any fermion line.

Thus we have

$$N^{\mu\nu} = \mathrm{Tr}[(\not\ell+\not k)\gamma^\mu \not\ell\gamma^\nu P_{\rm L}] \,. \tag{75.13}$$

The term in eq. (75.13) with $P_L \to 1/2$ simply yields half the result that we get in spinor electrodynamics with a Dirac field.

The term in eq. (75.13) with $P_L \to -1/2\gamma_5$, on the other hand, yields a vanishing contribution to $\Pi^{\mu\upsilon}(k)$. To see this, first note that

$$\begin{aligned} N^{\mu\nu} &\to -\tfrac{1}{2}\mathrm{Tr}[(\not\ell+\not k)\gamma^\mu \not\ell\gamma^\nu \gamma_5] \\ &= 2i\varepsilon^{\alpha\mu\beta\nu}(\ell+k)_\alpha \ell_\beta \\ &= 2i\varepsilon^{\alpha\mu\beta\nu}k_\alpha \ell_\beta \,. \end{aligned} \tag{75.14}$$

Thus we have

$$\begin{aligned} \Pi^{\mu\nu}(k) = \tfrac{1}{2}\,\Pi^{\mu\nu}(k)_{\rm Dirac} - 2g^2\varepsilon^{\alpha\mu\beta\nu}k_\alpha \int \frac{d^4\ell}{(2\pi)^4}\,\frac{\ell_\beta}{(\ell+k)^2\ell^2} \\ - (Z_3-1)(k^2 g^{\mu\nu} - k^\mu k^\nu) + O(g^4) \,. \end{aligned} \tag{75.15}$$

The integral is logarithmically divergent.

But, it carries a single vector index β, and the only vector it depends on is k.

Therefore, any Lorentz-invariant regularization must yield a result that is proportional to k_β.

This then vanishes when contracted with $\epsilon^{\alpha\mu\beta\nu}k_\alpha$.

We therefore conclude that, at the one-loop level, the contribution to $\Pi^{\mu\nu}(k)$ of a single charged Weyl field is half that of a Dirac field.

This is physically reasonable, since a Dirac field is equivalent to two charged Weyl fields.

Nothing interesting happens in the one-loop corrections to the fermion propagator, or the fermion–fermion–photon vertex.

There is simply an extra factor of P_L along the fermion line, which can be moved to the far right.

Except for this factor, the results exactly duplicate those of spinor electrodynamics.

All of this implies that a single Weyl field makes half the contribution of a Dirac field to the leading term in the beta function for the gauge coupling.

Next we turn to diagrams with three external photons, and no external fermions, shown in fig. (75.2).

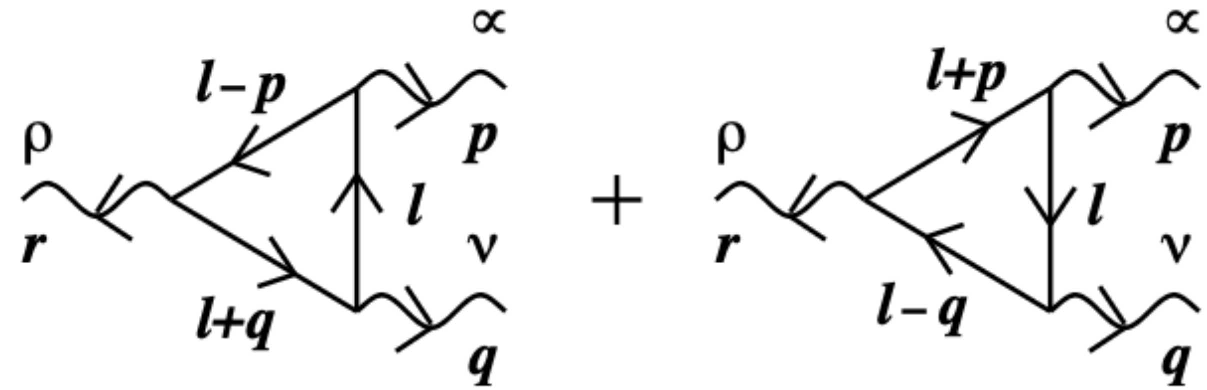


Figure 75.2: One-loop contributions to the three-photon vertex.

In spinor electrodynamics, the fact that the vector potential is odd under charge conjugation implies that the sum of these diagrams must vanish.

For the present case of a single Weyl field, there is no charge-conjugation symmetry, and so we must evaluate these diagrams.

The second diagram in fig. (75.2) is the same as the first, with p $\leftrightarrow$ q and $\mu \leftrightarrow \nu$.

Thus we have

$$i\mathbf{V}^{\mu\nu\rho}(p,q,r) = (-1)(ig)^3\left(\tfrac{1}{i}\right)^3 \int \frac{d^4\ell}{(2\pi)^4}\,\frac{N^{\mu\nu\rho}}{(\ell-p)^2\ell^2(\ell+q)^2} + (p,\mu \leftrightarrow q,\nu) + O(g^5)\ , \tag{75.16}$$

$$N^{\mu\nu\rho} = \mathrm{Tr}[(-\not{\ell}+\not{p})\gamma^\mu(-\not{\ell})\gamma^\nu(-\not{\ell}-\not{q})\gamma^\rho P_\mathrm{L}]\ . \tag{75.17}$$

The term in eq. (75.17) with $P_L \rightarrow 1/2$ simply yields half the result that we get in spinor electrodynamics with a Dirac field, which gives a vanishing contribution to $B^{\mu\upsilon\rho}(p, q, r)$.

Hence, we can make the replacement PL → -1/2 γ_5 in eq. (75.17).

Then, after cancelling some minus signs, we have

$$N^{\mu\nu\rho} \to \tfrac{1}{2}\mathrm{Tr}[(\not{\ell}-\not{p})\gamma^\mu\not{\ell}\gamma^\nu(\not{\ell}+\not{q})\gamma^\rho\gamma_5] \; . \tag{75.18}$$

We would now like to verify that $\mathbf{V}^{\mu\upsilon\rho}$(p, q, r) is gauge invariant.

We should have

$$p_\mu \mathbf{V}^{\mu\nu\rho}(p,q,r) = 0 \; , \tag{75.19}$$

$$q_\nu \mathbf{V}^{\mu\nu\rho}(p,q,r) = 0 \; , \tag{75.20}$$

$$r_\rho \mathbf{V}^{\mu\nu\rho}(p,q,r) = 0 \; . \tag{75.21}$$

Let us first check the last of these. From eq. (75.16) we find

$$\begin{aligned} r_\rho \mathbf{V}^{\mu\nu\rho}(p,q,r) = ig^3 \int \frac{d^4\ell}{(2\pi)^4} \, \frac{r_\rho N^{\mu\nu\rho}}{(\ell-p)^2\ell^2(\ell+q)^2} \\ + (p,\mu \leftrightarrow q,\nu) + O(g^5) \; , \end{aligned} \tag{75.22}$$

where

$$r_\rho N^{\mu\nu\rho} = \tfrac{1}{2}\mathrm{Tr}[(\not{\ell}-\not{p})\gamma^\mu\not{\ell}\gamma^\nu(\not{\ell}+\not{q})r_\rho\gamma^\rho\gamma_5] \; . \tag{75.23}$$

It will be convenient to use the cyclic property of the trace to rewrite eq. (75.23) as

$$r_\rho N^{\mu\nu\rho} = \tfrac{1}{2}\mathrm{Tr}[\not\ell\gamma^\nu(\not\ell+\not q)r_\rho\gamma^\rho(\not\ell-\not p)\gamma^\mu\gamma_5]\ . \tag{75.24}$$

To simplify eq. (75.24), we write $r_\rho\gamma^\rho = \not r = -(\not q+\not p) = -(\not\ell+\not q) + (\not\ell-\not p)$.

Then

$$\begin{aligned}(\not\ell+\not q)r_\rho\gamma^\rho(\not\ell-\not p) &= (\not\ell+\not q)[-(\not\ell+\not q) + (\not\ell-\not p)](\not\ell-\not p) \\ &= (\ell+q)^2(\not\ell-\not p) - (\ell-p)^2(\not\ell+\not q)\ .\end{aligned} \tag{75.25}$$

Now we have

$$\begin{aligned}r_\rho N^{\mu\nu\rho} &= \tfrac{1}{2}(\ell+q)^2\,\mathrm{Tr}[\not\ell\gamma^\nu(\not\ell-\not p)\gamma^\mu\gamma_5] - \tfrac{1}{2}(\ell-p)^2\,\mathrm{Tr}[\not\ell\gamma^\nu(\not\ell+\not q)\gamma^\mu\gamma_5] \\ &= -2i\varepsilon^{\alpha\nu\beta\mu}\Big[(\ell+q)^2\ell_\alpha(\ell-p)_\beta - (\ell-p)^2\ell_\alpha(\ell+q)_\beta\Big] \\ &= +2i\varepsilon^{\alpha\nu\beta\mu}\Big[(\ell+q)^2\ell_\alpha p_\beta + (\ell-p)^2\ell_\alpha q_\beta\Big]\end{aligned} \tag{75.26}$$

Putting eq. (75.26) into eq. (75.22), we get

$$\begin{aligned}r_\rho\mathbf{V}^{\mu\nu\rho}(p,q,r) &= -2g^3\,\varepsilon^{\alpha\nu\beta\mu}\int\frac{d^4\ell}{(2\pi)^4}\left[\frac{\ell_\alpha p_\beta}{\ell^2(\ell-p)^2} + \frac{\ell_\alpha q_\beta}{\ell^2(\ell+q)^2}\right] \\ &\quad + (p,\mu\leftrightarrow q,\nu) + O(g^5)\ ,\end{aligned} \tag{75.27}$$

Consider the first term in the integrand.

Because the only four-vector that it depends on is p, any Lorentz-invariant regularization of its integral must yield a result proportional to $p_\alpha p_\beta$.

Similarly, any Lorentz-invariant regularization of the integral of the second term must yield a result proportional to $q_\alpha q_\beta$.

Both $p_\alpha p_\beta$ and $q_\alpha q_\beta$ vanish when contracted with $\varepsilon^{\mu\alpha\upsilon\beta}$.

Therefore, we have shown that

$$r_\rho \mathbf{V}^{\mu\nu\rho}(p, q, r) = 0 \, , \tag{75.28}$$

as required by gauge invariance.

It might seem that now we are done: we can invoke symmetry among the external lines to conclude that we must also have $p_\mu \mathbf{V}^{\mu\upsilon\rho}(p, q, r) = 0$ and $q_\upsilon \mathbf{V}^{\mu\upsilon\rho}(p, q, r) = 0$.

However, eq. (75.16) is not manifestly symmetric on the exchanges $(p, \mu \leftrightarrow r, \rho)$ and $(q, \nu \leftrightarrow r, \rho)$.

So it still behooves us to compute either $p_\mu \mathbf{V}^{\mu\upsilon\rho}(p, q, r)$ or $q_\upsilon \mathbf{V}^{\mu\upsilon\rho}(p, q, r)$.

From eq. (75.16) we find

$$p_\mu \mathbf{V}^{\mu\nu\rho}(p,q,r) = ig^3 \int \frac{d^4\ell}{(2\pi)^4} \frac{p_\mu N^{\mu\nu\rho}}{(\ell-p)^2\ell^2(\ell+q)^2} + (p,\mu \leftrightarrow q,\nu) + O(g^5) \; , \tag{75.29}$$

where

$$p_\mu N^{\mu\nu\rho} = \tfrac{1}{2}\mathrm{Tr}[(\not\ell-\not p)p_\mu\gamma^\mu \not\ell \gamma^\nu(\not\ell+\not q)\gamma^\rho\gamma_5] \; . \tag{75.30}$$

To simplify eq. (75.30), we write $p_\mu\gamma^\mu = \not p = -(\not\ell-\not p) + \not\ell$.

Then

$$\begin{aligned} (\not\ell-\not p)p_\mu\gamma^\mu\not\ell &= (\not\ell-\not p)[-(\not\ell-\not p) + \not\ell]\not\ell \\ &= (\ell-p)^2\not\ell - \ell^2(\not\ell-\not p) \; . \end{aligned} \tag{75.31}$$

Now we have

$$\begin{aligned} p_\mu N^{\mu\nu\rho} &= \tfrac{1}{2}(\ell-p)^2\,\mathrm{Tr}[\not\ell\gamma^\nu(\not\ell+\not q)\gamma^\rho\gamma_5] - \tfrac{1}{2}\ell^2\,\mathrm{Tr}[(\not\ell-\not p)\gamma^\nu(\not\ell+\not q)\gamma^\rho\gamma_5] \\ &= -2i\varepsilon^{\alpha\nu\beta\rho}\Big[(\ell-p)^2\ell_\alpha q_\beta - \ell^2(\ell-p)_\alpha(\ell+q)_\beta\Big] \\ &= -2i\varepsilon^{\alpha\nu\beta\rho}\Big[(\ell-p)^2\ell_\alpha q_\beta - \ell^2(\ell-p)_\alpha(\ell-p+p+q)_\beta\Big] \\ &= -2i\varepsilon^{\alpha\nu\beta\rho}\Big[(\ell-p)^2\ell_\alpha q_\beta - \ell^2(\ell-p)_\alpha(p+q)_\beta\Big] \; . \end{aligned} \tag{75.32}$$

Putting eq. (75.32) into eq. (75.29), we get

$$p_\mu \mathbf{V}^{\mu\nu\rho}(p,q,r) = 2g^3\,\varepsilon^{\alpha\nu\beta\rho}\int\frac{d^4\ell}{(2\pi)^4}\left[\frac{\ell_\alpha q_\beta}{\ell^2(\ell+q)^2} - \frac{(\ell-p)_\alpha(p+q)_\beta}{(\ell-p)^2(\ell+q)^2}\right] \tag{75.33}$$

$$+ (p,\mu \leftrightarrow q,\nu) + O(g^5)\ ,$$

The first term on the right-hand side of eq. (75.33) must vanish, because any Lorentz-invariant regularization of the integral must yield a result proportional to $q_\alpha q_\beta$, and this vanishes when contracted with $\varepsilon^{\alpha\upsilon\beta\rho}$.

As for the second term, we can shift the loop momentum from ℓ to $\ell + p$, which results in

$$\frac{(\ell-p)_\alpha(p+q)_\beta}{(\ell-p)^2(\ell+q)^2} \rightarrow \frac{\ell_\alpha(p+q)_\beta}{\ell^2(\ell+p+q)^2}\ . \tag{75.34}$$

We can now use Lorentz invariance to argue that the integral of the right-hand side of eq. (75.34) must yield something proportional to $(p+q)_\alpha(p+q)_\beta$; this vanishes when contracted with with $\varepsilon^{\alpha\upsilon\beta\rho}$.

Thus, we have shown that $p_\mu \mathbf{V}^{\mu\upsilon\rho}(p, q, r) = 0$, as required by gauge invariance, provided that the shift of the loop momentum did not change the value of the integral.

This would of course be true if the integral was convergent.

Instead, however, the integral is linearly divergent, and so we must be more careful.

Consider a one-dimensional example of a linearly divergent integral: let

$$I(a) \equiv \int_{-\infty}^{+\infty} dx\, f(x+a) \,, \tag{75.35}$$

where $f(\pm\infty) = c_\pm$, with c_+ and c_- two finite constants.

If the integral converged, then I(a) would be independent of a.

In the present case, however, we can Taylor expand $f(x+a)$ in powers of a, and note that $f(\pm\infty) = c_\pm$ implies that every derivative of $f(x)$ vanishes at $x = \pm\infty$.

Thus we have

$$\begin{aligned} I(a) &= \int_{-\infty}^{+\infty} dx \left[f(x) + af'(x) + \tfrac{1}{2}a^2 f''(x) + \ldots \right] \\ &= I(0) + a(c_+ - c_-) \,. \end{aligned} \tag{75.36}$$

We see that I(a) is not independent of a.

Furthermore, even if we cannot assign a definite value to I(0) (because the integral is divergent), we can assign a definite value to the difference

$$I(a) - I(0) = a(c_+ - c_-) \,. \tag{75.37}$$

Now let us return to eqs. (75.33) and (75.34).

Define

$$f_\alpha(\ell) \equiv \frac{\ell_\alpha}{\ell^2(\ell+p+q)^2} \; . \tag{75.38}$$

Using Lorentz invariance, we can argue that

$$\int \frac{d^4\ell}{(2\pi)^4}\, f_\alpha(\ell) = A(p+q)_\alpha \; , \tag{75.39}$$

where A is a scalar that will depend on the regularization scheme.

Now consider

$$f_\alpha(\ell-p) = f_\alpha(\ell) - p^\beta \frac{\partial}{\partial \ell^\beta} f_\alpha(\ell) + \ldots \; . \tag{75.40}$$

The integral of the first term on the right-hand side of eq. (75.40) is given by eq. (75.39).

The integrals of the remaining terms can be converted to surface integrals at infinity.

Only the second term in eq. (75.40) falls off slowly enough to contribute.

To determine the value of its integral, we make a Wick rotation to euclidean space, which yields a factor of i as usual; then we have

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\partial}{\partial \ell^\beta} f_\alpha(\ell) = i \lim_{\ell\to\infty} \int \frac{dS_\beta}{(2\pi)^4}\, f_\alpha(\ell) \; , \tag{75.41}$$

where $dS_\beta = \ell^2 \ell_\beta d\Omega$ is a surface-area element, and $d\Omega$ is the differential solid angle in four dimensions.

We thus find

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\partial}{\partial\ell^\beta} f_\alpha(\ell) = i \lim_{\ell\to\infty} \int \frac{d\Omega}{(2\pi)^4} \frac{\ell_\beta \ell_\alpha}{(\ell{+}p{+}q)^2}$$

$$= i \frac{\Omega_4}{(2\pi)^4} \frac{1}{4} g_{\alpha\beta} \tag{75.42}$$

$$= \frac{i}{32\pi^2} g_{\alpha\beta} \ ,$$

where we used $\Omega_4 = 2\pi^2$.

Combining eqs. (75.38–75.42), we find

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{(\ell{-}p)_\alpha}{(\ell{-}p)^2(\ell{+}q)^2} = A(p{+}q)_\alpha - \frac{i}{32\pi^2} p_\alpha \ . \tag{75.43}$$

Using this in eq. (75.33), we find

$$p_\mu \mathbf{V}^{\mu\nu\rho}(p,q,r) = \frac{ig^3}{16\pi^2} \varepsilon^{\alpha\nu\beta\rho} p_\alpha (p{+}q)_\beta + (p,\mu \leftrightarrow q,\nu) + O(g^5)$$

$$= \frac{ig^3}{8\pi^2} \varepsilon^{\alpha\nu\beta\rho} p_\alpha q_\beta + O(g^5) \ . \tag{75.44}$$

An exactly analogous calculation results in

$$q_\nu \mathbf{V}^{\mu\nu\rho}(p,q,r) = \frac{ig^3}{8\pi^2} \varepsilon^{\alpha\rho\beta\mu} q_\alpha p_\beta + O(g^5) \ . \tag{75.45}$$

Eqs. (75.44) and (75.45) show that the three-photon vertex is not gauge invariant.

Since $r_\rho \mathbf{V}^{\mu\upsilon\rho}(p, q, r) = 0$, eqs. (75.44) and (75.45) also show that the three-photon vertex does not exhibit the expected symmetry among the external lines.

This is a puzzle, because the only asymmetric aspects of the diagrams in fig. (75.2) are the momentum labels on the internal lines.

The resolution of the puzzle lies in the fact that the integral in eq. (75.16) is linearly divergent, and so shifting the loop momentum changes its value.

To account for this, let us write $\mathbf{V}^{\mu\upsilon\rho}(p, q, r)$ with ℓ replaced with $\ell + a$, where a is an arbitrary linear combination of p and q.

We define

$$\mathbf{V}^{\mu\nu\rho}(p,q,r;a) \equiv \tfrac{1}{2} i g^3 \int \frac{d^4\ell}{(2\pi)^4} \frac{\mathrm{Tr}[(\not{\ell}+\not{a}-\not{p})\gamma^\mu(\not{\ell}+\not{a})\gamma^\nu(\not{\ell}+\not{a}+\not{q})\gamma^\rho\gamma_5]}{(\ell+a-p)^2(\ell+a)^2(\ell+a+q)^2} \tag{75.46}$$

$$+ (p,\mu \leftrightarrow q,\nu) + O(g^5)\ .$$

Our previous expression, eq. (75.16), corresponds to a = 0.

The integral in eq. (75.46) is linearly divergent, and so we can express the difference between $\mathbf{V}^{\mu\upsilon\rho}(p, q, r; a)$ and $\mathbf{V}^{\mu\upsilon\rho}(p, q, r; 0)$ as a surface integral.

Let us write

$$\mathbf{V}^{\mu\nu\rho}(p,q,r;a) = \tfrac{1}{2}ig^3 I_{\alpha\beta\gamma}(a)\,\mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\gamma\gamma^\rho\gamma_5] \\ + (p,\mu \leftrightarrow q,\nu) + O(g^5)\;, \tag{75.47}$$

where

$$I^{\alpha\beta\gamma}(a) \equiv \int \frac{d^4\ell}{(2\pi)^4}\,\frac{(\ell{+}a{-}p)^\alpha(\ell{+}a)^\beta(\ell{+}a{-}q)^\gamma}{(\ell{+}a{-}p)^2(\ell{+}a)^2(\ell{+}a{+}q)^2}\;. \tag{75.48}$$

Then we have

$$\begin{aligned} I_{\alpha\beta\gamma}(a) - I_{\alpha\beta\gamma}(0) &= a^\delta \int \frac{d^4\ell}{(2\pi)^4}\,\frac{\partial}{\partial\ell^\delta}\left[\frac{(\ell{-}p)_\alpha\ell_\beta(\ell{-}q)_\gamma}{(\ell{-}p)^2\ell^2(\ell{+}q)^2}\right] \\ &= ia^\delta \lim_{\ell\to\infty}\int \frac{d\Omega}{(2\pi)^4}\,\frac{\ell_\delta(\ell{-}p)_\alpha\ell_\beta(\ell{-}q)_\gamma}{(\ell{-}p)^2(\ell{+}q)^2} \\ &= ia^\delta\,\frac{\Omega_4}{(2\pi)^4}\,\frac{1}{24}\Big(g_{\delta\alpha}g_{\beta\gamma} + g_{\delta\beta}g_{\gamma\alpha} + g_{\delta\gamma}g_{\alpha\beta}\Big) \\ &= \frac{i}{192\pi^2}\Big(a_\alpha g_{\beta\gamma} + a_\beta g_{\gamma\alpha} + a_\gamma g_{\alpha\beta}\Big)\,. \end{aligned} \tag{75.49}$$

Using this in eq. (75.47), we get contractions of the form $g_{\alpha\beta}\gamma^\alpha\gamma^\mu\gamma^\beta = 2\gamma^\mu$.

The three terms in eq. (75.49) all end up contributing equally, and after using eq. (75.9) to compute the trace, we find

$$\mathbf{V}^{\mu\nu\rho}(p,q,r;a) - \mathbf{V}^{\mu\nu\rho}(p,q,r;0) = -\frac{ig^3}{16\pi^2}\,\varepsilon^{\mu\nu\rho\beta}a_\beta + (p,\mu \leftrightarrow q,\nu) + O(g^5)\ . \tag{75.50}$$

Since the Levi-Civita symbol is antisymmetric on $\mu \leftrightarrow \nu$, only the part of a that is antisymmetric on $p \leftrightarrow q$ contributes to $\mathbf{V}^{\mu\upsilon\rho}(p, q, r; a)$.

Therefore we will set a = c(p - q), where c is a numerical constant.

Then we have

$$\mathbf{V}^{\mu\nu\rho}(p,q,r;a) - \mathbf{V}^{\mu\nu\rho}(p,q,r;0) = -\frac{ig^3}{8\pi^2}\,c\,\varepsilon^{\mu\nu\rho\beta}(p-q)_\beta + O(g^5)\ . \tag{75.51}$$

Using this, along with eqs. (75.28), (75.44), and (75.45), and making some simplifying rearrangements of the indices and momenta on the right-hand sides (using p + q + r = 0), we find

$$p_\mu\mathbf{V}^{\mu\nu\rho}(p,q,r;a) = -\frac{ig^3}{8\pi^2}(1-c)\varepsilon^{\nu\rho\alpha\beta}q_\alpha r_\beta + O(g^5)\ , \tag{75.52}$$

$$q_\nu\mathbf{V}^{\mu\nu\rho}(p,q,r;a) = -\frac{ig^3}{8\pi^2}(1-c)\varepsilon^{\rho\mu\alpha\beta}r_\alpha p_\beta + O(g^5)\ , \tag{75.53}$$

$$r_\rho\mathbf{V}^{\mu\nu\rho}(p,q,r;a) = -\frac{ig^3}{8\pi^2}(2c)\varepsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta + O(g^5)\ . \tag{75.54}$$

We see that choosing $c = 1$ removes the anomalous right-hand side from eqs. (75.52) and (75.53), but it then necessarily appears in eq. (75.54).

Choosing $c = 1/3$ restores symmetry among the external lines, but now all three right-hand sides are anomalous. (This is what results from dimensional regularization of this theory with $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.)

We have therefore failed to construct a gauge-invariant U(1) theory with a single charged Weyl field.

Consider now a U(1) gauge theory with several left-handed Weyl fields ψ_i, with charges Q_i, so that the covariant derivative of ψ_i is $(\partial\mu - igQ_iA_\mu)\psi_i$.

Then each of these fields circulates in the loop in fig. (75.2), and each vertex has an extra factor of Q_i.

The right-hand sides of eqs. (75.52–75.54) are now multiplied by $\sum_i Q_i^3$.

And if $\sum_i Q_i^3$ happens to be zero, then gauge invariance is restored!

The simplest possibility is to have the ψ's come in pairs with equal and opposite charges. (In this case, they can be assembled into Dirac fields.)

But there are other possibilities as well: for example, one field with charge +2 and eight with charge -1.

Such a gauge theory is still chiral, but it is *anomaly free*. (It could be that further obstacles to gauge invariance arise with more external photons and/or more loops, but this turns out not to be the case. We will discuss this in section 77.)

All of this has a straightforward generalization to nonabelian gauge theories.

Suppose we have a single Weyl field in a (possibly reducible) representation R of the gauge group.

Then we must attach an extra factor of to the first diagram in fig. (75.2), and a factor of $\mathrm{Tr}(T^a_R T^c_R T^b_R)$ to the first diagram in fig. (75.2), and a factor of $\mathrm{Tr}(T^a_R T^b_R T^c_R)$ to the second; here the group indicies a, b, c go along with the momenta p, q, r, respectively.

Repeating our analysis shows that the diagrams with $P_L \to 1/2\gamma_5$ come with an extra factor of

$$\tfrac{1}{2}\mathrm{Tr}(\{T^a_{\mathrm{R}}, T^b_{\mathrm{R}}\}T^c_{\mathrm{R}}) = A(\mathrm{R})d^{abc} \tag{75.55}$$

Here d^{abc} is a completely symmetric tensor that is independent of the representation, and A(R) is the anomaly coefficient of R, introduced in section 70.

In order for this theory to exist, we must have A(R) = 0.

As shown in section 70, $A(\overline{\mathrm{R}}) = -A(\mathrm{R})$; thus a theory whose left-handed Weyl fields come in $\mathrm{R} \oplus \overline{\mathrm{R}}$ pairs is automatically anomaly free (as is one whose Weyl fields are all in real representations).

Otherwise, we must arrange the cancellation by hand.

For SU(2) and SO(N), all representations have A(R) = 0.

For SU(N) with N > 3, the fundamental representation has A(N) = 1, and most complex SU(N) representations R have A(R) ≠ 0.

So the cancellation is nontrivial.

We mention in passing two other kinds of anomalies: if we couple our theory to gravity, we can draw a triangle diagram with two gravitons and one gauge boson.

This diagram violates general coordinate invariance (the gauge symmetry of gravity).

If the gauge boson is from a nonabelian group, the diagram is accompanied by a factor of Tr $T^a_R = 0$, and so there is no anomaly.

If the gauge boson is from a U(1) group, the diagram is accompanied by a factor of $\sum_i Q_i$, and this must vanish to cancel the anomaly.

There is also a global anomaly that afflicts theories with an odd number of Weyl fermions in a pseudoreal representation, such as the fundamental representation of SU(2).

The global anomaly cannot be seen in perturbation theory; we will discuss it briefly in section 77.

76 Anomalies in Global Symmetries

In this section we will study anomalies in global symmetries that can arise in gauge theories that are free of anomalies in the local symmetries (and are therefore consistent quantum field theories).

A phenomenological application will be discussed in section 90.

The simplest example is electrodynamics with a massless Dirac field Ψ with charge Q = +1.

The lagrangian is

$$\mathcal{L} = i\overline{\Psi}\not{D}\Psi - \tfrac{1}{4}F^{a\mu\nu}F^{a}_{\mu\nu} \ , \tag{76.1}$$

where $\not{D} = \gamma^\mu D_\mu$ and $D_\mu = \partial_\mu - igA_\mu$. (We call the coupling constant g rather than e because we are using this theory as a formal example rather than a physical model.)

We can write Ψ in terms of two left-handed Weyl fields χ and ξ via

$$\Psi = \begin{pmatrix} \chi \\ \xi^\dagger \end{pmatrix} , \tag{76.2}$$

where χ has charge Q = +1 and ξ has charge Q = -1.

In terms of χ and ξ, the lagrangian is

$$\mathcal{L} = i\chi^\dagger\bar{\sigma}^\mu(\partial_\mu - igA_\mu)\chi + i\xi^\dagger\bar{\sigma}^\mu(\partial_\mu + igA_\mu)\xi - \tfrac{1}{4}F^{a\mu\nu}F^{a}_{\mu\nu} \ . \tag{76.3}$$

The lagrangian is invariant under a U(1) gauge transformation

$$\Psi(x) \to e^{-ig\Gamma(x)}\Psi(x) \ , \tag{76.4}$$

$$\overline{\Psi}(x) \to e^{+ig\Gamma(x)}\overline{\Psi}(x) \ , \tag{76.5}$$

$$A^\mu(x) \to A^\mu(x) - \partial^\mu\Gamma(x) \ . \tag{76.6}$$

In terms of the Weyl fields, eqs. (76.4) and (76.5) become

$$\chi(x) \to e^{-ig\Gamma(x)}\chi(x) \ , \tag{76.7}$$

$$\xi(x) \to e^{+ig\Gamma(x)}\xi(x) \ . \tag{76.8}$$

Because the fermion field is massless, the lagrangian is also invariant under a global symmetry in which χ and ξ transform with the same phase,

$$\chi(x) \to e^{+i\alpha}\chi(x) \ , \tag{76.9}$$

$$\xi(x) \to e^{+i\alpha}\xi(x) \ . \tag{76.10}$$

In terms of Ψ, this is

$$\Psi(x) \to e^{-i\alpha\gamma_5}\Psi(x) \ , \tag{76.11}$$

$$\overline{\Psi}(x) \to \overline{\Psi}(x)e^{-i\alpha\gamma_5} \ . \tag{76.12}$$

This is called *axial U(1) symmetry*, because the associated Noether current

$$j^\mu_{\mathrm{A}}(x) \equiv \overline{\Psi}(x)\gamma^\mu\gamma_5\Psi(x) \tag{76.13}$$

is an axial vector (that is, its spatial part is odd under parity).

Noether's theorem leads us to expect that this current is conserved: $\partial_\mu j^\mu{}_{\mathrm{A}} = 0$.

However, in this section we will show that the axial current actually has an *anomalous divergence*,

$$\partial_\mu j^\mu_{\rm A} = -\frac{g^2}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\ . \tag{76.14}$$

We will see in section 77 that eq. (76.14) is exact; there are no higher-order corrections.

We will demonstrate eq. (76.14) by making use of our results in section 75.

Consider the matrix element $<p, q \mid j^\rho_{\rm A}(z)|0\rangle$, where $< p, q \mid$ is a state of two outgoing photons with four-momenta p and q, and polarization vectors ϵ_μ and ϵ'_υ , respectively. (We omit the helicity label, which will play no essential role.)

Using the LSZ formula for photons (see section 67), we have

$$\langle p,q|j^\rho_{\rm A}(z)|0\rangle = (ig)^2\,\varepsilon_\mu\varepsilon'_\nu\int d^4x\,d^4y\,e^{-i(px+qy)}\langle 0|{\rm T}j^\mu(x)j^\nu(y)j^\rho_{\rm A}(z)|0\rangle\ , \tag{76.15}$$

where

$$j^\mu(x) \equiv \overline{\Psi}(x)\gamma^\mu\Psi(x) \tag{76.16}$$

is the Noether current corresponding to the U(1) gauge symmetry.

Since both $j^\mu(x)$ and $j^\mu{}_{\rm A}(x)$ are Noether currents, we expect the Ward identities

$$\frac{\partial}{\partial x^\mu}\langle 0|{\rm T}j^\mu(x)j^\nu(y)j^\rho_{\rm A}(z)|0\rangle = 0\ , \tag{76.17}$$

$$\frac{\partial}{\partial y^\nu}\langle 0|{\rm T}j^\mu(x)j^\nu(y)j^\rho_{\rm A}(z)|0\rangle = 0\ , \tag{76.18}$$

$$\frac{\partial}{\partial z^\rho}\langle 0|{\rm T}j^\mu(x)j^\nu(y)j^\rho_{\rm A}(z)|0\rangle = 0\ , \tag{76.19}$$

to be satisfied.

Note that there are no contact terms in eqs. (76.17–76.19), because both $j^\mu(x)$ and $j^\mu{}_A(x)$ are invariant under both U(1) transformations.

If we use eq. (76.19) in eq. (76.15), we see that we expect

$$\frac{\partial}{\partial z^\rho}\langle p,q|j^\rho_{\mathrm{A}}(z)|0\rangle = 0\ . \tag{76.20}$$

However, our experience in section 75 leads us to proceed more cautiously.

Let us define $C^{\mu\upsilon\rho}(p, q, r)$ via

$$\begin{aligned}(2\pi)^4\delta^4(p{+}q{+}r)C^{\mu\nu\rho}(p,q,r)& \\ \equiv \int d^4x\,d^4y\,d^4z\,e^{-i(px+qy+rz)}&\langle 0|\mathrm{T}j^\mu(x)j^\mu(y)j^\rho_{\mathrm{A}}(z)|0\rangle\ .\end{aligned} \tag{76.21}$$

Then we can rewrite eq. (76.15) as

$$\langle p,q|j^\rho_{\mathrm{A}}(z)|0\rangle = -g^2\varepsilon_\mu\varepsilon'_\nu C^{\mu\nu\rho}(p,q,r)e^{irz}\Big|_{r=-p-q}\ . \tag{76.22}$$

Taking the divergence of the current yields

$$\langle p,q|\partial_\rho j^\rho_{\mathrm{A}}(z)|0\rangle = -ig^2\varepsilon_\mu\varepsilon'_\nu r_\rho C^{\mu\nu\rho}(p,q,r)e^{irz}\Big|_{r=-p-q}\ . \tag{76.23}$$

The expected Ward identities become

$$p_\mu C^{\mu\nu\rho}(p,q,r) = 0\,, \tag{76.24}$$

$$q_\nu C^{\mu\nu\rho}(p,q,r) = 0\,, \tag{76.25}$$

$$r_\rho C^{\mu\nu\rho}(p,q,r) = 0\,. \tag{76.26}$$

To check eqs. (76.24–76.26), we compute $C^{\mu\upsilon\rho}$(p, q, r) with Feynman diagrams.

At the one-loop level, the contributing diagrams are exactly those we computed in section 75, except that the three vertex factors are now γ^μ , γ^υ , and $\gamma^\rho\gamma_5$, instead of ig γ^μ P_L, ig γ^υ P_L, and igγ^ρ P_L.

But, as we saw, the three P_L's can be combined into just one at the last vertex, and then this one can be replaced by -1/2γ_5 .

Thus, the vertex function i$\mathbf{V}^{\mu\upsilon\rho}$(p, q, r) of section 75 is related to $C^{\mu\upsilon\rho}$(p, q, r) by

$$i\mathbf{V}^{\mu\nu\rho}(p,q,r) = -\tfrac{1}{2}(ig)^3 C^{\mu\nu\rho}(p,q,r) + O(g^5)\,. \tag{76.27}$$

In section 75, we saw that we could choose a regularization scheme that preserved eqs. (76.24) and (76.25), but not also (76.26).

For the theory of this section, we definitely want to preserve eqs. (76.24) and (76.25), because these imply conservation of the current coupled to the gauge field, which is necessary for gauge invariance.

On the other hand, we are less enamored of eq. (76.26), because it implies conservation of the current for a mere global symmetry.

Using eq. (76.27) and our results from section 75, we find that preserving eqs. (76.24) and (76.25) results in

$$r_\rho C^{\mu\nu\rho}(p,q,r) = -\frac{i}{2\pi^2}\varepsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta + O(g^2) \tag{76.28}$$

in place of eq. (76.26). Using this in eq. (76.23), we find

$$\langle p,q|\partial_\rho j^\rho_{\rm A}(z)|0\rangle = -\frac{g^2}{2\pi^2}\varepsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta \varepsilon_\mu \varepsilon'_\nu e^{-i(p+q)z} + O(g^4)\ . \tag{76.29}$$

Now we come to the point.

The right-hand side of eq. (76.29) is exactly what we get in free-field theory for the matrix element of the right-hand side of eq. (76.14).

We conclude that eq. (76.14) is correct, up to possible higher-order corrections.

In the next section, we will see that eq. (76.14) is exact.

φ $'$ $\mathcal{T}$ $\mathbb{R}$ ρ $\mathcal{L}$ $\int$ $\bar{q}$ $\tilde{q}$ $\widetilde{\Delta}$ $\tilde{\mu}$ $\tilde{\varphi}$ $\widetilde{dk}$ $\overline{MS}$ $\overline{\eta}(x)$ $\overline{\eta}$ $\overline{u}$ $\overline{\Psi}$ the the the the the $(\overline{\eta},\eta,J)$ $\mathcal{F}$ $\overline{v}$

77 Anomalies and the Path Integral for Fermions

In the last section, we saw that in a U(1) gauge theory with a massless Dirac field Ψ with charge Q = +1, the axial vector current

$$j_{\rm A}^{\mu} = \overline{\Psi}\gamma^{\mu}\gamma_5\Psi \ , \tag{77.1}$$

which should (according to Noether's theorem) be conserved, actually has an anomalous divergence,

$$\partial_\mu j_{\rm A}^{\mu} = -\frac{g^2}{16\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \ . \tag{77.2}$$

In this section, we will derive eq. (77.2) directly from the path integral, using the *Fujikawa method*.

We will see that eq. (77.2) is exact; there are no higher-order corrections.

We can also consider a nonabelian gauge theory with a massless Dirac field Ψ in a (possibly reducible) representation R of the gauge group.

In this case, the triangle diagrams that we analyzed in the last section carry an extra factor of $\mathrm{Tr}(T^a_R T^b_R) = T(R)\delta^{ab}$, and we have

$$\partial_\mu j_{\rm A}^{\mu} = -\frac{g^2}{16\pi^2}T({\rm R})\varepsilon^{\mu\nu\rho\sigma}\partial_{[\mu}A^a_{\nu]}\partial_{[\rho}A^a_{\sigma]} + O(g^3) \ , \tag{77.3}$$

where $\partial_{[\mu}A^a{}_{\upsilon]} \equiv \partial_\mu A^a{}_\upsilon - \partial_\upsilon A^a{}_\mu$.

We expect the right-hand side of eq. (77.3) to be gauge invariant (since this theory is free of anomalies in the currents coupled to the gauge fields); this suggests that we should have

$$\partial_\mu j_{\mathrm{A}}^\mu = -\frac{g^2}{16\pi^2} T(\mathrm{R}) \varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \ , \tag{77.4}$$

where $F^a_{\mu\upsilon} = \partial_\mu A^a_\upsilon - \partial_\upsilon A^a_\mu + g f^{abc} A^b_\mu A^c_\upsilon$ is the nonabelian field strength.

We will see that eq. (77.4) is correct, and that there are no higher-order corrections.

We can write eq. (77.4) more compactly by using the matrix-valued gauge field

$$A_\mu \equiv T^a_{\mathrm{R}} A^a_\mu \tag{77.5}$$

and field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \ . \tag{77.6}$$

Then eq. (77.4) can be written as

$$\partial_\mu j_{\mathrm{A}}^\mu = -\frac{g^2}{16\pi^2} \varepsilon^{\mu\nu\rho\sigma} \, \mathrm{Tr}\, F_{\mu\nu} F_{\rho\sigma} \ . \tag{77.7}$$

We now turn to the derivation of eqs. (77.2) and (77.7).

We begin with the path integral over the Dirac field, with the gauge field treated as a fixed background, to be integrated later.

We have

$$Z(A) \equiv \int \mathcal{D}\Psi \, \mathcal{D}\overline{\Psi} \; e^{iS(A)} \; , \tag{77.8}$$

where

$$S(A) \equiv \int d^4x \; \overline{\Psi} i \not{D} \Psi \tag{77.9}$$

is the Dirac action, $i\not{D} = i\gamma^{\mu}D_{\mu}$ is the Dirac wave operator, and

$$D_{\mu} = \partial_{\mu} - igA_{\mu} \tag{77.10}$$

is the covariant derivative.

Here A_{μ} is either the U(1) gauge field, or the matrix-valued nonabelian gauge field of eq. (77.5), depending on the theory under consideration.

Our notation allows us to treat both cases simultaneously.

We can formally evaluate eq. (77.8) as a functional determinant,

$$Z(A) = \det(i\not{D}) \; . \tag{77.11}$$

However, this expression is not useful without some form of regularization.

We will take up this issue shortly.

Now consider an axial U(1) transformation of the Dirac field, but with a spacetime dependent parameter $\alpha(x)$:

$$\Psi(x) \to e^{-i\alpha(x)\gamma_5}\Psi(x) \ , \tag{77.12}$$

$$\overline{\Psi}(x) \to \overline{\Psi}(x)e^{-i\alpha(x)\gamma_5} \ . \tag{77.13}$$

We can think of eqs. (77.12) and (77.13) as a change of integration variable in eq. (77.8); then Z(A) should be independent of α(x).

The corresponding change in the action is

$$S(A) \to S(A) + \int d^4x \; j_{\mathrm{A}}^{\mu}(x)\partial_\mu \alpha(x) \ . \tag{77.14}$$

We can integrate by parts to write this as

$$S(A) \to S(A) - \int d^4x \; \alpha(x)\partial_\mu j_{\mathrm{A}}^{\mu}(x) \ . \tag{77.15}$$

f we assume that the measure $\mathcal{D}\Psi\mathcal{D}\overline{\Psi}$ is invariant under the axial U(1) transformation, then we have

$$Z(A) \to \int \mathcal{D}\Psi\,\mathcal{D}\overline{\Psi}\; e^{iS(A)}e^{-i\int d^4x\;\alpha(x)\partial_\mu j_{\mathrm{A}}^{\mu}(x)} \ . \tag{77.16}$$

This must be equal to the original expression for Z(A), eq. (77.8).

This implies that $\partial_\mu j^\mu{}_{\mathrm{A}}(x) = 0$ holds inside quantum correlation functions, up to contact terms, as discussed in section 22.

However, the assumption that the measure DΨ D$\overline{\Psi}$ is invariant under the axial U(1) transformation must be examined more closely.

The change of variable in eqs. (77.12) and (77.13) is implemented by the functional matrix

$$J(x,y) = \delta^4(x-y)e^{-i\alpha(x)\gamma_5} . \tag{77.17}$$

Because the path integral is over fermionic variables (rather than bosonic), we get a jacobian factor of $(\det J)^{-1}$ (rather than det J) for each of the transformations in eqs. (77.12) and (77.13), so that we have

$$\mathcal{D}\Psi\,\mathcal{D}\overline{\Psi} \to (\det J)^{-2}\,\mathcal{D}\Psi\,\mathcal{D}\overline{\Psi} . \tag{77.18}$$

Using log det J = Tr log J, we can write

$$(\det J)^{-2} = \exp\left[2i\int d^4x\,\alpha(x)\,\mathrm{Tr}\,\delta^4(x-x)\gamma_5\right], \tag{77.19}$$

where the explicit trace is over spin and group indices.

Like eq. (77.11), this expression is not useful without some form of regularization.

We could try to replace the delta-function with a gaussian; this is equivalent to

$$\delta^4(x-y) \to e^{\partial_x^2/M^2}\delta^4(x-y) , \tag{77.20}$$

where M is a regulator mass that we would take to infinity at the end of the calculation.

However, the appearance of the ordinary derivative ∂, rather than the covariant derivative D, implies that eq. (77.20) is not properly gauge invariant.

So, another possibility is

$$\delta^4(x-y) \to e^{D_x^2/M^2}\,\delta^4(x-y)\ . \tag{77.21}$$

However, eq. (77.21) presents us with a more subtle problem.

Our regularization scheme for eq. (77.19) should be compatible with our regularization scheme for eq. (77.11).

It is not obvious whether or not eq. (77.21) meets this criterion, because D^2 has no simple relation to $i\not{D}$.

To resolve this issue, we use

$$\delta^4(x-y) \to e^{(i\not{D}_x)^2/M^2}\,\delta^4(x-y) \tag{77.22}$$

to regulate the delta function in eq. (77.19).

To evaluate eq. (77.22), we write the delta function on the right-hand side of eq. (77.22) as a Fourier integral

$$\delta^4(x-y) \to \int \frac{d^4k}{(2\pi)^4}\, e^{(i\not{D}_x)^2/M^2}\, e^{ik(x-y)}\ . \tag{77.23}$$

Then we use $f(\partial)e^{ikx} = e^{ikx}f(\partial + ik)$; eq. (77.23) becomes

$$\delta^4(x-y) \to \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\, e^{(i\not{D}-\not{k})^2/M^2}\ , \tag{77.24}$$

where a derivative acting on the far right now yields zero.

We have

$$(i\not{D} - \not{k})^2 = \not{k}^2 - i\{\not{k}, \not{D}\} - \not{D}^2$$
$$= -k^2 - i\{\gamma^\mu, \gamma^\nu\}k_\mu D_\nu - \gamma^\mu\gamma^\nu D_\mu D_\nu \ . \tag{77.25}$$

Next we use $\gamma^\mu\gamma^\nu = \frac{1}{2}(\{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu]) = -g^{\mu\nu} - 2iS^{\mu\nu}$ to get

$$(i\not{D} - \not{k})^2 = -k^2 + 2ik\cdot D + D^2 + 2iS^{\mu\nu}D_\mu D_\nu \ . \tag{77.26}$$

In the last term, we can use the antisymmetry of $S^{\mu\upsilon}$ to replace $D_\mu D_\upsilon$ with $1/2[D_\mu, D_\upsilon] = -1/2\ igF_{\mu\upsilon}$, which yields

$$(i\not{D} - \not{k})^2 = -k^2 + 2ik\cdot D + D^2 + gS^{\mu\nu}F_{\mu\nu} \ . \tag{77.27}$$

We use eq. (77.27) in eq. (77.24), and then rescale k by M ; the result is

$$\delta^4(x-y) \to M^4 \int \frac{d^4k}{(2\pi)^4}\, e^{iMk(x-y)}\, e^{-k^2} e^{2ik\cdot D/M + D^2/M^2 + gS^{\mu\nu}F_{\mu\nu}/M^2} \ . \tag{77.28}$$

Thus we have

$$\mathrm{Tr}\,\delta^4(x-x)\gamma_5 \to M^4 \int \frac{d^4k}{(2\pi)^4}\, e^{-k^2}\, \mathrm{Tr}\, e^{2ik\cdot D/M + D^2/M^2 + gS^{\mu\nu}F_{\mu\nu}/M^2}\gamma_5 \ . \tag{77.29}$$

We can now expand the exponential in inverse powers of M; only terms up to M^{-4} will survive the $M \to \infty$ limit.

Furthermore, the trace over spin indices will vanish unless there are four or more gamma matrices multiplying γ_5.

Together, these considerations imply that the only term that can make a nonzero contribution is $1/2(gS^{\mu\upsilon}F_{\mu\upsilon})^2/M^4$.

Thus we find

$$\mathrm{Tr}\,\delta^4(x-x)\gamma_5 \rightarrow \tfrac{1}{2}g^2 \int \frac{d^4k}{(2\pi)^4}\, e^{-k^2} (\mathrm{Tr}\, F_{\mu\nu}F_{\rho\sigma})(\mathrm{Tr}\, S^{\mu\nu}S^{\rho\sigma}\gamma_5)\ , \tag{77.30}$$

where the first trace is over group indices (in the nonabelian case), and the second trace is over spin indices.

The spin trace is

$$\begin{aligned} \mathrm{Tr}\, S^{\mu\nu}S^{\rho\sigma}\gamma_5 &= \mathrm{Tr}\,(\tfrac{i}{2}\gamma^\mu\gamma^\nu)(\tfrac{i}{2}\gamma^\rho\gamma^\sigma)\gamma_5 \\ &= -\tfrac{1}{4}\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5 \\ &= i\varepsilon^{\mu\nu\rho\sigma}\ . \end{aligned} \tag{77.31}$$

To evaluate the integral over k in eq. (77.30), we analytically continue to euclidean spacetime; this results in an overall factor of i, as usual.

Then each of the four gaussian integrals gives a factor of $\pi^{1/2}$.

So we find

$$\mathrm{Tr}\,\delta^4(x-x)\gamma_5 \to -\frac{g^2}{32\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,F_{\mu\nu}F_{\rho\sigma}\ . \tag{77.32}$$

Using this in eq. (77.19), we get

$$(\det J)^{-2} = \exp\left[-\frac{ig^2}{16\pi^2}\int d^4x\,\alpha(x)\,\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,F_{\mu\nu}(x)F_{\rho\sigma}(x)\right]. \tag{77.33}$$

Including the transformation of the measure, eq. (77.18), in the transformation of the path integral, eq. (77.16), then yields

$$Z(A) \to \int \mathcal{D}\Psi\,\mathcal{D}\overline{\Psi}\;e^{iS(A)}e^{-i\int d^4x\;\alpha(x)\left[(g^2/16\pi^2)\varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\,F_{\mu\nu}(x)F_{\rho\sigma}(x)+\partial_\mu j^\mu_{\mathrm{A}}(x)\right]} \tag{77.34}$$

in place of eq. (77.16).

This must be equal to the original expression for Z(A), eq. (77.8).

This implies that eq. (77.7) holds inside quantum correlation functions, up to possible contact terms.

Note that this derivation of eq. (77.7) did not rely on an expansion in powers of g, and so eq. (77.7) is exact; there are no higher-order corrections.

This result is known as the *Adler-Bardeen theorem*.

It can also be (and originally was) established by a careful study of Feynman diagrams.

The Fujikawa method can be used to find the anomaly in the chiral gauge theories that we studied in section 75, but the analysis is more involved.

Here we will quote only the final result.

Consider a left-handed Weyl field in a (possibly reducible) representation R of the gauge group.

We define the chiral gauge current $j^{a\mu} \equiv \overline{\Psi}\, T^a{}_R \gamma^\mu P_L \Psi$.

Its covariant divergence (which should be zero, according to Noether's theorem) is given by

$$D_\mu^{ab} j^{b\mu} = \frac{g^2}{24\pi^2}\, \varepsilon^{\mu\nu\rho\sigma} \partial_\mu \mathrm{Tr}\Big[T^a_{\mathrm{R}}(A_\nu \partial_\rho A_\sigma - \tfrac{1}{2} i g A_\nu A_\rho A_\sigma)\Big]\,. \tag{77.35}$$

Note that the right-hand side of eq. (77.35) is not gauge invariant.

The anomaly spoils gauge invariance in chiral gauge theories, unless this right-hand side happens to vanish for group-theoretic reasons.

It can be shown that this occurs if and only if A(R) = 0, where A(R) is the anomaly coefficient of the representation R.

For comparison, note that eq. (77.7) can be written as

$$\partial_\mu j^\mu_{\mathrm{A}} = -\frac{g^2}{4\pi^2}\, \varepsilon^{\mu\nu\rho\sigma} \partial_\mu \mathrm{Tr}\Big[A_\nu \partial_\rho A_\sigma - \tfrac{2}{3} i g A_\nu A_\rho A_\sigma\Big]\,. \tag{77.36}$$

The relative value of the overall numerical prefactor in eqs. (77.35) and (77.36) is easy to understand: there is a minus one-half in eq. (77.35) from $P_L \rightarrow -1/2\, \gamma_5$, and a one-third from regularizing to preserve symmetry among the three external lines in the triangle diagram. (The relative coefficients of the second terms have no comparably simple explanation.)

Finally, a related but more subtle problem, known as a global anomaly, arises for theories with an odd number of Weyl fields in a pseudoreal representation, such as the fundamental representation of SU(2).

In this case, every gauge field configuration A_μ can be smoothly deformed into another gauge field configuration A' that has the same action, but has $Z(A') = -Z(A)$.

Thus, when we integrate over A, the contribution from A' cancels the contribution from A, and the result is zero.

Since its path integral is trivial, this theory does not exist.

78 Background Field Gauge

In the section, we will introduce a clever choice of gauge, background field gauge, that greatly simplifies the calculation of the beta function for Yang–Mills theory, especially at the one-loop level.

We begin with the lagrangian for Yang–Mills theory,

$$\mathcal{L}_{\mathrm{YM}} = -\tfrac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} \ , \tag{78.1}$$

where the field strength is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \ . \tag{78.2}$$

To evaluate the path integral, we must choose a gauge.

As we saw in section 71, one large class of gauges corresponds to choosing a gauge-fixing function $G^a(x)$, and adding $\mathcal{L}_{\rm gf} + \mathcal{L}_{\rm gh}$ to $\mathcal{L}_{\rm YM}$, where

$$\mathcal{L}_{\rm gf} = -\tfrac{1}{2}\xi^{-1} G^a G^a \ , \tag{78.3}$$

$$\mathcal{L}_{\rm gh} = \bar{c}^a \frac{\partial G^a}{\partial A^b_\mu} D^{bc}_\mu c^c \ . \tag{78.4}$$

Here $D^{bc}_\mu = \delta^{bc}\partial_\mu - ig(T^a_{\rm A})^{bc} A^a_\mu = \delta^{bc}\partial_\mu + g f^{bac} A^a_\mu$ is the covariant derivative in the adjoint representation, and c and $\bar{c}$ are the ghost and antighost fields.

The notation $\partial G^a/\partial A^b_\mu$ means that any derivatives that act on A^b_μ in G^a now act to the right in eq (78.4).

We get R_ξ gauge by choosing $G^a = \partial^\mu A^a_\mu$.

To get *background field gauge*, we first introduce a fixed, *classical background field* $\bar{A}^a_\mu(x)$, and the corresponding background covariant derivative,

$$\bar{D}_\mu \equiv \partial_\mu - igT^a_{\rm A} \bar{A}^a_\mu \ . \tag{78.5}$$

Then we choose

$$G^a = (\bar{D}^\mu)^{ab}(A-\bar{A})^b_\mu \, . \tag{78.6}$$

The ghost lagrangian becomes $\mathcal{L}_{\rm gh} = \bar{c}^a \bar{D}^{\mu ab} D^{bc}_\mu c^c$, or, after an integration by parts,

$$\mathcal{L}_{\rm gh} = -(\bar{D}^\mu \bar{c})^a (D_\mu c)^a \; , \tag{78.7}$$

where $(\bar{D}^\mu \bar{c})^a = \bar{D}^{\mu ab}\bar{c}^b$ and $(D_\mu c)^a = D^{ac}_\mu c^c$.

Under an infinitesimal gauge transformation, the change in the fields is

$$\delta_{\rm G} A^a_\mu(x) = -D^{ac}_\mu \theta^c(x) \; , \tag{78.8}$$

$$\delta_{\rm G} c^b(x) = -ig\theta^a(x)(T^a_{\rm A})^{bc} c^c(x) \; , \tag{78.9}$$

$$\delta_{\rm G} \bar{A}^a_\mu(x) = 0 \; . \tag{78.10}$$

The antighost $\bar{c}$ transforms in the same way as c (since the adjoint representation is real).

The background field $\bar{A}$ is fixed, and so does not change under a gauge transformation.

Of course, this means that $\mathcal{L}_{\rm gf}$ and $\mathcal{L}_{\rm gh}$ are not gauge invariant; their role is to fix the gauge.

We can, however, define a *background field gauge transformation*, under which only the background field transforms,

$$\delta_{\rm BG}\bar{A}^a_\mu(x) = -\bar{D}^{ac}_\mu\theta^c(x)\ , \tag{78.11}$$

$$\delta_{\rm BG}A^a_\mu(x) = 0\ , \tag{78.12}$$

$$\delta_{\rm BG}c^b(x) = 0\ . \tag{78.13}$$

Obviously, $\mathcal{L}_{\rm YM}$ is invariant under this transformation (since it does not involve the background field at all), but $\mathcal{L}_{\rm gf}$ and $\mathcal{L}_{\rm gh}$ are not.

However, $\mathcal{L}_{\rm gf}$ and $\mathcal{L}_{\rm gh}$ are invariant under the combined transformation $\delta_{\rm G+BG}$.

For $\mathcal{L}$gh, as given by eq. (78.7), this follows immediately from the fact that D_μ and $\bar{D}_\mu$ ave the same transformation property under the combined transformation, and that using covariant derivatives with all group indices contracted always yields a gauge-invariant expression.

To use this argument on $\mathcal{L}$gf , as given by eqs. (78.3) and (78.6), we need to show that $(A - A)^a_\mu$ transforms under the combined transformation in the same way as does an ordinary field in the adjoint representation, such as the ghost field in eq. (78.9). To show this, we write

$$\begin{aligned}\delta_{\rm G+BG}(A-\bar{A})^b_\mu &= -(D-\bar{D})^{ba}_\mu\theta^a \\ &= +ig(A-\bar{A})^c_\mu(T^c_{\rm A})^{ba}\theta^a \\ &= -ig\theta^a(T^a_{\rm A})^{bc}(A-\bar{A})^c_\mu\ .\end{aligned} \tag{78.14}$$

We used the complete antisymmetry of $(T^c{}_A)^{ba} = -if^{cba}$ to get the last line.

We see that $(A - \bar{A})^a{}_\mu$ transforms like an ordinary field in the adjoint representation, and so any expression that involves only covariant derivatives (either D_μ or $\bar{D}_\mu$) acting on this field, with all group indices contracted, is invariant under the combined transformation.

Therefore, the *complete lagrangian*, $\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{gf} + \mathcal{L}_{gh}$, is *invariant under the combined transformation*.

Now consider constructing the quantum action $\Gamma(A, c, \bar{c}; \bar{A})$.

Recall from section 21 that the quantum action can be expressed as the sum of all 1PI diagrams, with the external propagators replaced by the corresponding fields.

In a gauge theory, the quantum action is in general not gauge invariant, because we had to fix a gauge in order to carry out the path integral.

The quantum action thus depends on the choice of gauge, and hence (in the case of background field gauge) on the background field $\bar{A}$.

This is why we have written $\bar{A}$ as an argument of Γ, but separated by a semicolon to indicate its special role.

An important property of the quantum action is that it inherits all linear symmetries of the classical action; see problem 21.2.

In the present case, these symmetries include the combined gauge transformation δ_{G+BG}.

Therefore, the *quantum action* is also invariant under the *combined transformation*.

The quantum action takes its simplest form if we set the external field A equal to the background field $\bar{A}$.

Then, $\Gamma(\bar{A}, c, \bar{c}; \bar{A})$ is invariant under a gauge transformation of the form

$$\delta_{\mathrm{G+BG}} \bar{A}^a_\mu(x) = -\bar{D}^{ac}_\mu \theta^c(x) \,, \tag{78.15}$$

$$\delta_{\mathrm{G+BG}} c^b(x) = -ig\theta^a(x)(T^a_{\mathrm{A}})^{bc} c^c(x) \,. \tag{78.16}$$

This is now simply an ordinary gauge transformation, with A as the gauge field.

The quantum action can be expressed as the classical action, plus loop corrections.

For A = $\bar{A}$, we have

$$\Gamma(\bar{A}, c, \bar{c}; \bar{A}) = \int d^4x \left[-\tfrac{1}{4} \bar{F}^{a\mu\nu} \bar{F}^a_{\mu\nu} - (\bar{D}^\mu \bar{c})^a (\bar{D}_\mu c)^a \right] + \ldots \,, \tag{78.17}$$

where the ellipses stand for the loop corrections.

Note that Lgf has disappeared [because we set A =$\bar{A}$ in eq. (78.6)], and $\mathcal{L}_{\mathrm{gh}}$ has the form of a kinetic term for a complex scalar field in the adjoint representation.

This term is therefore manifestly gauge invariant, as is the $\bar{F}\bar{F}$ term.

The gauge invariance of the quantum action has an important consequence for the loop corrections.

In background field gauge, the renormalizing Z factors must respect the gauge invariance of the quantum action.

Therefore, using the notation of section 73, we must have

$$Z_1 = Z_2 \ , \tag{78.18}$$

$$Z_{1'} = Z_{2'} \ , \tag{78.19}$$

$$Z_3 = Z_{3g} = Z_{4g} \ . \tag{78.20}$$

Thus the relation between the bare and renormalized gauge couplings becomes

$$g_0^2 = Z_3^{-1} g^2 \tilde{\mu}^{\varepsilon} \ . \tag{78.21}$$

This relation now involves only Z_3.

We can therefore compute the beta function from Z3 alone.

This is the major advantage of background field gauge.

To compute the loop corrections, we need to evaluate 1PI diagrams in background-field gauge with the external propagators removed and replaced with external fields; the external gauge field should be set equal to the background field.

The easiest way to do this is to set

$$A = \bar{A} + \mathcal{A} \tag{78.22}$$

at the beginning, and to write the path integral in terms of $\mathcal{A}$.

Then the $\mathcal{A}$ field appears only on internal lines, and the $\bar{A}$ field only on external lines.

The gauge-fixing term now reads

$$\mathcal{L}_{\rm gf} = -\tfrac{1}{2}\xi^{-1}(\bar{D}^\mu \mathcal{A}_\mu)^a(\bar{D}^\nu \mathcal{A}_\nu)^a \ , \tag{78.23}$$

and the ghost term is given by eq. (78.7).

The Feynman rules that follow from $\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm gf} + \mathcal{L}_{\rm gh}$ are closely related to those we found in R_ξ gauge in section 72.

The ghost and gluon propagators are the same, and vertices involving all internal lines are also the same.

But if one or more gluon lines are external, then there are additional contributions to the vertices from $\mathcal{L}_{\rm gf}$ and $\mathcal{L}_{\rm gh}$.

Further simplifications arise at the one-loop level. Using eq. (78.22) in eq. (78.2), we find

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu \bar{A}^a_\nu - \partial_\nu \bar{A}^a_\mu + g f^{abc}\bar{A}^b_\mu \bar{A}^c_\nu \\ &\quad + \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu + g f^{abc}(\bar{A}^b_\mu \mathcal{A}^c_\nu + \mathcal{A}^b_\mu \bar{A}^c_\nu) + g f^{abc}\mathcal{A}^b_\mu \mathcal{A}^c_\nu \\ &= \bar{F}^a_{\mu\nu} + (\bar{D}_\mu \mathcal{A}_\nu)^a - (\bar{D}_\nu \mathcal{A}_\mu)^a + g f^{abc}\mathcal{A}^b_\mu \mathcal{A}^c_\nu \ . \end{aligned} \tag{78.24}$$

We then have

$$\begin{aligned} \mathcal{L}_{\rm YM} = -\tfrac{1}{4}\bar{F}^{a\mu\nu}\bar{F}^a_{\mu\nu} - \tfrac{1}{2}(\bar{D}^\mu \mathcal{A}^\nu)^a(\bar{D}_\mu \mathcal{A}_\nu)^a + \tfrac{1}{2}(\bar{D}^\mu \mathcal{A}^\nu)^a(\bar{D}_\nu \mathcal{A}_\mu)^a \\ - \tfrac{1}{2} g f^{abc}\bar{F}^{a\mu\nu}\mathcal{A}^b_\mu \mathcal{A}^c_\nu + \dots \ , \end{aligned} \tag{78.25}$$

where the ellipses stand for terms that are linear, cubic, or quartic in $\mathcal{A}$.

Vertices arising from terms linear in $\mathcal{A}$ cannot appear in a 1PI diagram, and the cubic and quartic vertices do not appear in the one-loop contribution to the $\bar{A}$ propagator.

The last term on the first line of eq. (78.25) can be usefully manipulated with some dummy-index relabelings and integrations by parts; we have

$$\begin{aligned}(\bar{D}^\mu \mathcal{A}^\nu)^a(\bar{D}_\nu \mathcal{A}_\mu)^a &= (\bar{D}^\nu \mathcal{A}_\mu)^c(\bar{D}^\mu \mathcal{A}_\nu)^c \\ &= -\mathcal{A}^b_\mu(\bar{D}^\nu \bar{D}^\mu)^{bc}\mathcal{A}^c_\nu \\ &= -\mathcal{A}^b_\mu(\bar{D}^\mu \bar{D}^\nu - [\bar{D}^\mu, \bar{D}^\nu])^{bc}\mathcal{A}^c_\nu \\ &= -\mathcal{A}^b_\mu(\bar{D}^\mu \bar{D}^\nu)^{bc}\mathcal{A}^c_\nu - ig(T^a_{\rm A})^{bc}\bar{F}^{a\mu\nu}\mathcal{A}^b_\mu\mathcal{A}^c_\nu \\ &= +(\bar{D}^\mu \mathcal{A}_\mu)^c(\bar{D}^\nu \mathcal{A}_\nu)^c - gf^{abc}\bar{F}^{a\mu\nu}\mathcal{A}^b_\mu\mathcal{A}^c_\nu \ .\end{aligned} \tag{78.26}$$

Now the first term on the right-hand side of eq. (78.26) has the same form as the gauge-fixing term.

If we choose $\xi = 1$, these two terms will cancel.

Setting $\xi = 1$, and including a renormalizing factor of Z_3, the terms of interest in the complete lagrangian become

$$\begin{aligned}\mathcal{L} = -\tfrac{1}{4}Z_3\bar{F}^{a\mu\nu}\bar{F}^a_{\mu\nu} - \tfrac{1}{2}Z_3(\bar{D}^\mu\mathcal{A}^\nu)^a(\bar{D}_\mu\mathcal{A}_\nu)^a - (\bar{D}^\mu\bar{c})^a(\bar{D}_\mu c)^a \\ - Z_3 g f^{abc}\bar{F}^{a\mu\nu}\mathcal{A}^b_\mu\mathcal{A}^c_\nu \ .\end{aligned} \tag{78.27}$$

In the ghost term, we have replaced Dμ with Dμ; the vertex corresponding to the dropped A term does not appear in the one-loop contribution to the A propagator.

Also, we can rescale A to absorb Z3 in all terms except the first; since A never appears on an external line, its normalization is irrelevant, and always cancels among propagators and vertices. (The same is true of the ghost field.)

The one-loop diagrams that contribute to the $\bar{A}$ propagator are shown in fig. (78.1).

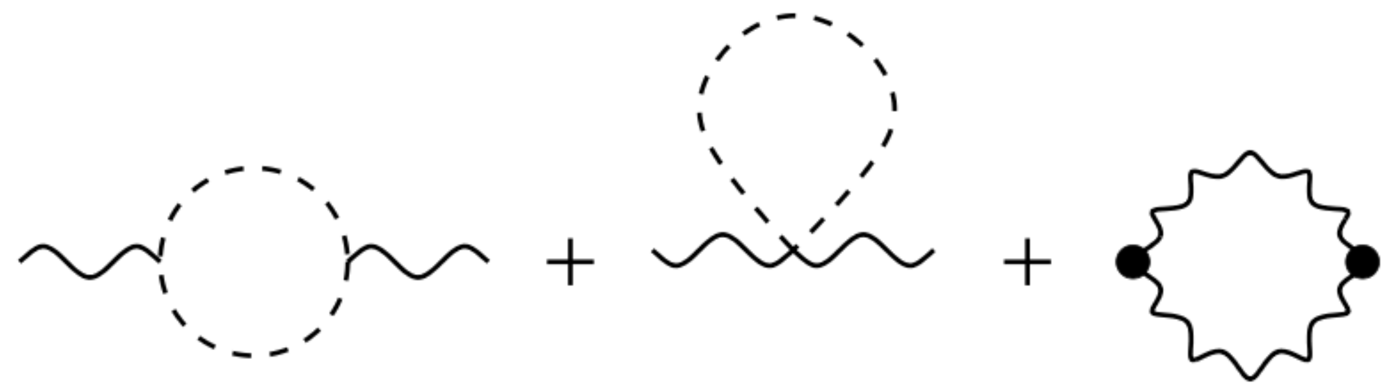


Figure 78.1: The one-loop contributions to the $\bar{A}$ propagator in background field gauge; the dashed lines can be either ghosts or internal $\mathcal{A}$ gauge fields. The dot denotes the $\bar{F}\mathcal{A}\mathcal{A}$ vertex.

The dashed lines in the first two diagrams represent either the $\mathcal{A}$ field or the ghost fields.

In either case, the second diagram vanishes, because it is proportional to $\int d^4\ell/\ell^2$, which is zero after dimensional regularization.

Note that the ghost term in eq. (78.27) has the form of a kinetic term for a complex scalar field in the adjoint representation.

One finds the contribution of a complex scalar field in a representation RCS to $\Pi(k^2)$ is

$$\Pi_{\rm CS}(k^2) = -\frac{g^2}{24\pi^2}\,T(\mathrm{R}_{\rm CS})\,\frac{1}{\varepsilon} + \text{finite}\ . \qquad (78.28)$$

The ghost contribution is minus this, with $R_{CS} \rightarrow A$. (The minus sign is from the closed ghost loop.)

Thus we have

$$\Pi_{\rm gh}(k^2) = +\frac{g^2}{24\pi^2}\,T(\mathrm{A})\,\frac{1}{\varepsilon} + \text{finite} + O(g^4)\,. \tag{78.29}$$

For reference we recall that the counterterm contribution is

$$\Pi_{\rm ct}(k^2) = -(Z_3-1)\,. \tag{78.30}$$

Next we consider the diagrams with $\mathcal{A}$ fields in the loop.

If the $\bar{F}\mathcal{A}\mathcal{A}$ interaction term was absent, the calculation would again be a familiar one; the $\bar{D}\mathcal{A}\bar{D}\mathcal{A}$ term in eq. (78.27) has the form of a kinetic term for a real scalar field that carries an extra index ν.

That this index is a Lorentz vector index is immaterial for the diagrammatic calculation; the index is simply summed around the loop, yielding an extra factor of d = 4.

There is also an extra factor of one-half (relative to the case of a complex scalar) because $\mathcal{A}$ is real rather than complex. (Equivalently, the diagram has a symmetry factor of S = 2 from exchange of the top and bottom internal propagators when they do not carry charge arrows.)

We thus have

$$\Pi_{\bar{D}\mathcal{A}\bar{D}\mathcal{A}}(k^2) = -\frac{g^2}{12\pi^2}\,T(\mathrm{A})\,\frac{1}{\varepsilon} + \text{finite} + O(g^4)\,. \tag{78.31}$$

If we now include the $\bar{F}\mathcal{A}\mathcal{A}$ interaction, we can think of $\bar{F}^a_{\mu\nu}$ as a constant external field.

We can then draw the third diagram of fig. (78.1), where and each dot denotes a vertex factor of $-2igf^{abc}\bar{F}^a_{\mu\nu}$. .

This vacuum diagram has a symmetry factor of S = 2 x 2: one factor of two for exchanging the top and bottom propagators, and one for exchanging the left and right sources.

Its contribution to the quantum action is

$$i\Gamma_{\bar{F}AA}/VT = \frac{1}{4}(-2igf^{acd}\bar{F}^a_{\mu\nu})(-2igf^{beg}\bar{F}^b_{\rho\sigma})\left(\tfrac{1}{i}\right)^2 \tilde{\mu}^\varepsilon \int \frac{d^d\ell}{(2\pi)^d}\,\frac{g_{\mu\rho}\delta_{ce}}{\ell^2}\,\frac{g_{\nu\sigma}\delta_{dg}}{\ell^2}$$

$$= g^2\,T(\mathrm{A})\bar{F}^{a\mu\nu}\bar{F}^a_{\mu\nu}\left(\frac{i}{8\pi^2\varepsilon} + \text{finite}\right), \tag{78.32}$$

where VT is the volume of spacetime.

Comparing this with the tree-level lagrangian $-\frac{1}{4}Z_3\bar{F}\bar{F}$, and recalling eq. (78.30), we see that eq. (78.32) is equivalent to a contribution to $\Pi(k^2)$ of

$$\Pi_{\bar{F}AA}(k^2) = +\frac{g^2}{2\pi^2}\,T(\mathrm{A})\,\frac{1}{\varepsilon} + \text{finite}\,. \tag{78.33}$$

There is also a one-loop diagram with one $\bar{D}A\bar{D}A$ vertex and one $\bar{F}AA$ vertex; however, contracting the vector indices on the $\bar{A}$ fields around the loop leads to a factor of $\bar{F}^{\mu\nu}g_{\mu\nu} = 0$.

Similarly, a one-loop diagram with a single $\bar{F}AA$ vertex vanishes.

We could also couple the gauge field to a Dirac fermion in the representation R_{DF}, and a complex scalar in the representation R_{CS}.

The corresponding contributions to $\Pi(k^2)$ were computed in section 73, and are given by eq. (78.28) and

$$\Pi_{\rm DF}(k^2) = -\frac{g^2}{6\pi^2}\,T(\mathrm{R}_{\rm DF})\,\frac{1}{\varepsilon} + \text{finite}\,, \tag{78.34}$$

Adding up eqs. (78.28), (78.29), (78.30), (78.31), (78.33), and (78.34), we find that finiteness of $\Pi(k^2)$ requires

$$Z_3 = 1+\frac{g^2}{24\pi^2}\Big[\big(+1-2+12\big)T(\mathrm{A})-4T(\mathrm{R}_{\rm DF})-T(\mathrm{R}_{\rm CS})\Big]\frac{1}{\varepsilon}+O(g^4) \tag{78.35}$$

in the $\overline{MS}$ renormalization scheme.

The analysis of section 28 now results in a beta function of

$$\beta(g) = -\frac{g^3}{48\pi^2}\Big[11T(\mathrm{A}) - 4T(\mathrm{R}_{\rm DF}) - T(\mathrm{R}_{\rm CS})\Big] + O(g^5)\,. \tag{78.36}$$

A Majorana fermion or a Weyl fermion makes half the contribution of a Dirac fermion in the same representation; a real scalar field makes half the contribution of a complex scalar field. (Majorana fermions and real scalars must be in real representations of the gauge group.)

In quantum chromodynamics, the gauge group is SU(3), and there are $n_F = 6$ flavors of quarks (which are Dirac fermions) in the fundamental representation.

We therefore have $T(A) = 3$, $T(R_{DF}) = 1/2 n_F$, and $T(R_{CS}) = 0$; therefore

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(11 - \tfrac{2}{3}n_F\right) + O(g^5)\ . \tag{78.37}$$

We see that the beta function is negative for $n_F \leq 16$, and so QCD is asymptotically free.

79 Gervais–Neveu Gauge

In section 78, we used background field gauge to set up the computation of a quantum action that is gauge invariant.

Given this quantum action, we can use it to compute scattering amplitudes via the corresponding tree diagrams, as discussed in section 19.

Since the ghost fields in the quantum action do not contribute to tree diagrams, we can simply drop all the ghost terms in the quantum action.

Because the quantum action computed in background field gauge is itself gauge invariant, it requires further gauge fixing to specify the gluon propagator and vertices.

We can choose whatever gauge is most convenient; for example, R_ξ gauge.

In principle, this gauge fixing involves introducing new ghost fields, but, once again, these do not contribute to tree diagrams, and so we can ignore them.

If we start with the tree-level approximation to the quantum action, then, in R_ξ gauge, we simply get the gluon propagator and vertices of section 72.

As we noted there, the complexity of the three- and four-gluon vertices in R_ξ gauge leads to long, involved computations of even simple processes like gluon-gluon scattering.

In this section, we will introduce another gauge, Gervais–Neveu gauge, that simplifies these tree-level computations.

We begin by specializing to the gauge group SU(N), and working with the matrix-valued field $A_\mu = A^a{}_\mu T^a$.

For later convenience, we will normalize the generators via

$$\mathrm{Tr}\, T^a T^b = \delta^{ab} \ . \tag{79.1}$$

With this choice, their commutation relations become

$$[T^a, T^b] = i\sqrt{2} f^{abc} T^c \ . \tag{79.2}$$

The tree-level action is specified by the Yang–Mills lagrangian

$$\mathcal{L}_{\mathrm{YM}} = -\tfrac{1}{4}\mathrm{Tr}\, F^{\mu\nu} F_{\mu\nu} \ , \tag{79.3}$$

where the matrix-valued field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\sqrt{2}}[A_\mu, A_\nu] \ . \tag{79.4}$$

Let us introduce the matrix-valued complex tensor

$$H_{\mu\nu} \equiv \partial_\mu A_\nu - \frac{ig}{\sqrt{2}} A_\mu A_\nu \ . \tag{79.5}$$

Then $F_{\mu\upsilon}$ is the antisymmetric part of $H_{\mu\upsilon}$

$$F_{\mu\nu} = H_{\mu\nu} - H_{\nu\mu} \ . \tag{79.6}$$

The Yang–Mills lagrangian can now be written as

$$\mathcal{L}_{\rm YM} = -\tfrac{1}{2}\mathrm{Tr}\Big(H^{\mu\nu}H_{\mu\nu} - H^{\mu\nu}H_{\nu\mu}\Big) \ . \tag{79.7}$$

To fix the gauge, we choose a matrix-valued gauge-fixing function G(x), and add to $\mathcal{L}_{\rm YM}$.

$$\mathcal{L}_{\rm gf} = -\tfrac{1}{2}\mathrm{Tr}\, GG \tag{79.8}$$

Here we have set the gauge parameter ξ to one, and ignored the ghost lagrangian (since, as we have already discussed, it does not affect tree diagrams).

The choice of G that yields Gervais–Neveu gauge is

$$G = H^\mu{}_\mu \ . \tag{79.9}$$

At first glance, this choice seems untenable, because we see from eq. (79.5) that this G (and hence $\mathcal{L}_{\rm gf}$) is not hermitian.

However, because the role of $\mathcal{L}_{\rm gf}$ is merely to fix the gauge, it is acceptable for $\mathcal{L}_{\rm gf}$ to be nonhermitian.

Combining eqs. (79.7) and (79.8), we get a total, gauge-fixed lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}\Big(H^{\mu\nu}H_{\mu\nu} - H^{\mu\nu}H_{\nu\mu} + H^{\mu}{}_{\mu}H^{\nu}{}_{\nu}\Big)\ . \tag{79.10}$$

Consider the terms in L with two derivatives.

After some integrations by parts, those from the third term in eq. (79.10) cancel those from the second, leading to

$$\mathcal{L}_{2\partial} = -\tfrac{1}{2}\mathrm{Tr}\,\partial^{\mu}A^{\nu}\partial_{\mu}A_{\nu}\ , \tag{79.11}$$

just as in R_ξ gauge with $\xi = 1$.

Now consider the terms with no derivatives.

Once again, those from the third term in eq. (79.10) cancel those from the second (after using the cyclic property of the trace), leading to

$$\mathcal{L}_{0\partial} = +\tfrac{1}{4}g^2\,\mathrm{Tr}\,A^{\mu}A^{\nu}A_{\mu}A_{\nu}\ . \tag{79.12}$$

Finally, we have the terms with one derivative,

$$\mathcal{L}_{1\partial} = +\frac{ig}{\sqrt{2}}\,\mathrm{Tr}\Big(\partial^\mu A^\nu A_\mu A_\nu - \partial^\mu A^\nu A_\nu A_\mu + \partial^\mu A_\mu A^\nu A_\nu\Big)\ . \tag{79.13}$$

Each derivative acts only on the field to its immediate right.

If we integrate by parts in the last term in eq. (79.13), we generate two terms; one of these cancels the first term in eq. (79.13), and the other duplicates the second.

Thus we have

$$\mathcal{L}_{1\partial} = -i\sqrt{2}g\,\mathrm{Tr}\,\partial^\mu A^\nu A_\nu A_\mu\ . \tag{79.14}$$

Combining eqs. (79.11), (79.12), and (79.14), we find

$$\mathcal{L} = \mathrm{Tr}\Big(-\tfrac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu - i\sqrt{2}g\,\partial^\mu A^\nu A_\nu A_\mu + \tfrac{1}{4}g^2 A^\mu A^\nu A_\mu A_\nu\Big)\ . \tag{79.15}$$

Because this lagrangian has a rather simple structure in terms of the matrix-valued field A_μ, it is helpful to stick with this notation, rather than trying to reexpress $\mathcal{L}$ in terms of $A^a{}_\mu = \mathrm{Tr}(T^a A_\mu)$.

In the next section, we explore the Feynman rules for a matrix-valued field in a simplified context.

80 The Feynman Rules for N x N Matrix

In section 79, we found that the lagrangian for SU(N) Yang–Mills theoryin Gervais–Neveu gauge is

$$\mathcal{L} = \mathrm{Tr}\Big(-\tfrac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu - i\sqrt{2}g\,\partial^\mu A^\nu A_\nu A_\mu + \tfrac{1}{4}g^2 A^\mu A^\nu A_\mu A_\nu\Big)\ , \tag{80.1}$$

where $A_\mu(x)$ is a traceless hermitian N x N matrix.

In this section, we will work out the Feynman rules for a simplified model of a scalar field that keeps the essence of the matrix structure.

Let B(x) be a hermitian N x N matrix that is *not* traceless.

Let T^a be a complete set of N^2 hermitian N x N matrices normalized according to

$$\mathrm{Tr}\,T^aT^b = \delta^{ab}\ . \tag{80.2}$$

We will take one of these matrices, T^{N^2}, to be proportional to the identity matrix; then eq. (80.2) requires the rest of the T^a's to be traceless.

We can expand B(x) in the T^a's, with coefficient fields $B^a(x)$,

$$B(x) = B^a(x)T^a\ , \tag{80.3}$$

$$B^a(x) = \mathrm{Tr}\,T^a B(x)\ , \tag{80.4}$$

where the repeated index in eq. (80.3) is implicitly summed over a = 1 to N^2.

Consider a lagrangian for B(x) of the form

$$\mathcal{L} = \mathrm{Tr}\Big(-\tfrac{1}{2}\partial^\mu B \partial_\mu B + \tfrac{1}{3} g B^3 - \tfrac{1}{4}\lambda B^4\Big)\ . \tag{80.5}$$

Using eqs. (80.2) and (80.3), we find an expression for $\mathcal{L}$ in terms of the coefficient fields,

$$\begin{aligned}\mathcal{L} = {} & -\tfrac{1}{2}\partial^\mu B^a \partial_\mu B^a + \tfrac{1}{3} g\, \mathrm{Tr}(T^a T^b T^c) B^a B^b B^c \\ & -\tfrac{1}{4}\lambda\, \mathrm{Tr}(T^a T^b T^c T^d) B^a B^b B^c B^d\ .\end{aligned} \tag{80.6}$$

It is easy to read off the Feynman rules from this form of $\mathcal{L}$.

The propagator for the coefficient field B^a is

$$\tilde{\Delta}^{ab}(k^2) = \frac{\delta^{ab}}{k^2 - i\epsilon}\ . \tag{80.7}$$

There is a three-point vertex with vertex factor $2ig\,\mathrm{Tr}(T^a T^b T^c)$, and a four-point vertex with vertex factor $-6i\lambda\,\mathrm{Tr}(T^a T^b T^c T^d)$.

This clearly leads to messy and complicated formulae for scattering amplitudes.

Instead, let us work with L in the form of eq. (80.5).

Writing the matrix indices explicitly, with one up and one down (and employing the rule that two indices can be contracted only if one is up and one is down), we have $B(x)_i{}^j = B^a(x)(T^a)_i{}^j$.

This implies that the propagator for $B_i{}^j$ is

$$\tilde{\Delta}_i{}^j{}_k{}^l(k^2) = \frac{(T^a)_i{}^j (T^a)_k{}^l}{k^2 - i\epsilon} \ . \tag{80.8}$$

Since the T^a matrices form a complete set, there is a completeness relation of the form $(T^a)_i{}^j\,(T^a)_k{}^l \leftrightarrow \delta_i{}^l \delta_k{}^j$.

To get the constant of proportionality, set j = k and l = i to turn the left-hand side into $(T^a)_i k (T^a) k i = \mathrm{Tr}(T^a T^a)$, and the right-hand side into $\delta_i{}^i \delta_k{}^k = N^2$.

From eq. (80.2) we have $\mathrm{Tr}(T^aT^a) = \delta^{aa} = N^2$.

So the constant of proportionality is one, and

$$(T^a)_i{}^j (T^a)_k{}^l = \delta_i{}^l \delta_k{}^j \ . \tag{80.9}$$

We can represent the B propagator with a double-line notation, as shown in fig. (80.1).

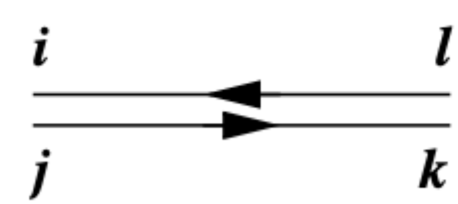
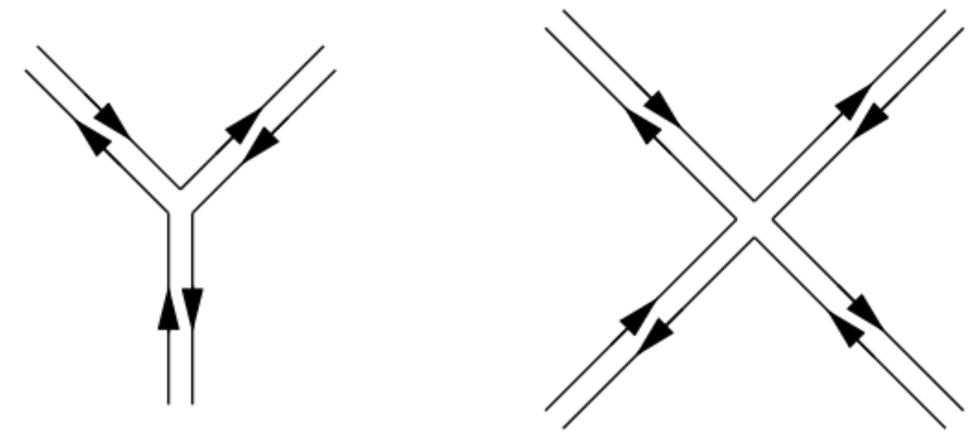


Figure 80.1: The double-line notation for the propagator of a hermitian matrix field.

The arrow on each line points from an up index to a down index.

Since the interactions are simple matrix products, with an up index from one field contracted with a down index from an adjacent field, the vertices follow the pattern shown in fig. (80.2).

Figure 80.2: 3- and 4-point vertices in the double-line notation.



Since an n-point vertex of this type has only an n-fold cyclic symmetry (rather than an n!-fold permutation symmetry), the vertex factor is i times the coefficient of $\mathrm{Tr}(B^n)$ in $\mathcal{L}$ times n (rather than n!).

Thus, for the lagrangian of eq. (80.5), the 3- and 4-point vertex factors are ig and $-i\lambda$.

Now consider a scattering process.

Particles corresponding to the coefficient fields labeled by the indices a_1 and a_2 (and with four-momenta k_1 and k_2) scatter into particles corresponding to the coefficient fields labeled by the indices a_3 and a_4 (and with four-momenta k_3 and k_4).

We wish to compute the scattering amplitude for this process, at tree level.

There are 18 contributing Feynman diagrams.

Three are shown in fig. (80.3); the remaining 15 are obtained by making noncylic permutations of the labels 1, 2, 3, 4 (equivalent to making unrestricted permutations of 2, 3, 4).

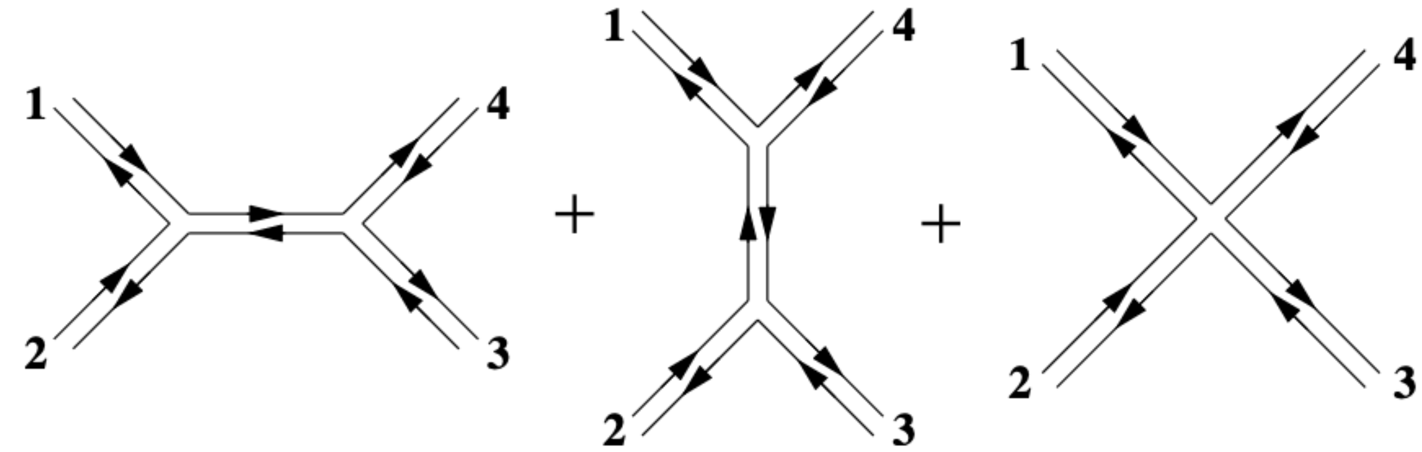


Figure 80.3: Tree diagrams with four external lines. Five more diagrams of each of these three types, with the external labels 2, 3, and 4 permuted, also contribute.

For simplicity, we will treat all external momenta as outgoing; then k^0_1 and k^0_2 are negative, and $k_1 + k_2 + k_3 + k_4 = 0$.

Each external line carries a factor of T^{a_i}, with its matrix indices contracted by following the arrows backward through the diagrams.

Omitting the $i\epsilon$'s in the propagators (which are not relevant for tree diagrams), the resulting tree-level amplitude is

$$i\mathcal{T} = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\left(\frac{(ig)^2(-i)}{(k_1+k_2)^2} + \frac{(ig)^2(-i)}{(k_1+k_4)^2} - i\lambda\right) \tag{80.10}$$
$$+ \Big((234) \to (342), (423), (243), (432), (324)\Big).$$

More generally, we can see that the value of any tree-level diagram with n external lines is proportional to $\mathrm{Tr}(T^{a_{i_1}} \ldots T^{a_{i_n}})$.

If the diagram is drawn in planar fashion (that is, with no crossed lines), then the ordering of the a_i indices in the trace is determined by the cyclic ordering of the labels on the external lines (which we take to be counterclockwise).

Then, each internal line contributes a factor of $-i/k^2$, each 3-point vertex a factor of ig, and each four-point vertex a factor of $-i\lambda$.

These are the *color-ordered* Feynman rules for this theory.

Return now to i $\mathcal{T}$ as given by eq. (80.10).

Suppose that we wish to square this amplitude, and sum over all possible particle types for each incoming or outgoing particle.

We then have to evaluate expressions like

$$\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})[\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3})]^* \ , \tag{80.11}$$

with all repeated indices summed.

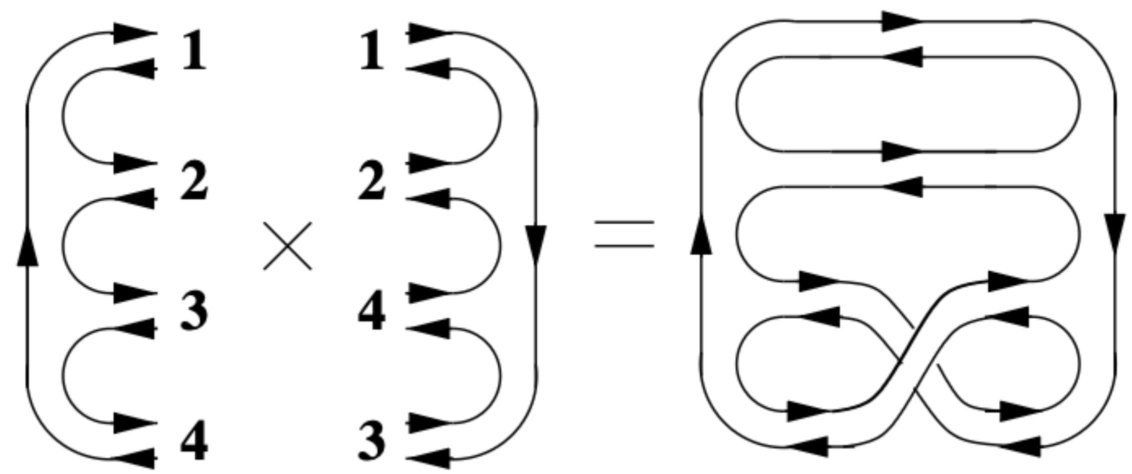
Using the hermiticity of the T^a matrices, we have

$$[\mathrm{Tr}(T^{a_1}\dots T^{a_n})]^* = \mathrm{Tr}(T^{a_n}\dots T^{a_1}) \ , \tag{80.12}$$

It is then easiest to evaluate eq. (80.11) diagrammatically, as shown in fig. (80.4).

φ $'$ $\mathcal{T}$ $\mathbb{R}$ ρ $\mathcal{L}$ $\int$ $\bar{q}$ $\tilde{q}$ $\widetilde{\Delta}$ $\tilde{\mu}$ $\tilde{\varphi}$ $\widetilde{dk}$ $\overline{MS}$ $\overline{\eta}(x)$ $\overline{\eta}$ $\overline{u}$ $\overline{\Psi}$ the the the the $(\overline{\eta}, \eta, J)$ $\mathcal{F}$ $\overline{v}$

Figure 80.4: Evaluation of $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})[\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3})]^*$, with all repeated indices summed. Each of the two closed single-line loops yields a factor of $\delta_i{}^i = N$.



Each closed single-line loop yields a factor of $\delta_i{}^i = N$.

The result is that the absolute square of any particular trace yields a factor of N^4, and the product of any trace times the complex conjugate of any other different trace yields a factor of N^2.

The coefficient of both $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$ and $\mathrm{Tr}(T^{a_1}T^{a_4}T^{a_3}T^{a_2})$ in eq. (80.10) is

$$A_3 \equiv \frac{g^2}{(k_1+k_2)^2} + \frac{g^2}{(k_1+k_4)^2} - \lambda\,. \tag{80.13}$$

Similarly, the coefficient of both $\mathrm{Tr}(T^{a_1}T^{a_3}T^{a_4}T^{a_2})$ and $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_4}T^{a_3})$ is

$$A_4 \equiv \frac{g^2}{(k_1+k_3)^2} + \frac{g^2}{(k_1+k_2)^2} - \lambda\,, \tag{80.14}$$

and of both $\mathrm{Tr}(T^{a_1}T^{a_4}T^{a_2}T^{a_3})$ and $\mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2}T^{a_4})$ is

$$A_2 \equiv \frac{g^2}{(k_1+k_4)^2} + \frac{g^2}{(k_1+k_3)^2} - \lambda\,. \tag{80.15}$$

Thus we have

$$\begin{aligned}\sum_{a_1,a_2,a_3,a_4} |\mathcal{T}|^2 &= (2N^4+2N^2)\sum_j |A_j|^2 + 4N^2\sum_{j\neq k} A_j^* A_k \\ &= (2N^4-2N^2)\sum_j |A_j|^2 + 4N^2(\sum_j A_j^*)(\sum_k A_k)\ ,\end{aligned} \tag{80.16}$$

where j and k are summed over 2, 3, 4.

Now suppose we wish to impose the condition that the matrix field B is traceless: Tr B = 0.

This means that we eliminate the component field with a = N^2, corresponding to the matrix $T^{N^2} = N^{-1/2}I$.

We must also eliminate T^{N^2} from the sum in eq. (80.9), leading to

$$(T^a)_i{}^j (T^a)_k{}^l = \delta_i{}^l \delta_k{}^j - \tfrac{1}{N}\delta_i{}^j \delta_k{}^l\ . \tag{80.17}$$

This can all be done diagrammatically by replacing the propagator in fig. (80.1) with the one in fig. (80.5). (The kinematic factor, $-i/k^2$, is unchanged.)

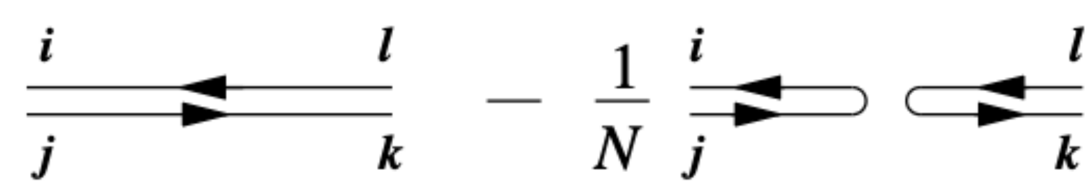


Figure 80.5: The propagator for a traceless hermitian field.

Fig. (80.5) must now be used as the internal propagator in the diagrams of fig. (80.3).

Also, when we multiply one diagram by the complex conjugate of another in the computation of $\sum_{a_1 \ldots a_n} |\mathcal{T}|^2$, we must use the propagator of fig. (80.5) to connect the external line of one diagram with the matching external line of the complex conjugate diagram.

Although these computations are still straightforward, they can become considerably more involved.

81 Scattering in Quantum Chromodynamics

In section 79, we found that the lagrangian for SU(N) Yang–Mills theory in Gervais–Neveu gauge is

$$\mathcal{L} = \mathrm{Tr}\Big(-\tfrac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu - i\sqrt{2}g\,\partial^\mu A^\nu A_\nu A_\mu + \tfrac{1}{4}g^2 A^\mu A^\nu A_\mu A_\nu\Big)\,, \tag{81.1}$$

where $A_\mu(x)$ is a traceless hermitian N x N matrix.

For quantum chromodynamics, N = 3, but we will leave N unspecified in our calculations.

In section 80, we worked out the *color-ordered Feynman rules* for a scalar matrix field; the same technology applies here as well.

In particular, we draw each tree diagram in planar fashion (that is, with no crossed lines).

Then the cyclic, counterclockwise ordering $i_1 \ldots i_n$ of the external lines fixes the color factor as $\mathrm{Tr}\,(T^{a_{i_1}} \ldots T^{a_{i_n}})$, where the generator matrices are normalized via $\mathrm{Tr}(T^aT^b) = \delta^{ab}$.

The tree-level n-gluon scattering amplitude is then written as

$$\mathcal{T} = g^{n-2} \sum_{\substack{\text{noncyclic}\\ \text{perms}}} \mathrm{Tr}(T^{a_1} \dots T^{a_n}) A(1, \dots, n) \; , \tag{81.2}$$

where we have pulled out the coupling constant dependence, and A(1, . . . , n) is a partial amplitude that we compute with the color-ordered Feynman rules.

The partial amplitudes are cyclically symmetric,

$$A(2, \dots, n, 1) = A(1, 2, \dots, n) \; . \tag{81.3}$$

The sum in eq. (81.2) is over all *noncyclic* permutations of 1 . . . n, which is equivalent to a sum over *all* permutations of 2 . . . n.

From the first term in eq. (81.1), we see that the gluon propagator is simply

$$\tilde{\Delta}_{\mu\nu}(k) = \frac{g_{\mu\nu}}{k^2 - i\epsilon} \; . \tag{81.4}$$

Here we have left out the matrix indices since we have already accounted for them with the color factor in eq. (81.2).

The second and third terms in eq. (81.1) yield three- and four-gluon vertices.

The three-gluon vertex factor (again without the matrix indices) is

$$i\mathbf{V}_{\mu\nu\rho}(p,q,r) = i(-i\sqrt{2}g)(-ip_\rho g_{\mu\nu}) + [\,2 \text{ cyclic permutations of } (\mu,p),(\nu,q),(\rho,r)\,] = -i\sqrt{2}g(p_\rho g_{\mu\nu} + q_\mu g_{\nu\rho} + r_\nu g_{\rho\mu})\ , \tag{81.5}$$

where the four-momenta p, q, and r are all taken to be outgoing.

The four-gluon vertex factor is simply

$$i\mathbf{V}_{\mu\nu\rho\sigma} = ig^2 g_{\mu\rho} g_{\nu\sigma}\ . \tag{81.6}$$

However, in the context of the color-ordered rules, it is simpler to designate the outgoing four-momentum on each external line as k_i, and contract the vector index with the corresponding polarization vector ε_i. (For now we suppress the helicity label $\lambda = \pm$.)

In this notation, the vertex factors become

$$i\mathbf{V}_{123} = -i\sqrt{2}g\Big[(\varepsilon_1\varepsilon_2)(k_1\varepsilon_3) + (\varepsilon_2\varepsilon_3)(k_2\varepsilon_1) + (\varepsilon_3\varepsilon_1)(k_3\varepsilon_2)\Big]\ , \tag{81.7}$$

$$i\mathbf{V}_{1234} = +ig^2(\varepsilon_1\varepsilon_3)(\varepsilon_2\varepsilon_4)\ , \tag{81.8}$$

where the external lines are numbered sequentially, counterclockwise around the vertex. (Of course, if an attached line is internal, the corresponding polarization vector is simply a placeholder for an internal propagator.)

The color-ordered three-point vertex, eq. (81.7), is antisymmetric on the reversal $123 \simeq 321$, while the four-point vertex, eq. (81.8), is symmetric on the reversal $1234 \simeq 4321$.

This implies the *reflection identity*

$$A(n, \ldots, 2, 1) = (-1)^n A(1, 2, \ldots, n) \,, \tag{81.9}$$

which will be useful later.

It is clear from eqs. (81.7) and (81.8) that every term in any tree-level scattering amplitude is proportional to products of polarization vectors with each other, or with external momenta. (Actually, this follows directly from Lorentz invariance, and the fact that the scattering amplitude is linear in each polarization.)

We get one momentum factor from each three-point vertex.

Since every tree diagram with n external lines has no more than n–2 vertices, there are no more than n–2 momenta to contract with the n polarizations.

Therefore, every term in every tree-level amplitude must include at least one product of two polarization vectors.

Then, if the product of every possible pair of polarization vectors vanishes, the tree-level amplitude for that process is zero.

We will now show that this is indeed the case if all, or all but one, of the external gluons have the same helicity. (Here we are using the semantic convention of section 60: the helicity of an external gluon is specified relative to the outgoing four-momentum k_i that labels the corresponding external line. If that gluon is actually incoming—as indicated by a negative value of k^0_i —then its physical helicity is opposite to its labeled helicity.)

To proceed, we recall from section 60 some formulae for products of polarization vectors in the spinor-helicity formalism,

$$\varepsilon_+(k;q)\cdot\varepsilon_+(k';q') = \frac{\langle q\,q'\rangle\,[k\,k']}{\langle q\,k\rangle\,\langle q'\,k'\rangle}\,, \tag{81.10}$$

$$\varepsilon_-(k;q)\cdot\varepsilon_-(k';q') = \frac{[q\,q']\,\langle k\,k'\rangle}{[q\,k]\,[q'\,k']}\,, \tag{81.11}$$

$$\varepsilon_+(k;q)\cdot\varepsilon_-(k';q') = \frac{\langle q\,k'\rangle\,[k\,q']}{\langle q\,k\rangle\,[q'\,k']}\,. \tag{81.12}$$

the first argument of each ε is the momentum of the corresponding line; the second argument is an arbitrary reference momentum.

Recall that the twistor producs $\langle q\ k\rangle$ and $[q\ k]$ are antisymmetric, and hence $\langle q\ q\rangle = [q\ q] = 0$.

Using this fact in eq. (81.10), we see that choosing the same reference momentum q for all positive-helicity polarizations results in a vanishing product for any pair of them.

Furthermore, if we choose this q equal to the momentum k' of a negative-helicity gluon, eq. (81.12) tells us that the product of its polarization with that of any positive-helicity gluon also vanishes.

Thus, if all, or all but one, of the external gluons have positive helicity, all possible polarization products are zero, and hence the tree-level scattering amplitude is also zero.

Thus we have shown that

$$A(1^{\pm}, 2^{+}, \ldots, n^{+}) = 0 \,, \tag{81.13}$$

where the superscripts are the helicities.

Of course, the same is true if all, or all but one, of the helicities are negative,

$$A(1^{\pm}, 2^{-}, \ldots, n^{-}) = 0 \,. \tag{81.14}$$

Now we turn to the calculation of some nonzero tree-level partial amplitudes, beginning with A(1−, 2−, 3+, 4+).

The contributing color-ordered Feynman diagrams are shown in fig. (81.1).

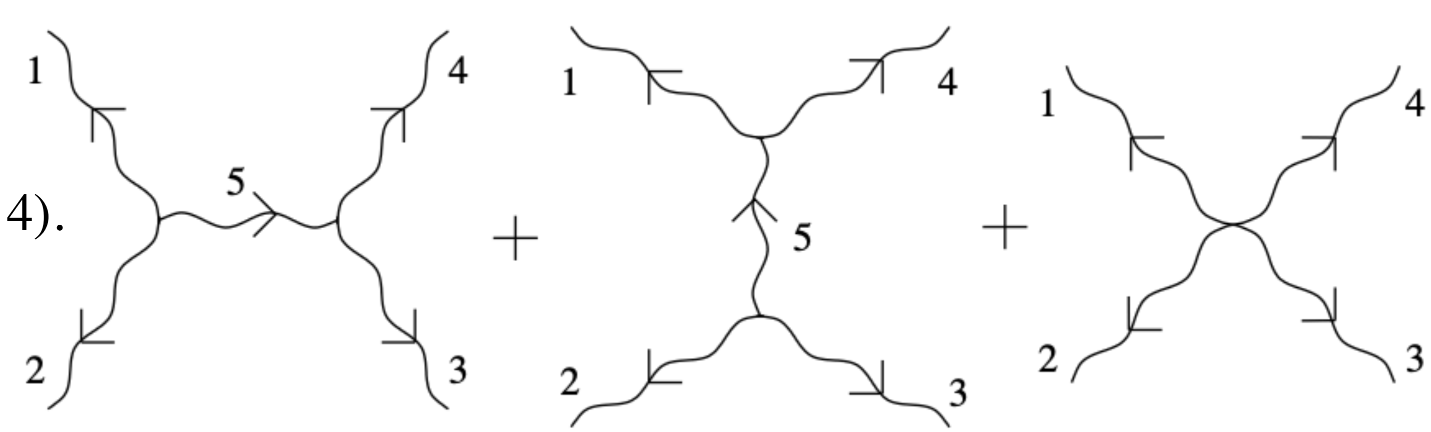


Figure 81.1: Diagrams for the partial amplitude A(1, 2, 3, 4).

We choose the reference momenta to be $q_1 = q_2 = k_3$ and $q_3 = q_4 = k_2$.

Then all polarization products vanish, with the exception of

$$\begin{aligned}
\varepsilon_1 \cdot \varepsilon_4 &= \varepsilon_-(k_1, q_1) \cdot \varepsilon_+(k_4, q_4) \\
&= \varepsilon_-(k_1, k_3) \cdot \varepsilon_+(k_4, k_2) \\
&= \frac{\langle 2\,1\rangle\,[4\,3]}{\langle 2\,4\rangle\,[3\,1]} \ .
\end{aligned} \tag{81.15}$$

With this choice of the reference momenta, the third diagram in fig. (81.1) obviously vanishes, because it has a factor of $\varepsilon_1 \cdot \varepsilon_3 = 0$ (and also, for good measure, $\varepsilon_2 \cdot \varepsilon_4 = 0$).

Now consider the 235 vertex in the second diagram; we have

$$\mathbf{V}_{235} \propto (\varepsilon_2 \varepsilon_3)(k_2 \varepsilon_5) + (\varepsilon_3 \varepsilon_5)(k_3 \varepsilon_2) + (\varepsilon_5 \varepsilon_2)(k_5 \varepsilon_3) \ , \tag{81.16}$$

where ε_5 is a placeholder for an internal propagator.

The first term in eq. (81.16) vanishes because $\varepsilon_2 \cdot \varepsilon_3 = 0$.

The second term vanishes because $k_3 = q_2$ and $q_2 \cdot \varepsilon_2 = 0$.

Finally, the third term vanishes because $k_5 = -k_2 - k_3 = -q_3 - k_3$, and $q_3 \cdot \varepsilon_3 = k_3 \cdot \varepsilon_3 = 0$.

Hence the 235 vertex vanishes, and therefore so does the second diagram.

That leaves only the first diagram.

We then have

$$ig^2 A(1^-,2^-,3^+,4^+) = (i\mathbf{V}_{125})(i\mathbf{V}_{345'})\Big|_{\varepsilon_5^\mu \varepsilon_5^\nu \,\to\, ig^{\mu\nu}/s_{12}}\,, \tag{81.17}$$

where 5′ means the momentum is $-k_5$ rather than k_5, and

$$s_{12} \equiv -(k_1+k_2)^2 = \langle 1\,2\rangle\,[2\,1]\;. \tag{81.18}$$

We have

$$i\mathbf{V}_{125} = -i\sqrt{2}g\Big[(\varepsilon_1\varepsilon_2)(k_1\varepsilon_5) + (\varepsilon_2\varepsilon_5)(k_2\varepsilon_1) + (\varepsilon_5\varepsilon_1)(k_5\varepsilon_2)\Big]\,, \tag{81.19}$$

but the first term vanishes because $\varepsilon_1 \cdot \varepsilon_2 = 0$.

Similarly, the first term of

$$i\mathbf{V}_{345'} = -i\sqrt{2}g\Big[(\varepsilon_3\varepsilon_4)(k_3\varepsilon_5) + (\varepsilon_4\varepsilon_5)(k_4\varepsilon_3) + (\varepsilon_5\varepsilon_3)(-k_5\varepsilon_4)\Big] \tag{81.20}$$

also vanishes.

When we take the product of these two vertices, and replace the internal polarizations with the propagator, as indicated in eq. (81.17), only the product of the third term of eq. (81.19) with the second term of eq. (81.20) is nonzero; all other terms include a vanishing product of polarizations.

We get

$$ig^2 A(1^-,2^-,3^+,4^+) = (-i\sqrt{2}g)^2(i/s_{12})(\varepsilon_1\varepsilon_4)(k_5\varepsilon_2)(k_4\varepsilon_3)\ . \qquad (81.21)$$

Since $k_5 = -k_1 - k_2$, and $k_2 \cdot \varepsilon_2 = 0$, we have $k_5 \cdot \varepsilon_2 = -k_1 \cdot \varepsilon_2$.

We evaluate $k_1 \cdot \varepsilon_2$ and $k_4 \cdot \varepsilon_3$ via the general formulae

$$p{\cdot}\varepsilon_+(k;q) \;=\; \frac{\langle q\,p\rangle\,[p\,k]}{\sqrt{2}\,\langle q\,k\rangle}\ , \qquad (81.22)$$

$$p{\cdot}\varepsilon_-(k;q) \;=\; \frac{[q\,p]\,\langle p\,k\rangle}{\sqrt{2}\,[q\,k]}\ . \qquad (81.23)$$

Setting $q_2 = k_3$ and $q_3 = k_2$, we get

$$k_1{\cdot}\varepsilon_2 \;=\; \frac{[3\,1]\,\langle 1\,2\rangle}{\sqrt{2}\,[3\,2]}\ , \qquad (81.24)$$

$$k_4{\cdot}\varepsilon_3 \;=\; \frac{\langle 2\,4\rangle\,[4\,3]}{\sqrt{2}\,\langle 2\,3\rangle}\ . \qquad (81.25)$$

Using eqs. (81.15), (81.18), (81.24), and (81.25) in eq. (81.21), and using antisymmetry of the twistor products to cancel common factors, we get

$$A(1^-,2^-,3^+,4^+) = \frac{\langle 2\,1\rangle\,[4\,3]^2}{[2\,1]\,[3\,2]\,\langle 2\,3\rangle}\ . \qquad (81.26)$$

We can make our result look nicer by multiplying the numerator anddenominator by $\langle 3\ 4\rangle$.

In the numerator, we use

$$\langle 3\,4\rangle\,[4\,3] = s_{34} = s_{12} = \langle 1\,2\rangle\,[2\,1]\ , \tag{81.27}$$

and cancel the [2 1] with the one in the denominator.

Now multiply the numerator and denominator by $\langle 4\ 1\rangle$, and use a momentum-conservation identity to replace $\langle 4\ 1\rangle\ [4\ 3]$ in the numerator with $-\langle 2\ 1\rangle\ [2\ 3]$, and cancel the [2 3] with the [3 2] in the denominator (which yields a minus sign).

Finally, multiply the numerator and denominator by $\langle 1\ 2\rangle$ to get

$$A(1^-,2^-,3^+,4^+) = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}\ . \tag{81.28}$$

This is our final result for $A(1^-, 2^-, 3^+, 4^+)$.

Now, using cyclic symmetry, we can get any partial amplitude where the two negative helicities are adjacent; for example,

$$A(1^+,2^-,3^-,4^+) = \frac{\langle 2\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}\ . \tag{81.29}$$

We must still calculate one partial amplitude where the negative helicities are not adjacent, such as $A(1^-, 2^+ 3^-, 4^+)$.

Once we have it, we can use cyclic symmetry to get all the remaining partial amplitudes.

Before turning to this calculation, let us consider the problem of squaring the total amplitude and summing over colors.

Because the generator matrices are traceless, we should (as we discussed in section 80) use the completeness relation

$$(T^a)_i{}^j (T^a)_k{}^l = \delta_i{}^l \delta_k{}^j - \tfrac{1}{N} \delta_i{}^j \delta_k{}^l \; . \tag{81.30}$$

However, recall that the Yang–Mills field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \tfrac{ig}{\sqrt{2}} [A_\mu, A_\nu] \; . \tag{81.31}$$

If we allow a generator matrix proportional to the identity, which corresponds to a gauge group of U(N) rather than SU(N), then this extra U(1) generator commutes with every other generator.

Thus the U(1) field does not appear in the commutator term in eq. (81.31).

Since it is this commutator term that is responsible for the interaction of the gluons, the U(1) field is a free field.

Therefore, any scattering amplitude involving the associated particle (which we will call the *fictitious photon*) must be zero.

Thus, if we write a scattering amplitude in the form of eq. (81.2), and replace one of the T^a's with the identity matrix, the result must be zero.

This *decoupling of the fictitious photon* allows us to use the much simpler completeness relation

$$(T^a)_i{}^j (T^a)_k{}^l = \delta_i{}^l \delta_k{}^j - \tfrac{1}{N} \delta_i{}^j \delta_k{}^l \ . \tag{81.32}$$

in place of eq. (81.30).

There is no need to subtract the U(1) generator from the sum over the generators, as we did in eq. (81.30), because the terms involving it vanish anyway.

The decoupling of the fictitious photon is useful in another way.

Let us apply it to the case of n = 4, and set $T^{a_4} \alpha$ I in eq. (81.2).

Then we have

$$\begin{aligned} 0 = {} & \mathrm{Tr}(T^{a_1} T^{a_2} T^{a_3}) \Big[A(1,2,3,4) + A(1,2,4,3) + A(1,4,2,3) \Big] \\ & + \mathrm{Tr}(T^{a_1} T^{a_3} T^{a_2}) \Big[A(1,3,2,4) + A(1,3,4,2) + A(1,4,3,2) \Big] \ . \end{aligned} \tag{81.33}$$

The content of each square bracket must vanish.

Requiring this of the first term yields

$$A(1,2,3,4) = - A(1,2,4,3) - A(1,4,2,3) \ . \tag{81.34}$$

Assigning some helicities, this reads

$$A(1^-,2^+,3^-,4^+) = -A(1^-,2^+,4^+,3^-) - A(1^-,4^+,2^+,3^-)\,. \tag{81.35}$$

Note that we have now expressed a partial amplitude with nonadjacent negative helicities in terms of partial amplitudes with adjacent negative helicities, which we have already calculated.

Thus we have

$$\begin{aligned}
A(1^-,2^+,3^-,4^+) &= -\left[\frac{\langle 3\,1\rangle^4}{\langle 3\,1\rangle\langle 1\,2\rangle\langle 2\,4\rangle\langle 4\,3\rangle} + \frac{\langle 3\,1\rangle^4}{\langle 3\,1\rangle\langle 1\,4\rangle\langle 4\,2\rangle\langle 2\,3\rangle}\right] \\
&= -\frac{\langle 1\,3\rangle^3}{\langle 2\,4\rangle}\left[\frac{1}{\langle 1\,2\rangle\langle 3\,4\rangle} + \frac{1}{\langle 1\,4\rangle\langle 2\,3\rangle}\right] \\
&= -\frac{\langle 1\,3\rangle^3}{\langle 2\,4\rangle}\left[\frac{\langle 1\,4\rangle\langle 2\,3\rangle + \langle 1\,2\rangle\langle 3\,4\rangle}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 1\,4\rangle\langle 2\,3\rangle}\right] \\
&= -\frac{\langle 1\,3\rangle^3}{\langle 2\,4\rangle}\left[\frac{-\langle 1\,3\rangle\langle 4\,2\rangle}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 1\,4\rangle\langle 2\,3\rangle}\right],
\end{aligned} \tag{81.36}$$

where the last line follows from the so-called Schouten identity..

A final clean-up yields

$$A(1^-,2^+,3^-,4^+) = \frac{\langle 1\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}\,. \tag{81.37}$$

Now that we have all the partial amplitudes, we can compute the color- summed $|T|^2$.

There are only three partial amplitudes that are not related by either cyclic permutations, eq. (81.3), or reflections, eq. (81.9); we can take these to be

$$A_3 \equiv A(1,2,3,4)\ , \tag{81.38}$$

$$A_4 \equiv A(1,3,4,2)\ , \tag{81.39}$$

$$A_2 \equiv A(1,4,2,3)\ , \tag{81.40}$$

where the subscript on the left-hand side is the third argument on the right-hand side. (Switching the second and fourth arguments is equivalent to a reflection and a cyclic permutation, and so leaves the partial amplitude unchanged.)

This mimics the notation we used at the end of section 80, and we can apply our result from there to the color sum,

$$\sum_{\text{colors}} |\mathcal{T}|^2 = 2N^2(N^2{-}1)g^4 \sum_j |A_j|^2 + 4N^2 g^4 \Big(\sum_j A_j^*\Big)\Big(\sum_k A_k\Big)\ , \tag{81.41}$$

where i and k are summed over 2, 3, 4. In the present case, however, eq. (81.34) is equivalent to $\sum_j A_j = 0$, so the second term in eq. (81.41) vanishes.

Our result for the color-summed squared amplitude is then

$$\sum_{\text{colors}} |\mathcal{T}|^2 = 2N^2(N^2{-}1)g^4 \Big(|A_2|^2 + |A_3|^2 + |A_4|^2\Big)\ . \tag{81.42}$$

For the case where 1 and 2 are the incoming gluons, and 3 and 4 are the outgoing gluons, we can write this in terms of the usual Mandelstam variables $s = s_{12} = s_{34}$, $t = s_{13} = s_{24}$, and $u = s_{14} = s_{23}$ by recalling that $|\langle 1\ 2\rangle|^2 = |[1\ 2]|^2 = |s_{12}|$, etc.

Let us take the case where gluons 1 and 2 have negative helicity, and 3 and 4 have positive helicity.

In this case, we see from eqs. (81.28) and (81.37) that the numerator in every nonvanishing partial amplitude is $\langle 1\ 2\rangle^4$.

Then we get

$$\sum_{\text{colors}} |\mathcal{T}|^2_{1^-2^-3^+4^+} = 2N^2(N^2{-}1)g^4 s^4\left(\frac{1}{s^2t^2}+\frac{1}{t^2u^2}+\frac{1}{u^2s^2}\right). \tag{81.43}$$

We can also sum over helicities.

There are six patterns of two positive and two negative helicities; $--++$ and $++--$ yield a factor of s^4, $-+-+$ and $+-+-$ yield t^4, and $-++-$ and $+--+$ yield u^4.

The helicity sum is therefore

$$\sum_{\substack{\text{colors}\\ \text{helicities}}} |\mathcal{T}|^2 = 4N^2(N^2{-}1)g^4(s^4+t^4+u^4)\left(\frac{1}{s^2t^2}+\frac{1}{t^2u^2}+\frac{1}{u^2s^2}\right). \tag{81.44}$$

Of course, we really want to average (rather than sum) over the initial colors and helicities; to do so we must divide eq. (81.44) by $4(N^2-1)^2$.

Next we turn to scattering of quarks and gluons.

We consider a single type of massless quark: a Dirac field in the N representation of SU(N).

The lagrangian for this field is $\mathcal{L} = i\overline{\Psi}\not{D}\Psi$, where the covariant derivative is $D_\mu = \partial_\mu - (ig/\sqrt{2})A_\mu$.

Thus the color-ordered vertex factor is

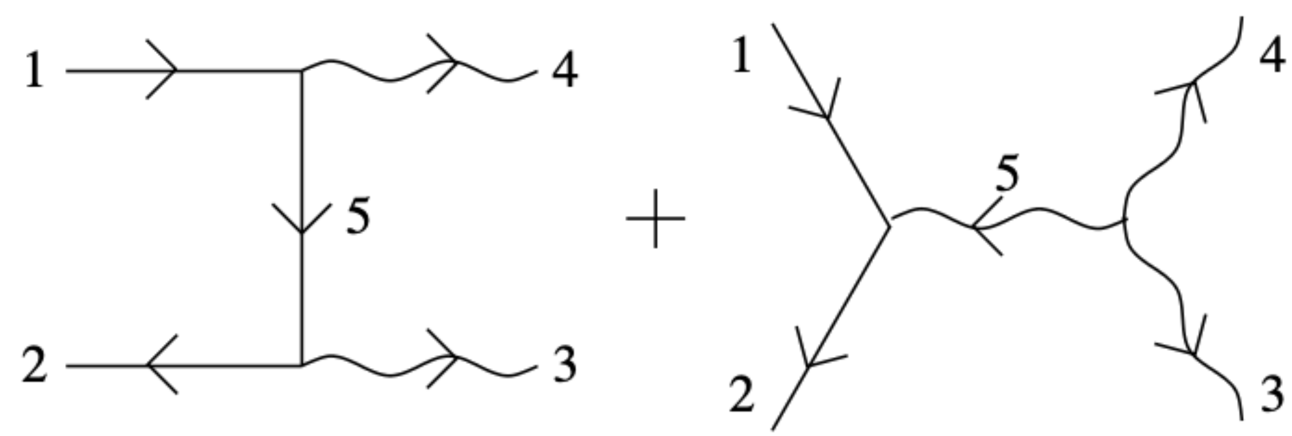
$$i\mathbf{V}^\mu = (ig/\sqrt{2})\gamma^\mu \; . \qquad (81.45)$$

To get the color factor, we use the double-line notation, with a single line for the quark.

As an example, consider the process of $\bar{q}q \rightarrow$ gg (and its crossing-related cousins).

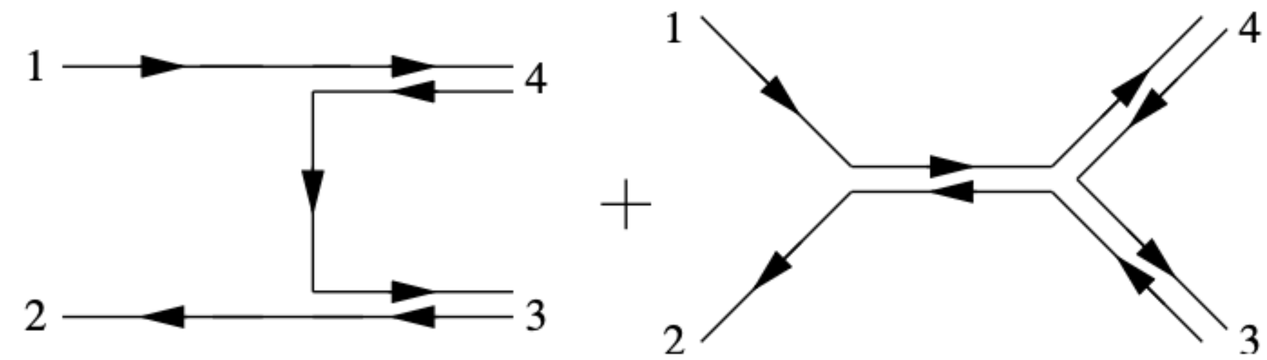
The contributing color-ordered tree diagrams are shown in fig. (81.2).

Figure 81.2: Color-ordered diagrams for $\bar{q}q$ gg scattering.



The corresponding double-line diagrams are shown in fig. (81.3); the quark is represented by a single line, with an arrow direction that matches its charge arrow.

Figure 81.3: The double-line version of fig. (81.2); the associated color factor is $(T^{a_3}T^{a_4})_{i_2}{}^{i_1}$.



To get the color factor, we start with line 2, and follow the arrows backwards; the result is $(T^{a_3}T^{a_4})_{i_2}{}^{i_1}$.

The complete amplitude can then be written as

$$\mathcal{T} = g^2\Big[(T^{a_3}T^{a_4})_{i_2}{}^{i_1}A(1_{\bar{q}},2_q,3,4) + (T^{a_4}T^{a_3})_{i_2}{}^{i_1}A(1_{\bar{q}},2_q,4,3)\Big]\ , \qquad (81.46)$$

where $A(1_{\bar{q}},2_q,3,4)$ is the appropriate partial amplitude.

The subscripts q and $\bar{q}$ indicate the labels that correspond to an outgoing quark and outgoing antiquark, respectively.

From our results for spinor electrodynamics in section 60, we know that a nonzero amplitude requires opposite helicities on the two ends of any fermion line.

Consider, then, the case of $\mathcal{T}_{-+\lambda_3\lambda_4}$.

The partial amplitude corresponding to the diagrams of fig. (81.2) is

$$
\begin{aligned}
ig^2 A(1^-_{\bar q}, 2^+_q, 3, 4) = {} & (ig/\sqrt{2})^2 (1/i)[2|\not\varepsilon_3(-\not p_5/p_5^2)\not\varepsilon_4|1\rangle \\
& + (ig/\sqrt{2})[2|\not\varepsilon_5|1\rangle i\mathbf{V}_{345}\Big|_{\varepsilon_5^\mu\varepsilon_5^\nu \to ig^{\mu\nu}/s_{12}} .
\end{aligned}
\tag{81.47}
$$

Suppose both gluons have positive helicity.

Then using

$$
\not\varepsilon_+(k;q) = \frac{\sqrt{2}}{\langle q\, k\rangle}\Big(|k]\langle q| + |q\rangle[k| \Big) , \tag{81.48}
$$

we can get both lines of eq. (81.47) to vanish by choosing $q_3 = q_4 = p_1$.

Similarly, if both gluons have negative helicity, then using

$$
\not\varepsilon_-(k;q) = \frac{\sqrt{2}}{[q\, k]}\Big(|k\rangle[q| + |q]\langle k| \Big) , \tag{81.49}
$$

we can get both lines of eq. (81.47) to vanish by choosing $q_3 = q_4 = p_2$.

So the gluons must have opposite helicities to get a nonzero amplitude.

Consider, then, the case of $\lambda_3 = +$ and $\lambda_4 = -$.

We can get $\mathbf{V}_{345}$ to vanish by choosing $q_3 = k_4$ and $q_4 = k_3$.

The partial amplitude is then given by just the first line of eq. (81.47),

$$A(1^-_{\bar{q}}, 2^+_q, 3^+, 4^-) = \tfrac{1}{2}\,[2|\not{\varepsilon}_{3+}(\not{p}_1 + \not{k}_4)\not{\varepsilon}_{4-}|1\rangle/(-s_{14})\ . \tag{81.50}$$

With $q_3 = k_4$ and $q_4 = k_3$, we have

$$\not{\varepsilon}_{3+} = \frac{\sqrt{2}}{\langle 4\,3\rangle}\Big(|4\rangle[3| + |3]\langle 4|\Big)\ , \tag{81.51}$$

$$\not{\varepsilon}_{4-} = \frac{\sqrt{2}}{[3\,4]}\Big(|4\rangle[3| + |3]\langle 4|\Big)\ . \tag{81.52}$$

Using the identity

$$\not{p} = -\,|p\rangle[p| - |p]\langle p|\ , \tag{81.53}$$

eq. (81.50) becomes

$$A(1^-_{\bar{q}}, 2^+_q, 3^+, 4^-) = \frac{[2\,3]\,\langle 4\,1\rangle\,[1\,3]\,\langle 4\,1\rangle}{\langle 4\,3\rangle\,[3\,4]\,s_{14}}\ . \tag{81.54}$$

In the numerator, we use $[1\ 3]\ \langle 4\ 1\rangle = -[2\ 3]\ \langle 4\ 2\rangle$.

In the denominator, we set $s_{14} = s_{23} = \langle 2\ 3\rangle\ [3\ 2]$.

Then we multiply the numerator and denominator by $\langle 1\ 2\rangle$, and use $[2\ 3]\ \langle 1\ 2\rangle = -[4\ 3]\ \langle 1\ 4\rangle$ in the numerator.

Finally we multiply both by $\langle 1\ 4\rangle$, and rearrange to get

$$A(1^-_{\bar{q}}\,,2^+_q\,,3^+\,,4^-) = \frac{\langle 1\,4\rangle^3\,\langle 2\,4\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}\,. \tag{81.55}$$

An analogous calculation yields

$$A(1^-_{\bar{q}}\,,2^+_q\,,3^-\,,4^+) = \frac{\langle 1\,3\rangle^3\,\langle 2\,3\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}\,. \tag{81.56}$$

The remaining nonzero amplitudes are related by complex conjugation.

Now that we have all the partial amplitudes, we can compute the color-summed $|T|^2$.

To do so, we multiply eq. (81.46) by its complex conjugate, and use hermiticity of the generator matrices to get

$$\begin{aligned}\sum_{\text{colors}} |\mathcal{T}|^2 = g^4\Big[&\mathrm{Tr}(T^aT^bT^bT^a)\Big(|A_3|^2+|A_4|^2\Big)\\ &+\mathrm{Tr}(T^aT^bT^aT^b)\Big(A_3^*A_4+A_4^*A_3\Big)\Big],\end{aligned} \tag{81.57}$$

where $A_3 \equiv A(1_{\bar{q}}, 2_q, 3, 4)$ and $A_4 \equiv A(1_{\bar{q}}, 2_q, 4, 3)$.

The traces are easily evaluated with the double-line technique of section 80; because the fictitious photon couples to the quark, we must use eq. (81.30) to project it out.

The traces in eq. (81.57) are also easily evaluated with the group- theoretic methods of section 70, with the normalization that the index of the fundamental representation is one: T(N) = 1.

Either way, the results are

$$\mathrm{Tr}(T^aT^bT^bT^a) = +(N^2-1)^2/N\ , \tag{81.58}$$

$$\mathrm{Tr}(T^aT^bT^aT^b) = -(N^2-1)/N\ . \tag{81.59}$$

The sum over the four possible helicity patterns (−++−, −+−+, +−−+, +−+−) is straightforward.

Now that we have calculated these scattering amplitudes for quarks and gluons, an important questions arises: why did we bother to do it?

Quarks and gluons are confined inside colorless bound states, the hadrons, and so apparently cannot appear as incoming and outgoing particles in a scattering event.

To answer this question, suppose we collide two hadrons with a center- of-mass energy $E = \sqrt{s}$ large enough so that the QCD coupling g is small when renormalized in the $\overline{MS}$ scheme with $\mu = E$. (In the real world, we have $\alpha \equiv g^2/4\pi = 0.12$ for $\mu = M_Z = 91$ GeV.)

Then we can think of each hadron as being made up of a loose collection of quarks and gluons, and these parts of a hadron, or partons, can be treated as independent participants in scattering processes.

In order to extract quantitative results for hadron scattering (maybe later), we need to know how each hadron's energy and momentum is shared among its partons.

This is described by parton distribution functions.

At present, these cannot be calculated from first principles, but they have to satisfy a variety of consistency conditions that can be derived from perturbation theory, and that relate their values at different energies.

These conditions are well satisfied by current experimental data.

82 Wilson Loops, Lattice Theory, and Confinement

In this section, we will construct a gauge-invariant operator, the Wilson loop, whose vacuum expectation value (VEV for short) can diagnose whether or not a gauge theory exhibits confinement.

A theory is confining if all finite-energy states are invariant under a global gauge transformation.

U(1) gauge theory—quantum electrodynamics—is not confining, because there are finite-energy states (such as the state of a single electron) that have nonzero electric charge, and hence change by a phase under a global gauge transformation.

Confinement is a nonperturbative phenomenon; it cannot be seen at any finite order in the kind of weak-coupling perturbation theory that we have been doing. (This is why we had no trouble calculating quark and gluon scattering amplitudes.)

In this section, we will introduce lattice gauge theory, in which spacetime is replaced by a discrete set of points; the inverse lattice spacing 1/a then acts as an ultraviolet cutoff (see section 29).

This cutoff theory can be analyzed at strong coupling, and, as we will see, in this regime the VEV of the Wilson loop is indicative of confinement.

The outstanding question is whether this phenomenon persists as we simultaneously lower the coupling and increase the ultraviolet cutoff (with the relationship between the two governed by the beta function), or whether we encounter a phase transition, signaled by a sudden change in the behavior of the Wilson loop VEV.

We take the gauge group to be SU(N).

Consider two spacetime points x^μ and $x^\mu + \varepsilon^\mu$, where ε^μ is infinitesimal.

Define the Wilson link

$$W(x{+}\varepsilon, x) \equiv \exp[ig\varepsilon^\mu A_\mu(x)] \ , \tag{82.1}$$

where $A_\mu(x)$ is an N x N matrix-valued traceless hermitian gauge field.

Since ε is infinitesimal, we also have

$$W(x{+}\varepsilon, x) = I + ig\varepsilon^\mu A_\mu(x) + O(\varepsilon^2) \ . \tag{82.2}$$

Let us determine the behavior of the Wilson link under a gauge transformation.

Using the gauge transformation of $A_\mu(x)$ from section 69, we find

$$W(x{+}\varepsilon, x) \to 1 + ig\varepsilon^\mu U(x)A_\mu(x)U^\dagger(x) - \varepsilon^\mu U(x)\partial_\mu U^\dagger(x) \ , \tag{82.3}$$

where U(x) is a spacetime-dependent special unitary matrix.

Since $UU^\dagger = 1$, we have $-U\,\partial_\mu U^\dagger = +(\partial_\mu U)U^\dagger$; thus we can rewrite eq. (82.3) as

$$W(x+\varepsilon,x) \to \Big((1+\varepsilon^\mu\partial_\mu)U(x)\Big)U^\dagger(x) + ig\varepsilon^\mu U(x)A_\mu(x)U^\dagger(x)\ . \tag{82.4}$$

In the first term, we can use $(1 + \varepsilon^\mu\partial_\mu)U(x) = U(x+\varepsilon) + O(\varepsilon^2)$.

In the second term, which already contains an explicit factor of ε^μ, we can replace U(x) with U(x+ε) at the cost of an $O(\varepsilon^2)$ error.

Then we get

$$W(x+\varepsilon,x) \to U(x+\varepsilon)\Big(1 + ig\varepsilon^\mu A_\mu(x)\Big)U^\dagger(x)\ , \tag{82.5}$$

which is equivalent to

$$W(x+\varepsilon,x) \to U(x+\varepsilon)W(x+\varepsilon,x)U^\dagger(x)\ . \tag{82.6}$$

Note also that eq. (82.1) implies $W^\dagger(x+\varepsilon, x) = W(x-\varepsilon, x)$.

We can shift x to x + ε at the cost of an $O(\varepsilon^2)$ error, and so

$$W^\dagger(x+\varepsilon,x) = W(x,x+\varepsilon)\ , \tag{82.7}$$

which is consistent with eq. (82.6).

Now consider multiplying together a string of Wilson links, specified by a starting point x and n sequential infinitesimal displacement vectors ε_j.

The ordered set of ε's defines a path P through spacetime that starts at x and ends at $y = x + \varepsilon_1 + \ldots + \varepsilon_n$.

The *Wilson line* for this path is

$$W_P(y,x) \equiv W(y,y-\varepsilon_n)\ldots W(x+\varepsilon_1+\varepsilon_2,x+\varepsilon_1)W(x+\varepsilon_1,x)\ . \tag{82.8}$$

Using eq. (82.6) and the unitarity of U(x), we see that, under a gauge transformation, the Wilson line transforms as

$$W_P(y,x) \to U(y)W_P(y,x)U^\dagger(x)\ . \tag{82.9}$$

Also, since hermitian conjugation reverses the order of the product in eq. (82.8), using eq. (82.7) yields

$$W_P^\dagger(y,x) = W_{-P}(x,y)\ , \tag{82.10}$$

where –P denotes the reverse of the path P.

Now consider a path that returns to its starting point, forming a closed, oriented curve C in spacetime.

The Wilson loop is the trace of the Wilson line for this path,

$$W_C \equiv \mathrm{Tr}\, W_C(x,x)\ . \tag{82.11}$$

Using eq. (82.9), we see that the Wilson loop is *gauge invariant*,

$$W_C \to W_C\ . \tag{82.12}$$

Also, eq. (82.10) implies

$$W_C^\dagger = W_{-C}\ , \tag{82.13}$$

where −C denotes the curve C traversed in the opposite direction.

To gain some intuition, we will calculate $\langle 0|W_C|0\rangle$ for U(1) gauge theory, without charged fields.

This is simply a free-field theory, and the calculation can be done exactly.

In order to avoid dealing with iϵ issues, it is convenient to make a Wick rotation to euclidean spacetime (see section 29).

The action is then

$$S = \int d^4x \; \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} \; , \tag{82.14}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The VEV of the Wilson loop is now given by the path integral

$$\langle 0|W_C|0\rangle = \int \mathcal{D}A \; e^{ig \oint_C dx_\mu A_\mu} e^{-S} \; . \tag{82.15}$$

If we formally identify $g \oint_C dx_\mu$ as a current $J_\mu(x)$, we can apply our results for the path integral from section 57.

After including a factor of i from the Wick rotation, we get

$$\langle 0|W_C|0\rangle = \exp\left[-\tfrac{1}{2}g^2 \oint_C dx_\mu \oint_C dy_\nu\, \Delta_{\mu\nu}(x-y)\right], \tag{82.16}$$

where $\Delta_{\mu\nu}(x-y)$ is the photon propagator in euclidean spacetime.

In Feynman gauge, we have

$$\begin{aligned}
\Delta_{\mu\nu}(x-y) &= \delta_{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2} \\
&= \delta_{\mu\nu} \frac{4\pi}{(2\pi)^4} \int_0^\infty \frac{k^3\,dk}{k^2} \int_0^\pi d\theta\, \sin^2\theta\, e^{ik|x-y|\cos\theta} \\
&= \delta_{\mu\nu} \frac{4\pi}{(2\pi)^4} \int_0^\infty \frac{k^3\,dk}{k^2} \frac{\pi J_1(k|x-y|)}{k|x-y|} \\
&= \frac{\delta_{\mu\nu}}{4\pi^2(x-y)^2} \int_0^\infty du\, J_1(u) \\
&= \frac{\delta_{\mu\nu}}{4\pi^2(x-y)^2}\,,
\end{aligned} \tag{82.17}$$

where $J_1(u)$ is a Bessel function.

Since $\Delta_{\mu\nu}(x-y)$ depends only on $x-y$, the double line integral in eq. (82.16) will yield a factor of the perimeter P of the curve C.

There is also an ultraviolet divergence as x approaches y; we will cut this off at a length scale a.

The result is then

$$\langle 0|W_C|0\rangle = \exp[-(\tilde{c}g^2/a)P]\,, \tag{82.18}$$

where $\tilde{c}$ is a numerical constant that depends on the shape of C and the details of the cutoff procedure.

This behavior of the Wilson loop in euclidean spacetime—exponential decay with the length of the perimeter—is called the *perimeter law*.

It is indicative of unconfined charges.

We can gain more insight into the meaning of $\langle 0|W_C|0\rangle$ by taking C to be a rectangle, with length T in the time direction and R in a space direction, where $a \ll R \ll T$. (Of course, in euclidean spacetime, the choice of the time direction is an arbitrary convention.)

The reason for this particular shape is that the current $g\oint_C dx_\mu$ corresponds to a point charge moving along the curve C.

When the particle is moving backwards in time, the associated minus sign is equivalent to a change in the sign of its charge.

So when we compute $\langle 0|W_C|0\rangle$, we are doing the path integral in the presence of a pair of point charges with opposite sign, separated by a distance R, that exists for a time T .

On general principles (see section 6), this path integral is proportional to $\exp(-E_{\text{pair}}T)$, where E_{pair} is the ground-state energy of the quantum electrodynamic field in the presence of the charged particle pair.

We now turn to the calculation.

If both x and y are on the same side of the rectangle, we find

$$\int_0^L \int_0^L \frac{dx\,dy}{(x-y)^2} = 2L/a - 2\ln(L/a) + O(1)\ , \tag{82.19}$$

where L is the length of the side (either R or T), and the O(1) term is a numerical constant that depends on the details of the short-distance cutoff.

If x and y are on perpendicular sides, the double line integral is zero, because then $dx \cdot dy = 0$.

If x is on one short side and y on the other, the integral evaluates to R^2/T^2, and this we can neglect.

Finally, if x is on one long side and y is on the other, we have

$$\int_0^T \int_0^T \frac{dx\,dy}{(x-y)^2 + R^2} = \pi T/R - 2\ln(T/R) - 2 + O(R^2/T^2)\ . \tag{82.20}$$

Adding up all these contributions, we find in the limit of large T that

$$\oint_C \oint_C \frac{dx \cdot dy}{(x-y)^2} = \Big(4/a - 2\pi/R\Big)T + O(\ln T)\ . \tag{82.21}$$

Combining this with eqs. (82.16) and (82.17), and setting $\alpha = g^2/4\pi$, we find

$$\langle 0|W_C|0\rangle = \exp\left[-\left(\frac{2\alpha}{\pi a} - \frac{\alpha}{R}\right)T\right]. \qquad (82.22)$$

Comparing this with the general expectation $\langle 0|W_C|0\rangle \propto \exp(-E_{\text{pair}}T)$, we find a cutoff dependent contribution to E_{pair} that represents a divergent self-energy for each point particle, plus the Coulomb potential energy for the pair, $V(R) = -\alpha/R$.

In the nonabelian case, where there are interactions among the gluons, we must expand everything in powers of g.

Then we find

$$\langle 0|W_C|0\rangle = \text{Tr}\left[1 - \tfrac{1}{2}g^2 T^a T^a \oint_C dx_\mu \oint_C dy_\nu\, \Delta_{\mu\nu}(x-y) + O(g^4)\right]. \qquad (82.23)$$

Since T^aT^a equals the quadratic Casimir C(N) for the fundamental representation (times an identity matrix), we see that to leading order in g^2 we simply reproduce the results of the abelian case, but with $g^2 \to g^2 C(N)$.

We can also consider a Wilson loop in a different representation by setting $A_\mu(x) = A^a_\mu(x)T^a_R$.

Then, at leading order, we get a factor of C(R) instead of C(N).

Perturbative corrections can be computed via standard Feynman diagrams with gluon lines that, in position space, have at least one end on the curve C.

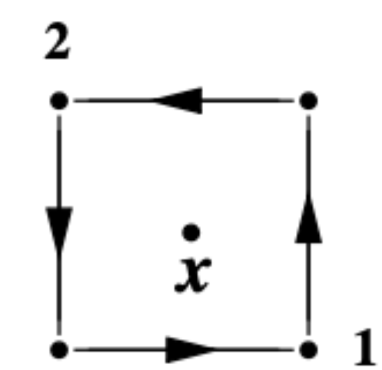
Next we turn to a strong coupling analysis.

We begin by constructing a lattice action for nonabelian gauge theory.

Consider a hypercubic lattice of points in four-dimensional euclidean spacetime, with a *lattice spacing a* between nearest-neighbor points.

The smallest Wilson loop we can make on this lattice goes around an elementary square or *plaquette*, as shown in fig. (82.1).

Figure 82.1: The minimal Wilson loop on a hypercubic lattice goes around an elementary plaquette; this one lies in the 1-2 plane.



Let ε_1 and ε_2 be vectors of length a in the 1 and 2 directions, and let x be the point at the center of the plaquette.

Using the center of each link as the argument of the gauge field, and using the lower-left corner as the starting point, we have (multiplying the Wilson links from right to left along the path)

$$W_{\rm plaq} = {\rm Tr}\, e^{-igaA_2(x-\varepsilon_1/2)} e^{-igaA_1(x+\varepsilon_2/2)} e^{+igaA_2(x+\varepsilon_1/2)} e^{+igaA_1(x-\varepsilon_2/2)} \ . \qquad (82.24)$$

If we now treat the gauge field as smooth and expand in a, we get

$$W_{\rm plaq} = {\rm Tr}\, e^{-igaA_2(x)+iga^2\partial_1 A_2(x)/2+\ldots}\, e^{-igaA_1(x)-iga^2\partial_2 A_1(x)/2+\ldots} \\ \times\, e^{+igaA_2(x)+iga^2\partial_1 A_2(x)/2+\ldots}\, e^{+igaA_1(x)-iga^2\partial_2 A_1(x)/2+\ldots} . \tag{82.25}$$

Next we use $e^A e^B = e^{A+B+[A,B]/2+\ldots}$ to combine the two exponential factors on the first line of eq. (82.25), and also the two exponential factors on the second line.

Then we use this formula once again to combine the two results.

We get

$$W_{\rm plaq} = {\rm Tr}\, e^{+iga^2(\partial_1 A_2-\partial_2 A_1-ig[A_1,A_2])+\ldots} , \tag{82.26}$$

where all fields are evaluated at x.

If we now take $W_{\rm plaq} + W_{-\rm plaq}$ and expand the exponentials, we find

$$W_{\rm plaq} + W_{-\rm plaq} = 2N - g^2 a^4\, {\rm Tr} F_{12}^2 + \ldots\ , \tag{82.27}$$

where $F_{12} = \partial_1 A_2 - \partial_2 A_1 - ig[A_1, A_2]$ is the Yang–Mills field strength.

From eq. (82.27), we conclude that an appropriate action for Yang–Mills theory on a euclidean spacetime lattice is

$$S = -\frac{1}{2g^2} \sum_{\rm plaq} W_{\rm plaq}\ , \tag{82.28}$$

where the sum includes both orientations of all plaquettes.

Each W_{plaq} is expressed as the trace of the product of four special unitary N x N matrices, one for each oriented link in the plaquette.

If U is the matrix associated with one orientation of a particular link, then $U^\dagger$ is the matrix associated with the opposite orientation of that link.

The path integral for this *lattice gauge theory* is

$$Z = \int \mathcal{D}U \; e^{-S} \; , \tag{82.29}$$

where

$$\mathcal{D}U = \prod_{\text{links}} dU_{\text{link}} \; , \tag{82.30}$$

and dU is the Haar measure for a special unitary matrix.

The Haar measure is invariant under the transformation U → VU , where V is a constant special unitary matrix, and is normalized via $\int dU = 1$; this fixes it uniquely.

For N ≥ 3, it obeys

$$\int dU \; U_{ij} = 0 \; , \tag{82.31}$$

$$\int dU \; U_{ij} U_{kl} = 0 \; , \tag{82.32}$$

$$\int dU \; U_{ij} U^*_{kl} = \tfrac{1}{N} \delta_{ik} \delta_{jl} \; , \tag{82.33}$$

which is all we will need to know.

Now consider a Wilson loop, expressed as the trace of the product of the U 's associated with the oriented links that form a closed curve C.

For simplicity, we take this curve to lie in a plane.

We have

$$\langle 0|W_C|0\rangle = Z^{-1}\int \mathcal{D}U\, W_C\, e^{-S}\,. \tag{82.34}$$

We will evaluate eq. (82.34) in the strong coupling limit by expanding e^{-S} in powers of $1/g^2$.

At zeroth order, $e^{-S} \to 1$; then eq. (82.31) tells us that the integral over every link in C vanishes.

To get a nonzero result, we need to have a corresponding $U^\dagger$ from the expansion of e^{-S}.

This can only come from a plaquette containing that link.

But then the integral over the other links of this plaquette will vanish, unless there is a compensating $U^\dagger$ for each of them.

We conclude that a nonzero result for $\langle 0|W_C|0\rangle$ requires us to fill the interior of C with plaquettes from the expansion of e^{-S}.

Since each plaquette is accompanied by a factor of $1/g^2$, we have

$$\langle 0|W_C|0\rangle \sim (1/g^2)^{A/a^2} \ , \tag{82.35}$$

where A is the area of the surface bounded by C, and A/a^2 is the number of plaquettes in this surface.

Eq. (82.35) yields the area law for a Wilson loop,

$$\langle 0|W_C|0\rangle \propto e^{-\tau A} \ , \tag{82.36}$$

where

$$\tau = c(g)/a^2 \tag{82.37}$$

is the *string tension*.

In the strong coupling limit, $c(g) = \ln(g^2) + O(1)$.

The area law for the Wilson loop implies confinement.

To see why, let us again consider a rectangular loop with area $A = RT$.

Comparing eq. (82.36) with the general expectation $\langle 0|W_C|0\rangle \leftrightarrow \exp(-E_{\text{pair}}T)$, we see that $E_{\text{pair}} = V(R) = \tau R$.

This corresponds to a linear potential between nonabelian point charges in the fundamental representation.

It takes an infinite amount of energy to separate these charges by an infinite distance; the charges are therefore confined.

The coefficient τ of R in V (R) is called the string tension because a linear potential is what we get from two points joined by a string with a fixed energy per unit length; the energy per unit length of a string is its tension.

The string tension τ is a physical quantity that should remain fixed as we remove the cutoff by lowering a.

Thus lowering a requires us to lower g.

The outstanding question is whether c(g) reaches zero at a finite, nonzero value of g.

If so, at this point there is a phase transition to an unconfined phase with zero string tension.

This has been proven to be the case for abelian gauge theory (which also exhibits an area law at strong coupling, by the identical argument).

In nonabelian gauge theory, on the other hand, analytic and numerical evidence strongly suggests that c(g) remains nonzero for all nonzero values of g.

At small a and small g, the behavior of g as a function of a is governed by the beta function, $\beta(g) = -a\, dg/da$. (The minus sign arises because the ultraviolet cutoff is a^{-1}.)

Requiring τ to be independent of a yields

$$c'(g) = -\tfrac{1}{2}\beta(g)c(g) \; . \tag{82.38}$$

At small g, we have

$$\beta(g) = -b_1 g^3 + O(g^5) \; , \tag{82.39}$$

thewhere b1 = 11N/48π2 for SU(N) gauge theory without quarks. Solving for c(g) yields

$$c(g) = C\exp(1/b_1 g^2) \; , \tag{82.40}$$

where C is an integration constant, which is nonzero if there is no phase transition. In this case, at small g, the string tension has the form

$$\tau = C\exp(1/b_1 g^2)/a^2 \; . \tag{82.41}$$

We then want to take the continuum limit of a → 0 and g → 0 with τ held fixed.

Note that eq. (82.41) shows that the string tension, at weak coupling, is not analytic in g, and so cannot be computed via the Taylor expansion in g that is provided by conventional weak-coupling perturbation theory.

Instead, the path integrals of eqs. (82.29) and (82.34) can be performed on a finite-size lattice via numerical integration.

The limiting factor in such a calculation is computer resources.

83 Chiral Symmetry Breaking

In the previous section, we discussed confinement in Yang–Mills theory without quarks.

In the real world, there are six different flavors of quark; see Table 1.

Table 1: The six flavors of quark. Each flavor is represented by a Dirac field in the 3 representation of the color group SU(3). Q is the electric charge in units of the proton charge. Masses are approximate, and are $\overline{MS}$ parameters. For the u, d, and s quarks, the $\overline{MS}$ scale μ is taken to be 2 GeV. For the c, b, and t quarks, μ is taken to be equal to the corresponding mass; e.g., the bottom quark mass is 4.3 GeV when μ = 4.3 GeV.

name	symbol	mass (GeV)	Q
up	u	0.0017	$+2/3$
down	d	0.0039	$-1/3$
strange	s	0.076	$-1/3$
charm	c	1.3	$+2/3$
bottom	b	4.3	$-1/3$
top	t	178	$+2/3$

Each flavor has a different mass, and is represented by a Dirac field in the fundamental or 3 representation of the color group SU(3).

Such a Dirac field is equivalent to two left-handed Weyl fields, one in the fundamental representation, and one in the antifundamental or 3 representation.

The lightest quarks are the up and down quarks, with masses of a few MeV.

These masses are small, in the following sense.

The gauge coupling g of QCD becomes large at low energies.

If we truncate the beta function after some number of terms (in practice, four or fewer), and integrate it, we find that g becomes infinite at some finite, nonzero value of the $\overline{MS}$ parameter μ; this value is called Λ_{QCD}.

Measurements of the strength of the gauge coupling at high energies imply $\Lambda_{QCD} \sim 0.2$ GeV.

The up and down quark masses are much less than Λ_{QCD}.

We can therefore begin with the approximation that the up and down quarks are massless.

The mass of the strange quark is also somewhat less than Λ_{QCD}.

It is sometimes useful (though clearly less justified) to treat the strange quark as massless as well.

If we are interested in hadron physics at energies below ~1 GeV, we can ignore the charm, bottom, and top quarks entirely; we will also ignore the strange quark for now.

Let us, then, consider QCD with $n_F = 2$ flavors of massless quarks.

We then have left-handed Weyl fields $\chi_{\alpha i}$, where $\alpha = 1, 2, 3$ is a color index for the 3 representation, and $i = 1, 2$ is a flavor index, and left-handed Weyl fields $\xi^{\alpha\bar{\imath}}$, where $\alpha = 1, 2, 3$ is a color index for the $\bar{3}$ representation, and $\bar{\imath} = 1, 2$ is a flavor index; we distinguish this flavor index from the one for the χ's by putting a bar over it, and we write it as a superscript for later notational convenience.

We suppress the undotted spinor index carried by both χ and ξ.

The lagrangian is

$$\mathcal{L} = i\chi^{\dagger\alpha i}\,\bar{\sigma}^\mu (D_\mu)_\alpha{}^\beta\,\chi_{\beta i} + i\xi^\dagger_{\bar{\imath}\alpha}\,\bar{\sigma}^\mu (\bar{D}_\mu)^\alpha{}_\beta\,\xi^{\beta\bar{\imath}} - \tfrac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}\ , \tag{83.1}$$

where $D_\mu = \partial_\mu - igT^a_3 A^a_\mu$ and $\bar{D}_\mu = \partial_\mu - igT^a_{\bar{3}} A^a_\mu$, with $(T^a_{\bar{3}})^\alpha{}_\beta = -(T^a_3)_\beta{}^\alpha$.

In addition to the SU(3) color gauge symmetry, this lagrangian has a global U(2) x U(2) flavor symmetry: $\mathcal{L}$ is invariant under

$$\chi_{\alpha i} \to L_i{}^j \chi_{\alpha j}\ , \tag{83.2}$$

$$\xi^{\alpha\bar{\imath}} \to (R^*)^{\bar{\imath}}{}_{\bar{\jmath}}\,\xi^{\alpha\bar{\jmath}}\ , \tag{83.3}$$

where L and R* are independent 2 x 2 constant unitary matrices. (The complex conjugation of R is a notational convention that turns out to be convenient.)

In terms of the Dirac field

$$\Psi_{\alpha i} = \begin{pmatrix} \chi_{\alpha i} \\ \xi^\dagger_{\alpha\bar{\imath}} \end{pmatrix}, \tag{83.4}$$

eqs. (83.2) and (83.3) read

$$P_{\mathrm{L}}\Psi_{\alpha i} \to L_i{}^j P_{\mathrm{L}}\Psi_{\alpha i}\ , \tag{83.5}$$

$$P_{\mathrm{R}}\Psi_{\alpha\bar{\imath}} \to R_{\bar{\imath}}{}^{\bar{\jmath}} P_{\mathrm{R}}\Psi_{\alpha\bar{\jmath}}\ , \tag{83.6}$$

where $P_{L,R} = 1/2\ (1 \mp \gamma_5)$.

Thus the global flavor symmetry is often called $U(2)_L$ x $U(2)_R$.

A symmetry that treats the left- and right-handed parts of a Dirac field differently is said to be *chiral*.

However, there is an anomaly in the axial U(1) symmetry corresponding to $L = R^* = e^{i\alpha}I$ (which is equivalent to $\Psi \to e^{-i\alpha\gamma 5}\, \Psi$ for the Dirac field).

Thus the nonanomalous global flavor symmetry is $SU(2)_L$ x $SU(2)_R$ x $U(1)_V$, where V stands for vector.

The $U(1)_V$ transformation corresponds to $L = R= e^{-i\alpha I}$, or equivalently $\Psi \to e^{-i\alpha}\Psi$

The corresponding conserved charge is quark number , the number of quarks minus the number of antiquarks; this is one third of the baryon number, the number of baryons minus the number of antibaryons. (Baryons are color-singlet bound states of three quarks; the proton and neuton are baryons. Mesons are color-singlet bound states of a quark and an antiquark; pions are mesons.)

Thus, $U(1)_V$ results in classification of hadrons by their baryon number.

How is the $SU(2)_L$ x $SU(2)_R$ symmetry realized in nature?

The vector subgroup $SU(2)_V$, obtained by setting R = L in eq. (83.3), is known as isotopic spin or isospin symmetry.

Hadrons clearly come in representations of $SU(2)_V$: the lightest spin-one-half hadrons (the proton, mass 0.938 GeV, and the neutron, mass 0.940 GeV) form a doublet or 2 representation, while the lightest spin-zero hadrons (the π^0, mass 0.135 GeV, and the $\pi^{\pm}$, mass 0.140 GeV) form a triplet or 3 representation.

Isospin is not an exact symmetry; it is violated by the small mass difference between the up and down quarks, and by electromagnetism.

Thus we see small differences in the masses of the hadrons assigned to a particular isotopic multiplet.

The role of the axial part of the $SU(2)_L \times SU(2)_R$ symmetry, obtained by setting $R= L^{\dagger}$ in eq. (83.3), is harder to identify.

The hadrons do not appear to be classified into multiplets by a second SU(2) symmetry group.

In particular, there is no evidence for a classification that distinguishes the left- and right-handed components of spin-one-half hadrons like the proton and neutron.

Reconciliation of these observations with the $SU(2)_L \times SU(2)_R$ symmetry of the underlying lagrangian is only possible if the axial generators are spontaneously broken.

The three pions (which have spin zero, odd parity, and are by far the lightest hadrons) are then identified as the corresponding Goldstone bosons.

They are not exactly massless (and hence are sometimes called pseudogoldstone bosons) because the $SU(2)_L \times SU(2)_R$ symmetry is, as we just discussed, not exact.

To spontaneously break the axial part of the SU(2)$_L$ x SU(2)$_R$, some operator that transforms nontrivially under it must acquire a nonzero vacuum expectation value, or VEV for short.

To avoid spontaneous breakdown of Lorentz invariance, this operator must be a Lorentz scalar, and to avoid spontaneous breakdown of the SU(3) gauge symmetry, it must be a color singlet.

Since we have no fundamental scalar fields that could acquire a nonzero VEV, we must turn to *composite fields* instead.

The simplest candidate is $\chi^a_{\alpha i}\xi_a^{\alpha\bar{\jmath}} = \overline{\Psi}^{\alpha\bar{\jmath}} P_{\rm L}\Psi_{\alpha i}$, where a is an undotted spinor index. (The product of two fields is generically singular, and a renormalization scheme must be specified to define it.)

We assume that

$$\langle 0|\chi^a_{\alpha i}\xi_a^{\alpha\bar{\jmath}}|0\rangle = -v^3\delta_i{}^{\bar{\jmath}} \ , \qquad (83.7)$$

where v is a parameter with dimensions of mass.

Its numerical value depends on the renormalization scheme; for $\overline{MS}$ with μ = 2 GeV, $v \simeq 0.23$ GeV.

To see that this *fermion condensate* does the job of breaking the axial generators of SU(2)$_L$ x SU(2)$_R$ while preserving the vector generators, we note that, under the transformation of eqs. (83.2) and (83.3),

$$\begin{aligned} \langle 0|\chi^a_{\alpha i}\xi_a^{\alpha\bar{\jmath}}|0\rangle &\to L_i{}^k (R^*)^{\bar{\jmath}}{}_{\bar{n}} \langle 0|\chi^a_{\alpha k}\xi_a^{\alpha\bar{n}}|0\rangle \\ &\to -v^3 (LR^\dagger)_i{}^{\bar{\jmath}} \ , \end{aligned} \qquad (83.8)$$

where we used eq. (83.7) to get the second line.

If we take R = L, corresponding to an $SU(2)_V$ transformation, the right-hand side of eq. (83.8) is unchanged from its value in eq. (83.7).

This signifies that $SU(2)_V$ [and also $U(1)_V$] is unbroken.

However, for a more general transformation with R ≠ L, the right-hand side of eq. (83.8) does not match that of eq. (83.7), signifying the spontaneous breakdown of the axial generators.

Eq. (83.7) is *nonperturbative*: $\langle 0|\chi^{a}_{\alpha i}\xi^{\alpha\bar{\jmath}}_{a}|0\rangle$ vanishes at tree level.

Perturbative corrections then also vanish, because of the chiral flavor symmetry of the lagrangian.

Thus the value of v is not accessible in perturbation theory.

On general grounds, we expect $v \sim \Lambda_{QCD}$, since Λ_{QCD} is the only mass scale in the theory when the quarks are massless.

Similarly, Λ_{QCD} sets the scale for the masses of all the hadrons that are not pseudogoldstone bosons, including the proton and neutron.

We can construct a low-energy effective lagrangian for the three pseudogoldstone bosons (to be identified as the pions) in the following way.

We allow the orientation in flavor space of the VEV of $\chi^a_{\alpha i}\xi^{\alpha\bar{j}}_a$ to vary slowly asa function of spacetime.

That is, in place of eq. (83.7), we write

$$\langle 0|\chi^a_{\alpha i}(x)\xi^{\alpha\bar{j}}_a(x)|0\rangle = -v^3 U_i{}^{\bar{j}}(x)\ , \tag{83.9}$$

where U(x) is a spacetime dependent unitary matrix.

We can write it as

$$U(x) = \exp[2i\pi^a(x)T^a/f_\pi]\ , \tag{83.10}$$

where $T^a = 1/2\ \sigma^a$ with a = 1, 2, 3 are the generator matrices of SU(2), $\pi^a(x)$ are three real scalar fields to be identified with the pions, and f_π is a parameter with dimensions of mass, the pion decay constant.

We do not include a fourth generator matrix proportional to the identity, since the corresponding field would be the Goldstone boson for the $U(1)_A$ symmetry that is eliminated by the anomaly.

Equivalently, we require det U(x) = 1.

We will think of U(x) as an effective, low energy field.

Its lagrangian should be the most general one that is consistent with the underlying $SU(2)_L$ x $SU(2)_R$ symmetry.

Under a general $SU(2)_L$ x $SU(2)_R$ transformation, we have

$$U(x) \to LU(x)R^\dagger \;, \tag{83.11}$$

where L and R are independent special unitary matrices.

We can organize the terms in the effective lagrangian for U(x) (also known as the chiral lagrangian) by the number of derivatives they contain.

Because $U^\dagger U = 1$, there are no terms with no derivatives.

There is one term with two (all others being equivalent after integrations by parts),

$$\mathcal{L} = -\tfrac{1}{4} f_\pi^2 \,\mathrm{Tr}\, \partial^\mu U^\dagger \partial_\mu U \;. \tag{83.12}$$

If we substitute in eq. (83.10) for U, and expand in inverse powers of f_π , we find

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu \pi^a \partial_\mu \pi^a + \tfrac{1}{6} f_\pi^{-2}(\pi^a\pi^a\partial^\mu\pi^b\partial_\mu\pi^b - \pi^a\pi^b\partial^\mu\pi^b\partial_\mu\pi^a) + \ldots \;. \tag{83.13}$$

Thus the pion fields are conventionally normalized, and they have interactions that are dictated by the general form of eq. (83.12).

These interactions lead to Feynman vertices that contain factors of momenta p divided by f_π.

Therefore, we can think of p/f_π as an expansion parameter.

Of course, we should also add to $\mathcal{L}$ all possible inequivalent terms with four or more derivatives, with coefficients that include inverse powers of f_π .

These will lead to more vertices, but their effects will be suppressed by additional powers of p/f_π.

Comparison with experiment then yields $f_\pi = 92.4$ MeV. (In practice, the value of f_π is more readily determined from the decay rate of the pion via the weak interaction; see section 90)

This value for f_π may seem low; it is, for example, less than the mass of the “almost masless” pions.

However, it turns out that tree and loop diagrams contribute roughly equally to any particular process if each extra derivative in $\mathcal{L}$ is accompanied by a factor of $(4\pi f_\pi)^{-1}$ rather that f_π^{-1} , and each loop momentum is cut off at $4\pi f_\pi$.

Thus it is $4\pi f\pi \sim 1$ GeV that sets the scale of the interactions, rather than $f_\pi \sim 100$ MeV.

Now let us consider the effect of including the small masses for the up and down quarks.

The most general mass term we can add to the lagrangian is

$$\begin{aligned}
\mathcal{L}_{\text{mass}} &= -\xi^{\alpha\bar{\jmath}} M_{\bar{\jmath}}{}^{i} \chi_{\alpha i} + \text{h.c.} \\
&= -M_{\bar{\jmath}}{}^{i} \chi_{\alpha i} \xi^{\alpha\bar{\jmath}} + \text{h.c.} \qquad (83.14)\\
&= -\text{Tr}\, M \chi_\alpha \xi^\alpha + \text{h.c.}\ ,
\end{aligned}$$

where M is a complex 2 x 2 matrix.

By making an $SU(2)_L \times SU(2)_R$ transformation, we can bring M to the form

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} e^{-i\theta/2} , \qquad (83.15)$$

where m_u and m_d are real and positive.

We cannot remove the overall phase θ, however, without making a forbidden $U(1)_A$ transformation.

A nonzero value of θ has physical consequences, as we will discuss in section 94.

For now, we note that experimental observations fix $|\theta| < 10^{-9}$, and so we will set $\theta = 0$ on this phenomenological basis.

Next, we replace $\chi_\alpha \xi^\alpha$ in eq. (83.14) with its spacetime dependent VEV, eq. (83.7).

The result is a term in the chiral lagrangian that incorporates the leading effect of the quark masses,

$$\mathcal{L}_{\text{mass}} = v^3 \, \text{Tr}(MU + M^\dagger U^\dagger) \, . \qquad (83.16)$$

Here we continue to distinguish M and $M^\dagger$, even though, with $\theta = 0$, they are the same matrix.

If we think of M as transforming as $M \to RML^\dagger$, while U transforms as $U \to LUR^\dagger$, then $\text{Tr}\, MU$ is formally invariant.

We then require all terms in the chiral lagrangian to exhibit this formal invariance.

If we expand $\mathcal{L}_{\text{mass}}$ in inverse powers of f_π, and use $M^\dagger = M$, we find

$$\begin{aligned} \mathcal{L}_{\text{mass}} &= -4(v^3/f_\pi^2)\,\text{Tr}(MT^aT^b)\pi^a\pi^b + \ldots \\ &= -2(v^3/f_\pi^2)\,\text{Tr}(M\{T^a,T^b\})\pi^a\pi^b + \ldots \\ &= -(v^3/f_\pi^2)(\text{Tr}\,M)\pi^a\pi^a + \ldots\ . \end{aligned} \tag{83.17}$$

We used the SU(2) relation $\{T^a,T^b\}=1/2\delta^{ab}$ to get the last line.

From eq. (83.17), we see that all three pions have the same mass, given by the *Gell-Mann–Oakes–Renner relation*,

$$m_\pi^2 = 2(m_u + m_d)v^3/f_\pi^2\ . \tag{83.18}$$

On the right-hand side, the quark masses and v^3 depend on the renormalization scheme, but their product does not.

In the real world, electromagnetic interactions raise the mass of the $\pi^\pm$ slightly above that of the π^0.

This framework is easily expanded to include the strange quark.

The three pions (π^+, π^-, mass 0.140 GeV; π^0, mass 0.135 GeV), the four kaons (K^+, K^-, mass 0.494 GeV; K^0, $\overline{K}^0$, mass 0.498 GeV), and the eta (η, mass 0.548 GeV) are identified as the eight expected Goldstone bosons.

We can assemble them into the hermitian matrix

$$\Pi \equiv \pi^a T^a / f_\pi = \frac{1}{2f_\pi}\begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\,\overline{K}{}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}. \tag{83.19}$$

The second line of eq. (83.17) still applies, but now the T^a's are the generators of SU(3), and M includes a third diagonal entry for the strange quark mass(The details are straightforward).

Next we turn to the coupling of the pions to the nucleons (the proton and neutron).

We define a Dirac field N_i, where N_1 = p (the proton) and N_2 = n (the neutron).

We assume that, under an $SU(2)_L$ x $SU(2)_R$ transformation,

$$P_L N_i \to L_i{}^j P_L N_j \ , \tag{83.20}$$

$$P_R N_{\bar{\imath}} \to R_{\bar{\imath}}{}^{\bar{\jmath}} P_R N_{\bar{\jmath}} \ . \tag{83.21}$$

The standard Dirac kinetic term $i\overline{N}\partial\!\!\!/ N$ is then $SU(2)_L$ x $SU(2)_R$ invariant, but the standard mass term $m_N \overline{N}N$ is not. (Here m_N is the value of the nucleon mass in the limit of zero up and down quark masses.)

However, we can construct an invariant mass term by including appropriate factors of U and $U^\dagger$,

$$\mathcal{L}_{\text{mass}} = -m_N \overline{N}(U^\dagger P_{\text{L}} + U P_{\text{R}})N \; . \tag{83.22}$$

There is one other parity, time-reversal, and $SU(2)_L \times SU(2)_R$ invariant term with one derivative.

Including this term, we have

$$\begin{aligned} \mathcal{L} = {} & i\overline{N}\partial\!\!\!/ N - m_N \overline{N}(U^\dagger P_{\text{L}} + U P_{\text{R}})N \\ & - \tfrac{1}{2}(g_{\text{A}} - 1) i\overline{N}\gamma^\mu (U\partial_\mu U^\dagger P_{\text{L}} + U^\dagger \partial_\mu U P_{\text{R}})N \; , \end{aligned} \tag{83.23}$$

where $g_A = 1.27$ is the axial vector coupling.

Its value is determined from the decay rate of the neutron via the weak interaction; see section 90.

The form of the lagrangian in eq. (83.23) is somewhat awkward.

It can be simplified by first defining

$$u(x) \equiv \exp[\, i\pi^a(x)T^a/f_\pi] \, , \tag{83.24}$$

so that $U(x) \equiv u^2(x)$.

Then we define a new nucleon field

$$\mathcal{N} \equiv (u^\dagger P_{\text{L}} + u P_{\text{R}})N \; . \tag{83.25}$$

This is a field redefinition.

Equivalently, using the unitarity of u, we have

$$N = (uP_{\rm L} + u^\dagger P_{\rm R})\mathcal{N} \; . \qquad (83.26)$$

Using eq. (83.26) in eq. (83.23), along with the identities $\partial_\mu U = (\partial_\mu u)u + u(\partial_\mu u)$, $(\partial_\mu u^\dagger)u = -u^\dagger(\partial_\mu u)$, etc., we ultimately find

$$\mathcal{L} = i\overline{\mathcal{N}}\partial\!\!\!/\mathcal{N} - m_N\overline{\mathcal{N}}\mathcal{N} + \overline{\mathcal{N}}v\!\!\!/\mathcal{N} - g_{\rm A}\overline{\mathcal{N}}a\!\!\!/\gamma_5\mathcal{N} \; , \qquad (83.27)$$

where we have defined the hermitian vector fields

$$v_\mu \equiv \tfrac{1}{2}i[u^\dagger(\partial_\mu u) + u(\partial_\mu u^\dagger)] \; , \qquad (83.28)$$

$$a_\mu \equiv \tfrac{1}{2}i[u^\dagger(\partial_\mu u) - u(\partial_\mu u^\dagger)] \; . \qquad (83.29)$$

If we now expand u and u† in inverse powers of f_π, we get

$$\mathcal{L} = i\overline{\mathcal{N}}\partial\!\!\!/\mathcal{N} - m_N\overline{\mathcal{N}}\mathcal{N} + (g_{\rm A}/f_\pi)\partial_\mu\pi^a\overline{\mathcal{N}}T^a\gamma^\mu\gamma_5\mathcal{N} + \ldots \; . \qquad (83.30)$$

We can integrate by parts in the interaction term to put the derivative on the fields.

Then, if we consider a process where an off-shell pion is scattered by an on-shell nucleon, we can use the Dirac equation to replace the derivatives of $\overline{\mathcal{N}}$ and $\mathcal{N}$ with factors of m_N.

We then find a coupling of the pion to an on-shell nucleon of the form

$$\mathcal{L}_{\pi\overline{N}N} = -ig_{\pi\overline{N}N}\pi^a\overline{\mathcal{N}}\sigma^a\gamma_5\mathcal{N}\,, \tag{83.31}$$

where we have set $T^a = 1/2\sigma^a$, and identified the *pion-nucleon coupling constant*,

$$g_{\pi\overline{N}N} = g_{\rm A} m_N / f_\pi \,. \tag{83.32}$$

The value of $g_{\pi\overline{N}N}$ can be determined from measurements of the neutron-proton scattering cross section, assuming that it is dominated by pion exchange; the result is $g_{\pi\overline{N}N} = 13.5$.

Eq. (83.32), known as the *Goldberger–Treiman relation*, is then satisfied to within about 5%.

84 Spontaneous Breaking of Gauge Symmetries

Consider scalar electrodynamics, specified by the lagrangian

$$\mathcal{L} = -(D^\mu\varphi)^\dagger D_\mu\varphi - V(\varphi) - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \tag{84.1}$$

where φ is a complex scalar field, $D_\mu = \partial_\mu - igA_\mu$, and

$$V(\varphi) = m^2\varphi^\dagger\varphi + \tfrac{1}{4}\lambda(\varphi^\dagger\varphi)^2\,. \tag{84.2}$$

(We call the gauge coupling constant g rather than e because we are using this theory as a formal example rather than a physical model.)

So far we have always taken $m^2 > 0$, but now let us consider $m^2 < 0$.

We analyzed this model in the absence of the gauge field in section 32.

Classically, the field has a nonzero vacuum expectation value (VEV for short), given by

$$\langle 0|\varphi(x)|0\rangle = \tfrac{1}{\sqrt{2}}v \; , \tag{84.3}$$

where we have made a global U(1) transformation to set the phase of the VEV to zero, and

$$v = (4|m^2|/\lambda)^{1/2} \; . \tag{84.4}$$

We therefore write

$$\varphi(x) = \tfrac{1}{\sqrt{2}}(v + \rho(x))e^{-i\chi(x)/v} \; , \tag{84.5}$$

where $\rho(x)$ and $\chi(x)$ are real scalar fields.

The scalar potential depends only on ρ, and is given by

$$V(\varphi) = \tfrac{1}{4}\lambda v^2\rho^2 + \tfrac{1}{4}\lambda v\rho^3 + \tfrac{1}{16}\lambda\rho^4 \; . \tag{84.6}$$

Since χ does not appear in the potential, it is massless; it is the Goldstone boson for the spontaneously broken U(1) symmetry.

The big difference in the gauge theory is that we can make a gauge transformation that shifts the phase of $\varphi(x)$ by an arbitrary spacetime function.

We can use this gauge freedom to set $\chi(x) = 0$; this choice is called unitary gauge.

Using eq. (84.5) with $\chi(x) = 0$ in eq. (84.1), we have

$$\begin{aligned} -(D^\mu\varphi)^\dagger D_\mu\varphi &= -\tfrac{1}{2}(\partial^\mu\rho + ig(v+\rho)A^\mu)(\partial_\mu\rho - ig(v+\rho)A_\mu) \\ &= -\tfrac{1}{2}\partial^\mu\rho\partial_\mu\rho - \tfrac{1}{2}g^2(v+\rho)^2 A^\mu A_\mu \ . \end{aligned} \tag{84.7}$$

Expanding out the last term, we see that the gauge field now has a mass

$$M = gv \ . \tag{84.8}$$

This is the *Higgs mechanism*: the Goldstone boson disappears, and the gauge field acquires a mass.

Note that this leaves the counting of particle spin states unchanged: a massless spin-one particle has two spin states, but a massive one has three.

The Goldstone boson has become the third or longitudinal state of the now-massive gauge field.

A scalar field whose VEV breaks a gauge symmetry is generically called a *Higgs field*.

This generalizes in a straightforward way to a nonabelian gauge theory.

Consider a complex scalar field φ in a representation R of the gauge group.

The kinetic term for φ is $-(D_\mu\varphi)^\dagger D_\mu\varphi$, where the covariant derivative is $(D_\mu\varphi)_i = \partial_\mu\varphi_i - ig_a A^a_\mu (T^a_{\mathrm{R}})_i{}^j\varphi_j$, and the indices i and j run from 1 to d(R).

We assume that φ acquires a VEV

$$\langle 0|\varphi_i(x)|0\rangle = \tfrac{1}{\sqrt{2}}v_i \ , \tag{84.9}$$

where the value of v_i is determined (up to a global gauge transformation) by minimizing the potential.

If we replace φ by its VEV in $-(D_\mu\varphi)^\dagger D_\mu\varphi$, we find a mass term for the gauge fields,

$$\mathcal{L}_{\rm mass} = -\tfrac{1}{2}(M^2)^{ab}A^{a\mu}A^b_\mu \ , \tag{84.10}$$

where the mass-squared matrix is

$$(M^2)^{ab} = \tfrac{1}{2}g^2 v_i^* \{T^a_{\rm R}, T^b_{\rm R}\}_{ij}v_j \ . \tag{84.11}$$

The anticommutator appears because $A^{a\mu}A^b_\mu$ is symmetric on $a \leftrightarrow b$, and so we replaced $T^a_R T^b_R$ with $1/2\ \{T^a_R\ , T^b_R\}$.

If the field φ is real rather than complex (which is possible only if R is a real representation), then we remove the factor of root-two from the right-hand side of eq. (84.9), but this is compensated by an extra factor of one-half from the kinetic term for a real scalar field; thus eq. (84.11) holds as written.

If there is more than one gauge group, then the g^2 in eq. (84.11) is replaced by $g_a g_b$, where g_a is the coupling constant that goes along with the generator T^a, and all generators of all gauge groups are included in the mass-squared matrix.

Recall from section 32 that a generator T^a is spontaneously broken if $(T^a{}_R)_{ij}\, v_j \neq 0$.

From eq. (84.11), we see that gauge fields corresponding to broken generators get a mass, while those corresponding to unbroken generators do not.

The unbroken generators (if any) form a gauge group with massless gauge fields.

The massive gauge fields (and all other fields) form representations of this unbroken group.

Let us work out some simple examples.

Consider the gauge group SU(N), with a complex scalar field φ in the fundamental representation.

We can make a global SU(N) transformation to bring the VEV entirely into the last component, and furthermore make it real.

Any generator $(T^a)_i{}^j$ that does not have a nonzero entry in the last column will remain unbroken.

These generators form an unbroken SU(N−1) gauge group.

There are three classes of broken generators: those with $(T^a)_i{}^N = 1/2$ for $i \neq N$ (there are N−1 of these); those with $(T^a)_i{}^N = -1/2\, i$ for $i \neq N$ (there are also N−1 of these), and finally the single generator T^{N^2-1} $= [2N(N-1)]^{-1/2}\, \mathrm{diag}(1, \ldots, 1, -(N-1))$.

The gauge fields corresponding to the generators in the first two classes get a mass $M = 1/2\, g\upsilon$; we can group them into a complex vector field that transforms in the fundamental representation of the unbroken SU(N−1) subgroup.

The gauge field corresponding to $T^{N^2-\dot{1}}$ gets a mass $M = [(N-1)/2N\,]^{1/2}\, g\upsilon$; it is a singlet of SU(N−1).

Consider the gauge group SO(N), with a real scalar field in the fundamental representation.

We can make a global SO(N) transformation to bring the VEV entirely into the last component.

Any generator $(T^a)_i{}^j$ that does not have a nonzero entry in the last column will remain unbroken.

These generators form an unbroken SO(N−1) subgroup. There are N−1 broken generators, those with $(T^a)_i{}^N = -i$ for $i \neq N$.

The corresponding gauge fields get a mass $M = g\upsilon$; they form a fundamental representation of the unbroken SO(N−1) subgroup.

In the case N = 3, this subgroup is SO(2), which is equivalent to U(1).

Consider the gauge group SU(N), with a real scalar field Φ^a in the adjoint representation.

It will prove more convenient to work with the matrix-valued field $\Phi = \varphi^a T^a$; the covariant derivative of Ω is $D_\mu\Phi = \omega_\mu\Phi - igA^a{}_\mu[T^a,\Phi]$, and the VEV of φ is a traceless hermitian N x N matrix V .

Thus the mass-squared matrix for the gauge fields is $(M^2)^{ab} = -1/2\, g^2\, \mathrm{Tr}\{[T^a,V], [T^b,V]\}$.

We can make a global SU(N) transformation to bring V into diagonal form.

Suppose the diagonal entries consist of N_1 υ_1's, followed by N_2 υ_2's, etc., where $\upsilon_1 < \upsilon_2 < \dots$, and $\sum_i N_i \upsilon_i = 0$.

Then all generators whose nonzero entries lie entirely within the ith block commute with V , and hence form an unbroken SU(Ni) subgroup.

Furthermore, the linear combination of diagonal generators that is proportional to V also commutes with V , and forms a U(1) subgroup.

Thus the unbroken gauge group is $SU(N_1) \times SU(N_2) \times \dots \times U(1)$.

The gauge coupling constants for the different groups are all the same, and equal to the original SU(N) gauge coupling constant.

As a specific example, consider the case of SU(5), which has 24 generators.

Let the diagonal entries of V be given by $(-1/3\,, -1/3\,, -1/3\,, +1/2\,, +1/2\,)\upsilon$.

The unbroken subgroup is then SU(3) x SU(2) x U(1).

The number of broken generators is 24 – 8 – 3 – 1 = 12.

The generator of the U(1) subgroup is $T^{24} = \mathrm{diag}(-1/3\ c, -1/3\ c, -1/3\ c, +1/2\ c, +1/2\ c)$, where $c^2 = 3/5$.

Under the unbroken SU(3) x SU(2) x U(1) subgroup, the 5 representation of SU(5) transforms as

$$5 \to (3, 1, -\tfrac{1}{3}) \oplus (1, 2, +\tfrac{1}{2})\ . \tag{84.12}$$

Here the last entry is the value of T^{24}/c.

The $\overline{5}$ of SU(5) then transforms as

$$\overline{5} \to (\overline{3}, 1, +\tfrac{1}{3}) \oplus (1, 2, -\tfrac{1}{2})\ . \tag{84.13}$$

To find out how the adjoint or 24 representation of SU(5) transforms under the SU(3) x SU(2) x U(1) subgroup, we use the SU(5) relation

$$5 \otimes \overline{5} = 24 \oplus 1\ . \tag{84.14}$$

From eqs. (84.12) and (84.13), we have

$$5 \otimes \overline{5} \to [(3, 1, -\tfrac{1}{3}) \oplus (1, 2, +\tfrac{1}{2})] \otimes [(\overline{3}, 1, +\tfrac{1}{3}) \oplus (1, 2, -\tfrac{1}{2})]\ . \tag{84.15}$$

If we expand this out, and compare with eq. (84.14), we see that

$$\begin{aligned} 24 \to\ & (8, 1, 0) \oplus (1, 3, 0) \oplus (1, 1, 0) \\ & \oplus (3, 2, -\tfrac{5}{6}) \oplus (\overline{3}, 2, +\tfrac{5}{6})\ . \end{aligned} \tag{84.16}$$

The first line on the right-hand side of eq. (84.16) is the adjoint representation of SU(3) x SU(2) x U(1); the corresponding gauge fields remain massless.

The second line shows us that the gauge fields corresponding to the twelve broken generators can be grouped into a complex vector field in the representation (3, 2,–5/6).

Since it is an irreducible representation of the unbroken subgroup, all twelve vectors fields must have the same mass.

This mass is most easily computed from $(M^2)^{44} = -g^2 \,\mathrm{Tr}([T^4, V][T^4, V])$, where we have defined $(T^4)_{ij} = 1/2\, (\delta_i{}^1\delta_j{}^4 + \delta_i{}^4\delta_j{}^1)$; the result is $M = 5/(6\sqrt{2})\, gv$.

85 Spontaneously Broken Abelian Gauge Theory

Consider scalar electrodynamics, specified by the lagrangian

$$\mathcal{L} = -(D^\mu\varphi)^\dagger D_\mu\varphi - V(\varphi) - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \;, \qquad (85.1)$$

where φ is a complex scalar field and $D_\mu = \partial_\mu - igA_\mu$.

We choose

$$V(\varphi) = \tfrac{1}{4}\lambda(\varphi^\dagger\varphi - \tfrac{1}{2}v^2)^2 \;, \qquad (85.2)$$

which yields a nonzero VEV for φ.

We therefore write

$$\varphi(x) = \tfrac{1}{\sqrt{2}}(v + \rho(x))e^{-i\chi(x)/v} \;, \qquad (85.3)$$

where $\rho(x)$ and $\chi(x)$ are real scalar fields.

The scalar potential depends only on ρ, and is given by

$$V(\varphi) = \tfrac{1}{4}\lambda v^2\rho^2 + \tfrac{1}{4}\lambda v\rho^3 + \tfrac{1}{16}\lambda\rho^4\ . \tag{85.4}$$

We can now make a gauge transformation to set

$$\chi(x) = 0\ . \tag{85.5}$$

This is *unitary gauge*.

The kinetic term for φ becomes

$$-(D^\mu\varphi)^\dagger D_\mu\varphi = -\tfrac{1}{2}\partial^\mu\rho\partial_\mu\rho - \tfrac{1}{2}g^2(v+\rho)^2A^\mu A_\mu\ . \tag{85.6}$$

We see that the gauge field has acquired a mass

$$M = gv\ . \tag{85.7}$$

The terms in $\mathcal{L}$ that are quadratic in A_μ are

$$\mathcal{L}_0 = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}M^2A^\mu A_\mu\ . \tag{85.8}$$

The equation of motion that follows from eq. (85.8) is

$$[(-\partial^2 + M^2)g^{\mu\nu} + \partial^\mu\partial^\nu]A_\nu = 0\ . \tag{85.9}$$

If we act with ∂_μ on this equation, we get

$$M^2\partial^\nu A_\nu = 0\ . \tag{85.10}$$

If we now use eq. (85.10) in eq. (85.9), we find that each component of A_ν obeys the Klein-Gordon equation,

$$(-\partial^2 + M^2)A_\nu = 0 \,. \tag{85.11}$$

The general solution of eqs. (85.9) and (85.10) is

$$A^\mu(x) = \sum_{\lambda=-,0,+} \int \widetilde{dk} \left[\varepsilon_\lambda^{\mu*}(k)a_\lambda(k)e^{ikx} + \varepsilon_\lambda^\mu(k)a_\lambda^\dagger(k)e^{-ikx}\right], \tag{85.12}$$

where the polarization vectors must satisfy $k_\mu \varepsilon^\mu{}_\lambda(k) = 0$.

In the rest frame, where $k = (M, 0, 0, 0)$, we choose the polarization vectors to correspond to definite spin along the $\hat{z}$ axis,

$$\begin{aligned} \varepsilon_+(0) &= \tfrac{1}{\sqrt{2}}(0,1,-i,0) \;, \\ \varepsilon_-(0) &= \tfrac{1}{\sqrt{2}}(0,1,+i,0) \;, \\ \varepsilon_0(0) &= (0,0,0,1) \;. \end{aligned} \tag{85.13}$$

More generally, the three polarization vectors along with the timelike unit vector k^μ/M form an orthonormal and complete set,

$$k\cdot\varepsilon_\lambda^\mu(k) = 0 \,, \tag{85.14}$$

$$\varepsilon_{\lambda'}(k)\cdot\varepsilon_\lambda^*(k) = \delta_{\lambda'\lambda} \,, \tag{85.15}$$

$$\sum_{\lambda=-,0,+} \varepsilon_\lambda^{\mu*}(k)\varepsilon_\lambda^\nu(k) = g^{\mu\nu} + \frac{k^\mu k^\nu}{M^2} \,. \tag{85.16}$$

Since the lagrangian of eq. (85.8) has no manifest gauge invariance, quantization is straightforward.

The coefficients $a^\dagger_\lambda(k)$ and $a_\lambda(k)$ become particle creation and annihilation operators in the usual way, and the propagator of the A_μ field is given by

$$
\begin{aligned}
i\langle 0|\mathrm{T}A^\mu(x)A^\nu(y)|0\rangle &= \int \frac{d^4k}{(2\pi)^4}\,\frac{e^{ik(x-y)}}{k^2+M^2-i\epsilon}\,\sum\nolimits_\lambda \varepsilon_\lambda^{\mu *}(k)\varepsilon_\lambda^\nu(k) \\
&= \int \frac{d^4k}{(2\pi)^4}\,\frac{e^{ik(x-y)}}{k^2+M^2-i\epsilon}\left(g^{\mu\nu}+\frac{k^\mu k^\nu}{M^2}\right).
\end{aligned}
\tag{85.17}
$$

The interactions of the massive vector field A_μ with the real scalar field ρ can be read off of eq. (85.6).

The self-interactions of the ρ field can be read off of eq. (85.4).

The resulting Feynman rules can be used for tree-level calculations.

Loop calculations are more subtle.

We have imposed the gauge condition $\chi(x) = 0$, which corresponds to inserting a functional delta function $\prod_x \delta(\chi(x))$ into the path integral.

In order to integrate over χ, we must make a change of integration variables from Re φ and Im φ to ρ and χ; this is simply a transformation from cartesian to polar coordinates, analogous to $dxdy = rdrd\varphi$.

So we must include a factor analogous to r in the functional measure; this factor is

$$\begin{aligned}\prod_x \Big(v+\rho(x)\Big) &= \det(v+\rho) \\ &\propto \det(1+v^{-1}\rho) \\ &\propto \int \mathcal{D}\bar{c}\,\mathcal{D}c\; e^{-im_{\rm gh}^2 \int d^4x\; \bar{c}(1+v^{-1}\rho)c}\,. \end{aligned} \tag{85.18}$$

In the last line, we have written the functional determinant as an integral over ghost fields.

We see that they have no kinetic term, and we have chosen the overall normalization of their action so that their mass is m_{gh}, where m_{gh} is an arbitrary mass parameter.

Thus the momentum-space propagator for the ghosts is simply $\tilde{\Delta}(k^2) = 1/m_{\rm gh}^2$.

We also see that there is a ghost-ghost-scalar vertex, with vertex factor $-im^2_{gh}v^{-1}$, but there is no interaction between the ghosts and the vector field.

This seems like a fairly convenient gauge for loop calculations, but there is a complication.

The fact that the ghost propagator is independent of the momentum means that additional internal ghost propagators do not help the convergence of loop-momentum integrals.

The same is true of vector-field propagators; from eq. (85.17) we see that, in momentum space, the propagator scales like $1/M^2$ in the limit that all components of k become large.

Thus, in unitary gauge, loop diagrams with arbitrarily many external lines diverge.

This makes it difficult to establish renormalizability.

A gauge that does not suffer from this problem is a generalization of R_ξ gauge (and in fact this name has traditionally been applied only to this generalization).

We begin by using a cartesian basis for φ,

$$\varphi = \tfrac{1}{\sqrt{2}}(v + h + ib) \ , \tag{85.19}$$

where h and b are real scalar fields.

In terms of h and b, the potential is

$$V(\varphi) = \tfrac{1}{4}\lambda v^2 h^2 + \tfrac{1}{4}\lambda v h(h^2 + b^2) + \tfrac{1}{16}\lambda(h^2 + b^2)^2 \ , \tag{85.20}$$

and the covariant derivative of φ is

$$D_\mu\varphi = \tfrac{1}{\sqrt{2}}\Big[(\partial_\mu h + gbA_\mu) + i(\partial_\mu b - g(v{+}h)A_\mu)\Big] \ . \tag{85.21}$$

Thus the kinetic term for φ becomes

$$-(D^\mu\varphi)^\dagger D_\mu\varphi = -\tfrac{1}{2}(\partial_\mu h + gbA_\mu)^2 - \tfrac{1}{2}(\partial_\mu b - g(v{+}h)A_\mu)^2 \ . \tag{85.22}$$

Expanding this out, and rearranging, we get

$$\begin{aligned} -(D^\mu\varphi)^\dagger D_\mu\varphi = {} & -\tfrac{1}{2}\partial_\mu h\partial_\mu h - \tfrac{1}{2}\partial_\mu b\partial_\mu b - \tfrac{1}{2}g^2v^2A^\mu A_\mu + gvA^\mu\partial_\mu b \\ & + gA^\mu(h\partial_\mu b - b\partial_\mu h) \\ & - gvhA^\mu A_\mu - \tfrac{1}{2}g^2(h^2+b^2)A^\mu A_\mu \ . \end{aligned} \tag{85.23}$$

The first line on the right-hand side of eq. (85.23) contains all the terms that are quadratic in the fields.

The first two are the kinetic terms for the h and b fields.

The third is the mass term for the vector field.

The fourth is an annoying cross term between the vector field and the derivative of b.

In abelian gauge theory, in the absence of spontaneous symmetry breaking, we fix R_ξ gauge by adding to $\mathcal{L}$ the gauge-fixing and ghost terms

$$\mathcal{L}_{\rm gf} + \mathcal{L}_{\rm gh} = -\tfrac{1}{2}\xi^{-1}G^2 - \bar{c}\,\frac{\delta G}{\delta\theta}\,c\,, \tag{85.24}$$

where $G = \partial^\mu A_\mu$, and $\theta(x)$ parameterizes an infinitesimal gauge transformation,

$$A_\mu \to A_\mu - \partial_\mu\theta\,, \tag{85.25}$$

$$\varphi \to \varphi - ig\theta\varphi\,. \tag{85.26}$$

With $G = \partial^\mu A_\mu$, we have $\partial G/\partial\theta = -\partial^2$.

Thus the ghost fields have no interactions, and can be ignored.

In the presence of spontaneous symmetry breaking, we choose instead

$$G = \partial^\mu A_\mu - \xi g v b\,, \tag{85.27}$$

which reduces to $\partial^\mu A_\mu$ when $v = 0$.

Multiplying out G^2, we have

$$\begin{aligned}\mathcal{L}_{\rm gf} &= -\tfrac{1}{2}\xi^{-1}\partial^\mu A_\mu \partial^\nu A_\nu + gvb\partial^\mu A_\mu - \tfrac{1}{2}\xi g^2 v^2 b^2 \\ &= -\tfrac{1}{2}\xi^{-1}\partial^\mu A_\nu \partial^\nu A_\mu - gvA_\mu\partial^\mu b - \tfrac{1}{2}\xi g^2 v^2 b^2 \ ,\end{aligned} \tag{85.28}$$

where we integrated by parts in the first two terms to get the second line.

Note that the second term on the second line of eq. (85.28) cancels the annoying last term on the first line of eq. (85.23).

Also, the last term on the second line of eq. (85.28) gives a mass $\xi^{1/2}M$ to the b field.

We must still evaluate $\mathcal{L}_{\rm gh}$.

To do so, we first translate eq. (85.26) into

$$h \to h + g\theta b \ , \tag{85.29}$$

$$b \to b - g\theta(v+h) \ . \tag{85.30}$$

Then we have

$$\frac{\delta G}{\delta\theta} = -\partial^2 + \xi g^2 v(v+h) \ . \tag{85.31}$$

From eq. (85.24) we see that the ghost lagrangian is

$$\begin{aligned}\mathcal{L}_{\rm gh} &= -\bar{c}\Big[-\partial^2 + \xi g^2 v(v+h)\Big]c \\ &= -\partial^\mu\bar{c}\partial_\mu c - \xi g^2 v^2 \bar{c}c - \xi g^2 v h\bar{c}c \ .\end{aligned} \tag{85.32}$$

We see from the second term that the ghost has acquired the same mass as the b field, $\xi^{1/2}M$.

Now let us examine the vector field.

Including $\mathcal{L}_{\rm gf}$, the terms in $\mathcal{L}$ that are quadratic in the vector field can be written as

$$\mathcal{L}_0 = -\tfrac{1}{2}A_\mu\Big[g^{\mu\nu}(-\partial^2 + M^2) + (1-\xi^{-1})\partial^\mu\partial^\nu\Big]A_\nu \ . \tag{85.33}$$

In momentum space, this reads

$$\tilde{\mathcal{L}}_0 = -\tfrac{1}{2}\tilde{A}_\mu(-k)\Big[(k^2+M^2)g^{\mu\nu} + (1-\xi^{-1})k^\mu k^\nu\Big]\tilde{A}_\nu(k) \ . \tag{85.34}$$

The kinematic matrix is

$$\begin{aligned}\Big[\ldots\Big] &= (k^2+M^2)g^{\mu\nu} + (1-\xi^{-1})k^\mu k^\nu \\ &= (k^2+M^2)\Big(P^{\mu\nu}(k) + k^\mu k^\nu/k^2\Big) + (1-\xi^{-1})k^\mu k^\nu \\ &= (k^2+M^2)P^{\mu\nu}(k) + \xi^{-1}(k^2+\xi M^2)k^\mu k^\nu/k^2 \ ,\end{aligned} \tag{85.35}$$

where $P^{\mu\upsilon}(k) = g^{\mu\upsilon} - k^\mu k^\upsilon/k^2$ projects onto the transverse subspace; $P^{\mu\upsilon}(k)$ and $k^\mu k^\upsilon/k^2$ are orthogonal projection matrices.

Using this fact, it is easy to invert eq. (85.35) to get the propagator for the massive vector field in R_ξ gauge,

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{P^{\mu\nu}(k)}{k^2 + M^2 - i\epsilon} + \frac{\xi\, k^\mu k^\nu/k^2}{k^2 + \xi M^2 - i\epsilon} \,. \tag{85.36}$$

We see that the transverse components of the vector field propagate with mass M , while the longitudinal component propagates with the same mass as the b and ghost fields, $\xi^{1/2}M$.

Eq. (85.36) simplifies greatly if we choose $\xi = 1$; then we have

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu}}{k^2 + M^2 - i\epsilon} \quad (\xi = 1) \,. \tag{85.37}$$

On the other hand, leaving ξ as a free parameter allows us to check that all ξ dependence cancels out of any physical scattering amplitude.

Since their masses depend on ξ, the ghosts, the b field, and the longitudinal component of the vector field must all represent unphysical particles that do not appear in incoming or outgoing states.

To summarize, in R_ξ gauge we have the physical h field with mass-squared $m^2_h = 1/2\, \lambda\upsilon^2$ and propagator $1/(k^2 + m^2_h)$, the unphysical b field with propagator $1/(k^2 + \xi M^2)$, the ghost fields $\bar{c}$ and c with propagator $1/(k^2 + \xi M^2)$, and the vector field with the propagator of eq. (85.36).

For external vectors, the polarizations are still given by eq. (85.13), and obey the sum rules of eq. (85.16).

The mass parameter M is given by $M = g\upsilon$.

The interactions of these fields are governed by

$$\begin{aligned}\mathcal{L}_1 = &-\tfrac{1}{4}\lambda v h(h^2+b^2) - \tfrac{1}{16}\lambda(h^2+b^2)^2 \\ &+ gA^\mu(h\partial_\mu b - b\partial_\mu h) - gvhA^\mu A_\mu - \tfrac{1}{2}g^2(h^2+b^2)A^\mu A_\mu \\ &- \xi g^2 v h\bar{c}c \,.\end{aligned} \tag{85.38}$$

It is interesting to consider the limit $\xi \to \infty$.

In this limit, the vector propagator in R_ξ gauge, eq. (85.36), turns into the massive vector propagator of eq. (85.17),

$$\tilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu} + k^\mu k^\nu/M^2}{k^2+M^2-i\epsilon} \qquad (\xi = \infty)\,. \tag{85.39}$$

The b field becomes infinitely heavy, and we can drop it. (Equivalently, its propagator goes to zero.)

The ghost fields also become infinitely heavy, but we must be more careful with them because their interaction term, the last line of eq. (85.38), also contains a factor of ξ.

The vertex factor for this interaction is$-i\xi g^2\upsilon = -i(\xi M^2)\upsilon^{-1}$.

Note that this is the same vertex factor that we found in unitary gauge for the interaction between the ϱ field and the ghost fields; see eq. (85.18) and take $m^2_{gh} = \xi M^2$.

Thus we cannot drop the ghost fields, but we can take their propagator to be $1/m2_{gh}$ rather than $1/(k^2 + m^2_{gh})$, since $k^2 \ll m^2_{gh}= \xi M^2$ in the limit $\xi \to \infty$.

This is the ghost propagator that we found in unitary gauge.

We conclude that the R_ξ gauge in the limit $\xi \to \infty$ is equivalent to unitary gauge.

Of course, in this limit, we reencounter the problems with divergent diagrams that led us to consider alternative gauge choices in the first place.

For practical loop calculations, R_ξ gauge with $\xi = 1$ is typically the most convenient.

In the next section, we consider R_ξ gauge for nonabelian theories.

86 Spontaneously Broken Nonabelian Gauge Theory

In the previous section, we worked out the lagrangian for a U(1) gauge theory with spontaneous symmetry breaking in R_ξ gauge.

In this section, we extend this analysis to a general nonabelian gauge theory.

As in section 85, it will be convenient to work with real scalar fields.

We therefore decompose any complex scalar fields into pairs of real ones, and organize all the real scalar fields into a big list ϕ_i, $i = 1, \ldots, N$.

These real scalar fields form a (possibly reducible) representation R of the gauge group.

Let T^a be the gauge-group generator matrices that act on ϕ; they are linear combinations of the generators of the SO(N) group that rotates all components of ϕ_i into each other.

Because these SO(N) generators are hermitian and antisymmetric, so are the T^a's. Thus $i(T^a)_{ij}$ is a real, antisymmetric matrix.

The lagrangian for our theory can now be written as

$$\mathcal{L} = -\tfrac{1}{2}D^\mu\phi D_\mu\phi - V(\phi) - \tfrac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}\ , \qquad (86.1)$$

where

$$(D_\mu\phi)_i = \partial_\mu\phi_i - ig_a A^a_\mu(T^a)_{ij}\phi_j \qquad (86.2)$$

is the covariant derivative, and the adjoint index a runs over all generators of all gauge groups.

Because ϕ_i and A^a_μ are real fields, and $i(T^a)_{ij}$ is a real matrix, $(D_\mu\phi)_i$ is real.

Now we suppose that the potential $V(\phi)$ is minimized when ϕ has a VEV

$$\langle 0|\phi_i(x)|0\rangle = v_i \ . \tag{86.3}$$

A generator T^a is unbroken if $(Ta)_{ij}v_j = 0$, and broken if $(Ta)_{ij}v_j \neq 0$.

Each broken generator results in a massless Goldstone boson.

To see this, we note that the potential must be invariant under a global gauge transformation,

$$V((1-i\theta^a \mathcal{T}^a)\phi) = V(\phi) \ . \tag{86.4}$$

Expanding to linear order in the infinitesimal parameter θ, we find

$$\frac{\partial V}{\partial \phi_j}(\mathcal{T}^a)_{jk}\phi_k = 0 \ . \tag{86.5}$$

We differentiate eq. (86.5) with respect to ϕ_k to get

$$\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}(\mathcal{T}^a)_{jk}\phi_k + \frac{\partial V}{\partial \phi_j}(\mathcal{T}^a)_{jk} = 0 \ . \tag{86.6}$$

Now set $\phi_i = v_i$; then $\partial V/\partial \phi_i$ vanishes, because ϕi = vi minimizes V(ϕ).

Also, we can identify

$$(m^2)_{ij} = \left.\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right|_{\phi_i = v_i} \tag{86.7}$$

as the mass-squared matrix for the scalars (after spontaneous symmetry breaking).

Thus eq. (86.6) becomes

$$(m^2)_{ij}(\mathcal{T}^a v)_j = 0 \,. \tag{86.8}$$

We see that if $T^a v \neq 0$, then $T^a v$ is an eigenvector of the mass-squared matrix with eigenvalue zero.

So there is a zero eigenvalue for every linearly independent broken generator.

Let us write

$$\phi_i(x) = v_i + \chi_i(x) \,, \tag{86.9}$$

where χ_i is a real scalar field.

The covariant derivative of ϕ becomes

$$(D_\mu \phi)_i = \partial_\mu \chi_i - i g_a A^a_\mu (\mathcal{T}^a)_{ij}(v+\chi)_j \tag{86.10}$$

It is now convenient to define a set of real antisymmetric matrices

$$(\tau^a)_{ij} \equiv i g_a (\mathcal{T}^a)_{ij} \,, \tag{86.11}$$

and the real rectangular matrix

$$F^a{}_i \equiv (\tau^a)_{ij} v_j \,. \tag{86.12}$$

We can now write

$$(D_\mu\phi)_i = \partial_\mu\chi_i - A^a_\mu(F^a + \tau^a\chi)_i \ . \tag{86.13}$$

The kinetic term for ϕ becomes

$$\begin{aligned} -\tfrac{1}{2}D^\mu\phi D_\mu\phi = {} & -\tfrac{1}{2}\partial^\mu\chi_i\partial_\mu\chi_i - \tfrac{1}{2}(F^a{}_iF^b{}_i)A^{a\mu}A^b_\mu + F^a{}_iA^a_\mu\partial^\mu\chi_i \\ & + A^a_\mu\chi_i(\tau^a)_{ij}\partial^\mu\chi_j - A^{a\mu}A^b_\mu F^a{}_i(\tau^b)_{ij}\chi_j \\ & - \tfrac{1}{2}A^{a\mu}A^b_\mu\chi_i(\tau^a\tau^b)_{ij}\chi_j \ . \end{aligned} \tag{86.14}$$

We see (from the second term on the right-hand side) that the mass-squared matrix for the vector fields is

$$(M^2)^{ab} = F^a{}_iF^b{}_i = (FF^{\mathrm{T}})^{ab} \ . \tag{86.15}$$

A theorem of linear algebra states that every real rectangular matrix can be written as

$$F^a{}_i = S^{ab}(M^b\delta^b{}_j)R_{ji} \ , \tag{86.16}$$

where S and R are orthogonal matrices, and the diagonal entries M^a are real and nonnegative.

From eq. (86.15) we see that these diagonal entries are the masses of the vector fields.

The vector fields of definite mass are then given by $\tilde{A}^a_\mu = S^{ba}A^b_\mu$.

Now we are ready to fix R_ξ gauge.

To do so, we add to $\mathcal{L}$ the gauge-fixing and ghost terms

$$\mathcal{L}_{\rm gf} + \mathcal{L}_{\rm gh} = -\tfrac{1}{2}\xi^{-1}G^a G^a - \bar{c}^a \, \frac{\delta G^a}{\delta\theta^b} \, c^b \;, \tag{86.17}$$

where we choose

$$G^a = \partial^\mu A^a_\mu - \xi F^a{}_i \chi_i \;. \tag{86.18}$$

Then we have

$$\begin{aligned} \mathcal{L}_{\rm gf} &= -\tfrac{1}{2}\xi^{-1}\partial^\mu A^a_\mu \partial^\nu A^a_\nu + F^a{}_i\chi_i\partial^\mu A^a_\mu - \tfrac{1}{2}\xi(F^a{}_iF^a{}_j)\chi_i\chi_j \\ &= -\tfrac{1}{2}\xi^{-1}\partial^\mu A^a_\nu \partial^\nu A^a_\mu - F^a{}_i A^a_\mu \partial^\mu\chi_i - \tfrac{1}{2}\xi(F^a{}_iF^a{}_j)\chi_i\chi_j \;. \end{aligned} \tag{86.19}$$

We integrated by parts in the first two terms to get the second line.

Note that the second term on the second line of eq. (86.19) cancels the annoying last term on the first line of eq. (86.14).

Also, the last term on the second line of eq. (86.19) makes a contribution to the mass-squared matrix for the χ fields,

$$\xi(M^2)_{ij} = \xi F^a{}_i F^a{}_j = \xi(F^{\rm T}F)_{ij} \;. \tag{86.20}$$

Eq. (86.16) tells us that the eigenvalues of this matrix are $\xi^{1/2}M^a$, where M^a are the vector-boson masses.

The mass-squared matrix ξM^2 should be added to the mass-squared matrix m^2 that we get from the potential, eq. (86.7).

Note that eqs. (86.8) and (86.12) imply that $(m^2)_{ij} F^a{}_j = 0$; eq. (86.20) then yields $(m^2)_i(\xi M^2)_{jk} = 0$.

Thus these two contributions to the mass-squared matrix of the scalar fields live in orthogonal subspaces.

The scalar fields of definite mass are $\tilde{\chi}_i = R_{ij}\chi_j$, where the block of R in the m^2 subspace is chosen to diagonalize m^2.

The m^2 subspace consists of the physical, massive scalars, and the ξM^2 subspace consists of the unphysical Goldstone bosons; these are the fields that would be set to zero in unitary gauge.

We must still evaluate $\mathcal{L}_{gh}$ - to do so, we recall that $\theta^a(x)$ parameterizes an infinitesimal gauge transformation,

$$A^a_\mu \to A^a_\mu - D^{ab}_\mu \theta^b \ , \tag{86.21}$$

$$\chi_i \to -\theta^a (\tau^a)_{ij}(v+\chi)_j \ . \tag{86.22}$$

Thus we have

$$\begin{aligned} \frac{\delta G^a}{\delta \theta^b} &= -\partial^\mu D^{ab}_\mu + \xi F^a{}_j (\tau^b)_{jk}(v+\chi)_k \\ &= -\partial^\mu D^{ab}_\mu + \xi F^a{}_j F^b{}_j + \xi F^a{}_j (\tau^b)_{jk}\chi_k \\ &= -\partial^\mu D^{ab}_\mu + \xi (M^2)^{ab} + \xi F^a{}_j (\tau^b)_{jk}\chi_k \ , \end{aligned} \tag{86.23}$$

and so the ghost lagrangian is

$$\mathcal{L}_{\rm gh} = -\partial^\mu \overline{c}^a D_\mu^{ab} c^b - \xi (M^2)^{ab} \overline{c}^a c^b - \xi F^a{}_j (\tau^b)_{jk} \chi_k \overline{c}^a c^b \ . \tag{86.24}$$

The ghost fields of definite mass are $\tilde{c}^a = S^{ba} c^b$ and $\tilde{\overline{c}}^a = S^{ba} \overline{c}^b$.

The complete gauge-fixed lagrangian is now given by eqs. (86.1), (86.14) (86.19), and (86.24).

We can rewrite it in terms of the fields of definite mass.

This results in the replacements

$$F^a{}_i \to M^a \delta^a{}_i \ , \tag{86.25}$$

$$(\tau^a)_{ij} \to S^{ab} (R^{\rm T} \tau^b R)_{ij} \ , \tag{86.26}$$

$$f^{abc} \to S^{ad} S^{be} S^{cg} f^{deg} \tag{86.27}$$

throughout $\mathcal{L}$.

The Feynman rules then follow in the usual way.

87: The Standard Model: Gauge and Higgs Sector

We now turn to the construction of the Standard Model of elementary particles, also called the Glashow–Weinberg–Salam model.

This is the complete (except for gravity) quantum field theory that appears to describe our world.

It can be succinctly specified as a gauge theory with gauge group SU(3) x SU(2) x U(1), with left-handed Weyl fields in three copies of the representation

$$(1, 2, -1/2) \oplus (1, 1, +1) \oplus (3, 2, +1/6) \oplus (3, 1, -2/3) \oplus (3, 1, +1/3),$$

and a complex scalar field in the representation $(1, 2, -1/2)$.

Here the last entry of each triplet gives the value of the U(1) charge, known as hypercharge.

The lagrangian includes all terms of mass dimension four or less that are allowed by the gauge symmetries and Lorentz invariance.

We will construct the Standard Model over several sections.

We begin with the electroweak part of the gauge group, SU(2) x U(1), and the complex scalar field ϕ, known as the Higgs field, in the representation $(2, -1/2)$.

The Higgs field acquires a nonzero VEV that spontaneously breaks SU(2) x U(1) to U(1); the unbroken U(1) is identified as electromagnetism.

We begin with the covariant derivative of the Higgs field φ,

$$(D_\mu\varphi)_i = \partial_\mu\varphi_i - i[g_2 A^a_\mu T^a + g_1 B_\mu Y]_i{}^j\varphi_j \ , \tag{87.1}$$

where $T^a = 1/2\sigma^a$ and $Y = -1/2\ I$; Y is the hypercharge generator.

It will prove useful to write out $g_2A^a_\mu T^a + g_1B_\mu Y$ in matrix form,

$$g_2 A^a_\mu T^a + g_1 B_\mu Y = \frac{1}{2}\begin{pmatrix} g_2A^3_\mu - g_1B_\mu & g_2(A^1_\mu - iA^2_\mu) \\ g_2(A^1_\mu + iA^2_\mu) & -g_2A^3_\mu - g_1B_\mu \end{pmatrix}. \tag{87.2}$$

Now suppose that φ has a potential

$$V(\varphi) = \tfrac{1}{4}\lambda(\varphi^\dagger\varphi - \tfrac{1}{2}v^2)^2 \ . \tag{87.3}$$

This potential gives φ a nonzero VEV.

We can make a global gauge transformation to bring this VEV entirely into the first component, and furthermore make it real, so that

$$\langle 0|\varphi(x)|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix}. \tag{87.4}$$

The kinetic term for φ is $-(D^\mu\varphi)^\dagger D_\mu\varphi$.

After replacing φ by its VEV, we find a mass term for the gauge fields,

$$\mathcal{L}_{\rm mass} = -\tfrac{1}{8}v^2\,(1\,,\;\;0)\begin{pmatrix} g_2A^3_\mu - g_1B_\mu & g_2(A^1_\mu - iA^2_\mu) \\ g_2(A^1_\mu + iA^2_\mu) & -g_2A^3_\mu - g_1B_\mu \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{87.5}$$

To diagonalize this mass-squared matrix, we first define the *weak mixing angle*

$$\theta_{\rm W} \equiv \tan^{-1}(g_1/g_2)\ , \tag{87.6}$$

and the fields

$$W^{\pm}_\mu \equiv \tfrac{1}{\sqrt{2}}(A^1_\mu \mp iA^2_\mu)\ , \tag{87.7}$$

$$Z_\mu \equiv c_{\rm W}A^3_\mu - s_{\rm W}B_\mu\ , \tag{87.8}$$

$$A_\mu \equiv s_{\rm W}A^3_\mu + c_{\rm W}B_\mu\ , \tag{87.9}$$

where $s_W \equiv \sin\theta_W$, $c_W \equiv \cos\theta_W$.

In terms of these fields, eq. (87.5) becomes

$$\begin{aligned}\mathcal{L}_{\rm mass} &= -\tfrac{1}{8}g_2^2v^2\,(1\,,\;\;0)\begin{pmatrix} \frac{1}{c_{\rm W}}Z_\mu & \sqrt{2}\,W^+_\mu \\ \sqrt{2}\,W^-_\mu & \ldots \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= -(g_2v/2)^2\,W^{+\mu}W^-_\mu - \tfrac{1}{2}(g_2v/2c_{\rm W})^2\,Z^\mu Z_\mu \\ &= -M_{\rm W}^2W^{+\mu}W^-_\mu - \tfrac{1}{2}M_{\rm Z}^2Z^\mu Z_\mu\ ,\end{aligned} \tag{87.10}$$

where we have identified

$$M_{\rm W} = g_2 v/2 \,, \tag{87.11}$$

$$M_{\rm Z} = M_{\rm W}/\cos\theta_{\rm W} \,. \tag{87.12}$$

The observed masses of the $W^\pm$ and Z^0 particles are $M_W = 80.4$ GeV and $M_Z = 91.2$ GeV.

Eq. (87.12) then implies $\cos\theta_W = 0.882$, or, as it is more usually expressed, $\sin^2\theta_W = 0.223$.

Note that the A_μ field remains massless; this signifies that there is an unbroken U(1) subgroup.

We will identify this unbroken U(1) with the gauge group of electromagnetism.

Before introducing leptons and quarks (which we do in sections 87 and 88), let us work out the complete lagrangian for the gauge and Higgs fields, in unitary gauge.

This is sufficient for tree-level calculations.

The two complex components of the ϕ field yield four real scalar fields; three of these become the longitudinal components of the $W^\pm$ and Z^0.

The remaining scalar field must be able to account for shifts in the overall scale of φ.

Thus we can write, in unitary gauge,

$$\varphi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} v + H(x) \\ 0 \end{pmatrix}, \tag{87.13}$$

where H is a real scalar field; the corresponding particle is the *Higgs boson*.

The potential now reads

$$V(\varphi) = \tfrac{1}{4}\lambda v^2 H^2 + \tfrac{1}{4}\lambda v H^3 + \tfrac{1}{16}\lambda H^4 \ . \qquad (87.14)$$

We see that the mass of the Higgs boson is given by $m^2_H = 1/2\ \lambda v^2$.

The Higgs boson has been observed; its mass is $m_H = 125.35$ GeV as predicted by the theory!

The kinetic term for H comes from the kinetic term for φ, and is the usual one for a real scalar field, $-1/2\ \partial^\mu H\ \partial_\mu H$.

Finally, recall that the mass term for the gauge fields, eq. (87.10), is proportional to v^2.

Hence it should be multiplied by a factor of $(1 + v^{-1}H)^2$.

Now we have to work out the kinetic terms for the gauge fields.

We have

$$\mathcal{L} = -\tfrac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} - \tfrac{1}{4}B^{\mu\nu}B_{\mu\nu} \ , \qquad (87.15)$$

where

$$F^1_{\mu\nu} = \partial_\mu A^1_\nu - \partial_\nu A^1_\mu + g_2(A^2_\mu A^3_\nu - A^2_\nu A^3_\mu)\ , \tag{87.16}$$

$$F^2_{\mu\nu} = \partial_\mu A^2_\nu - \partial_\nu A^2_\mu + g_2(A^3_\mu A^1_\nu - A^3_\nu A^1_\mu)\ , \tag{87.17}$$

$$F^3_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu + g_2(A^1_\mu A^2_\nu - A^1_\nu A^2_\mu)\ , \tag{87.18}$$

$$B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu\ . \tag{87.19}$$

Next, form the combinations $F^1{}_{\mu\upsilon} \pm iF^2{}_{\mu\upsilon}$.

Using eq. (87.7), we find

$$\tfrac{1}{\sqrt{2}}(F^1_{\mu\nu} - iF^2_{\mu\nu}) = D_\mu W^+_\nu - D_\nu W^+_\mu\ , \tag{87.20}$$

$$\tfrac{1}{\sqrt{2}}(F^1_{\mu\nu} + iF^2_{\mu\nu}) = D^\dagger_\mu W^-_\nu - D^\dagger_\nu W^-_\mu\ , \tag{87.21}$$

where we have defined a covariant derivative that acts on $W^+{}_\mu$,

$$\begin{aligned} D_\mu &\equiv \partial_\mu - ig_2 A^3_\mu \\ &= \partial_\mu - ig_2(s_\mathrm{W} A_\mu + c_\mathrm{W} Z_\mu)\ . \end{aligned} \tag{87.22}$$

If we identify A_μ as the electromagnetic vector potential, and assign electric charge Q = +1 (in units of the proton charge) to the W^+, then we see from eq. (87.22) that we must identify the electromagnetic coupling constant e as

$$e = g_2 \sin\theta_\mathrm{W}\ . \tag{87.23}$$

Here we are adopting the convention that e is positive. (In our treatment of quantum electrodynamics, we used the convention that e is negative, but that is less convenient in the present context.)

We also have

$$\begin{aligned} F^3_{\mu\nu} &= \partial_\mu A^3_\nu - \partial_\nu A^3_\mu - ig_2(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) \\ &= s_W F_{\mu\nu} + c_W Z_{\mu\nu} - ig_2(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) \,, \end{aligned} \tag{87.24}$$

$$B_{\mu\nu} = c_W F_{\mu\nu} - s_W Z_{\mu\nu} \,, \tag{87.25}$$

where $F_{\mu\upsilon} = \partial_\mu A_\upsilon - \partial_\upsilon A_\mu$ is the usual electromagnetic field strength, and

$$Z_{\mu\nu} \equiv \partial_\mu Z_\nu - \partial_\nu Z_\mu \tag{87.26}$$

is the abelian field strength associated with the Z_μ field.

Now we can assemble all of this into the complete lagrangian for the electroweak gauge fields and the Higgs boson in unitary gauge.

We will express g_2 in terms of e and θ_W via $g_2 = e/\sin\theta_W$, and λ in terms of m_H and υ via $\lambda = 2m^2_H/\upsilon^2$.

We ultimately get

$$
\begin{aligned}
\mathcal{L} = {} & -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{4}Z^{\mu\nu}Z_{\mu\nu} - D^{\dagger\mu}W^{-\nu}D_\mu W^+_\nu + D^{\dagger\mu}W^{-\nu}D_\nu W^+_\mu \\
& + ie(F^{\mu\nu} + \cot\theta_{\rm W} Z^{\mu\nu})W^+_\mu W^-_\nu \\
& - \tfrac{1}{2}(e^2/\sin^2\theta_{\rm W})(W^{+\mu}W^-_\mu W^{+\nu}W^-_\nu - W^{+\mu}W^+_\mu W^{-\nu}W^-_\nu) \\
& - (M_{\rm W}^2 W^{+\mu}W^-_\mu + \tfrac{1}{2}M_{\rm Z}^2 Z^\mu Z_\mu)(1+v^{-1}H)^2 \\
& - \tfrac{1}{2}\partial^\mu H \partial_\mu H - \tfrac{1}{2}m_{\rm H}^2 H^2 - \tfrac{1}{2}m_{\rm H}^2 v^{-1}H^3 - \tfrac{1}{8}m_{\rm H}^2 v^{-2}H^4 \ ,
\end{aligned}
\tag{87.27}
$$

where

$$
D_\mu = \partial_\mu - ie(A_\mu + \cot\theta_{\rm W} Z_\mu) \ . \tag{87.28}
$$

With the W^+_μ field assigned electric charge Q = +1, this lagrangian exhibits manifest electromagnetic gauge invariance.

The full underlying SU(2) x U(1) gauge invariance is not manifest, however, because we have fixed unitary gauge.

88: The Standard Model: Lepton Sector

Leptons are spin-one-half particles that are singlets of the color group.

There are six different flavors of lepton; see Table 2.

Table 2: The six flavors of lepton. Q is the electric charge in units of the proton charge. Each charged flavor is represented by a Dirac field, each neutral flavor by a Majorana field (or, equivalently, a left-haned Weyl field). Neutrino masses are exactly zero in the Standard Model.

name	symbol	mass (MeV)	Q
electron	e	0.511	-1
electron neutrino	ν_e	0	0
muon	μ	105.7	-1
muon neutrino	ν_μ	0	0
tau	τ	1777	-1
tau neutrino	ν_τ	0	0

The six flavors are naturally grouped into *three families or generations*: e and ν_e, μ and ν_μ, τ and ν_τ.

Let us begin by describing a single lepton family, the electron and its neutrino.

We introduce left-handed Weyl fields ℓ and $\bar{e}$ in the representations (2,–1/2) and (1, +1) of SU(2) x U(1).

Here the bar over the e in the field $\bar{e}$ is part of the name of the field, and does not denote any sort of conjugation.

The covariant derivatives of these fields are

$$(D_\mu \ell)_i = \partial_\mu \ell_i - i g_2 A^a_\mu (T^a)_i{}^j \ell_j - i g_1 (-\tfrac{1}{2}) B_\mu \ell_i \ , \qquad (88.1)$$

$$D_\mu \bar{e} = \partial_\mu \bar{e} - i g_1 (+1) B_\mu \bar{e} \ , \qquad (88.2)$$

and their kinetic terms are

$$\mathcal{L}_{\text{kin}} = i \ell^{\dagger i} \bar{\sigma}^\mu (D_\mu \ell)_i + i \bar{e}^\dagger \bar{\sigma}^\mu D_\mu \bar{e} \ . \qquad (88.3)$$

The representation (2,–1/2) ⊕ (1, +1) for the left-handed Weyl fields is complex; hence the gauge theory is chiral, and therefore parity violating.

We cannot write down a mass term involving ℓ and/or $\bar{e}$ because there is no gauge-group singlet contained in any of the products

$$\begin{aligned}(2,-\tfrac{1}{2}) &\otimes (2,-\tfrac{1}{2})\ , \\ (2,-\tfrac{1}{2}) &\otimes (1,+1)\ , \\ (1,+1) &\otimes (1,+1)\ .\end{aligned} \tag{88.4}$$

However, we are able to write down a Yukawa coupling of the form

$$\mathcal{L}_{\text{Yuk}} = -y\varepsilon^{ij}\varphi_i\ell_j\bar{e} + \text{h.c.}\ , \tag{88.5}$$

where φ is the Higgs field in the (2,–1/2) representation that we introduced in the last section, and y is the Yukawa coupling constant.

A gauge-invariant Yukawa coupling is possible because there is a singlet on the right-hand side of

$$(2,-\tfrac{1}{2}) \otimes (2,-\tfrac{1}{2}) \otimes (1,+1) = (1,0) \oplus (3,0)\ . \tag{88.6}$$

There are no other gauge-invariant terms involving ℓ or $\bar{e}$ that have mass dimension four or less.

Hence there are no other terms that we could add to $\mathcal{L}$ while preserving renormalizability.

We add eqs. (88.3) and (88.5) to the lagrangian for φ and the gauge fields that we worked out in the last section.

In unitary gauge, we replace φ_1 with $1/\sqrt{2}\,(v + H)$, where H is the real scalar field representing the physical Higgs boson, and φ_2 with zero.

The Yukawa term becomes

$$\mathcal{L}_{\text{Yuk}} = -\frac{1}{\sqrt{2}} y(v+H)(\ell_2 \bar{e} + \text{h.c.})\ . \tag{88.7}$$

It is now convenient to assign new names to the SU(2) components of ℓ,

$$\ell = \begin{pmatrix} \nu \\ e \end{pmatrix}\ . \tag{88.8}$$

We will rely on context to distinguish the field e from the electromagnetic coupling constant e.

Then eq. (88.7) becomes

$$\begin{aligned} \mathcal{L}_{\text{Yuk}} &= -\frac{1}{\sqrt{2}} y(v+H)(e\bar{e} + \bar{e}^\dagger e^\dagger) \\ &= -\frac{1}{\sqrt{2}} y(v+H)\overline{\mathcal{E}}\mathcal{E} \end{aligned} \tag{88.9}$$

where we have defined a Dirac field for the electron,

$$\mathcal{E} \equiv \begin{pmatrix} e \\ \bar{e}^\dagger \end{pmatrix}. \tag{88.10}$$

We see that the electron has acquired a mass

$$m_e = \frac{yv}{\sqrt{2}} \, . \tag{88.11}$$

The neutrino has remained massless

We can describe the neutrino with a Majorana field

$$\mathcal{N} \equiv \begin{pmatrix} \nu \\ \nu^\dagger \end{pmatrix} . \tag{88.12}$$

However, it is often more convenient to work with

$$\mathcal{N}_\mathrm{L} \equiv P_\mathrm{L} \mathcal{N} = \begin{pmatrix} \nu \\ 0 \end{pmatrix} , \tag{88.13}$$

where $P_L = 1/2\ (1-\gamma_5)$.

We can think of $\mathcal{N}_L$ as a Dirac field; for example, the neutrino kinetic term $i\nu^\dagger\bar{\sigma}^\mu\partial_\mu\nu$ can be written as $i\overline{\mathcal{N}}_\mathrm{L}\partial\!\!\!/\mathcal{N}_\mathrm{L}$.

Now we return to eqs. (88.1) and (88.2), and express the covariant derivatives in the terms of the $W^\pm_\mu$, Z_μ, and A_μ fields.

From our results in section 87, we have

$$g_2 A^1_\mu T^1 + g_2 A^2_\mu T^2 = \frac{g_2}{\sqrt{2}} \begin{pmatrix} 0 & W^+_\mu \\ W^-_\mu & 0 \end{pmatrix} \tag{88.14}$$

and

$$\begin{aligned} g_2 A^3_\mu T^3 + g_1 B_\mu Y &= \tfrac{e}{s_\mathrm{W}}(s_\mathrm{W} A_\mu + c_\mathrm{W} Z_\mu) T^3 + \tfrac{e}{c_\mathrm{W}}(c_\mathrm{W} A_\mu - s_\mathrm{W} Z_\mu) Y \\ &= e(A_\mu + \cot\theta_\mathrm{W} Z_\mu) T^3 + e(A_\mu - \tan\theta_\mathrm{W} Z_\mu) Y \\ &= e(T^3 + Y) A_\mu + e(\cot\theta_\mathrm{W} T^3 - \tan\theta_\mathrm{W} Y) Z_\mu \,. \end{aligned} \tag{88.15}$$

Since we identify A_μ as the electromagnetic field and e as the electromagnetic coupling constant (with the convention that e is positive), we identify

$$Q = T^3 + Y \tag{88.16}$$

as the generator of electric charge.

Then, since

$$T^3\nu = +\tfrac{1}{2}\nu\,, \quad T^3 e = -\tfrac{1}{2}e\,, \quad T^3\bar{e} = 0\,, \tag{88.17}$$

$$Y\nu = -\tfrac{1}{2}\nu\,, \quad Y e = -\tfrac{1}{2}e\,, \quad Y\bar{e} = +\bar{e}\,, \tag{88.18}$$

we see from eq. (88.16) that

$$Q\nu = 0\,, \qquad Qe = -e\,, \qquad Q\bar{e} = +\bar{e}\,. \tag{88.19}$$

This is just the set of electric charge assignments that we expect for the electron and the neutrino.

Then (since the action of Q on the fields is more familiar than the action of Y) it is convenient to replace Y in eq. (88.15) with $Q - T^3$.

We find

$$\begin{aligned} g_2 A^3_\mu T^3 + g_1 B_\mu Y &= eQA_\mu + e[(\cot\theta_W + \tan\theta_W)T^3 - \tan\theta_W Q]Z_\mu \\ &= eQA_\mu + \frac{e}{s_W c_W}(T^3 - s_W^2 Q)Z_\mu \,. \end{aligned} \tag{88.20}$$

In terms of the four-component fields, we have

$$(g_2 A^3_\mu T^3 + g_1 B_\mu Y)\mathcal{E} = \Big[-eA_\mu + \frac{e}{s_W c_W}(-\tfrac{1}{2}P_L + s_W^2)Z_\mu\Big]\mathcal{E} \,, \tag{88.21}$$

$$(g_2 A^3_\mu T^3 + g_1 B_\mu Y)\mathcal{N}_L = \frac{e}{s_W c_W}(+\tfrac{1}{2})Z_\mu \mathcal{N}_L \,. \tag{88.22}$$

tUsing eqs. (88.14) and (88.21–88.22) in eqs. (88.1–88.3), we find the couplings of the gauge fields to the leptons,

$$\mathcal{L}_{\rm int} = \tfrac{1}{\sqrt{2}} g_2 W^+_\mu J^{-\mu} + \tfrac{1}{\sqrt{2}} g_2 W^-_\mu J^{+\mu} + \frac{e}{s_W c_W} Z_\mu J^\mu_Z + eA_\mu J^\mu_{\rm EM} \,, \tag{88.23}$$

where we have defined the currents

$$J^{+\mu} \equiv \overline{\mathcal{E}}_L \gamma^\mu \mathcal{N}_L \,, \tag{88.24}$$

$$J^{-\mu} \equiv \overline{\mathcal{N}}_L \gamma^\mu \mathcal{E}_L \,, \tag{88.25}$$

$$J^\mu_Z \equiv J^\mu_3 - s_W^2 J^\mu_{\rm EM} \,, \tag{88.26}$$

$$J_3^\mu \equiv \tfrac{1}{2}\overline{\mathcal{N}}_{\rm L}\gamma^\mu \mathcal{N}_{\rm L} - \tfrac{1}{2}\overline{\mathcal{E}}_{\rm L}\gamma^\mu \mathcal{E}_{\rm L} \;, \tag{88.27}$$

$$J_{\rm EM}^\mu \equiv -\overline{\mathcal{E}}\gamma^\mu \mathcal{E} \;. \tag{88.28}$$

In many cases, we are interested in scattering amplitudes for leptons whose momenta are all well below the $W^\pm$ and Z^0 masses.

In this case, we can integrate the $W^\pm{}_\mu$ and Z_μ fields out of the path integral, as discussed in section 29.

We get the leading term (in a double expansion in powers of the gauge couplings and inverse powers of M_W and M_Z) by ignoring the kinetic energy and other interactions of the $W^\pm{}_\mu$ and Z_μ fields, solving the equations of motion for them that follow from $\mathcal{L}_{\rm mass} + \mathcal{L}_{\rm int}$, where $\mathcal{L}_{\rm int}$, is given by eq. (88.23) and

$$\mathcal{L}_{\rm mass} = -M_{\rm W}^2 W^{+\mu} W_\mu^- - \tfrac{1}{2} M_{\rm Z}^2 Z^\mu Z_\mu \;, \tag{88.29}$$

and finally substituting the solutions back into $\mathcal{L}_{\rm mass} + \mathcal{L}_{\rm int}$.

This is equivalent to evaluating tree-level Feynman diagrams with a single $W^\pm$ or Z^0 exchanged, with the propagator $g^{\mu\nu}/M^2{}_{W,Z}$.

The result is

$$
\begin{aligned}
\mathcal{L}_{\rm eff} &= \frac{g_2^2}{2M_{\rm W}^2} J^{+\mu} J^-_\mu + \frac{e^2}{2s_{\rm W}^2 c_{\rm W}^2 M_{\rm Z}^2} J_{\rm Z}^\mu J_{{\rm Z}\mu} \\
&= \frac{e^2}{2s_{\rm W}^2 M_{\rm W}^2} (J^{+\mu} J^-_\mu + J_{\rm Z}^\mu J_{{\rm Z}\mu}) \\
&= 2\sqrt{2}\, G_{\rm F} (J^{+\mu} J^-_\mu + J_{\rm Z}^\mu J_{{\rm Z}\mu}) \,.
\end{aligned}
\tag{88.30}
$$

We used $e = g_2 \sin \theta_W$ and $M_W = M_Z \cos \theta_W$ to get the second line, and we defined the *Fermi constant*

$$
G_{\rm F} \equiv \frac{e^2}{4\sqrt{2}\sin^2\theta_{\rm W} M_{\rm W}^2}
\tag{88.31}
$$

in the third line.

We can use $\mathcal{L}_{\rm eff}$ to compute the tree-level scattering amplitude for processes like $\nu_e e^- \to \nu_e e^-$.

Having worked out the interactions of a single lepton generation, we now examine what happens when there is more than one of them.

Let us consider the fields ℓ_{iI} and $\bar{e}_I$, where I = 1, 2, 3 is a generation index.

The kinetic term for all these fields is

$$
\mathcal{L}_{\rm kin} = i\ell_I^{\dagger i} \bar{\sigma}^\mu (D_\mu)_i{}^j \ell_{jI} + i\bar{e}_I^\dagger \bar{\sigma}^\mu D_\mu \bar{e}_I \ ,
\tag{88.32}
$$

where the repeated generation index is summed.

The most general Yukawa term we can write down now reads

$$\mathcal{L}_{\text{Yuk}} = -\varepsilon^{ij}\varphi_i \ell_{jI} y_{IJ} \bar{e}_J + \text{h.c.} \;, \tag{88.33}$$

where y_{IJ} is a complex 3 x 3 matrix, and the generation indices are summed.

We can make unitary transformations in generation space on the fields: $\ell_I \rightarrow L_{IJ}\, \ell_J$ and $\bar{e}_I \rightarrow \bar{E}_{IJ}\bar{e}_J$, where L and $\bar{E}$ are independent unitary matrices.

The kinetic terms are unchanged, and the Yukawa matrix y is replaced with $L^{\mathrm{T}} y \bar{E}$.

We can choose L and E so that $L^{\mathrm{T}} y \bar{E}$. is diagonal with positive real entries y_I .

The charged leptons $\mathcal{E}_I$ then have masses $m_{e_I} = y_I v/\sqrt{2}$, ;and the neutrinos remain massless.

In the currents, eqs. (88.24–88.28), we simply add a generation index I to each field, and sum over it.

Let us work out the details for one process of particular importance: muon decay, $\mu^- \rightarrow e^- \overline{\nu}_e \nu_\mu$.

Let the four-component fields be E for the electron, $\mathcal{M}$ for the muon, $\mathcal{N}_e$ for the electron neutrino, and $\mathcal{N}_m$ for the muon neutrino.

Only the charged currents $J^{\pm}{}_{\mu}$ are relevant; the neutral current $J^{\pm}{}_{\chi}$ and the electromagnetic current $J^{\mu}{}_{\mathrm{EM}}$ do not contribute.

Ignoring the τ terms in the charged currents, we have

$$J^{+\mu} = \overline{\mathcal{E}}_{\mathrm{L}}\gamma^{\mu}\mathcal{N}_{e\mathrm{L}} + \overline{\mathcal{M}}_{\mathrm{L}}\gamma^{\mu}\mathcal{N}_{m\mathrm{L}} \ , \tag{88.34}$$

$$J^{-\mu} = \overline{\mathcal{N}}_{e\mathrm{L}}\gamma^{\mu}\mathcal{E}_{\mathrm{L}} + \overline{\mathcal{N}}_{m\mathrm{L}}\gamma^{\mu}\mathcal{M}_{\mathrm{L}} \ . \tag{88.35}$$

The relevant term in the effective interaction is

$$\mathcal{L}_{\mathrm{eff}} = 2\sqrt{2}\,G_{\mathrm{F}}(\overline{\mathcal{E}}_{\mathrm{L}}\gamma^{\mu}\mathcal{N}_{e\mathrm{L}})(\overline{\mathcal{N}}_{m\mathrm{L}}\gamma_{\mu}\mathcal{M}_{\mathrm{L}}) \ . \tag{88.36}$$

This can be simplified by means of a Fierz identity

$$\mathcal{L}_{\mathrm{eff}} = -4\sqrt{2}\,G_{\mathrm{F}}(\overline{\mathcal{M}}^{\mathrm{C}}P_{\mathrm{L}}\mathcal{N}_{e})(\overline{\mathcal{E}}P_{\mathrm{R}}\mathcal{N}_{m}^{\mathrm{C}}) \ . \tag{88.37}$$

Assigning momenta as shown in fig. (88.1), and using the usual Feynman rules for incoming and outgoing particles and antiparticles, the scattering amplitude is

$$\begin{aligned} \mathcal{T} &= -4\sqrt{2}\,G_{\mathrm{F}}(u_1^{\mathrm{T}}\mathcal{C}P_{\mathrm{L}}v_2')(\overline{u}_3'P_{\mathrm{R}}\mathcal{C}\overline{u}_1'^{\mathrm{T}}) \\ &= -4\sqrt{2}\,G_{\mathrm{F}}(\overline{v}_1P_{\mathrm{L}}v_2')(\overline{u}_3'P_{\mathrm{R}}v_1') \ . \end{aligned} \tag{88.38}$$

Taking the complex conjugate, and using $\overline{P_{\mathrm{L}}} = P_{\mathrm{R}}$, we find

$$\langle|\mathcal{T}|^2\rangle = 64G_{\mathrm{F}}^2(p_1p_2')(p_1'p_3') \ . \tag{88.39}$$

Multiplying eqs. (88.38) and (88.39), summing over final spins and averaging over the initial spin, we get

$$\begin{aligned}\langle|\mathcal{T}|^2\rangle = \tfrac{1}{2}(4\sqrt{2})^2 G_{\mathrm{F}}^2\, \mathrm{Tr}[(-\not{p}_1-m_\mu)P_{\mathrm{L}}(-\not{p}'_2)P_{\mathrm{R}}] \\ \times\, \mathrm{Tr}[(-\not{p}'_3+m_e)P_{\mathrm{R}}(-\not{p}'_1)P_{\mathrm{L}}]\ .\end{aligned} \tag{88.40}$$

The traces are easily evaluated, with the result

$$\langle|\mathcal{T}|^2\rangle = 64G_{\mathrm{F}}^2(p_1p'_2)(p'_1p'_3)\ . \tag{88.41}$$

We get the decay rate Γ by multiplying $\langle|\mathcal{T}|^2\rangle$ by $d\mathrm{LIPS}_3(p_1)$ and integrating over $p'_{1,2,3}$.

We can work out the result (in the limit $m_e \ll m_\mu$),

$$\Gamma = \frac{G_{\mathrm{F}}^2 m_\mu^5}{192\pi^3}\ . \tag{88.42}$$

After including one-loop corrections from electromagnetism, and accounting for the nonzero electron mass, the measured muon decay rate is used to determine the value of G_F, with the result $G_F = 1.166 \times 10^{-5}\ \mathrm{GeV}^{-2}$

89 The Standard Model: Quark Sector

Quarks are spin-one-half particles that are triplets of the color group.

There are six different flavors of quark; see Table 1 in section 83.

The six flavors are naturally grouped into three families or generations: u and d, c and s, t and b.

Let us begin by describing a single quark family, the up and down quarks.

We introduce left-handed Weyl fields q, $\bar{u}$, and $\bar{d}$ in the representations (3, 2, + 1/6), (3, 1,–2/3), and (3, 1, + 1/3) of SU(3) x SU(2) x U(1).

Here the bar over the letter in the fields $\bar{u}$ and $\bar{d}$ *is part of the name of the field*, and does not denote any sort of conjugation.

The covariant derivatives of these fields are

$$(D_\mu q)_{\alpha i} = \partial_\mu q_{\alpha i} - ig_3 A^a_\mu (T^a_3)_\alpha{}^\beta q_{\beta i} - ig_2 A^a_\mu (T^a_2)_i{}^j q_{\beta j} \\ - ig_1(+\tfrac{1}{6})B_\mu q_{\alpha i} \,, \tag{89.1}$$

$$(D_\mu \bar{u})^\alpha = \partial_\mu \bar{u}^\alpha - ig_3 A^a_\mu (T^{\bar{a}}_3)^\alpha{}_\beta \bar{u}^\beta - ig_1(-\tfrac{2}{3})B_\mu \bar{u}^\alpha \,, \tag{89.2}$$

$$(D_\mu \bar{d})^\alpha = \partial_\mu \bar{d}^\alpha - ig_3 A^a_\mu (T^{\bar{a}}_3)^\alpha{}_\beta \bar{d}^\beta - ig_1(+\tfrac{1}{3})B_\mu \bar{d}^\alpha \,. \tag{89.3}$$

We rely on context to distinguish the SU(3) gauge fields from the SU(2) gauge fields.

The kinetic terms for q, $\bar{u}$, and $\bar{d}$ are

$$\mathcal{L}_{\rm kin} = iq^{\dagger\alpha i}\bar{\sigma}^\mu (D_\mu q)_{\alpha i} + i\bar{u}^\dagger_\alpha \bar{\sigma}^\mu (D_\mu \bar{u})^\alpha + i\bar{d}^\dagger_\alpha \bar{\sigma}^\mu (D_\mu \bar{d})^\alpha \tag{89.4}$$

The representation $(3, 2, +1/6) \oplus (3, 1, -2/3) \oplus (3, 1, +1/3)$ for the left-handed Weyl fields is complex; hence the gauge theory is chiral, and therefore parity violating.

We cannot write down a mass term involving q, $\bar{u}$, and/or $\bar{d}$ because there is no gauge-group singlet contained in any of the products of their representations.

But we are able to write down Yukawa couplings of the form

$$\mathcal{L}_{\text{Yuk}} = -y'\varepsilon^{ij}\varphi_i q_{\alpha j}\bar{d}^{\alpha} - y''\varphi^{\dagger i}q_{\alpha i}\bar{u}^{\alpha} + \text{h.c.}\,, \tag{89.5}$$

where φ is the Higgs field in the $(1, 2, -1/2)$ representation that we introduced in section 87, and y′ and y″ are the Yukawa coupling constants.

These gauge-invariant Yukawa couplings are possible because there are singlets on the right-hand sides of

$$(1, 2, -\tfrac{1}{2}) \otimes (3, 2, +\tfrac{1}{6}) \otimes (\bar{3}, 1, +\tfrac{1}{3}) = (1, 1, 0) \oplus \ldots\,, \tag{89.6}$$

$$(1, 2, +\tfrac{1}{2}) \otimes (3, 2, +\tfrac{1}{6}) \otimes (\bar{3}, 1, -\tfrac{2}{3}) = (1, 1, 0) \oplus \ldots\,. \tag{89.7}$$

There are no other gauge-invariant terms involving q, $\bar{u}$, or $\bar{d}$ that havemass dimension four or less.

Hence there are no other terms that we could add to $\mathcal{L}$ while preserving renormalizability.

In unitary gauge, we replace φ_1 with $1/\sqrt{2}\,(v + H)$, where H is the real scalar field representing the physical Higgs boson, and φ_2 with zero.

The Yukawa term becomes

$$\mathcal{L}_{\rm Yuk} = -\tfrac{1}{\sqrt{2}} y'(v+H) q_{\alpha 2} \bar{d}^{\alpha} - \tfrac{1}{\sqrt{2}} y''(v+H) q_{\alpha 1} \bar{u}^{\alpha} + \text{h.c.}\ . \tag{89.8}$$

It is now convenient to assign new names to the SU(2) components of q,

$$q = \begin{pmatrix} u \\ d \end{pmatrix} . \tag{89.9}$$

Then eq. (89.8) becomes

$$\begin{aligned} \mathcal{L}_{\rm Yuk} &= -\tfrac{1}{\sqrt{2}} y'(v+H)(d_\alpha \bar{d}^{\alpha} + \bar{d}^{\dagger}_{\alpha} d^{\dagger\alpha}) - \tfrac{1}{\sqrt{2}} y''(v+H)(u_\alpha \bar{u}^{\alpha} + \bar{u}^{\dagger}_{\alpha} u^{\dagger\alpha}) \\ &= -\tfrac{1}{\sqrt{2}} y'(v+H)\overline{\mathcal{D}}^{\alpha}\mathcal{D}_\alpha - \tfrac{1}{\sqrt{2}} y''(v+H)\overline{\mathcal{U}}^{\alpha}\mathcal{U}_\alpha\ , \end{aligned} \tag{89.10}$$

where we have defined Dirac fields for the down and up quarks,

$$\mathcal{D}_\alpha \equiv \begin{pmatrix} d_\alpha \\ \bar{d}^{\dagger}_{\alpha} \end{pmatrix}, \qquad \mathcal{U}_\alpha \equiv \begin{pmatrix} u_\alpha \\ \bar{u}^{\dagger}_{\alpha} \end{pmatrix}. \tag{89.11}$$

We see from eq. (89.10) that the up and down quarks have acquired masses

$$m_d = \frac{y' v}{\sqrt{2}}\ , \qquad m_u = \frac{y'' v}{\sqrt{2}}\ . \tag{89.12}$$

Now we return to eqs. (89.1–89.3), and express the covariant derivatives in the terms of the $W^{\pm}_{\mu}$, Z_{μ}, and A_{μ} fields.

From our results in section 88, we have

$$g_2 A^1_\mu T^1 + g_2 A^2_\mu T^2 = \frac{g_2}{\sqrt{2}} \begin{pmatrix} 0 & W^+_\mu \\ W^-_\mu & 0 \end{pmatrix}, \tag{89.13}$$

$$g_2 A^3_\mu T^3 + g_1 B_\mu Y = eQA_\mu + \frac{e}{s_{\rm W} c_{\rm W}}(T^3 - s^2_{\rm W} Q) Z_\mu \,, \tag{89.14}$$

where

$$Q = T^3 + Y \tag{89.15}$$

is the generator of electric charge.

Then, since

$$T^3 u = +\tfrac{1}{2} u \,, \quad T^3 d = -\tfrac{1}{2} d \,, \quad T^3 \bar{u} = 0 \,, \qquad T^3 \bar{d} = 0 \,, \tag{89.16}$$

$$Y u = +\tfrac{1}{6} u \,, \quad Y d = +\tfrac{1}{6} d \,, \quad Y \bar{u} = -\tfrac{2}{3} \bar{u} \,, \quad Y \bar{d} = +\tfrac{1}{3} \bar{d} \,, \tag{89.17}$$

we see from eq. (89.15) that

$$Q u = +\tfrac{2}{3} u \qquad Q d = -\tfrac{1}{3} d \,, \quad Q \bar{u} = -\tfrac{2}{3} \bar{u} \,, \quad Q \bar{d} = +\tfrac{1}{3} \bar{d} \,. \tag{89.18}$$

This is just the set of electric charge assignments that we expect for the up and down quarks.

In terms of the four-component fields, we have

$$(g_2 A^3_\mu T^3 + g_1 B_\mu Y)\mathcal{U} = \Big[+\tfrac{2}{3}eA_\mu + \tfrac{e}{s_W c_W}(+\tfrac{1}{2}P_L - \tfrac{2}{3}s^2_W)Z_\mu\Big]\mathcal{U}\ , \tag{89.19}$$

$$(g_2 A^3_\mu T^3 + g_1 B_\mu Y)\mathcal{D} = \Big[-\tfrac{1}{3}eA_\mu + \tfrac{e}{s_W c_W}(-\tfrac{1}{2}P_L + \tfrac{1}{3}s^2_W)Z_\mu\Big]\mathcal{D}\ , \tag{89.20}$$

Using eqs. (89.13) and (89.19–89.20) in eqs. (89.1–89.4), we find the couplings of the electroweak gauge fields to the quarks,

$$\mathcal{L}_{\rm int} = \tfrac{1}{\sqrt{2}}g_2 W^+_\mu J^{-\mu} + \tfrac{1}{\sqrt{2}}g_2 W^-_\mu J^{+\mu} + \tfrac{e}{s_W c_W}Z_\mu J^\mu_Z + eA_\mu J^\mu_{\rm EM}\ , \tag{89.21}$$

where we have defined the currents

$$J^{+\mu} \equiv \overline{\mathcal{D}}_L \gamma^\mu \mathcal{U}_L\ , \tag{89.22}$$

$$J^{-\mu} \equiv \overline{\mathcal{U}}_L \gamma^\mu \mathcal{D}_L\ , \tag{89.23}$$

$$J^\mu_Z \equiv J^\mu_3 - s^2_W J^\mu_{\rm EM}\ , \tag{89.24}$$

$$J^\mu_3 \equiv \tfrac{1}{2}\overline{\mathcal{U}}_L \gamma^\mu \mathcal{U}_L - \tfrac{1}{2}\overline{\mathcal{D}}_L \gamma^\mu \mathcal{D}_L\ , \tag{89.25}$$

$$J^\mu_{\rm EM} \equiv +\tfrac{2}{3}\overline{\mathcal{U}}\gamma^\mu\mathcal{U} - \tfrac{1}{3}\overline{\mathcal{D}}\gamma^\mu\mathcal{D}\ . \tag{89.26}$$

Having worked out the interactions of a single quark generation, we now examine what happens when there is more than one of them.

Let us consider the fields $q_{\alpha iI}$, $\bar{u}_I$, and $\bar{d}_I$, , where I = 1, 2, 3 is a generation index.

The kinetic term for all these fields is

$$\mathcal{L}_{\rm kin} = iq^{\dagger\alpha i I}\bar{\sigma}^{\mu}(D_{\mu})_{\alpha i}{}^{\beta j}q_{\beta j I} + i\bar{u}^{\dagger}_{\alpha I}\bar{\sigma}^{\mu}(D_{\mu})^{\alpha}{}_{\beta}\bar{u}^{\beta}_{I} + i\bar{d}^{\dagger}_{\alpha I}\bar{\sigma}^{\mu}(D_{\mu})^{\alpha}{}_{\beta}\bar{d}^{\beta}_{I} \ , \tag{89.27}$$

where the repeated generation index is summed.

The most general Yukawa term we can write down now reads

$$\mathcal{L}_{\rm Yuk} = -\varepsilon^{ij}\varphi_i q_{\alpha j I} y'_{IJ}\bar{d}^{\alpha}_{J} - \varphi^{\dagger i} q_{\alpha i I} y''_{IJ}\bar{u}^{\alpha}_{J} + \text{h.c.} \ , \tag{89.28}$$

where y'_{IJ} and y''_{IJ} are complex 3 x 3 matrices, and the generation indices are summed.

In unitary gauge, this becomes

$$\mathcal{L}_{\rm Yuk} = -\tfrac{1}{\sqrt{2}}(v+H)d_{\alpha I}y'_{IJ}\bar{d}^{\alpha}_{J} - \tfrac{1}{\sqrt{2}}(v+H)u_{\alpha I}y''_{IJ}\bar{u}^{\alpha}_{J} + \text{h.c.} \ . \tag{89.29}$$

We can make unitary transformations in generation space on the fields:

$$d_I \to D_{IJ}d_J,\ \bar{d}_I \to \bar{D}_{IJ}\bar{d}_J,\ u_I \to U_{IJ}u_J,\ \text{and}\ \bar{u}_I \to \bar{U}_{IJ}\bar{u}_J,$$

where U , D, $\bar{U}$ and $\bar{D}$ are independent unitary matrices.

The kinetic terms are unchanged (except for the couplings to the $W^{\pm}$, as we will discuss momentarily), and the Yukawa matrices y′ and y″ are replaced with $D^{\rm T}y'\bar{D}$ and $U^{\rm T}y''\bar{U}$.

the

We can choose D, $\bar{D}$, U, and $\bar{U}$ so that $D^{\mathrm{T}}y'\bar{D}$ and $U^{\mathrm{T}}y''\bar{U}$ are diagonal with positive real entries y'_I and y''_I .

The down quarks $\mathcal{D}_I$ then have masses $m_{d_I} = y'_I\, v/\sqrt{2}$, and the up quarks UI have masses $m_{u_I} = y''_I\, v/\sqrt{2}$.

In the neutral currents $J^\mu{}_3$ and $J^\mu{}_{EM}$, we simply add a generation index I to each field, and sum over it.

The charged currents are more complicated, however; they become

$$J^{+\mu} = \overline{\mathcal{D}}_{\mathrm{L}I}(V^\dagger)_{IJ}\gamma^\mu\mathcal{U}_{\mathrm{L}J}\ , \tag{89.30}$$

$$J^{-\mu} = \overline{\mathcal{U}}_{\mathrm{L}I}V_{IJ}\gamma^\mu\mathcal{D}_{\mathrm{L}K}\ , \tag{89.31}$$

where $V \equiv U^\dagger D$ is the *Cabibbo–Kobayashi–Maskawa matrix* (or *CKM matrix* for short).

Note that we did not have this complication in the lepton sector, because there we had only one Yukawa term.

A 3 x 3 unitary matrix has 9 real parameters.

However, we are still free to make the independent phase rotations $\mathcal{D}_I \rightarrow e^{i\alpha_I}\mathcal{D}_I$ and $\mathcal{U}_I \rightarrow e^{i\beta_I}\mathcal{U}_I$, as these leave the kinetic and mass terms invariant.

These phase changes allow us to make the first row and column of V_{IJ} real, eliminating 5 of the 9 parameters.

The remaining four can be chosen as θ_1 (the Cabibbo angle), θ_2, θ_3, and δ, where

$$V = \begin{pmatrix} c_1 & +s_1c_3 & +s_1s_3 \\ -s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta} & c_1c_2s_3 + s_2c_3e^{i\delta} \\ -s_1s_2 & c_1s_2c_3 + c_2s_3e^{i\delta} & c_1s_2s_3 - c_2c_3e^{i\delta} \end{pmatrix}, \qquad (89.32)$$

and $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$.

The measured values of these angles are $s_1 = 0.224$, $s_2 = 0.041$, $s_3 = 0.016$, and $\delta = 40^\circ$.

Note that the charged currents now have some terms with a phase factor $e^{i\delta}$, and some without.

Since the time-reversal operator T is antiunitary ($T^{-1}iT=-i$), the charged currents do not transform in a simple way under time reversal.

This implies that the charged current terms in $\mathcal{L}_{int}$ are is not time-reversal invariant; hence the electroweak interactions violate time-reversal symmetry.

Since CP T is always a good symmetry, time-reversal violation is equivalent to CP violation; ε is therefore sometimes called the CP violating phase.

At high energies, we can use our results to compute electroweak contributions to scattering amplitudes involving quarks.

This is because, at high energies, the SU(3) coupling g_3 is weak; we can, for example, reliably compute the decay rates of the $W^{\pm}$ and Z^0 into quarks, because $\alpha_3(M_Z) \equiv g^2_3(M_Z)/4\pi = 0.12$ is small enough to make QCD loop corrections a few-percent effect.

To understand low-energy processes such as neutron decay, we must first write the currents in terms of hadron fields.

We take this up in the next section.

For now, we simply note that the terms in the charged currents that involve only up and down quarks are

$$J^{+\mu} = c_1 \overline{\mathcal{D}}_{\mathrm{L}} \gamma^\mu \mathcal{U}_{\mathrm{L}} \ , \tag{89.33}$$

$$J^{-\mu} = c_1 \overline{\mathcal{U}}_{\mathrm{L}} \gamma^\mu \mathcal{D}_{\mathrm{L}} \ , \tag{89.34}$$

where c_1 is the cosine of the Cabibbo angle.

90 Electroweak Interactions of Hadrons

Now that we know how quarks couple to the electroweak gauge fields, we can use this information to obtain amplitudes for various processes involving hadrons.

We will focus on three of the most important: neutron decay, $n \to pe^-\bar{\nu}$; charged pion decay, $\pi^- \to \mu^-\bar{\nu}_\mu$; and neutral pion decay, $\pi^0 \to \gamma\gamma$.

Recall from section 83 the chiral lagrangian for pions and nucleons,

$$
\begin{aligned}
\mathcal{L} = &-\tfrac{1}{4}f_\pi^2\,\mathrm{Tr}\,\partial^\mu U^\dagger\partial_\mu U + v^3\,\mathrm{Tr}(MU + M^\dagger U^\dagger) \\
&+ i\overline{N}\partial\!\!\!/ N - m_N\overline{N}(U^\dagger P_\mathrm{L} + UP_\mathrm{R})N \\
&- \tfrac{1}{2}(g_\mathrm{A}{-}1)i\overline{N}\gamma^\mu(U\partial_\mu U^\dagger P_\mathrm{L} + U^\dagger\partial_\mu U P_\mathrm{R})N\ ,
\end{aligned}
\tag{90.1}
$$

where $U(x) = \exp[2i\pi^a(x)T^a/f_\pi]$, π^a is the pion field, N is the nucleon field, f_π is the pion decay constant, M is the quark mass matrix, v^3 is the value of the quark condensate, m_N is the nucleon mass, and g_A is the axial vector coupling.

The electroweak gauge group SU(2) x U(1) is a subgroup of the $SU(2)_L$ x $SU(2)_R$ x $U(1)_V$ flavor group that we have in the limit of zero quark mass.

It will prove convenient to go through the formal procedure of gauging the full flavor group, and only later identifying the electroweak subgroup.

We therefore define matrix-valued gauge fields $l_\mu(x)$ and $r_\mu(x)$ that transform as

$$
l_\mu \to L l_\mu L^\dagger + iL\partial_\mu L^\dagger\ , \tag{90.2}
$$

$$
r_\mu \to R r_\mu R^\dagger + iR\partial_\mu R^\dagger\ . \tag{90.3}
$$

Here L(x) and R(x) are 2 x 2 unitary matrices that correspond to a general $SU(2)_L$ x $SU(2)_R$ x $U(1)_V$ gauge transformation; we restrict the U(1) part of the transformation to the vector subgroup by requiring det L = det R and Tr l_μ = Tr r_μ.

The transformation rules for the pion and nucleon fields are

$$U \to LUR^\dagger \;, \tag{90.4}$$

$$N_\mathrm{L} \to LN_\mathrm{L} \;, \tag{90.5}$$

$$N_\mathrm{R} \to RN_\mathrm{R} \;, \tag{90.6}$$

where $N_L \equiv P_L N$ and $N_R \equiv P_R N$ are the left- and right-handed parts of the nucleon field.

We can make the chiral lagrangian gauge invariant (except for terms involving the quark masses) by replacing ordinary derivatives with appropriate covariant derivatives.

We determine the covariant derivative of each field by requiring it to transform in the same way as the field itself; for example, $D_\mu U \to L(D_\mu U)R^\dagger$.

We thus find

$$D_\mu U = \partial_\mu U - il_\mu U + iUr_\mu \;, \tag{90.7}$$

$$D_\mu U^\dagger = \partial_\mu U^\dagger + iU^\dagger l_\mu - ir_\mu U^\dagger \;, \tag{90.8}$$

$$D_\mu N_\mathrm{L} = (\partial_\mu - il_\mu)N_\mathrm{L} \;, \tag{90.9}$$

$$D_\mu N_\mathrm{R} = (\partial_\mu - ir_\mu)N_\mathrm{R} \;. \tag{90.10}$$

Making the substitution $\partial \to D$ in $\mathcal{L}$, we learn how the pions and nucleons couple to these gauge fields.

As in section 83, it is more convenient to work with the nucleon field $\mathcal{N}$, defined via

$$N = (uP_{\rm L} + u^\dagger P_{\rm R})\mathcal{N} \ , \tag{90.11}$$

where $u^2 = U$.

Making this transformation, we ultimately find

$$\begin{aligned}\mathcal{L} = -\tfrac{1}{4}f_\pi^2\,{\rm Tr}(&\partial^\mu U^\dagger\partial_\mu U - il^\mu U\overleftrightarrow{\partial}_{\!\mu} U^\dagger - ir^\mu U^\dagger\overleftrightarrow{\partial}_{\!\mu} U \\ &+ l^\mu l_\mu + r^\mu r_\mu - 2l^\mu U r_\mu U^\dagger) \\ &+ v^3\,{\rm Tr}(MU + M^\dagger U^\dagger) + i\overline{\mathcal{N}}\partial\!\!\!/\mathcal{N} - m_N\overline{\mathcal{N}}\mathcal{N} \\ &+ \overline{\mathcal{N}}(v\!\!\!/ + \tfrac{1}{2}\tilde{l\!\!\!/} + \tfrac{1}{2}\tilde{r\!\!\!/}\,)\mathcal{N} - g_{\rm A}\overline{\mathcal{N}}(a\!\!\!/ + \tfrac{1}{2}\tilde{l\!\!\!/} - \tfrac{1}{2}\tilde{r\!\!\!/}\,)\gamma_5\mathcal{N} \ ,\end{aligned} \tag{90.12}$$

where

$$v_\mu \equiv \tfrac{1}{2}i[u^\dagger(\partial_\mu u) + u(\partial_\mu u^\dagger)] \ , \tag{90.13}$$

$$a_\mu \equiv \tfrac{1}{2}i[u^\dagger(\partial_\mu u) - u(\partial_\mu u^\dagger)] \ , \tag{90.14}$$

$$\tilde{l}_\mu \equiv u^\dagger l_\mu u \ , \tag{90.15}$$

$$\tilde{r}_\mu \equiv u r_\mu u^\dagger \ . \tag{90.16}$$

It is now convenient to set

$$l_\mu = l_\mu^a T^a + b_\mu \ , \tag{90.17}$$

$$r_\mu = r_\mu^a T^a + b_\mu \ . \tag{90.18}$$

We have normalized b_μ so that the corresponding charge is baryon number.

The SU(2) gauge fields of the Standard Model can now be identified as

$$g_2 A^a_\mu = l^a_\mu \ , \tag{90.19}$$

and the electromagnetic gauge field as

$$eA_\mu = l^3_\mu + r^3_\mu + \tfrac{1}{2} b_\mu \ . \tag{90.20}$$

Eq. (90.20) follows from reconciling eqs. (90.9) and (90.10) with the requirement that the electromagnetic covariant derivatives of the proton field $p = \mathcal{N}_1$ and the neutron field $n = \mathcal{N}_2$ be given by $(\partial_\mu - ieA_\mu)p$ and $\partial_\mu n$.

We can now find the hadronic parts of the currents that couple to the gauge fields by differentiating $\mathcal{L}$ with respect to them, and then setting them to zero.

We find

$$\begin{aligned} J^{a\mu}_{\rm L} &= (\partial\mathcal{L}/\partial l^a_\mu)\Big|_{l=r=0} \\ &= \tfrac{1}{4} i f_\pi^2 \,\mathrm{Tr}\, T^a U \overleftrightarrow{\partial^\mu} U^\dagger + \tfrac{1}{2}\overline{\mathcal{N}} u^\dagger T^a \gamma^\mu (1 - g_{\rm A}\gamma_5) u \mathcal{N} \\ &= +\tfrac{1}{2} f_\pi \partial^\mu \pi^a - \tfrac{1}{2}\varepsilon^{abc}\pi^b \partial^\mu \pi^c + \tfrac{1}{2}\overline{\mathcal{N}} T^a \gamma^\mu (1 - g_{\rm A}\gamma_5)\mathcal{N} + \ldots \end{aligned} \tag{90.21}$$

$$
\begin{aligned}
J_{\mathrm{R}}^{a\mu} &= (\partial\mathcal{L}/\partial r_\mu^a)\Big|_{l=r=0} \\
&= \tfrac14 i f_\pi^2 \,\mathrm{Tr}\, T^a U^\dagger \overleftrightarrow{\partial^\mu} U + \tfrac12 \overline{\mathcal{N}} u T^a \gamma^\mu (1+g_{\mathrm{A}}\gamma_5) u^\dagger \mathcal{N} \\
&= -\tfrac12 f_\pi \partial^\mu \pi^a - \tfrac12 \varepsilon^{abc}\pi^b \partial^\mu \pi^c + \tfrac12 \overline{\mathcal{N}} T^a \gamma^\mu (1+g_{\mathrm{A}}\gamma_5)\mathcal{N} + \ldots\ ,
\end{aligned}
\tag{90.22}
$$

$$
\begin{aligned}
J_{\mathrm{B}}^{\mu} &= (\partial\mathcal{L}/\partial b_\mu)\Big|_{l=r=0} \\
&= \overline{\mathcal{N}}\gamma^\mu \mathcal{N}\ .
\end{aligned}
\tag{90.23}
$$

In the third lines of eqs. (90.21) and (90.22), we have expanded in inverse powers of f_π.

We can now identify the currents that couple to the physical $W^\pm_\mu$, Z_μ, and A_μ fields as

$$
\begin{aligned}
J^{+\mu} &= c_1 (J_{\mathrm{L}}^{1\mu} - i J_{\mathrm{L}}^{2\mu}) \\
&= \tfrac{1}{\sqrt2} c_1 (f_\pi \partial^\mu \pi^+ + i\pi^0 \overleftrightarrow{\partial_\mu} \pi^+) + \tfrac12 c_1 \overline{n}\gamma^\mu (1-g_{\mathrm{A}}\gamma_5) p + \ldots\ ,
\end{aligned}
\tag{90.24}
$$

$$
\begin{aligned}
J^{-\mu} &= c_1 (J_{\mathrm{L}}^{1\mu} + i J_{\mathrm{L}}^{2\mu}) \\
&= \tfrac{1}{\sqrt2} c_1 (f_\pi \partial^\mu \pi^- - i\pi^0 \overleftrightarrow{\partial^\mu} \pi^-) + \tfrac12 c_1 \overline{p}\gamma^\mu (1-g_{\mathrm{A}}\gamma_5) n + \ldots\ ,
\end{aligned}
\tag{90.25}
$$

$$
J_{\mathrm{Z}}^{\mu} = J_3^\mu - s_{\mathrm{W}}^2 J_{\mathrm{EM}}^\mu\ ,
\tag{90.26}
$$

$$
\begin{aligned}
J_3^{\mu} &= J_{\mathrm{L}}^{3\mu} \\
&= \tfrac12 (f_\pi \partial^\mu \pi^0 + i\pi^+ \overleftrightarrow{\partial^\mu} \pi^-) \\
&\quad + \tfrac14 \overline{p}\gamma^\mu (1-g_{\mathrm{A}}\gamma_5) p - \tfrac14 \overline{n}\gamma^\mu (1-g_{\mathrm{A}}\gamma_5) n + \ldots\ ,
\end{aligned}
\tag{90.27}
$$

$$
\begin{aligned}
J^{\mu}_{\mathrm{EM}} &= J^{3\mu}_{\mathrm{L}} + J^{3\mu}_{\mathrm{R}} + \tfrac{1}{2} J^{\mu}_{\mathrm{B}} \\
&= i\pi^{+} \overleftrightarrow{\partial^{\mu}} \pi^{-} + \overline{p}\gamma^{\mu} p + \ldots \;,
\end{aligned}
\tag{90.28}
$$

where c_1 is the cosine of the Cabibbo angle, and the interactions are specified by

$$
\mathcal{L}_{\mathrm{int}} = \tfrac{1}{\sqrt{2}} g_2 W^{+}_{\mu} J^{-\mu} + \tfrac{1}{\sqrt{2}} g_2 W^{-}_{\mu} J^{+\mu} + \frac{e}{s_{\mathrm{W}} c_{\mathrm{W}}} Z_{\mu} J^{\mu}_{\mathrm{Z}} + e A_{\mu} J^{\mu}_{\mathrm{EM}} \,. \tag{90.29}
$$

For low-energy processes involving $W^{\pm}$ or Z^0 exchange, we can use the effective current-current interaction that we derived in section 88,

$$
\mathcal{L}_{\mathrm{eff}} = 2\sqrt{2}\, G_{\mathrm{F}} (J^{+\mu} J^{-}_{\mu} + J^{\mu}_{\mathrm{Z}} J_{\mathrm{Z}\mu}) \,. \tag{90.30}
$$

We should include both hadronic and leptonic contributions to the currents.

Consider charged pion decay, $\pi^{-} \to \mu^{-} \overline{\nu}_{\mu}$.

The relevant terms in the charged currents (neutral currents do not contribute) are

$$
J^{-\mu} = \tfrac{1}{\sqrt{2}} c_1 f_{\pi} \partial^{\mu} \pi^{-} \;, \tag{90.31}
$$

$$
J^{+\mu} = \tfrac{1}{2} \overline{\mathcal{M}} \gamma^{\mu} (1 - \gamma_5) \mathcal{N}_m \;, \tag{90.32}
$$

where $\mathcal{M}$ is the muon field and $\mathcal{N}_{\mathrm{m}}$ is the muon neutrino field.

The relevant term in the effective interaction is then

$$\mathcal{L}_{\text{eff}} = G_{\text{F}} c_1 f_\pi \partial^\mu \pi^- \overline{\mathcal{M}} \gamma_\mu (1-\gamma_5) \mathcal{N}_m \; . \tag{90.33}$$

The corresponding decay amplitude is

$$\mathcal{T} = G_{\text{F}} c_1 f_\pi k^\mu \, \overline{u}_1 \gamma_\mu (1-\gamma_5) v_2 \; , \tag{90.34}$$

where the four-momenta of the pion, muon, and antineutrino are k, p_1, and p_2.

Eq. (90.34) can be simplified by using $\not{k} = \not{p}_1 + \not{p}_2$ along with $\overline{u}_1 \not{p}_1 = -m_\mu \overline{u}_1$ and $\not{p}_2 v_2 = 0$; we get

$$\mathcal{T} = -G_{\text{F}} c_1 f_\pi m_\mu \overline{u}_1 (1-\gamma_5) v_2 \; . \tag{90.35}$$

We see that $\mathcal{T}$ is proportional to the muon mass; since $m_\mu \gg m_e$, decay to $\mu^- \overline{\nu}_\mu$ is preferred over decay to $e^- \overline{\nu}_e$.

Squaring $\mathcal{T}$ and summing over final spins, we find

$$\begin{aligned} \langle |\mathcal{T}|^2 \rangle &= (G_{\text{F}} c_1 f_\pi m_\mu)^2 (-8 p_1 \cdot p_2) \\ &= 4 (G_{\text{F}} c_1 f_\pi m_\mu)^2 (m_\pi^2 - m_\mu^2) \; . \end{aligned} \tag{90.36}$$

We used $-2p_1 \cdot p_2 = p^2_1 + p^2_2 - (p_1+p_2)^2 = -m^2_\mu + 0 + m^2_\pi$ to get the second line.

We now have

$$\begin{aligned}\Gamma &= \frac{1}{2m_\pi}\int d\text{LIPS}_2(k)\langle|\mathcal{T}|^2\rangle \\ &= \frac{|\mathbf{p}_1|}{8\pi m_\pi^2}\,\langle|\mathcal{T}|^2\rangle \\ &= \frac{G_\text{F}^2 c_1^2 f_\pi^2 m_\mu^2 m_\pi}{4\pi}\left(1-\frac{m_\mu^2}{m_\pi^2}\right)^2 ,\end{aligned} \tag{90.37}$$

where we used $|\mathbf{p}_1| = (m^2_\pi - m^2_\mu)/2m_\pi$ to get the last line.

Since we determine the value of GF from the decay rate of the muon (see section 88), the charged pion decay rate allows us to fix the value of $c_1 f_\pi$.

The value of c_1 can be determined from the rate for the decay process $\pi^- \to \pi^0 e^- \bar{\nu}_e,$ which is straightforwardly calculated.

The relevant hadronic term in the charged current is

$$J^{-\mu} = -\tfrac{1}{\sqrt{2}} i c_1 \pi^0 \overleftrightarrow{\partial^\mu} \pi^- \ , \tag{90.38}$$

which depends on c_1 but not f_π.

Comparison with experiment then yields $c_1 = 0.974$.

We note that the key feature of eq. (90.38) is that it involves spin-zero hadrons that are members of an isospin triplet; eq. (90.38) applies to any such hadrons, including nuclei.

Thus c_1 can also be measured in superallowed Fermi decays, which take a nucleus from one spin-zero state to another spin-zero state with the same parity in the same isospin multiplet.

Having thus determined c_1, the charged pion decay rate yields $f_\pi = 92.4$ MeV.

Next we consider neutron decay, $n \to pe^-\overline{\nu}_e$.

The relevant terms in the charged currents (neutral currents do not contribute) are

$$J^{-\mu} = \tfrac{1}{2}c_1\overline{p}\gamma^\mu(1-g_A\gamma_5)n \; , \tag{90.39}$$

$$J^{+\mu} = \tfrac{1}{2}\overline{\mathcal{E}}\gamma^\mu(1-\gamma_5)\mathcal{N}_e \; , \tag{90.40}$$

where $\mathcal{E}$ is the electron field and $\mathcal{N}_e$ is the electron neutrino field.

The relevant term in the effective interaction is then

$$\mathcal{L}_{\text{eff}} = \tfrac{1}{\sqrt{2}}G_F c_1\overline{p}\gamma^\mu(1-g_A\gamma_5)n\overline{\mathcal{E}}\gamma_\mu(1-\gamma_5)\mathcal{N}_e \; . \tag{90.41}$$

Consider a neutron with four-momentum $p_n = (m_n, \mathbf{0})$, and spin up along the z axis; the decay amplitude is

$$\mathcal{T} = \tfrac{1}{\sqrt{2}}G_F c_1[\overline{u}_p\gamma^\mu(1-g_A\gamma_5)u_n][\overline{u}_e\gamma_\mu(1-\gamma_5)v_{\overline{\nu}}] \; , \tag{90.42}$$

where $u_n\overline{u}_n = \tfrac{1}{2}(1-\gamma_5 \not{z})(-\not{p}_n+m_n)$.

We take the absolute square of $\mathcal{T}$ and sum over the final spins.

Since the maximum available kinetic energy is $m_n - m_p - m_e = 0.782$ MeV $\ll m_p$, the proton is nonrelativistic, and we can use the approximations $p_p \cdot p_e \simeq -m_p E_e$ and $p_p \cdot p_{\bar\nu} \simeq -m_p E_{\bar\nu}$ in addition to the exact formulae $p_n \cdot p_e = -m_n E_e$ and $p_n \cdot p_{\bar\nu} = -m_n E_{\bar\nu}$.

After a tedious but straightforward calculation, we find

$$\begin{aligned} \langle |\mathcal{T}|^2 \rangle &= 16\, G_F^2 c_1^2 (1 + 3g_A^2) m_n m_p E_e E_{\bar\nu} \\ &\quad \times \left[1 + a \frac{\mathbf{p}_e \cdot \mathbf{p}_{\bar\nu}}{E_e E_{\bar\nu}} + A \frac{\hat{\mathbf{z}} \cdot \mathbf{p}_e}{E_e} + B \frac{\hat{\mathbf{z}} \cdot \mathbf{p}_{\bar\nu}}{E_{\bar\nu}} \right], \end{aligned} \tag{90.43}$$

where the *correlation coefficients* are given by

$$a = \frac{1 - g_A^2}{1 + 3g_A^2}, \quad A = \frac{2g_A(1 - g_A)}{1 + 3g_A^2}, \quad B = \frac{2g_A(1 + g_A)}{1 + 3g_A^2}. \tag{90.44}$$

When we integrate over the final momenta to get the total decay rate, the correlation terms vanish, and so the rate is proportional to $G_F^2 c_1^2 (1 + 3g_A^2)$.

Since we get the value of $G_F^2 c_1^2$ from the rate for $\pi^- \to \pi^0 e^- \bar\nu_e$. the neutron decay rate allows us to determine $1 + 3g_A^2$.

To get the sign of g_A, we need a measurement of either A or B. (The antineutrino three-momentum can be determined from the electron and proton three-momenta.)

The result is that $g_A = +1.27$.

The measured values of the three correlation coefficients are all consistent with eq. (90.44).

Finally, we consider the decay of the neutral pion into two photons, $\pi^0 \to \gamma\gamma$.

None of the terms in our chiral lagrangian, eq. (90.12), couple a single π^0 to two photons.

Therefore, without adding more terms, this process does not occur at tree level.

However, at the one-loop level, we have the diagrams of fig. (90.1); a proton circulates in the loop.

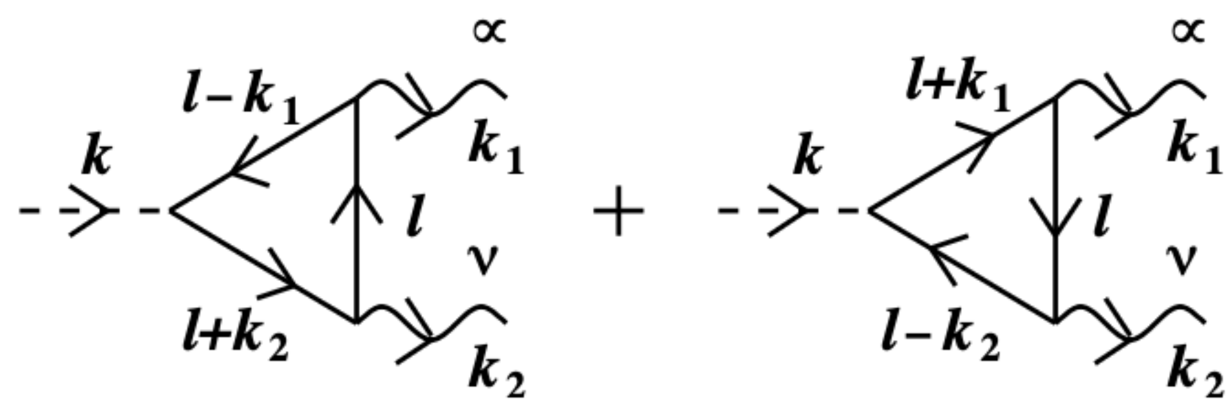


Figure 90.1: One-loop diagrams contributing to $\pi_0 \simeq \gamma\gamma$. The solid line is a proton.

Let us evaluate these diagrams.

In section 83, we found that the coupling of the π^0 to the nucleons is given by

$$\mathcal{L}_{\pi^0\overline{N}N} = -\tfrac{1}{2}(g_{\rm A}/f_\pi)\partial_\mu\pi^0(\overline{p}\gamma^\mu\gamma_5 p - \overline{n}\gamma^\mu\gamma_5 n) \; . \qquad (90.45)$$

This leads to a $\pi^0\overline{p}p$ vertex factor of $1/2\,(g_{\rm A}/f_\pi)k_\rho\gamma^\rho\gamma_5$.

The diagrams in fig. (90.1) are then identical to the diagrams we evaluated in section 76, and so the one-loop decay amplitude is

$$iT_{1-\text{loop}} = \tfrac{1}{2}(g_A/f_\pi)(ie)^2\varepsilon_{1\mu}\varepsilon_{2\nu}k_\rho C^{\mu\nu\rho}(k_1,k_2,k)\ , \tag{90.46}$$

where

$$k_\rho C^{\mu\nu\rho}(k_1,k_2,k) = -\frac{i}{2\pi^2}\varepsilon^{\mu\nu\alpha\beta}k_{1\alpha}k_{2\beta}\ . \tag{90.47}$$

Here we have chosen to renormalize so as to have $k_{1\mu}C^{\mu\upsilon\rho}(k_1, k_2, k) = 0$ and $k_{2\upsilon}C^{\mu\upsilon\rho}(k_1, k_2, k) = 0$; this is required by electromagnetic gauge invariance.

Combining eqs. (90.46) and (90.47), we get

$$T_{1-\text{loop}} = -\frac{g_A e^2}{4\pi^2 f_\pi}\,\varepsilon^{\alpha\mu\beta\nu}k_{1\alpha}\varepsilon_{1\mu}k_{2\beta}\varepsilon_{2\nu}\ . \tag{90.48}$$

This result is subject to higher-loop corrections.

Note that diagrams with extra internal pion lines attached to the nucleon loop are not suppressed by any small expansion parameter.

Thus we cannot trust the overall coefficient in eq. (90.48).

Note that this amplitude would arise at tree level from an interaction of the form $\mathcal{L}_{\pi 0\gamma\gamma} \propto \pi^0\varepsilon^{\alpha\mu\beta\upsilon}F_{\mu\alpha}F_{\upsilon\beta}$.

If we integrate out the nucleon fields to get an effective lagrangian for the pions and photons alone, such a term should appear.

There is a problem, however.

The $SU(2)_L \times SU(2)_R \times U(1)_V$ symmetry of the effective lagrangian implies that a pion field that has no derivatives acting on it must be accompanied by at least one factor of a quark mass.

For example, we could have $\mathcal{L}_{\pi^0\gamma\gamma} \propto i\,\mathrm{Tr}(MU - M^\dagger U^\dagger)\varepsilon^{\alpha\mu\beta\nu}F_{\mu\alpha}F_{\nu\beta}$.

The problem is that there are no quark-mass factors in eq. (90.48).

So we have an apparent contradiction between our explicit one-loop result, and a general argument based on symmetry.

This contradiction is resolved by noting that the electromagnetic gauge field results in an anomaly in the axial current $J^{3\mu}{}_A \equiv J^{3\mu}{}_L - J^{3\mu}{}_R$.

In terms of the quark doublet

$$\mathcal{Q} = \begin{pmatrix} \mathcal{U} \\ \mathcal{D} \end{pmatrix}, \tag{90.49}$$

this current is

$$\begin{aligned} J^{3\mu}_{\mathrm{A}} &= \overline{\mathcal{Q}}T^3\gamma^\mu\gamma_5\mathcal{Q} \\ &= \tfrac{1}{2}\overline{\mathcal{U}}\gamma^\mu\gamma_5\mathcal{U} - \tfrac{1}{2}\overline{\mathcal{D}}\gamma^\mu\gamma_5\mathcal{D}\,, \end{aligned} \tag{90.50}$$

where we have suppressed the color indices.

Using our results in sections 76 and 77, the anomalous divergence of this current is given by

$$\partial_\mu J_{\rm A}^{3\mu} = -\frac{e^2}{16\pi^2}\,{\rm Tr}(T^3Q^2)\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\;, \tag{90.51}$$

where

$$Q = \begin{pmatrix} +\frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} \tag{90.52}$$

is the electric charge matrix acting on the quark fields, and the trace includes a factor of three for color; we thus have

$${\rm Tr}(T^3Q^2) = 3\Big(\tfrac{1}{2}(+\tfrac{2}{3})^2 - \tfrac{1}{2}(-\tfrac{1}{3})^2\Big) = +\tfrac{1}{2}\;, \tag{90.53}$$

and so

$$\partial_\mu J_{\rm A}^{3\mu} = -\frac{e^2}{32\pi^2}\,\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\;. \tag{90.54}$$

This formula is exact in the limit of zero quark mass.

Now using eqs. (90.21) and (90.22), we can write the axial current in terms of the pion fields as

$$\begin{aligned} J_{\rm A}^{3\mu} &\equiv J_{\rm L}^{3\mu} - J_{\rm R}^{3\mu} \\ &= f_\pi \partial^\mu \pi^0 + \ldots\;. \end{aligned} \tag{90.55}$$

(We do not include the nucleon contribution because we are considering the effective lagrangian for pions and photons after integrating out the nucleons.)

From eq. (90.55) we have $\partial_\mu J^{3\mu}{}_A = f_\pi\, \partial^2\pi^0 + \ldots$.

Combining this with eq. (90.54), we get

$$-\partial^2\pi^0 = \frac{e^2}{32\pi^2 f_\pi}\, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + O(f_\pi^{-2}) \,. \qquad (90.56)$$

This equation of motion would follow from an effective lagrangian that included an interaction term of the form

$$\mathcal{L}_{\pi^0\gamma\gamma} = \frac{e^2}{32\pi^2 f_\pi}\, \pi^0 \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \,. \qquad (90.57)$$

This interaction leads to a $\pi^0 \to \gamma\gamma$ decay amplitude of

$$\mathcal{T} = -\frac{e^2}{4\pi^2 f_\pi}\, \varepsilon^{\mu\nu\rho\sigma} k_{1\mu} \varepsilon_{1\nu} k_{2\rho} \varepsilon_{2\sigma} \,. \qquad (90.58)$$

This amplitude receives no higher-order corrections in e^2, but is subject to quark-mass corrections; these are suppressed by powers of $m^2_\pi/(4\pi f_\pi)^2$.

Comparing eq. (90.58) with eq. (90.48), we see the one-loop result (which receives unsuppressed corrections) is too large by a factor of $g_A = 1.27$.

Squaring $\mathcal{T}$, summing over final spins, integrating over dLIPS2(k), and multiplying by a symmetry factor of one half (because there are two identical particles in the final state), we ultimately find that the decay rate is

$$\Gamma = \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi^2} \,. \tag{90.59}$$

This prediction is in agreement with the experimental result, which has an uncertainty of about 7%.

91 Neutrino Masses

Recall from sections 88 and 89 that a single generation of quarks and leptons consists of left-handed Weyl fields $q_{\alpha i}$, $\bar{u}^\alpha$, $\bar{d}^\alpha$, ℓ_i, and $\bar{e}$ in the representations (3, 2, + 1/6), ($\bar{3}$, 1,–2/3), ($\bar{3}$, 1, + 1/3), (1, 2,–1/2), and (1, 1, +1) of the gauge group SU(3) x SU(2) x U(1).

The Higgs field is a complex scalar φ_i in the representation (1, 2,–1/2).

The Yukawa couplings among these fields that are allowed by the gauge symmetry are

$$\mathcal{L}_{\text{Yuk}} = -y\varepsilon^{ij}\varphi_i\ell_j\bar{e} - y'\varepsilon^{ij}\varphi_i q_{\alpha j}\bar{d}^\alpha - y''\varphi^{\dagger i} q_{\alpha i}\bar{u}^\alpha + \text{h.c.}\,. \tag{91.1}$$

After the Higgs field acquires its VEV, these three terms give masses to the electron, down quark, and up quark, respectively.

The neutrino remains massless.

Thus, massless neutrinos are a prediction of the Standard Model.

However, there is now good experimental evidence that the three neutrinos actually have small masses.

The data implies that mass of the heaviest neutrino is in the range from 0.04 eV to 0.5 eV.

To account for this, we must extend the Standard Model.

Let us continue to consider a single generation.

We introduce a new left-handed Weyl field $\bar{\nu}$ in the representation $(1, 1, 0)$; this field does not couple to the gauge fields at all, and its kinetic term is simply $i\bar{\nu}^\dagger \bar{\sigma}^\mu \partial_\mu \bar{\nu}$. (The bar over the ν in the field $\bar{\nu}$ *is part of the name of the field*, and does not denote any sort of conjugation.)

With this new field, we can introduce a new Yukawa coupling of the form

$$\mathcal{L}_{\nu\,\mathrm{Yuk}} = -\tilde{y}\varphi^{\dagger i}\ell_i\bar{\nu} + \mathrm{h.c.}\,. \tag{91.2}$$

In unitary gauge, this becomes

$$\mathcal{L}_{\nu\,\mathrm{Yuk}} = -\tfrac{1}{\sqrt{2}}\tilde{y}(v+H)(\nu\bar{\nu} + \bar{\nu}^\dagger\nu^\dagger)\,. \tag{91.3}$$

We see that the neutrino mass is $\tilde{m} = \tilde{y}v/\sqrt{2}$.

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If this was the end of the story, we would have no understanding of why the neutrino mass is so much less than the other first-generation quark and lepton masses; we would simply have to take $\tilde{y}$ much less than y, y′, and y″.

However, because $\tilde{y}$ is in a real representation of the gauge group, we are allowed by the gauge symmetry to add a mass term of the form

$$\mathcal{L}_{\bar{\nu}\,\mathrm{mass}} = -\tfrac{1}{2}M(\bar{\nu}\bar{\nu} + \bar{\nu}^\dagger\bar{\nu}^\dagger)\ . \tag{91.4}$$

Here M is an arbitrary mass parameter.

In particular, it could be quite large.

Adding eqs. (91.3) and (91.4), we find a mass matrix of the form

$$\mathcal{L}_{\nu\,\bar{\nu}\,\mathrm{mass}} = -\tfrac{1}{2}\,(\nu \quad \bar{\nu}) \begin{pmatrix} 0 & \tilde{m} \\ \tilde{m} & M \end{pmatrix} \begin{pmatrix} \nu \\ \bar{\nu} \end{pmatrix} + \text{h.c.}\ . \tag{91.5}$$

If we take M » $\tilde{m}$, then the eigenvalues of this mass matrix are M and - $\tilde{m}^2/M$. (The sign of the smaller eigenvalue can be absorbed into the phase of the corresponding eigenfield.)

Thus, if $\tilde{m}$ is of the order of the electron mass, then $\tilde{m}^2/M$ is less than 1 eV if M is greater than 10^3 GeV.

So $\tilde{y}$ can be of the same order as the other Yukawa couplings, provided M is large.

This is called the *seesaw mechanism* for getting small neutrino masses.

The eigenfield corresponding to the smaller eigenvalue is mostly ν, and the eigenfield corresponding to the larger eigenvalue is mostly $\bar{\nu}$.

Another way to get this result is to integrate out the heavy $\bar{\nu}$ field at the beginning of our analysis.

We get the leading term (in an expansion in inverse powers of M) by ignoring the kinetic energy of the $\bar{\nu}$ field, solving the equation of motion for it that follows from $\mathcal{L}_{\bar{\nu}\,\mathrm{mass}}$ + $L_{\nu\mathrm{Yuk}}$, and finally substituting the solution back into $\mathcal{L}_{\bar{\nu}\,\mathrm{mass}}$ + $L_{\nu\mathrm{Yuk}}$.

The result is

$$\begin{aligned} \mathcal{L}_{\nu\mathrm{Yuk+mass}} &= \frac{\tilde{y}^2}{2M}\Big[(\varphi^{\dagger i}\ell_i)(\varphi^{\dagger j}\ell_j) + \mathrm{h.c.}\Big] \\ &= -\tfrac{1}{2}m_\nu(\nu\nu + \nu^\dagger\nu^\dagger)(1 + H/v)^2 \ , \end{aligned} \tag{91.6}$$

where

$$m_\nu \equiv -\frac{\tilde{m}^2}{M} = -\frac{\tilde{y}^2 v^2}{2M} \ . \tag{91.7}$$

Again, we can absorb the minus sign in eq. (91.7) by making the field re-definition $\nu \rightarrow i\nu$.

The seesaw mechanism has a straightforward extension to three generations.

Let us consider the fields ℓ_{iI}, $\bar{e}_I$, and $\bar{\nu}_I$, where I = 1, 2, 3 is a generation index.

The most general Yukawa and mass terms we can write down now read

$$\mathcal{L}_{\text{Yuk+mass}} = -\varepsilon^{ij}\varphi_i \ell_{jI} y_{IJ} \bar{e}_J - \varphi^{\dagger i} \ell_{iI} \tilde{y}_{IJ} \bar{\nu}_J - \tfrac{1}{2} M_{IJ} \bar{\nu}_I \bar{\nu}_J + \text{h.c.}\ , \tag{91.8}$$

where y_{IJ} and $\tilde{y}_{IJ}$ are complex 3 → 3 matrices, M_{IJ} is a complex symmetric 3 x 3 matrix, and the generation indices are summed.

In unitary gauge, this becomes

$$\mathcal{L}_{\text{Yuk+mass}} = -\tfrac{1}{\sqrt{2}}(v+H) e_I y_{IJ} \bar{e}_J - \tfrac{1}{\sqrt{2}}(v+H) \nu_I \tilde{y}_{IJ} \bar{\nu}_J - \tfrac{1}{2} M_{IJ} \bar{\nu}_I \bar{\nu}_J + \text{h.c.}\,. \tag{91.9}$$

We can now integrate out the $\bar{\nu}_I$ fields; eq. (91.9) is then replaced with

$$\mathcal{L}_{\text{Yuk+mass}} = -\tfrac{1}{\sqrt{2}}(v+H) e_I y_{IJ} \bar{e}_J - \tfrac{1}{2}(m_\nu)_{IJ}(\nu_I \nu_J + \nu_I^\dagger \nu_J^\dagger)(1+H/v)^2\ , \tag{91.10}$$

where we have defined the complex symmetric neutrino mass matrix

$$(m_\nu)_{IJ} \equiv -\tfrac{1}{2} v^2 (\tilde{y}^{\mathrm{T}} M^{-1} \tilde{y})_{IJ}\ . \tag{91.11}$$

We can make unitary transformations in generation space on the fields: $e_I \to E_{IJ}\, e_J$,, $\bar{e}_I \to \bar{E}_{IJ} \bar{e}_J$,, and $\nu_I \to N_{IJ}\, \nu_J$, where E, $\bar{E}$, and N are independent unitary matrices.

The kinetic terms are unchanged (except for the couplings to the $W^\pm$, as we will discuss momentarily), and the matrices y and m_ν are replaced with $E^{\mathrm{T}} y \bar{E}$ and $N^{\mathrm{T}} m_\nu N$.

We can choose the unitary matrices E, $\bar{E}$, and N so that $E^{\mathrm{T}} y \bar{E}$ and $N^{\mathrm{T}} m_\nu N$. are diagonal with positive real entries y_I and m_{ν_I}.

The neutrinos N_I then have masses m_{ν_I} , and the charged leptons ε_I have masses m_{e_I} = $y_I v/\sqrt{2}$.

In the neutral currents $J^\mu{}_3$ and $J^\mu{}_{\mathrm{EM}}$, we simply add a generation index I to each field, and sum over it.

The charged currents are more complicated, however; they become

$$J^{+\mu} = \overline{\mathcal{E}}_{\mathrm{L}I}(X^\dagger)_{IJ}\gamma^\mu \mathcal{N}_{\mathrm{L}J} \,, \tag{91.12}$$

$$J^{-\mu} = \overline{\mathcal{N}}_{\mathrm{L}I} X_{IJ}\gamma^\mu \mathcal{E}_{\mathrm{L}K} \,, \tag{91.13}$$

where $X \equiv N^\dagger E$ is the analog in the lepton sector of the CKM matrix V in the quark sector. One difference, though, between X and V is that the phases of the Majorana N_I fields are fixed by the requirement that the neutrino masses are real and positive.

Thus we cannot change these phases to make the first column of X real, as we did with V .

We are allowed to change the phases of the Dirac ε_I fields, so we can make the first row of X real.

Thus X has 9 – 3 = 6 parameters, two more than the CKM matrix V.

The presence of X in the charged currents leads to the phenomenon of *neutrino oscillations*.

A neutrino that is produced by scattering an electron off a target will be a linear combination $X_{IJ}\nu_J$ of the neutrinos of definite mass.

The different mass eigenstates propagate at different speeds, and then (in a subsequent scattering) may become (if there is enough energy) muons or taus rather than electrons.

It is the observation of neutrino oscillations that led to belief that neutrinos have mass.

This has now been observed.

Neutrinos are significantly lighter than other known particles, with their masses being less than 0.45 eV/c^2 as determined by the KATRIN experiment.

The exact mass of each neutrino flavor is still unknown, but it is established that they are all much lighter than electrons.