Quantum States/Postulates - Real Meaning of Formalism

Now we set up the formal axiomatic structure of quantum theory

and along way review most of stuff that we have already talked about.

Hopefully, you understand QM after this pass!

Must redevelop theoretical machinery of quantum physics.

Must build on experimental insight of present discussions

=> valid statements about nature of quantum objects

(no matter how confusing that nature seems to be).

Where Are We Now?

Experiment —> reality slightly out of focus,

until specific experiment forces picture to sharpen into one possibility or another.

Is being "**out of focus**" only an issue of the theory

and not a reflection of what is actually happening?

Maybe future experiments will expose physical variables (not included so far)

-> advanced theory that resolves what appears to be paradoxical

-> called so-called Hidden Variable theories

Or, maybe current experiments are saying something **genuine** about nature of reality.

Problem then is our **expectations** about reality.

Our view of reality

-> Experience centered on macroworld (that we live in).

Rocks always follow single path when thrown.

Common sense understanding colors view of reality, irrespective of scale

No guarantee that such applies outside the macroworld.

Experiments with photons/electrons —> they are **not** tiny rocks.

Consequently, one of our challenges is:

how a rock (made up of microworld objects) can behave in “common sense” way,

given the underlying strangeness of the quantum world.

Need to completely specify scheme for describing our experimental results in consistent way

—> some measure of predictability.

Need to develop quantum theory from 1st principles that can apply everywhere.

Covered ideas earlier and now expand on those discussions

—> all aspects of full theory will be clearly delineated.

Describing Quantum Systems

A classical state **cannot** describe experiments in the quantum world.

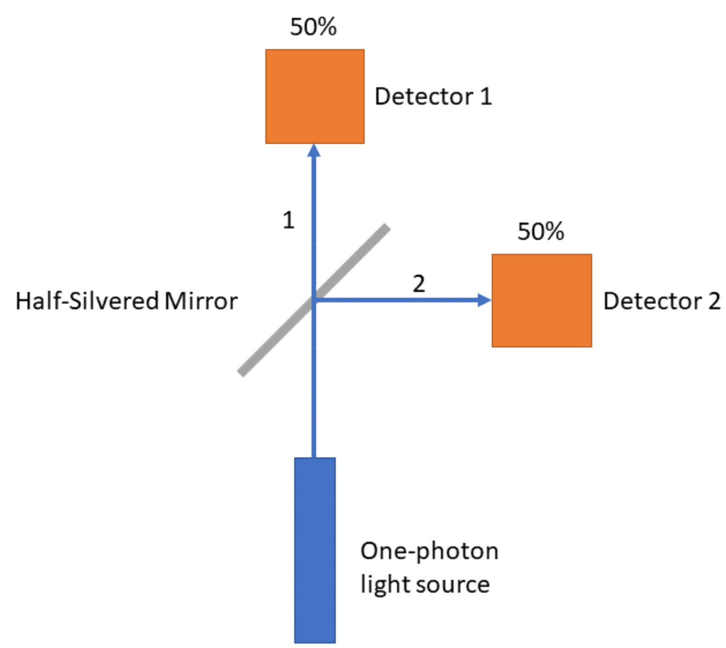
A classical state = **a list** —> values of various physical properties,
this cannot apply to a photon in an interference experiment
which is “apparently” traveling on 2 paths at same time. (“apparently” = “seems like”)

Any physical property determining the direction of electron through (S-G) experiment
is influenced by exact details of how experiment was set up(**the context**),
this runs completely contrary to classical idea that experiments
just reveal what is already there.

How can we construct a quantum state to replace the usual classical description of system.

It is important that certain basic characteristics are designed in from the beginning.

1. Inherent **randomness** must be represented,
i.e., description of photon arriving at half-silvered mirror
must allow photon to have equal chance of being transmitted
or reflected, without stating that it will definitely do one or the other.

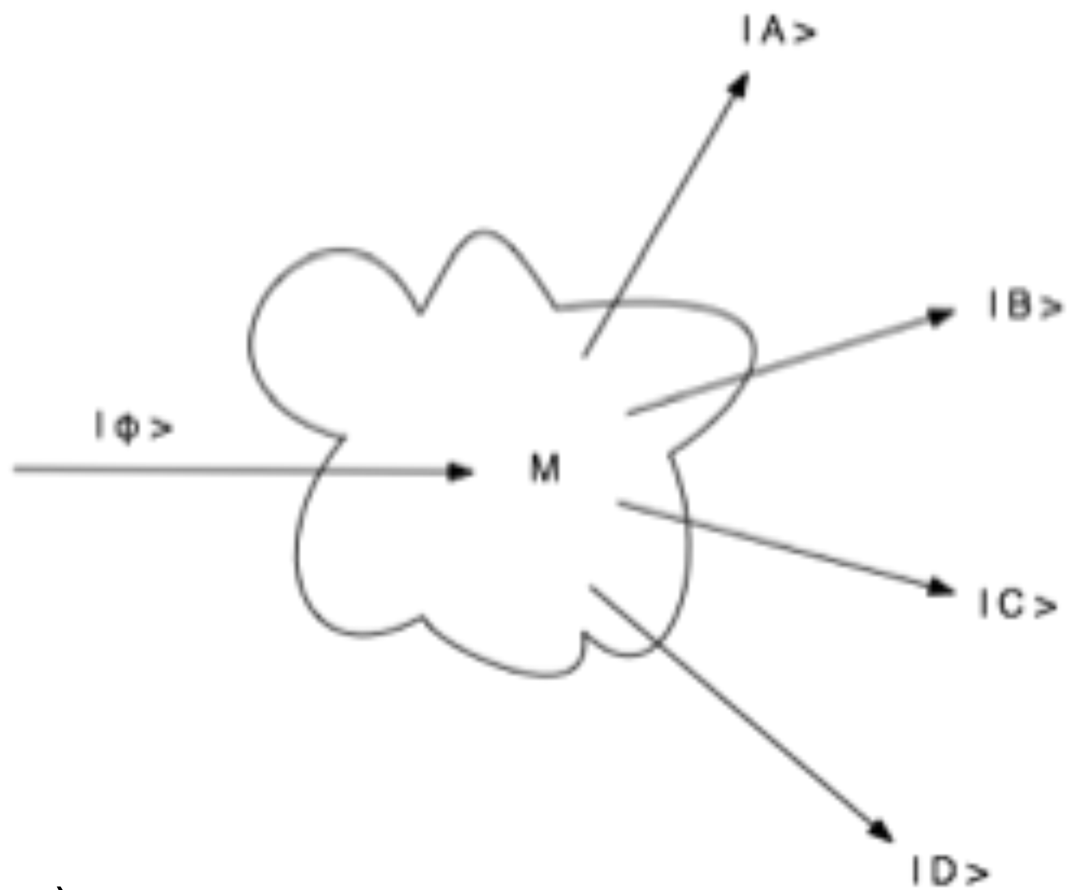


2. **Contextuality** of quantum behavior must be incorporated.
Experimental results are influenced by overall setup of devices.

If photon detector is placed beyond the half-silvered mirror, then
photon will either reflect or transmit. If no detector present, then
results -> that photon “explores” both possibilities.

3. Quantum systems exist in a mixed state that combines classical states in impossible ways
as we will see(have seen, e.g., simultaneously reflecting and transmitting at half-silvered
mirror), i.e., that combine classical states which are **incompatible** with each other!

Figure summarizes what needs to be achieved in quantum description.



LHS = system(electron),

in quantum state = $|\phi\rangle$ (a **ket**).

we are re-developing a mathematical language of QM

Electron interacts with measuring apparatus, M,

-> one of several possibilities can occur,

each with different probability.

Example:

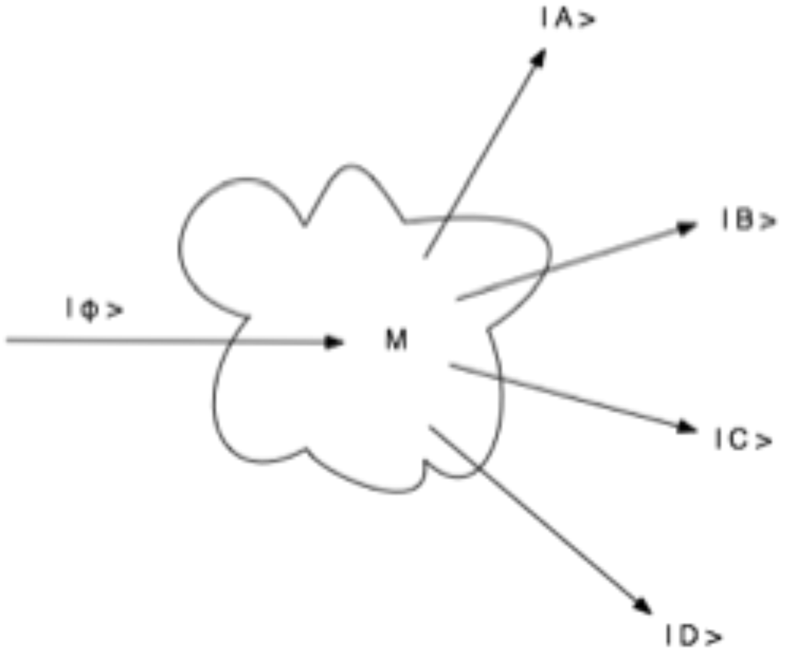
$|U\rangle$ state electron entering and interacting with a (LEFT,RIGHT) S-G magnet

-> emerges in $|L\rangle$ or $|R\rangle$ state.

We take most basic direct approach: No skipping dead ends……

It will enable us to learn important things as we proceed!

Assume the initial state takes the following form:



$$|\phi\rangle = p_1\,|A\rangle + p_2\,|B\rangle + p_3\,|C\rangle + p_4\,|D\rangle + \ldots\ldots\ldots$$

where numbers p_1,......,p_4,.... which somehow represent (do not know how yet) the probability that electron would end up in each state $|A\rangle$, $|B\rangle$, etc.

Not saying that numbers are probabilities; just related to probabilities(way to be determined).

Attractive formulation —> already catches some of the flavor of quantum behavior.

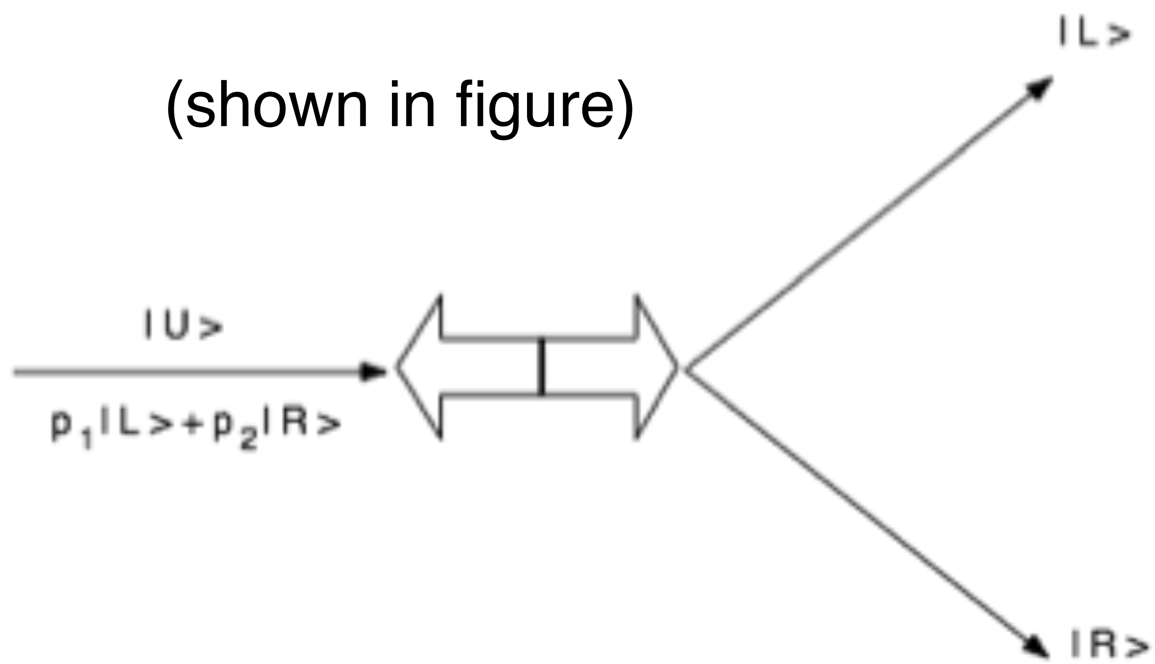
Seems to say that quantum state $|\phi\rangle$ at any time is made up of all possibilities $|A\rangle$, $|B\rangle$, etc, which may subsequently come about.

Example: S-G experiment, write

$$|U\rangle = p_1 \, |L\rangle + p_2 \, |R\rangle$$

to represent initial state of electron when approaching L-R magnet (next measurement)

(shown in figure)



After electron passed through magnet, it is no longer appropriate to describe it by state $|U\rangle$:

since now either in state $|L\rangle$ or $|R\rangle$.

Initial description **seems** to have “collapsed” (after measurement) into one(we observed) of two alternatives that it is “composed” of.

But REMEMBER other state ($|R\rangle$ or $|L\rangle$) is still somewhere(unobserved)

This way of expressing quantum states is **similar** to way probabilities are combined.

Imagine trying to calculate the average number of words per page in a book.

One way: count up number of words and divide by number of pages.

Equivalent way: group pages into sets (each page in set had same number of words).

Average then becomes

$$\text{average number of words} =$$

$$\frac{(\text{number pages}(700 \text{ words}))\text{x}700+(\text{number pages}(600 \text{ words}))\text{x}600+..}{\text{total number of pages}}$$

$$\text{average number of words} =$$

$$\frac{(\text{number pages}(700 \text{ words}))\text{x}700}{\text{total number of pages}} + \frac{(\text{number pages}(600 \text{ words}))\text{x}600}{\text{total number of pages}} + ..$$

$$\text{average number of words} = (\text{Probability of } 700 \text{ words})\text{x}700$$

$$+(\text{Probability of } 600 \text{ words})\text{x}600 + ..$$

Looks like formula that is used above when different final state possibilities are included.

$$|\phi\rangle = p_1 |A\rangle + p_2 |B\rangle + p_3 |C\rangle + p_4 |D\rangle + ...$$

Some standard probability rules are:

Given an event E_1 with probability P_1, event E_2 with probability P_2 and event E_3 with probability P_3, the **rules** of probability state

$$\text{Probability}(E_1 \; or \; E_2 \; or \; E_3) = P_1 + P_2 + P_3$$

If events correspond to measuring different values of physical property (e.g., V_1, V_2, and V_3), then average value of property after many trials is

$$\text{average value of } V = (V_1 \times P_1) + (V_2 \times P_2) + (V_3 \times P_3)$$

Again, that "looks like" the assumed form of the quantum mechanical state above.

If the two ideas **are** exactly the same, then terms p_1, p_2, p_3, etc in

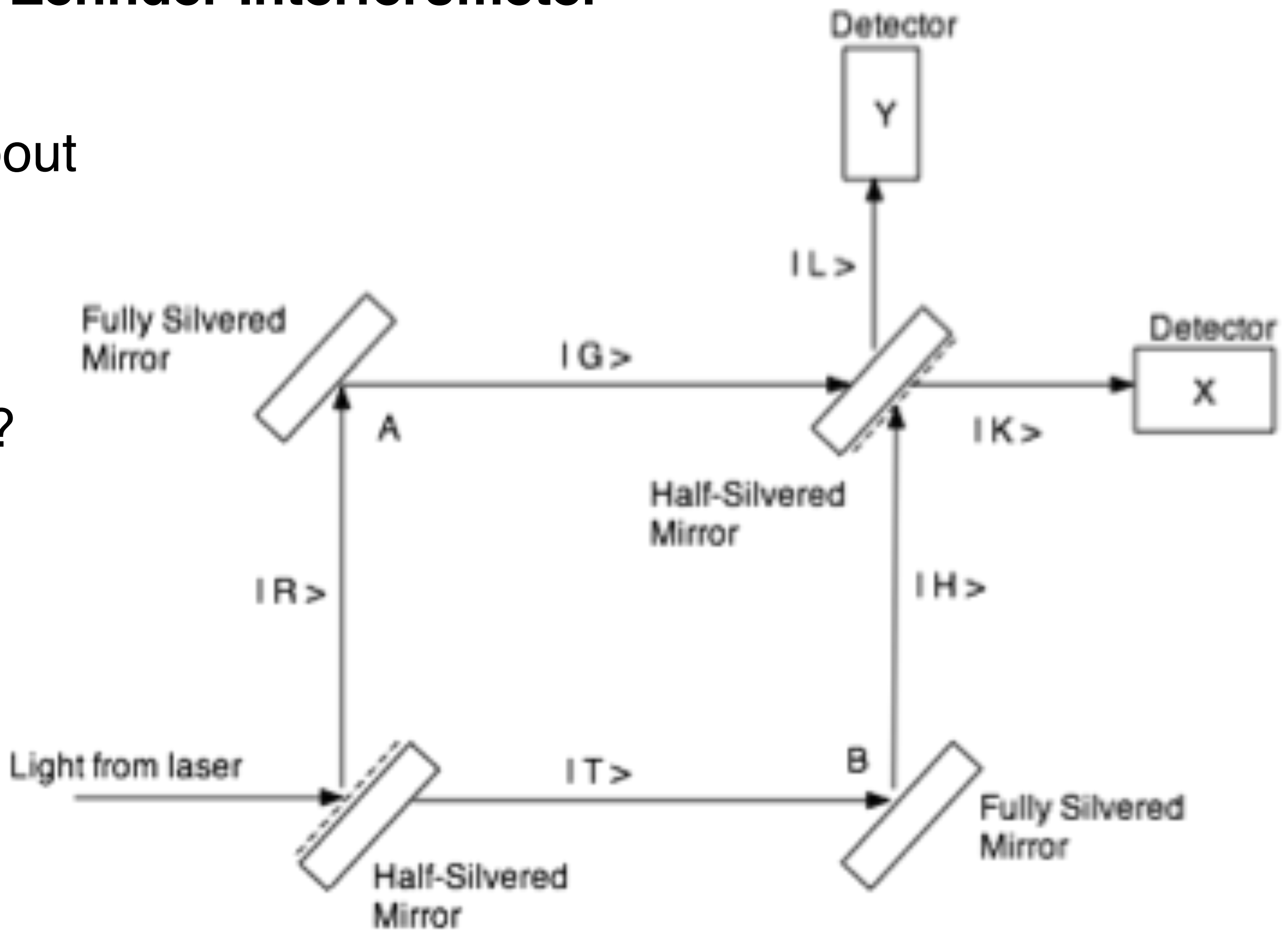
$$|\phi\rangle = p_1 |A\rangle + p_2 |B\rangle + p_3 |C\rangle + p_4 |D\rangle + ...$$

would have to be **actual** probabilities, but there is a problem with this idea.

Again, we let experiment show us what is wrong!

Very important to understand that Physics is based on experimental results!

Specific Example: Return to the Mach-Zehnder Interferometer



We have made an assumption(above) about

how to represent a quantum state.

Does it work in an experimental situation?

Choose a Mach-Zehnder interferometer.

Experiment **seems to need**

wave/particle descriptions of light

—-> will be a good test

of the quantum state assumption.

Assume intensity of laser beam turned down(make it a real QM experiment)

—-> only one photon(since 1980s) crossing entire experimental setup at any time.

At 1st half-silvered mirror, the single photon is **either** reflected **or** transmitted.

Our assumption (state = sum of things it could be) —> quantum state of arriving photon is

$$|\phi\rangle = a\,|T\rangle + b\,|R\rangle$$

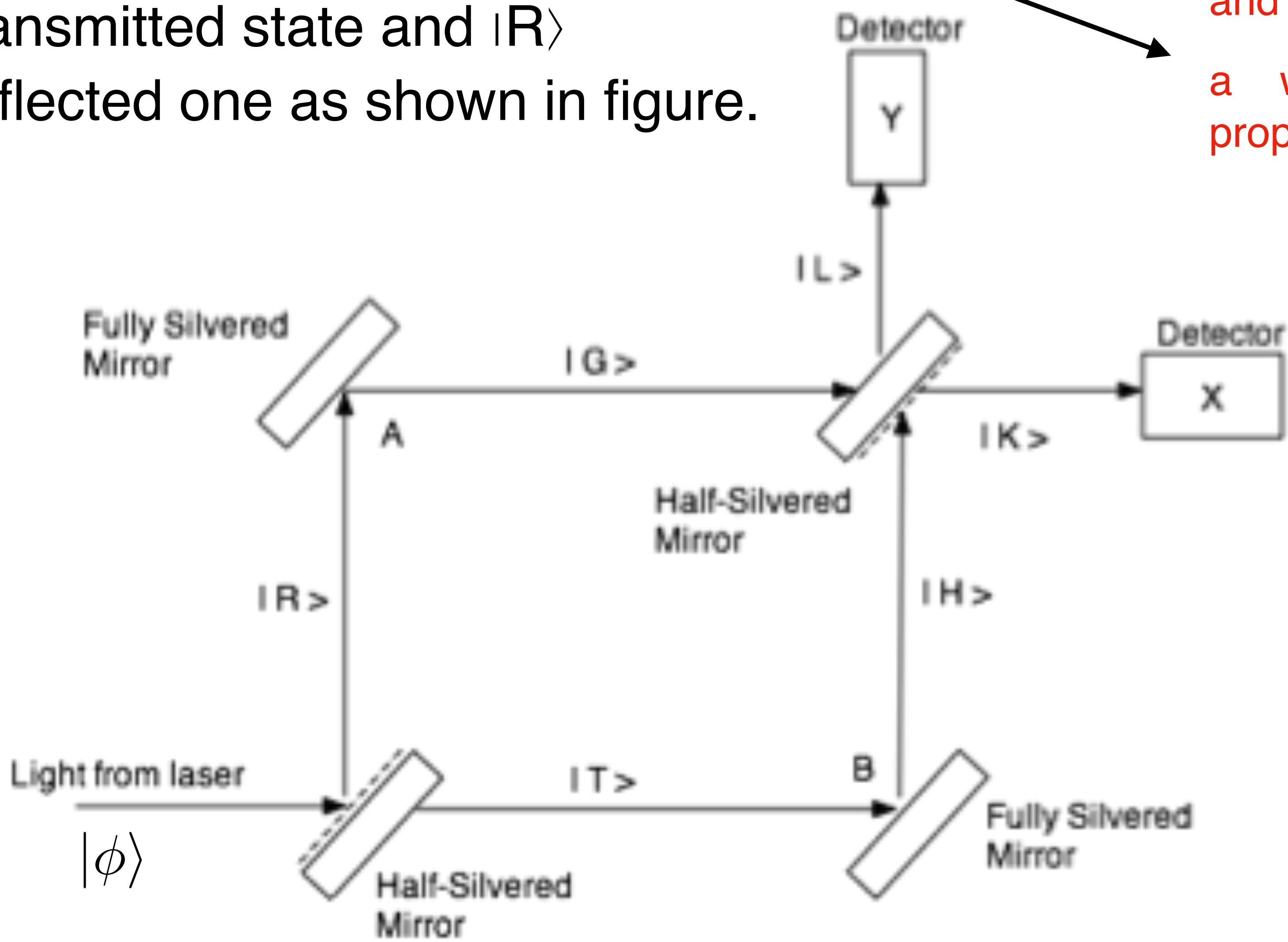
NOTE: No matter where mirror appears

b will always related to the **reflected** part property

and

a will is always related to the **transmitted** part property; real properties of the device!!

where $|T\rangle$ represents transmitted state and $|R\rangle$ reflected one as shown in figure.



Numbers a and b are related to probabilities that photon transmitted or reflected, respectively.

Numbers are determined by **construction** of mirror.

Part of state $|\phi\rangle$ designated $|R\rangle$ contains information about photons reflected at half-silvered mirror……..

Important: Unless we have measurement device on either arm of experiment, cannot say photon has either been transmitted or reflected

MUST MEASURE TO KNOW!! **That is th Quantum World!!**

Consequently,

we should not imply photon is going in any particular direction.

Alternative idea might say:

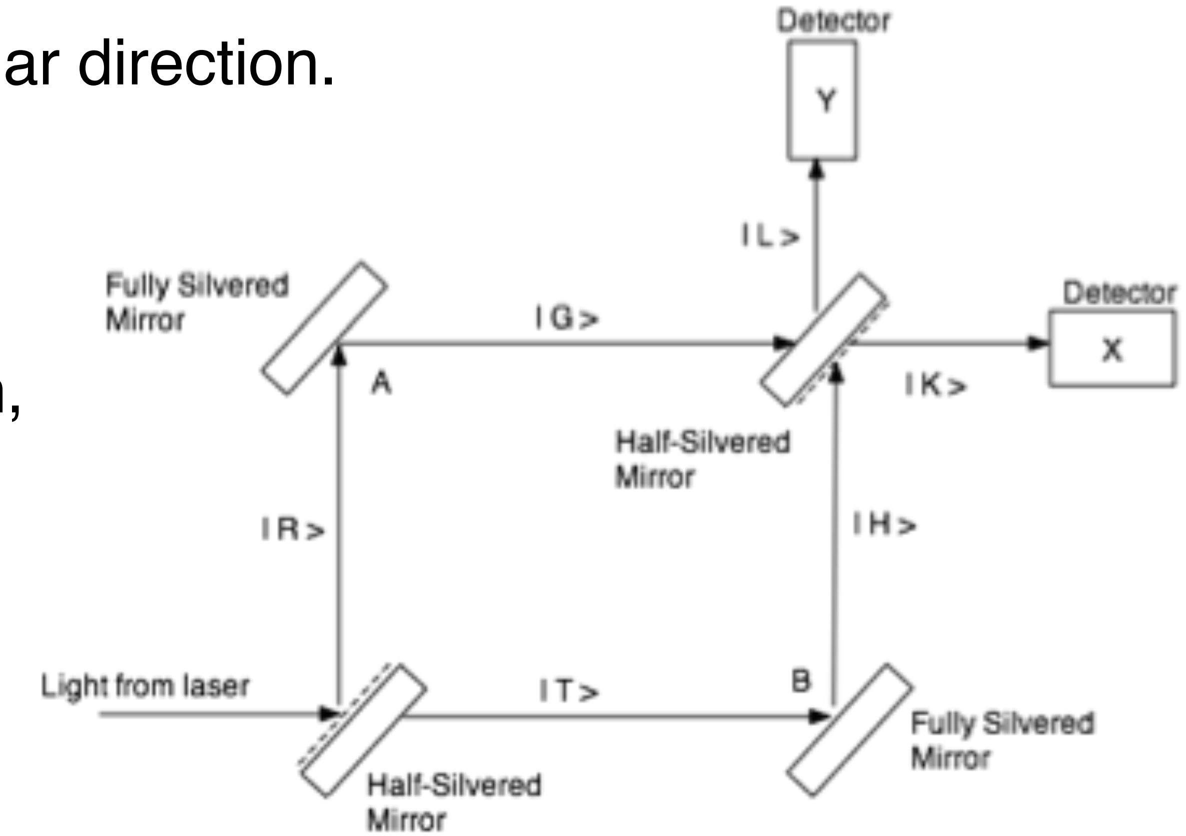
that reflected photon state moves along upper arm,

but states do not move - they are just math!

Settle on saying(is clearly awkward):

reflected photon property/information

meets fully silvered mirror at A



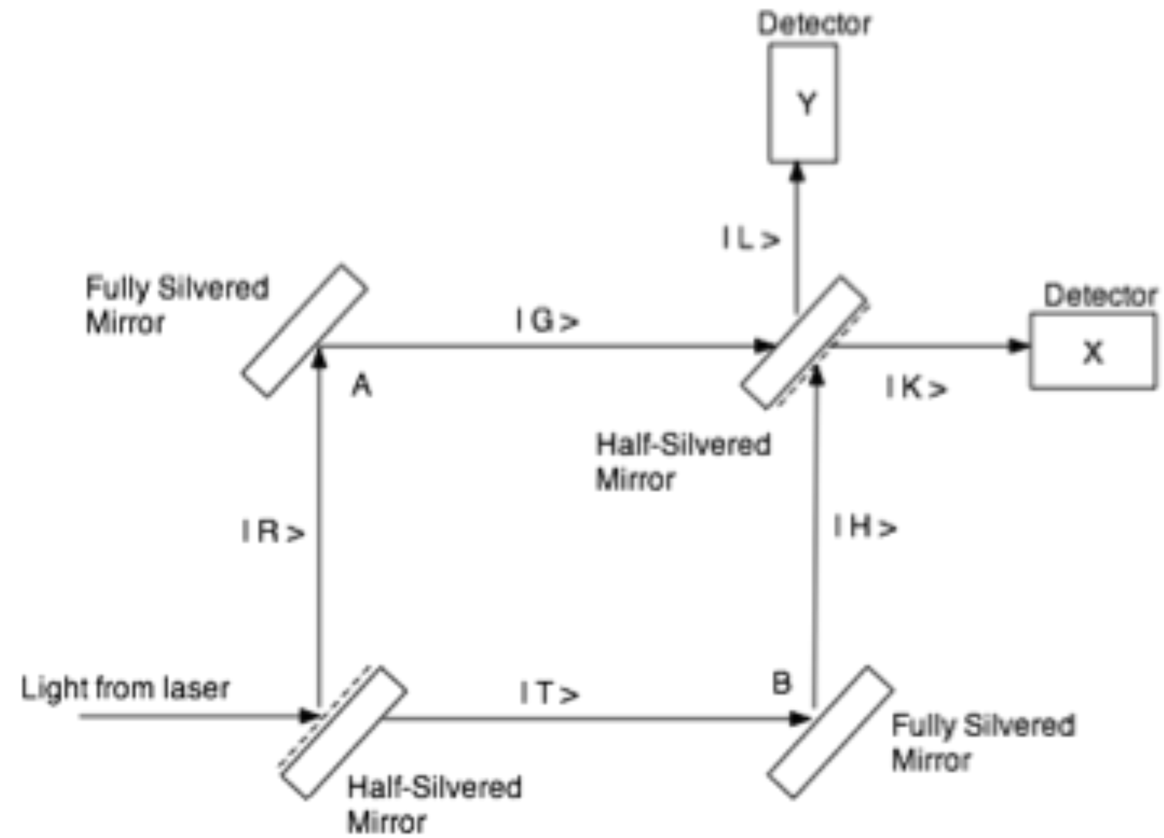
This illustrates is why we should only talk (use words) about actual measurement results and NOT at intermediate stages!!

Consequently, same arguments say the quantum state must change at mirror to $|G\rangle$.

At next half-silvered mirror, state becomes either one of transmission through or reflection up.

Figure shows:

Must be **careful** when constructing representation,



From point of view of state $|G\rangle$, $|K\rangle$ is the transmitted state and $|L\rangle$ is the reflected one.

So, have to write

$$|G\rangle = b\,|L\rangle + a\,|K\rangle$$

Used SAME FACTORS, a (transmission) and b (reflection), since 2nd mirror is identical to 1st half-silvered mirror and built the same way.

Meanwhile the other property of the photon is in state $|T\rangle$.

This property reaches mirror position B, where is reflected and changes into state $|H\rangle$.

However, our assumption now says must also write $|H\rangle$ in terms of $|K\rangle$ and $|L\rangle$

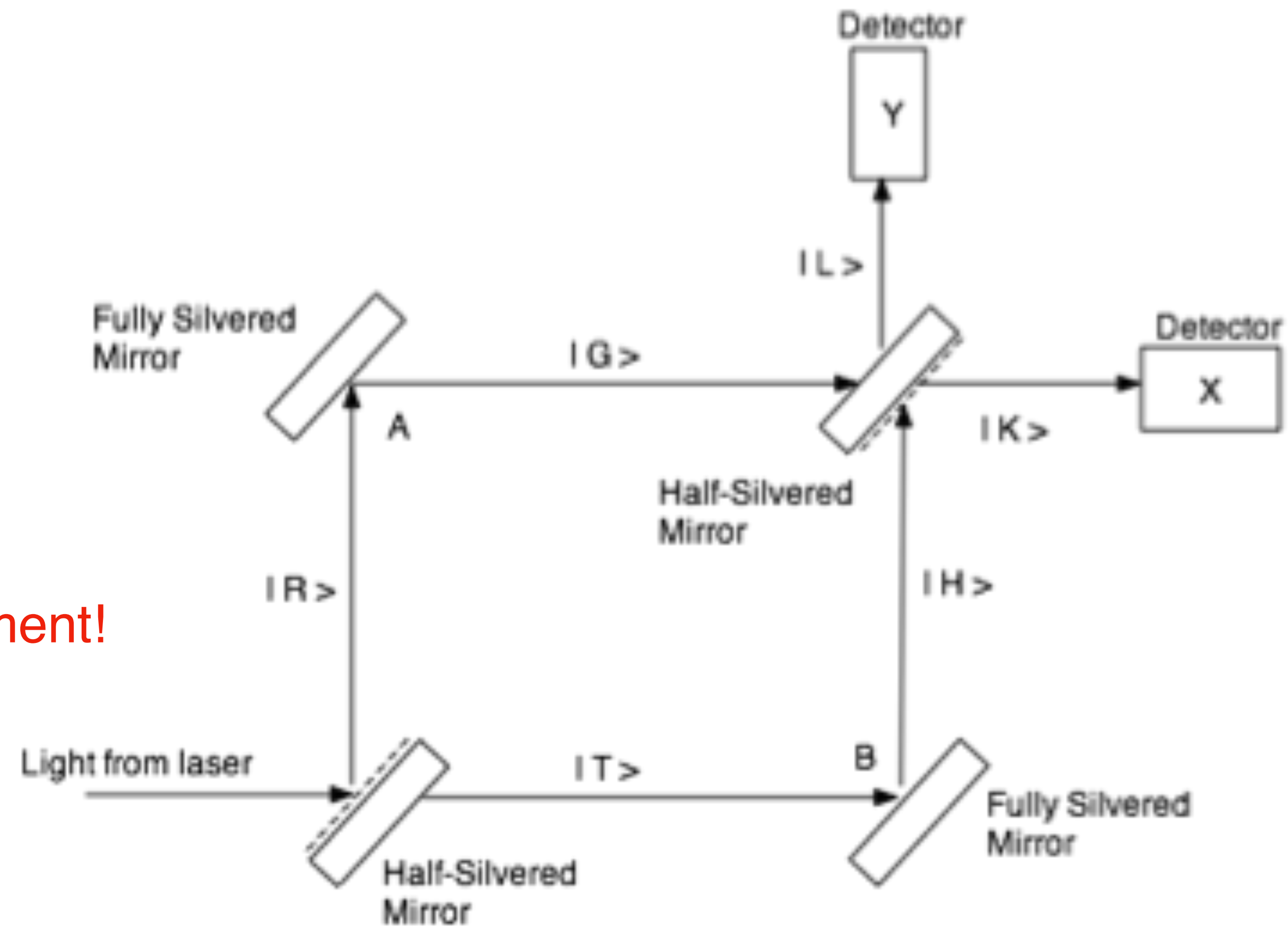
—> the possible outcomes when $|H\rangle$ interacts with the final half-silvered mirror.

Figure shows - be careful when constructing representation, from point of view of $|H\rangle$, $|L\rangle$ is transmitted state and $|K\rangle$ is reflected one.

So, have to write

$$|H\rangle = b\,|K\rangle + a\,|L\rangle$$

Have used same a and b in all cases



Now have $|H\rangle$ and $|G\rangle$ in terms of $|K\rangle$ and $|L\rangle$

—> can go back to original state $|\phi\rangle$

and write it in terms of $|K\rangle$ and $|L\rangle$.

what it can become after total experiment!

$$|\phi\rangle = a\,|T\rangle + b\,|R\rangle = a\,|H\rangle + b\,|G\rangle$$

$$= a\,(b\,|K\rangle + a\,|L\rangle) + b\,(b\,|L\rangle + a\,|K\rangle)$$

$$= ab\,|K\rangle + ab\,|K\rangle + a^2\,|L\rangle + b^2\,|L\rangle$$

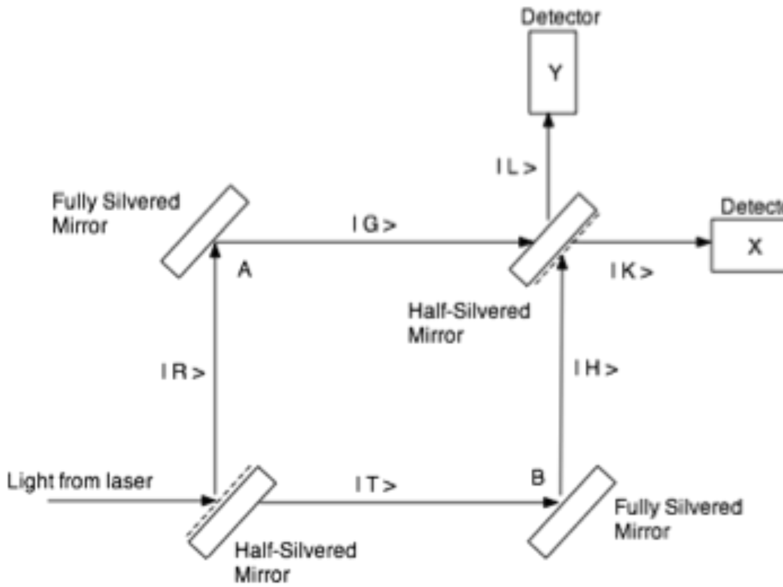
$$= 2ab\,|K\rangle + \left(a^2 + b^2\right)|L\rangle$$

This state represents effect of **entire** apparatus on the initial state within the context of our assumptions.

Have produced a representation of initial photon state $|\phi\rangle$

$$|\phi\rangle = 2ab\,|K\rangle + (a^2 + b^2)\,|L\rangle$$

in terms of two **possible final outcomes** $|K\rangle$ and $|L\rangle$.



Numbers 2ab and $(a^2 + b^2)$ must represent (in **some** way according to our assumption) probabilities that photon will be detected at X and Y, respectively, i.e., that it will end up with state $|K\rangle$ or state $|L\rangle$ for single photon.

Now, the **actual** experimental results are as follows:

If distances along two paths in detector are **equal**,

then **all** photons passing through device are picked up at detector X

and **none** reach Y.

Consequently, in this case, final state of photon in that situation cannot include the state $|L\rangle$.

If our formalism works, then we must have $(a^2 + b^2) = 0$ in that case —> an immediate problem.

If a and b are probabilities, then they must be positive numbers (≥ 0).

Since the square of positive number is positive(also the square of negative number),

there is no way we can obtain $(a^2 + b^2) = 0$.

Consequently, terms a and b cannot be probabilities.

Of course, could have $a = b = 0$

—> no physical sense,

i.e., says that half-silvered mirror does not work as observed in real experiments.

Should we abandon this approach for representing quantum systems?

Do not be too hasty.

Mathematicians looking at equation $(a^2 + b^2) = 0$ are not bothered at all.

—> realize immediately that a and b are just complex numbers.

Our assumption led to consistent representation of photon's state

on far side of Mach-Zehnder experiment —>

$$|\psi\rangle = (a^2 + b^2)\,|L\rangle + 2ab\,|K\rangle$$

with proviso that $|L\rangle$ state is never observed

if lengths of two arms in experiment equal.

Consequently, need to have $(a^2 + b^2) = 0$ —> possible puzzle.

But it can work fine if we are using complex numbers.

If $a = bi$ so that $a^2 = -b^2$, then $(a^2 + b^2) = 0$.

—-> A price to pay……..

If numbers multiplying states are complex numbers, what can they mean?

Clearly, they cannot directly represent a probability(as we were assuming),

which is a real, non-negative number.

We started with idea

that the numbers used to multiply states

were related to probability

that a state emerges as result of a measurement.

Our hope —> that numbers might actually be the probability is now

dashed by applying idea to Mach-Zehnder experiment.

that is why we do experiments

Must use complex numbers

if going to represent all possible experimental situations.

Move is required by experimental results! **That is the way Physics is supposed to work!**

That is way a theoretical physicist makes progress.

So the numbers cannot be probabilities

—> instead numbers are called probability amplitudes(as we saw in earlier discussions).

Therefore, we now have

$$|\phi\rangle = a_1\,|A\rangle + a_2\,|B\rangle + a_3\,|C\rangle + a_4\,|D\rangle + \ldots\ldots\ldots\ldots\ldots\ldots$$

where a_1, a_2, a_3, etc are probability amplitudes for states $|A\rangle$, $|B\rangle$, $|C\rangle$, etc.

How are probability amplitudes related to probabilities?

Probability obtained from an amplitude must have **all** factors of i removed.

Earlier we earlier learned a procedure that removes all factors of i from complex number:

—> multiplying number by its conjugate.

Possible interpretation (just a guess at this point): to convert probability amplitudes into probabilities, multiply amplitude by its complex conjugate.

—> **RULE 1**: If $|\phi\rangle = a_1 |A\rangle + a_2 |B\rangle + a_3 |C\rangle + a_4 |D\rangle + ...$ then

$Prob\,(|\phi\rangle \rightarrow |A\rangle) = a_1^* a_1 = |a_1|^2 \;,\; Prob\,(|\phi\rangle \rightarrow |B\rangle) = b_1^* b_1 = |b_1|^2 \;,\; etc$ and so on

—> a fundamental rule(postulate) of quantum theory as we saw earlier.

Rules(postulates) can't be proven mathematically.

Mathematics says what probability amplitudes mean —> it is then job of physics to use this fact.

Only way of doing it is to relate the mathematics to experimental results.

Thus, we assume this rule, use it to do calculations

and then check and see if all predictions are correct.

If all works out, then the rule gets accepted as correct.

In this case, the relationship between amplitudes and probabilities is a cornerstone of quantum theory; the success of whole theory relies on it being correct.

Quantum theory has been around for over 108 years now and it works, so we can regard this rule as being confirmed by experiments.

States in Stern-Gerlach(S-G) Experiment (a concrete example)

Now we apply these ideas to the S-G experiments we discussed earlier.

If send $|U\rangle$ state electron through (LEFT,RIGHT) magnet,

can emerge from either channel with equal probability. **RULE 1**

Similarly, send $|D\rangle$ state electron into (LEFT,RIGHT) magnet,

also emerges from either channel with equal probability. **RULE 1**

Using the assumption,

we write two quantum states $|U\rangle$ and $|D\rangle$ in following form (remember color/hardness)

$$|U\rangle = a\,|R\rangle + b\,|L\rangle \qquad\qquad |D\rangle = c\,|R\rangle + d\,|L\rangle$$

where a, b, c, and d are probability amplitudes.

What are the values of these numbers? Some clues to help us out.

$$|U\rangle = a\,|R\rangle + b\,|L\rangle \qquad\qquad |D\rangle = c\,|R\rangle + d\,|L\rangle$$

First, a must be different from c, and/or b must be different from d since $|U\rangle$ and $|D\rangle$ are different states although both are combinations of $|L\rangle$ and $|R\rangle$.

Second, if probability of emerging from either channel is same, then Rule 1 (amplitude absolute squared = probability) tells us solution that agrees with experiment is

$$aa^* = bb^* = cc^* = dd^* = \frac{1}{2}$$

—> correct result is:

$$|U\rangle = \frac{1}{\sqrt{2}}\,|R\rangle + \frac{1}{\sqrt{2}}\,|L\rangle \qquad\qquad |D\rangle = \frac{1}{\sqrt{2}}\,|R\rangle - \frac{1}{\sqrt{2}}\,|L\rangle$$

Remember color/hardness states.

Could switch +/– signs (an overall change in sign)

—> no change in physical content (remember Do Nothing box).

Are they the correct combinations? We can't prove that yet.

In derivation, we have also used following:

RULE 2: NORMALIZATION - If

$$|\phi\rangle = a_1 |A\rangle + a_2 |B\rangle + a_3 |C\rangle + a_4 |D\rangle + \ldots\ldots\ldots\ldots\ldots$$

then from earlier definitions

$$\langle\phi \,|\, \phi\rangle = |a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2 + \ldots. = 1$$ **(vector length)²**

Works for $|U\rangle$ state:

$$|a|^2 + |b|^2 = \frac{1}{2} + \frac{1}{2} = 1$$

Rule 2

—> total probability = sum of probability for each possibility in final state = 1.

i.e., **Something has to happen!**

If sum probabilities < 1, then it means there was a probability that something would happen that was not included in list of possibilities (states in linear combination), which violates Rule 1.

All state vectors in QM have length 1.

Rule 2 puts constraint on amplitudes

—> values must say state has been **normalized** to 1, or total probability =1

Digression: General Stern-Gerlach States

S-G experiments:

We used only positions of magnets at 90° with respect to one another - (U/D) versus (R/L);

It is clearly physically possible to have any orientation.

Consider beam of $|U\rangle$ state electrons arriving at an S-G magnet

with axis tilted at angle θ to vertical. (U/D) θ = 0° and (R/L) θ = 90°

Electrons still emerge from magnet

along one of the two paths as before.

Call states $|1\rangle$ and $|2\rangle$.

In this case, however, different number of electrons pass down each channel,

indicating(experimentally) that the probability amplitudes are not same:

In general, we have (a not necessarily = to b)

$$|U\rangle = a\,|1\rangle + b\,|2\rangle \quad \text{with} \quad aa^* + bb^* = |a|^2 + |b|^2 = 1$$

A more advanced QM course is able to show(we just quote result) that:

$$|U\rangle = \cos\left(\frac{\theta}{2}\right)|1\rangle + \sin\left(\frac{\theta}{2}\right)|2\rangle \qquad |D\rangle = \sin\left(\frac{\theta}{2}\right)|1\rangle - \cos\left(\frac{\theta}{2}\right)|2\rangle$$

where θ measures angle between axis of magnet and vertical.

These states are **consistent** with our earlier results.

i.e., send $|U\rangle$ and $|D\rangle$ states into (LEFT,RIGHT) magnet, $\theta = 90^\circ$ —> $\theta/2 = 45^\circ$.

Now, $$\sin(45^\circ) = \cos(45^\circ) = \frac{1}{\sqrt{2}}$$ —> get states from earlier.

Note $$\sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right) = 1$$ —> Rule 2 satisfied for any θ.

Some Further General Thoughts (*for the deep thinkers*) before continuing……

Summarizing: about quantum states:

Mathematical representation of initial quantum state $|\phi\rangle$,

—> an expansion (sum) over possible **final** quantum states as

$$|\phi\rangle = a_1\,|1\rangle + a_2\,|2\rangle + a_3\,|3\rangle + a_4\,|4\rangle + \ldots = \sum_n a_n\,|n\rangle$$

Amplitudes are collection of complex numbers related to probability that initial state $|\phi\rangle$ will "change" into (or emerge as) one of final states $|n\rangle$ as result of measurement.

Rule 1 gives relationship between amplitudes and probabilities.

List of possible final states(the possible measurement results)

called a **basis** —> **HOME space** of expansion in states

So now we know how to represent amplitudes and their meaning;

What about the basis states we are using?

How can we write down $|n\rangle$ in mathematical terms?

Is there some equation or formula for $|n\rangle$?

Up to now have simply written states such as $|\phi\rangle$ in terms of basis states,

and these in turn have been written as a combination of a further basis.

For example we wrote $|U\rangle$ as combination of $|L\rangle$ and $|R\rangle$.

In turn $|L\rangle$ written as combination of $|U\rangle$ and $|D\rangle$ as can be done for $|R\rangle$.

System seems to lead to regression of writing one thing in terms of another without actually getting anywhere.

Not entirely true, however, as we will see.

Always Remember the structure of a quantum state is a reflection

of the **contextuality** of quantum physics(from earlier).

State $|U\rangle$ can be written as

$$|U\rangle = \frac{1}{\sqrt{2}}\,|R\rangle + \frac{1}{\sqrt{2}}\,|L\rangle$$

in the **context** of approaching a (LEFT,RIGHT) magnet, i.e., if U electron **about to enter** a (LEFT,RIGHT) magnet

or state $|U\rangle$ can be written as

$$|U\rangle = \cos\left(\frac{\theta}{2}\right)|1\rangle + \sin\left(\frac{\theta}{2}\right)|2\rangle$$

in **context** of approaching a magnet at angle θ

i.e., if U electron **about to enter** a magnet oriented at angle θ

REMEMBER this RULE:
Always choose the appropriate language (basis states) before discussing an experiment i.e HOME SPACE
or
experimental results will not be understandable!

Each $|n\rangle$ state in the basis being used

represents a possible result of measurement in that context!.

We are still missing some way of extracting quantitative information about a physics property (observable) from state $|n\rangle$ (to come later)

What Are Quantum States?

According to picture built up so far, the quantum state of a system contains a series of complex numbers related to probability that system will “collapse” into new state when a measurement takes place.

Each new state represents a possible result of measurement, i.e., a path or quantitative value of physical variable.

However, this simple description of a quantum state hides a number of difficulties.

If we make a measurement on a single electron,

then result = a distinct value of physical property being measured.

But a single measurement **cannot** confirm a probability of finding electron with that value.

Must make same measurement many times

—> how often each specific value comes up

—> some practical difficulties,

i.e., how do we ensure a single electron is in **exactly** same state **every time** we make a measurement on it?

Better off using collection of electrons(ensemble), if can carefully put all in same initial state to start with, and then perform only one measurement on each.

But, if this is best way of carrying out measurement then we must ask what does the quantum state of a **single** system actually represent?

Given a collection of electrons in same state prior measurement, does quantum state describe each electron in collection or does state only meaningfully refer to collection as a whole?

This is more than just debate over terminology: it raises important questions about the nature of probability.

So let us expand our earlier discussion of probability somewhat.

Probabilities occur in many different ways.

Sometimes when we deal with probabilities, there exists a physical aspect of system that reflects this, i.e., throwing fair dice

—> each face coming up 1/6 of the time

because there are 6 faces to choose from.

However, if we have collection of balls in bag and half are red and other half are white, then probability of drawing red ball out of bag (no looking) is 1/2.

Here the probability is not a direct reflection of property of each ball.

Probability, in this case, only exists when the balls are placed in a collection.

i.e., the probability state describes only the collection and not the individual balls within it.

If quantum state = collection of systems, different probabilities might —> systems not quite identical —> amplitudes might not represent genuine unpredictability inherent to system,

i.e., they may simply express existence of ignorance of situation at a deeper level.

Maybe hidden variables exist(as said earlier) and if know values, exact predictions possible.

Collection of systems have various possible values of hidden variables, just don't know them.

Probability differences —> how many of each type was in collection.

However, if quantum state refers to single system, probabilities might reflect physical nature of system —> opens up **new way of looking at reality**, as we will see.

If quantum state of system represented by set of probability amplitudes, then we are describing the state in terms of what it can become as result of measurement.

After measurement, one of possibilities has taken place so system is in new state.

New state also best described in terms of what it can become after next measurement.

Therefore continually describing systems in terms of what they become or change into, never what they are.

Perhaps there is nothing more to describing what something is then saying what it can become or what it can do.

Somehow current state of quantum system has its future implicit within it.

Once measurement taken place, one of implicit possibilities becomes explicit.

This is a very abstract picture!

But, classical state of system is also abstract.

It is represented by a series of quantities gotten from the physical properties of system.
Only seems more real since speed, position, mass, etc., of object are familiar terms.

Meaning of quantum state is a philosophical question.

Provided different quantum descriptions allows calculations that correctly predict outcome of experiment, **there is no experimental way in which different ways of thinking about quantum state can be distinguished.**

Majority of physicists take very pragmatic view.

Quantum theory works: allows calculations, do experiments, and have a career.

Deeper questions about meaning rather fruitless (not accessible to experimental resolution).

This, however, is not me!

There was a distinct element of weirdness about all this that I wanted to understand.

Probability amplitudes seem detached from reality.

The normal puzzlement or the unease one feels when learning an unfamiliar subject

—> a genuine problem in QM:

How can everyday world of experience (seems deterministic) result from underlying quantum reality described by probability amplitudes?

We Need More Discussion on Amplitudes

Let us look at amplitudes in detail.

—> ways amplitudes are combined.

—> so we can make a significant step toward understanding the way QM works.

Amplitudes are key in QM

—> Link between theory/experiment.

Theorist uses QM to calculate amplitudes for given situation

and then experimenters set up that situation and make measurements.

Experiments costly and complicated.

Normally same particle passed through sequence of measuring systems,

each extracts information about situation.

Each measuring device also has some effect on state of system.

Set of amplitudes for effect of measuring device depend on outcome of previous measurement.

Thus, need to be able to trace amplitudes through sequence of situations.

Need know how to combine/convert amplitudes.

Basic rules for combining amplitudes

is implied by experimental results.

REMEMBER: Physics is an experimental science!

Rules are new postulates.

RULE 3: State changes(transitions) are governed by amplitudes.

Successive transitions

—> amplitudes **multiplied** together.

Suppose two **alternatives** possible

If the alternatives can be distinguished in experiment.

—> probabilities **add**

When two alternatives cannot be distinguished,

amplitudes add first and then

probability = absolute square of **total** amplitude.

Best way to understand this rule

is to see how it works **in practice**.

S-G experiments - initial state $|\phi\rangle$ $|\phi\rangle = a\,|U\rangle + b\,|D\rangle$

choose correct language
or go to HOME space

Have chosen $|U\rangle$ and $|D\rangle$ as basis for state expansion since 1st part of experiment = (UP,DOWN) magnet

—> (UP,DOWN) states —> HOME space

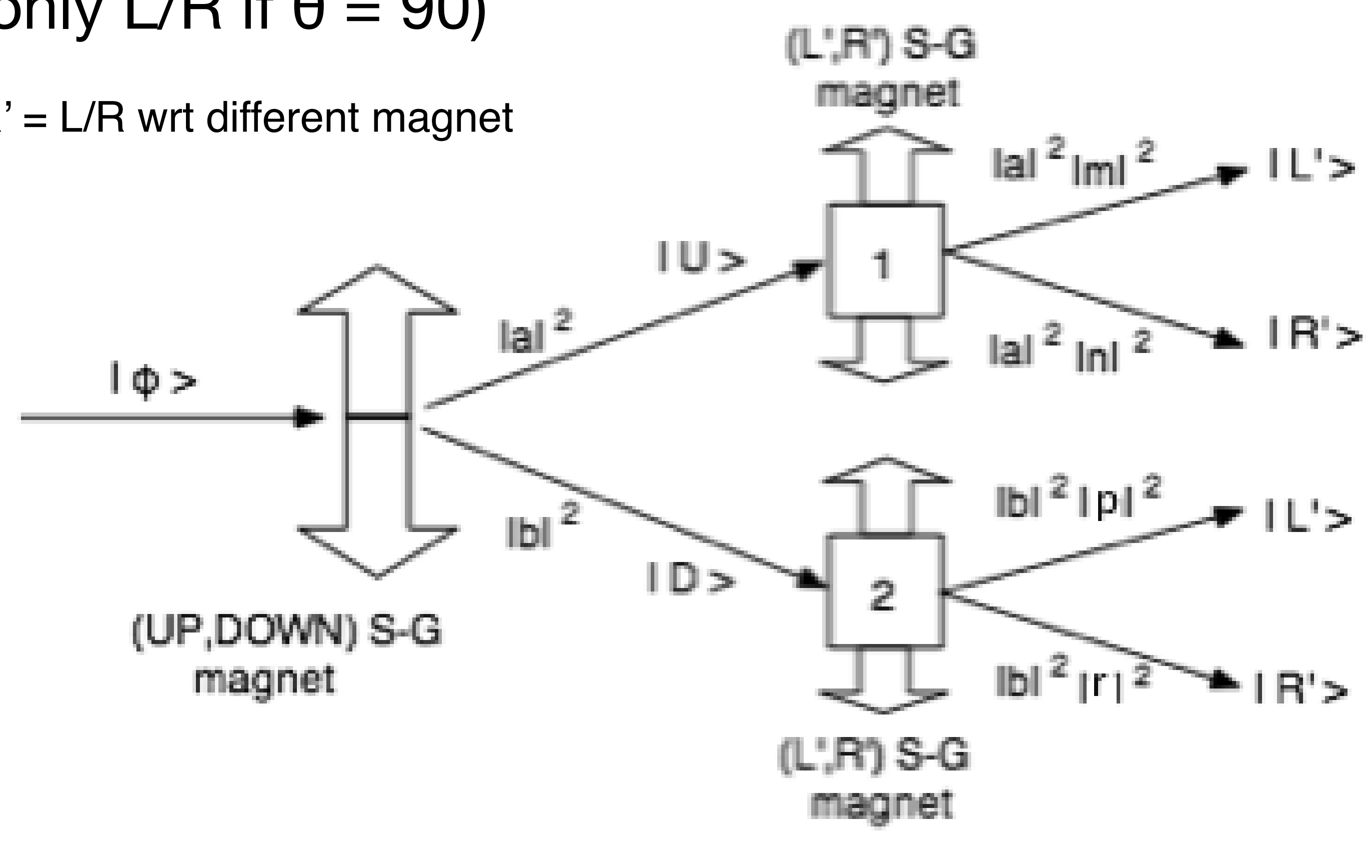
for 1st part of experiment (—> possible results of measurement).

Then, allow each beam from 1st magnet to pass into other magnets arranged at angle θ,

so emerging states are $|L'\rangle$ and $|R'\rangle$ —> expand $|U\rangle$ and $|D\rangle$ states in new basis $|L'\rangle$ and $|R'\rangle$:

(only L/R if θ = 90)

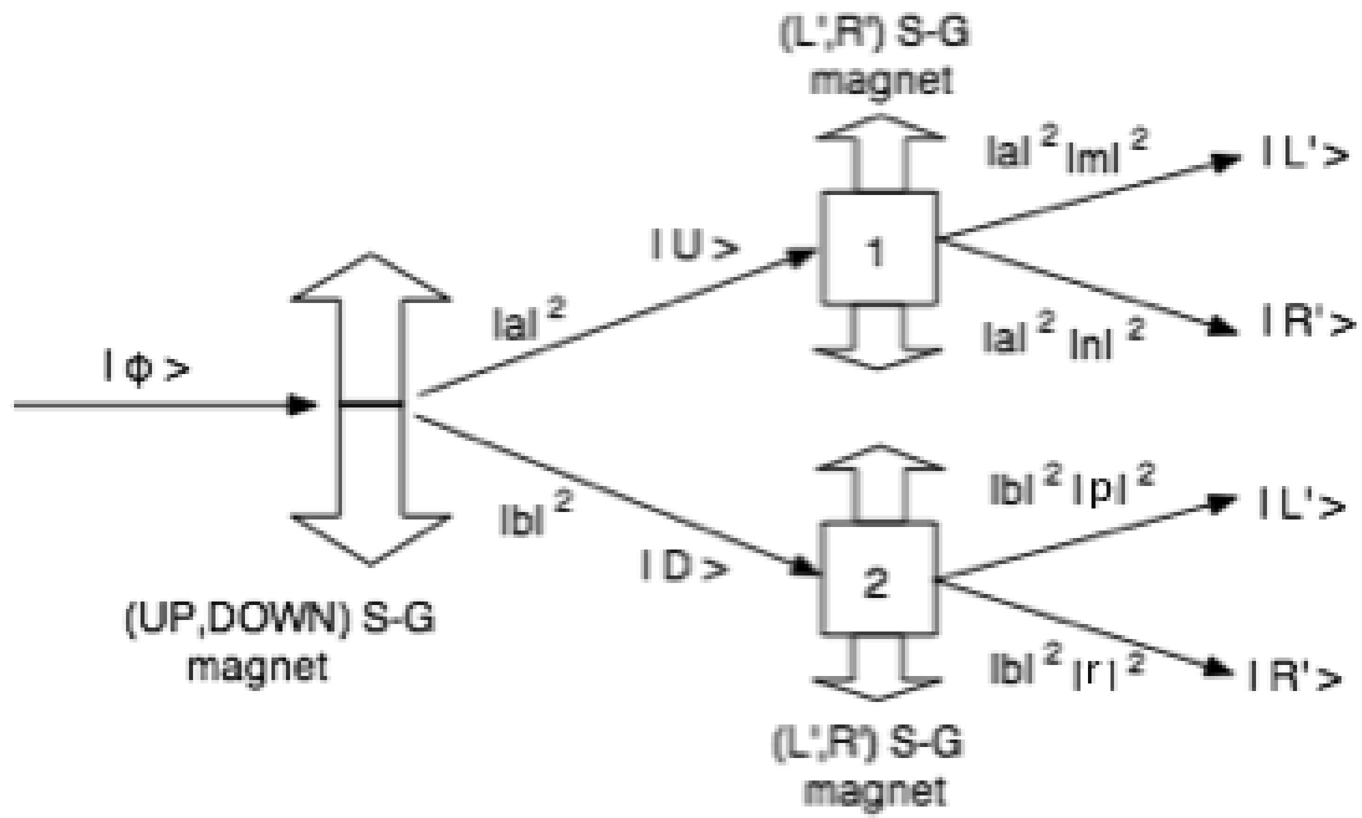
L'/R' = L/R wrt different magnet



DETAILS:

if $|\phi\rangle = a\,|U\rangle + b\,|D\rangle$ $\quad |U\rangle = m\,|L'\rangle + n\,|R'\rangle \quad , \quad |D\rangle = p\,|L'\rangle + r\,|R'\rangle$

then



At 1st magnet, beam of electrons in state $|\phi\rangle$ divides into UP beam containing fraction $|a|^2$ of original beam, and DOWN beam containing fraction $|b|^2$ of original beam.

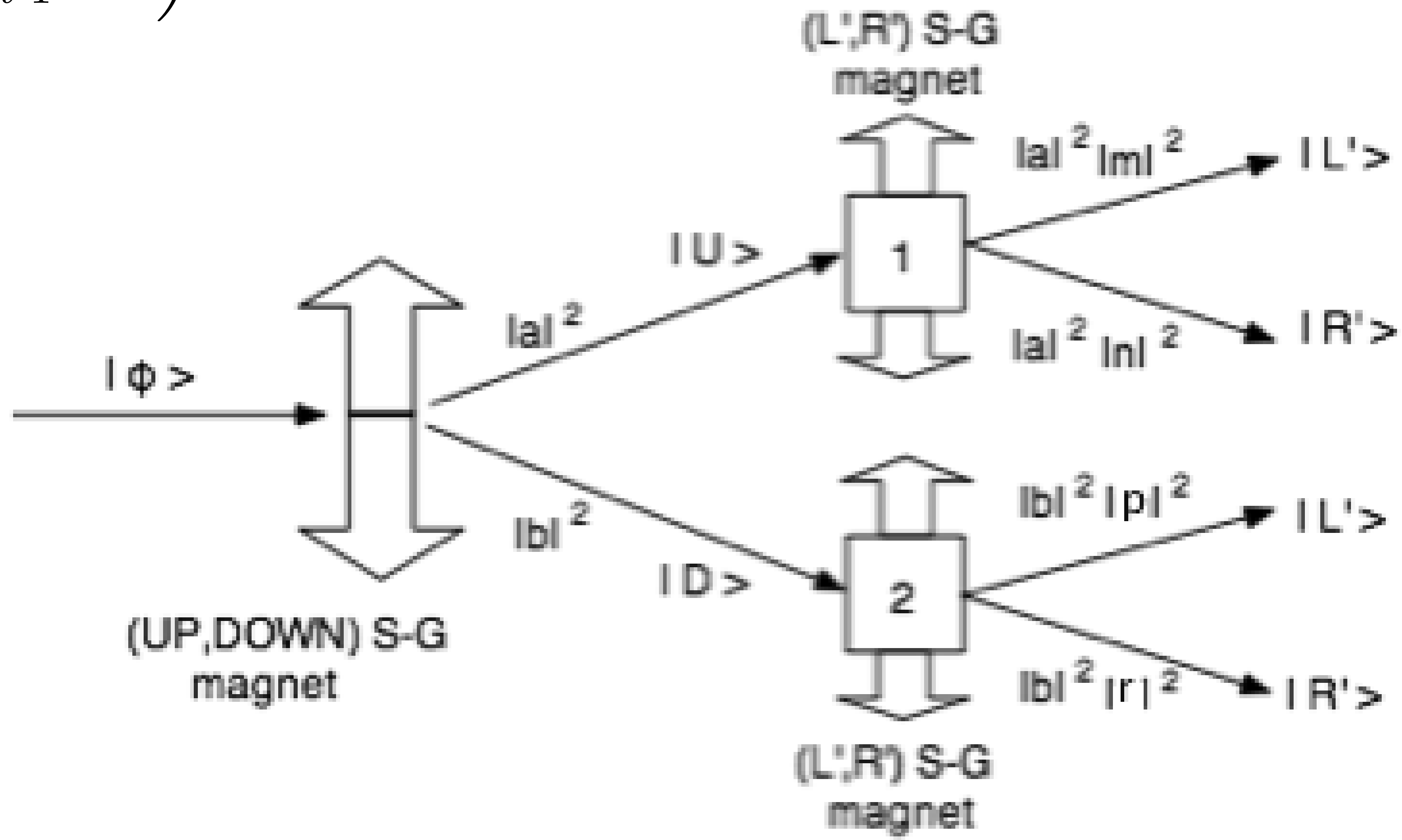
At 2nd magnet, $|U\rangle$ states collapse into either $|L'\rangle$ with probability $|m|^2$ or $|R'\rangle$ with probability $|n|^2$.

Same happens to $|D\rangle$ electrons.

(see figure)

Probability that electron starting in state $|\phi\rangle$ ends up in state $|L'\rangle$ having gone through magnet 1 is

$$Prob\left(|\phi\rangle \underset{via\ magnet\ 1}{\rightarrow} |L'\rangle\right) = |a|^2 \times |m|^2$$



Result obtained by considering fraction of original beam that makes it through each stage.

Remember Rule: **Successive** transitions —> amplitudes **multiplied** together.

Note, get exactly same result if constructed amplitude governing state change from $|\phi\rangle$ to $|L'\rangle$ as follows:

successive = multiply

$$amplitude\left(|\phi\rangle \underset{via\ magnet\ 1}{\rightarrow} |L'\rangle\right) = amplitude\left(|\phi\rangle \rightarrow |U\rangle\right) \times amplitude\left(|U\rangle \rightarrow |L'\rangle\right) = a \times m$$

then calculate probability by complex squaring amplitude, i.e.,

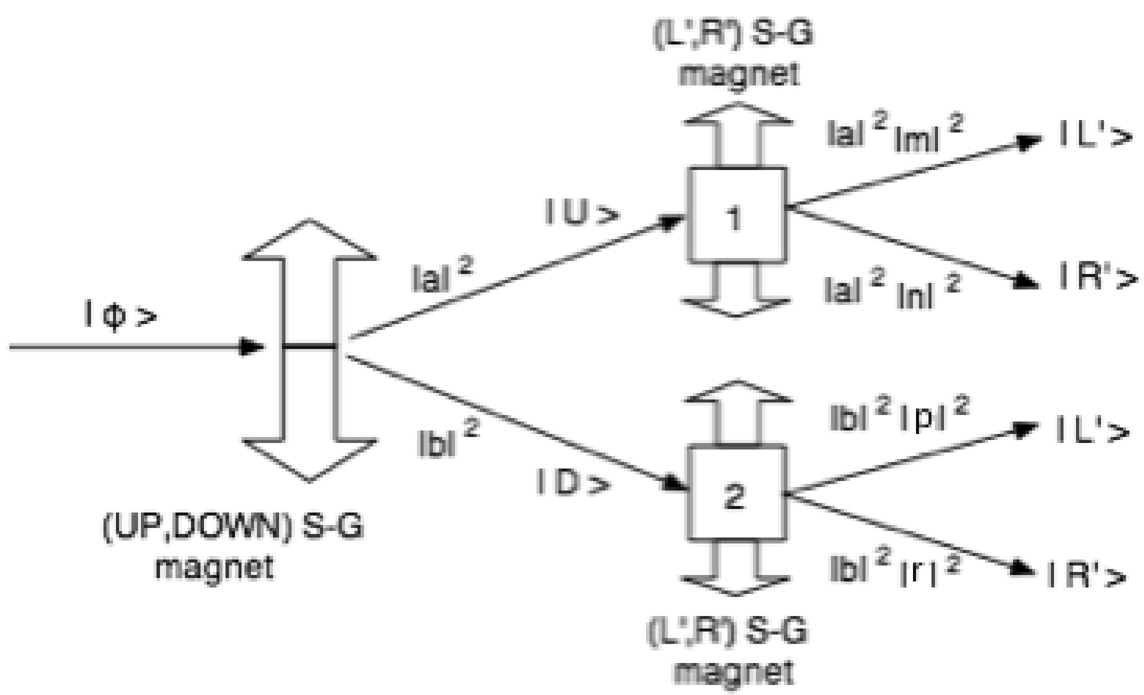
$$|a \times m|^2 = (a\,m) \times (a^*\,m^*) = aa^* \times mm^* = |a|^2 \times |m|^2$$

End result is just what Rule 3 said: when one transition follows another, amplitudes multiply.

So, 1st part of the rule works.

Now a different question: what would be probability of electron ending up in state $|L'\rangle$ if we are not worried about which (L′,R′) magnet it went through?

Look back at the last figure.



$$Prob\left(|\phi\rangle \underset{via\ magnet\ 1}{\rightarrow} |L'\rangle\right) = |a|^2 \times |m|^2$$

$$Prob\left(|\phi\rangle \underset{via\ magnet\ 2}{\rightarrow} |L'\rangle\right) = |b|^2 \times |p|^2$$

So overall probability is $\qquad Prob\,(|\phi\rangle \rightarrow |L'\rangle) = |a|^2 \times |m|^2 + |b|^2 \times |p|^2$

This is just the application of standard rule in probability calculations: when have one event OR another,

probabilities add.

Rule 3 states, when two alternatives are possible, probabilities add if alternatives can be distinguished in experiment.

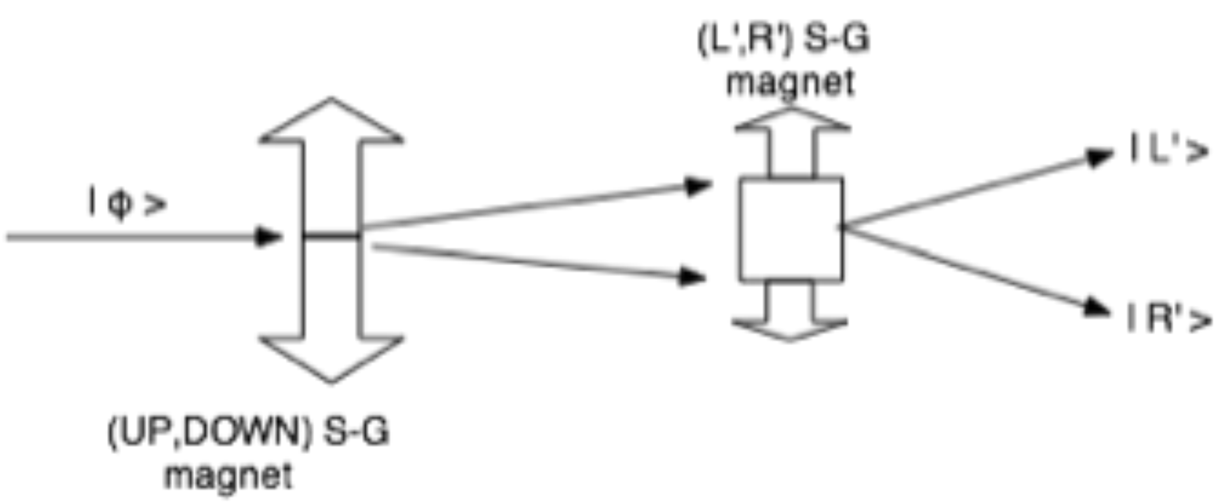
Crucial part here is phrase **"can be distinguished in experiment"**.

Information about which alternative a particular system follows has to be available,

EVEN IF we don't choose to use the information in the experiment(example later).

That just leaves us with the final part of Rule 3, which applies in situations where we **cannot** tell which alternative is happening.

Example:



Modify last experiment —> now 2nd magnet is pulled forward so **both beams** pass through 1 magnet.

Probability that electron state $|\phi\rangle$ ends up in state $|L'\rangle$ remembering that possibilities are now indistinguishable.

Rule 3 —> trace amplitude through each possible path individually

$$amplitude\left(|\phi\rangle \underset{top\ path}{\rightarrow} |L'\rangle\right) = a \times m \qquad amplitude\left(|\phi\rangle \underset{tbottom\ path}{\rightarrow} |L'\rangle\right) = b \times p$$

then add amplitudes together

$$amplitude\left(|\phi\rangle \underset{cannot\ tell\ which\ path}{\rightarrow} |L'\rangle\right) = a \times m + b \times p$$

Calculate probability by complex squaring total amplitude

$$Prob\left(|\phi\rangle \underset{cannot\ tell\ which\ path}{\rightarrow} |L'\rangle\right) = |a \times m + b \times p|^2$$

Final part of Rule 3 is a very important aspect of QM.

Rule that must add the amplitudes before complex squaring if paths indistinguishable has **no equivalent** in normal probability calculations or classical physics.

This is a theoretical representation of what happens in any interference experiment.

Adding amplitudes before squaring (when cannot tell which path followed)

—> **two amplitudes interfere**.

If both $a \times m$ and $b \times p$ are positive, then combined amplitude bigger.

If one negative, then amplitude decreases.

In an interference experiment, amplitudes change depending on length of path through experiment affects sign of amplitude

—> total probability gets bigger/smaller depending on path length.

This is exactly the property we are looking for to explain interference experiments(see later).

It is all about a Change of Basis or a Change of Language

Alternative indistinguishable path probability calculation.

Now we use purely mathematical manipulation.

Do in detail.

Learn/practice mathematical/algebraic procedures.

Idea that states can be expanded over any basis is an important aspect of QM

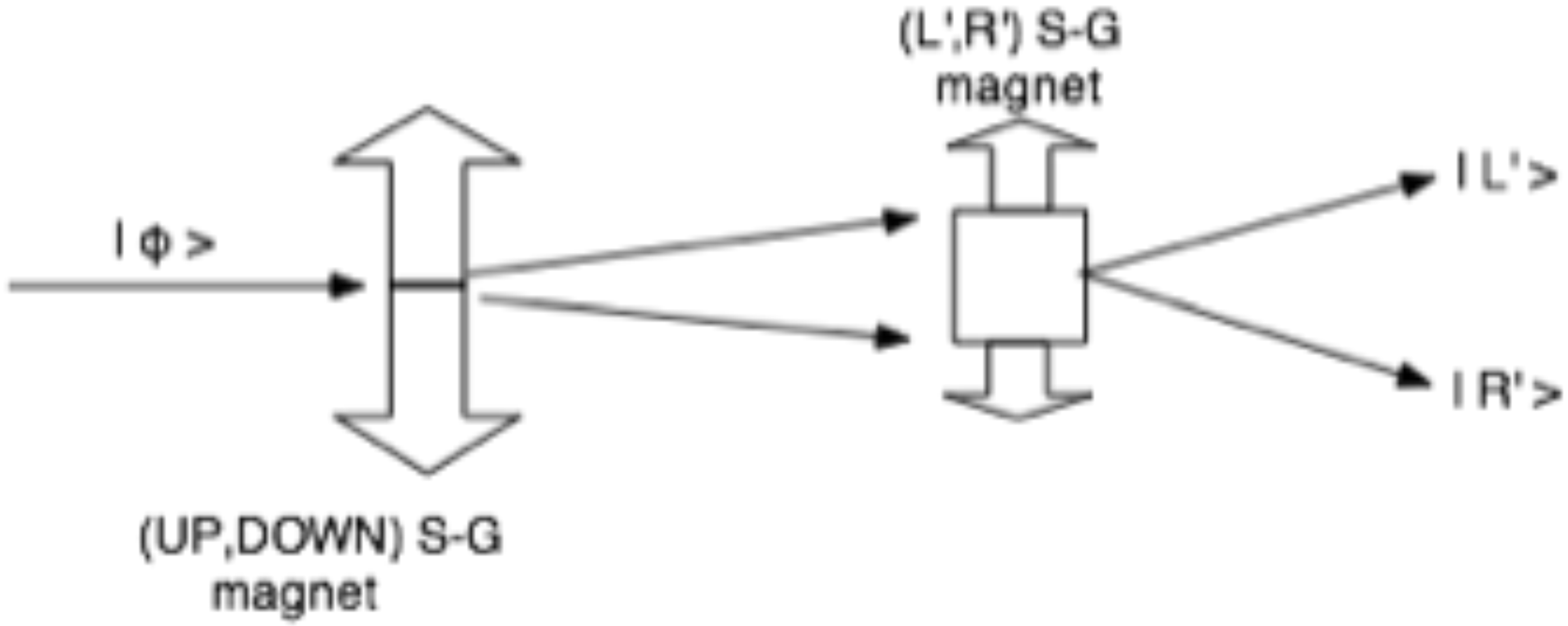
—> part of way that **contextuality** of QM is reflected in theory.

When we expand state, we always choose that basis useful in describing measurement **about to take place**

—> called going to HOME space(before measurement).

Every quantum state can be expanded in many ways, depending on what experiment is involved.

Illustrate this idea via 2nd look at indistinguishable path experiment .



1st thing electrons in $|\phi\rangle$ come to is (UP,DOWN) S-G magnet.

Expand state in $|U\rangle$ and $|D\rangle$ (the basis).

$$|\phi\rangle = a\,|U\rangle + b\,|D\rangle \quad \text{Expansion 1}$$

Electrons then hit (L$'$,R$'$) magnet,

—> sensible to expand $|U\rangle$ and $|D\rangle$ states in $|L'\rangle$ and $|R'\rangle$.

$$\begin{aligned} |U\rangle &= m\,|L'\rangle + n\,|R'\rangle \\ |D\rangle &= p\,|L'\rangle + q\,|R'\rangle \end{aligned} \qquad \text{Expansion 2}$$

Initial state is $|\phi\rangle$ and final state interested in is $|L'\rangle$ - **do not care about in-between!**

Need expansion of $|\phi\rangle$ in $|L'\rangle$ and $|R'\rangle$.

Get expansion by pulling together information already have.

Plug Expansion 2 into Expansion 1.

Do details,

$$|\phi\rangle = a\,|U\rangle + b\,|D\rangle = a\left[m\,|L'\rangle + n\,|R'\rangle\right] + b\left[p\,|L'\rangle + q\,|R'\rangle\right]$$

$$= (am + bp)\,|L'\rangle + (an + bq)\,|R'\rangle$$

Result very interesting, look carefully.

$$|\phi\rangle = (am + bp)\,|L'\rangle + (an + bq)\,|R'\rangle$$

Expansion —> amplitude for $|\phi\rangle$ to collapse to $|L'\rangle$ by end of experiment (also —> amplitude for $|\phi\rangle$ to collapse to $|R'\rangle$).

Look closely, amplitude (am + bp) = same result we calculated using Rule 3 earlier.

If paths distinguishable, Expansion 1 no longer valid.

Instead $|\phi\rangle$ turns into one of $|U\rangle$ or $|D\rangle$, whereas Expansion 1 assuming both valid at same time i.e., state has not collapsed so cannot tell which path is happening.

Same quantum state can be expanded in many ways, depending on experiment.

Also possible to switch from one basis to another with bit of algebra.

Expanding the Dirac Language to make things easier……..

for those of you trying to do the calculations

Have mainly used KETS $|..\rangle$ so now **reintroduce** BRAS $\langle..|$

Just an alternative(equivalent) way of representing/expanding state using complex conjugates of amplitudes rather than amplitudes themselves

$$\langle\phi| = a_1^* \langle 1| + a_2^* \langle 2| + a_3^* \langle 3| + ... + a_n^* \langle n|$$

when $$|\phi\rangle = a_1 |1\rangle + a_2 |2\rangle + a_3 |3\rangle + \cdots + a_n |n\rangle$$

KETS and BRAS contain same information.

RULE 4: If system starts in $|\phi\rangle$ and ends in $|\psi\rangle$, then amplitude for transition is calculated by taking bra of final state and acting on ket of initial state

(mathematically = "Braket" = inner product <—> use × symbol for the moment)

$$amplitude\,(|\phi\rangle \to |\psi\rangle) = \langle\psi| \times |\phi\rangle$$

Start in $$|\phi\rangle = a\,|U\rangle + b\,|D\rangle$$

Rule 4 —> amplitude for collapse of $|\phi\rangle$ into $|U\rangle$ is

$$amplitude\,(|\phi\rangle \to |U\rangle) = \langle U| \times |\phi\rangle = \langle U| \times (a\,|U\rangle + b\,|D\rangle)$$

Expand out

$$amplitude\,(|\phi\rangle \to |U\rangle) = \langle U| \times (a\,|U\rangle + b\,|D\rangle) = a\,\langle U| \times |U\rangle + b\,\langle U| \times |D\rangle$$

But we already know the answer to this question………

Definition of state expansion —> amplitude for $|\phi\rangle$ changing into $|U\rangle$ is a .

$$|\phi\rangle = a\,|U\rangle + b\,|D\rangle \quad \text{Expansion 1}$$

If rule consistent with already known answer, then must have

$$\langle U| \times |U\rangle = 1 \quad , \quad \langle U| \times |D\rangle = 0$$

This is all totally consistent.

According to Rule 4, $\langle U|\times|U\rangle$ = amplitude for $|U\rangle$ to change into $|U\rangle$, which for normalized states = 1.

However, $|U\rangle$ cannot change into $|D\rangle$ (not directly), so $\langle U | \times |D\rangle = 0$.

Two statements correspond to the orthonormality of (UP,DOWN) basis states (earlier)

(remember Hardness/color).

Hence

$$amplitude\,(|\phi\rangle \to |U\rangle) = a\,\langle U| \times |U\rangle + b\,\langle U| \times |D\rangle = a \times 1 + b \times 0 = a$$

Simplification: Bra acting on Ket, leave multiplication sign out

—> write ⟨ψ | φ⟩ not ⟨ψ| × |φ⟩ <—> **Dirac Braket**.

Orthogonal Bases

$$\langle \phi \,|\, \phi \rangle = \{a_1^* \langle 1| + a_s^* \langle 2| + ... + a_n^* \langle n|\}\{a_1 |1\rangle + a_2 |2\rangle + ... + a_n |n\rangle$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i^* a_j \langle i \,|\, j \rangle$$

all possible pairs in sum!

States |1⟩, |2⟩, |3⟩, etc = basis (abbreviation {|n⟩}).

If choose orthonormal basis = basis states, two things follow.

1. Terms ⟨n | m⟩, n not equal m vanish since ⟨n | m⟩ = 0

2. Terms ⟨1|1⟩ or ⟨n|m⟩, n = m, are ⟨1|1⟩=⟨n|n⟩=⟨m|m⟩=1, etc.

Remember $\langle n \,|\, m \rangle = \delta_{nm}$

Our rules **directly reflect** experimental facts.

If states $|1\rangle$, $|2\rangle$, $|3\rangle$, etc, represent different measurement results and experiment has separated out distinct paths or quantitative values of a physical variable, then one state cannot overlap(exist at same time) with another (they have nothing in common - they are **orthogonal**).

Hence amplitude for transition from $|m\rangle$ to $|n\rangle$ ($m \neq n$) is **zero** for basis states.

—> **physical basis** for orthonormality of basis states corresponding to measurements.

Physicists describe any two states where $\langle n \mid m\rangle = 0$ as **orthogonal** states.

In good basis set, all states are orthogonal to one another and collection is called an **orthogonal basis**.

Assuming basis $\{|n\rangle\}$ is orthogonal set, calculation of $\langle \phi|\phi\rangle$ reduces nicely to

$$\begin{aligned}\langle\phi\,|\,\phi\rangle &= \{a_1^*\langle 1| + a_s^*\langle 2| + ... + a_n^*\langle n|\}\{a_1|1\rangle + a_2|2\rangle + ... + a_n|n\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^* a_j \langle i\,|\,j\rangle \quad = 1\end{aligned}$$

Now illustrate algebraically better way to do calculation.

Use this new way later.

Details presented for the mathematically inclined …………

Let

$$|\phi\rangle = a_1\,|1\rangle + a_2\,|2\rangle + \cdots + a_K\,|K\rangle = \sum_{i=1}^{K} a_i\,|i\rangle$$

K = size (dimension) of basis

$$\langle\phi| = a_1^*\,\langle 1| + a_2^*\,\langle 2| + \cdots + a_K^*\,\langle K| = \sum_{j=1}^{K} a_j^*\,\langle j|$$

$$\langle\phi\,|\,\phi\rangle = \left(\sum_{j=1}^{K} a_j^*\,\langle j|\right)\left(\sum_{i=1}^{K} a_i\,|i\rangle\right) = \sum_{j=1}^{K}\sum_{i=1}^{K} a_i a_j^*\,\langle j\,|\,i\rangle$$

Basis is orthonormal —> $\langle j\,|\,i\rangle = \delta_{ij}$ (Kronecker delta) and therefore

$$\langle\phi\,|\,\phi\rangle = \sum_{j=1}^{K}\sum_{i=1}^{K} a_i a_j^* \delta_{ij} = \sum_{i=1}^{K} a_i a_i^* = \sum_{i=1}^{K} |a_i|^2$$

or

$$\langle \phi \,|\, \phi \rangle = a_1^* a_1 + a_2^* a_2 + a_3^* a_3 + ... + a_K^* a_K = 1$$

Dirac language really is the Language(elegant) appropriate to QM ......

As starting point write $|\phi\rangle = a\,|U\rangle + b\,|D\rangle$, which now becomes

$$|\phi\rangle = \langle U \mid \phi\rangle\,|U\rangle + \langle D \mid \phi\rangle\,|D\rangle$$

Since

$$\langle U \mid \phi\rangle = a\,\langle U \mid U\rangle + b\,\langle U \mid D\rangle = a \times 1 + b \times 0 = a$$

$$\langle D \mid \phi\rangle = a\,\langle D \mid U\rangle + b\,\langle D \mid D\rangle = a \times 0 + b \times 1 = b$$

Then write $|U\rangle$ and $|D\rangle$ in $|L'\rangle$ and $|R'\rangle$

$$|U\rangle = m\,|L'\rangle + n\,|R'\rangle = \langle L' \mid U\rangle\,|L'\rangle + \langle R' \mid U\rangle\,|R'\rangle$$

$$|D\rangle = p\,|L'\rangle + q\,|R'\rangle = \langle L' \mid D\rangle\,|L'\rangle + \langle R' \mid D\rangle\,|R'\rangle$$

and substitute into $|\phi\rangle$ expansion.

Earlier -> expression for amplitude governing transition $|\phi\rangle \rightarrow |L'\rangle$, or (am + bp)=(ma + pb).

In Dirac language this is

$$|\phi\rangle = \langle U \mid \phi\rangle \, |U\rangle + \langle D \mid \phi\rangle \, |D\rangle$$

$$|\phi\rangle = \langle U \mid \phi\rangle \left(\langle L' \mid U\rangle \, |L'\rangle + \langle R' \mid U\rangle \, |R'\rangle\right) \quad + \langle D \mid \phi\rangle \left(\langle L' \mid D\rangle \, |L'\rangle + \langle R' \mid D\rangle \, |R'\rangle\right)$$

$$|\phi\rangle = \left[\langle L' \mid U\rangle \langle U \mid \phi\rangle + \langle L' \mid D\rangle \langle D \mid \phi\rangle\right] |L'\rangle \; + \left[\langle R' \mid U\rangle \langle U \mid \phi\rangle + \langle R' \mid D\rangle \langle D \mid \phi\rangle\right] |R'\rangle$$

$$|\phi\rangle = [ma + pb] \, |L'\rangle + [na + qb] \, |R'\rangle$$

so that

$$\begin{aligned} amplitude\left(|\phi\rangle \underset{cannot\ tell\ which\ path}{\rightarrow} |L'\rangle\right) &= \langle L' \mid U\rangle \langle U \mid \phi\rangle + \langle L' \mid D\rangle \langle D \mid \phi\rangle \\ &= ma + pb \end{aligned}$$

Beautifully illustrates how amplitudes combine together. Look closely.

1st term = amplitude takes $|\phi\rangle \rightarrow |L'\rangle$ via intermediate state $|U\rangle$.

2nd term = amplitude takes $|\phi\rangle \rightarrow |L'\rangle$ via intermediate state $|D\rangle$.

Expression = example of another important rule.

RULE 5: Any amplitude governing transition from initial state to final state via a **particular** intermediate state can be written

$$\langle final\ state \,|\, initial\ state\rangle = \langle final\ state \,|\, intermediate\ state\rangle \langle intermediate\ state \,|\, initial\ state\rangle$$

If intermediate states (labelled by i) are indistinguishable, have to add all possible terms up to get overall amplitude

$$\langle final\ state \,|\, initial\ state\rangle = \sum_i \left[\langle final\ state \,|\, i\rangle \langle i \,|\, initial\ state\rangle\right]$$

Look carefully: I really have just inserted an IDENTITY operator

$$\sum_i |i\rangle \langle i| = \hat{I}$$

Rule 5 = extension of Rule 3(more formal way).

Going the Other Way

Dirac language —> amplitude for transition between initial state and final state is

$$\langle final\ state \mid initial\ state \rangle$$

Of course, always possible that process goes in opposite direction from final to initial state.

Don't get confused about words initial and final; call them i and j.

RULE 6 (very important):

Amplitude from $|i\rangle$ to $|j\rangle$ is complex conjugate of amplitude from $|j\rangle$ to $|i\rangle$.

$$\langle i \mid j \rangle = \langle j \mid i \rangle^*$$

$$\langle i \mid j \rangle = (a^* \langle U| + b^* \langle D|)\,(c\,|U\rangle + d\,|D\rangle) = a^*c + b^*d$$

$$\langle j \mid i \rangle^* = [(c^* \langle U| + d^* \langle D|)\,(a\,|U\rangle + b\,|D\rangle)]^* \;= [c^*a + d^*b]^* = a^*c + b^*d = \langle i \mid j \rangle$$

QM Measurement (1st of many passes - we will get it correct by end of class!)

Now for most profound puzzle in QM, the process of measurement takes quantum state, expressed as collection of possibilities, and makes one of the possibilities ACTUAL.

Without measurement, we could not relate theory to reality.

Despite the crucial nature of measurement, there still exist disputes about how it actually takes place in microworld - I will eliminate all the disputes as we proceed.

Embracing Change

Change is built into structure of quantum state.

We have seen how state can be expanded over set of basis states representing different possible **outcomes** of experiment.

Once experiment performed, original state has **collapsed** into one of basis states.

State collapse = peculiar process(more later), but is not only way quantum states change.

Most of time for quantum systems that exist (not in experiments),things can and do change.

Theory must also describe normal processes where things interact, change and develop.

Let us now look at ordinary time development and, then come back to the more mysterious, changes due to measurement.

Types of States

Quantum states divide into two groups.

State $|U\rangle$ —> particle with definite value of property —> eigenstate of some operator.

State $|\phi\rangle$ = combination of basis states —> does not have definite value for measurement

—> mixed states or superpositions (exist as pure and non-pure types).

Eigenstates —> systems with definite value(physical property).

Know object in eigenstate, can predict with absolute certainty a measurement result.

Remember: sent electrons in $|U\rangle$ into (UP,DOWN) S-G magnet —> electrons emerged top path.

Add 2nd (UP,DOWN) measurement —> electrons emerged top path.

Another property of eigenstate: make measurement of physical property of eigenstate, then another measurement —> no state change —> Repeated measurement postulate.

If we choose to measure property different from that determined by eigenstate —> cannot predict with certainty what will happen, i.e., send $|U\rangle$ electron into (LEFT,RIGHT) magnet,

—> do not know on which path it will emerge.

Eigenstates can have more than one physical property at same time i.e., magenta electron with definite position.

Can measure either property, predict with certainty what will happen, and not change state during measurement

—> has definite values of both physical quantities.

Mixed States $|R\rangle = \frac{1}{\sqrt{2}}\left(|U\rangle + |D\rangle\right)$ —> pure mixed state —> superposition !

Choose to measure electron in this state using (UP,DOWN) S-G magnet, get either $|U\rangle$ or $|D\rangle$ with 50:50 probability.

After long run of experiments, expect to observe $|U\rangle$ and $|D\rangle$ with equal frequency.

Mixed states —> trouble when we want understand what QM says about the world.

Mixture is often not allowed in classical situation i.e., Mach-Zehnder device —> ended up with mixed state that combined **two different paths** through device.

Classical particle follows one path and cannot seem to be in two places at same time.

Mixed state is unavoidable in quantum analysis however.

Otherwise, we couldn't explain experimental fact that no photons ended up in one of detectors (for case of equal path lengths).

Quantum mixed states = central puzzle of quantum theory.

They are not some sort of average.

They do not describe an existence i.e., a blend of separate states(classical mixed state).

When we observe a quantum mixed state

—> mixture collapses into one of the component states.

Mixture —> relative probability of finding component states

Mixed states —> set of **tendencies**(or **potentialities**) for something to happen.

State loaded with possibilities and measurement —> one possibility = actuality.

Physicists disagree on the extent to which they believe propensity or latency in state is related to physical nature of object.

Distinction between mixed state and eigenstate is not an absolute divide.

$|U\rangle$ and $|D\rangle$ eigenstates for (UP,DOWN) measurements, but mixed states for (LEFT,RIGHT) measurements.

Similarly, $|L\rangle$ and $|R\rangle$ eigenstates for (LEFT,RIGHT) measurements, but mixed states for (UP,DOWN) measurements.

Expectation Values

Suppose we have a large number electrons **all** in same state $|U\rangle$, and send some at (UP,DOWN) S-G device.

Know(absolute certainty) results always are UP, i.e., $|U\rangle$ = eigenstate of (UP,DOWN).

Send remainder of electrons at (LEFT,RIGHT) device —> now ability to predict is more limited.

Given specific electron —> half the time emerges from LEFT channel and half the time emerges from RIGHT channel.

No way of telling what happens to each electron —> how **randomness** finds expression in quantum world.

We can, however, say something about the average.

Let RIGHT = +1 and LEFT = −1, then for any set of measurements average value for electrons is close to zero (except for randomness and/or experimental precision).

Now consider set of electrons in state $|\phi\rangle$ where

$$|\phi\rangle = a\,|L\rangle + b\,|R\rangle \quad with \quad |a|^2 + |b|^2 = 1 \quad but \quad a \neq b$$

Electrons subjected to (LEFT,RIGHT) measurement, find fraction = $|a|^2$ out LEFT and fraction = $|b|^2$ out RIGHT.

No knowledge of each individual electron, average value —> interesting result.

If $|a|^2$ —> result +1 and $|b|^2$ —> result −1, then average is

$$average\ value = |a|^2 \times (1) + |b|^2 \times (-1) = |a|^2 - |b|^2$$

Quantum theory: average(set of measurements) on collection of identically prepared systems (same state) = expectation value as we showed earlier.

Expectation value applies to a set of measurements.

For set of electrons, each electron —> either (+1) or (–1) —> not necessarily equal to expectation value, which belongs to set as whole.

Some physicists say that the connection between expectation values and the results of set of measurements

—> view that states represent collections of systems

—-> electrons in state $|\phi\rangle$

—-> not saying each individual electron in state $|\phi\rangle$

Ensemble Interpretation

but whole set(ensemble) has a state

—> puts to rest the view that state applies to an individual system!

Some sense in that view.

Since can never tell from single measurement which state that system is in.

Sending electron at (UP,DOWN) magnet and comes out UP —> cannot distinguish $|U\rangle$ as initial state —> UP always and $|R\rangle$ as initial state —> made actual as $|U\rangle$ by measurement.

No help to repeat experiment on same electron;

once emerged from UP channel = state $|U\rangle$ (no matter how it started).

Must have a set of electrons and measure each separately.

Finding one comes out DOWN channel —> can tell difference between $|U\rangle/|D\rangle$.

Would not allow us to tell difference between $|R\rangle/|L\rangle$, however.

Have to look at (LEFT,RIGHT) property for that.

So, clearly there is something to idea that states refer only to collections of systems.

Such view

—> instrumentalist interpretation of quantum theory.

Realist, however, wants to know what happening to each individual electron.

—> Now need to consider Operators

Need to be able to tell if state is mixed state or eigenstate with respect to a measurement.

Mathematical machinery of quantum theory must allow for that, otherwise can only construct state after the measurement.

An **operator** takes mathematical expression and transforms it into something else.

In quantum theory operators take one state into another where process is governed by strict rules for each operator.

Many different operators in quantum theory —> different jobs.

Most important —> operators representing process of measuring a physical property.

Example = (UP,DOWN) S-G operator, $\hat{S}_z$, (UP-DOWN direction is z–axis).

Role of operator = pull out of state information about how state will react to (UP,DOWN) measurement.

Rules that govern how it works are simple.

$|U\rangle$ and $|D\rangle$ eigenstates of (UP,DOWN) measurements —> (UP,DOWN) measurement or corresponding operator does not change these states.

Thus, $$\hat{S}_z |U\rangle = +1 |U\rangle \quad and \quad \hat{S}_z |D\rangle = -1 |D\rangle$$

Operator takes eigenstate and multiplies by number equal to value found(on some meter) by measurement of particle in that state.

Action of operator = pull value from state.

Remember assigned values (+1) to UP and (–1) to DOWN —> Definition of meaning of operator in this context.

Value multiplying the state = value experiment reveals with certainty = eigenvalue of eigenstate.

Remember complete set of eigenstates for operator for physical variable always = basis set. (Hermitian operators).

If state concerned not eigenstate of vertical(z-axis) S-G measurements, then $\hat{S}_z$ makes a mess., i.e.,

$$\hat{S}_z |R\rangle = \hat{S}_z \left(\frac{1}{\sqrt{2}} |U\rangle + \frac{1}{\sqrt{2}} |D\rangle \right) = \left(\frac{1}{\sqrt{2}} \hat{S}_z |U\rangle + \frac{1}{\sqrt{2}} \hat{S}_z |D\rangle \right) = \left(\frac{1}{\sqrt{2}} |U\rangle - \frac{1}{\sqrt{2}} |D\rangle \right) = |L\rangle$$

Action of operator on states (not eigenstates) —> important role.

Look at this mathematical construction

$$\begin{aligned} \langle R| \hat{S}_z |R\rangle &= \left(\frac{1}{\sqrt{2}} \langle U| + \frac{1}{\sqrt{2}} \langle D| \right) \hat{S}_z \left(\frac{1}{\sqrt{2}} |U\rangle + \frac{1}{\sqrt{2}} |D\rangle \right) \\ &= \left(\frac{1}{\sqrt{2}} \langle U| + \frac{1}{\sqrt{2}} \langle D| \right) \left(\frac{1}{\sqrt{2}} \hat{S}_z |U\rangle + \frac{1}{\sqrt{2}} \hat{S}_z |D\rangle \right) \\ &= \left(\frac{1}{\sqrt{2}} \langle U| + \frac{1}{\sqrt{2}} \langle D| \right) \left(\frac{1}{\sqrt{2}} |U\rangle - \frac{1}{\sqrt{2}} |D\rangle \right) \\ &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \langle U \mid U\rangle - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \langle D \mid D\rangle = \frac{1}{2} - \frac{1}{2} = 0 \end{aligned}$$

Used fact $\langle U \mid D\rangle = \langle D \mid U\rangle = 0$ (property of basis states).

average value of UP-DOWN operator in the $|R\rangle$ state, i.e., it is equal parts UP and DOWN!

Not an especially useful calculation?

Don't jump to conclusions.

Do again using $|\phi\rangle$ instead of $|R\rangle$.

Then

$$\begin{aligned}\langle\phi|\,\hat{S}_z\,|\phi\rangle &= \left(a^*\langle U| + b^*\langle D|\right)\hat{S}_z\left(a\,|U\rangle + b\,|D\rangle\right)\\ &= \left(a^*\langle U| + b^*\langle D|\right)\left(a\hat{S}_z\,|U\rangle + b\hat{S}_z\,|D\rangle\right)\\ &= \left(a^*\langle U| + b^*\langle D|\right)\left(a\,|U\rangle - b\,|D\rangle\right)\\ &= a^*a\,\langle U \mid U\rangle - b^*b\,\langle D \mid D\rangle \;= |a|^2 - |b|^2\end{aligned}$$

average value of UP-DOWN operator in the $|\phi\rangle$ state, i.e., it is probability $|a|^2$ for +1 and probability $|b|^2$ for -1.

as we found earlier!

Mathematical constructions $\langle\phi|\,\hat{S}_z\,|\phi\rangle$ or $\langle R|\,\hat{S}_z\,|R\rangle$ —> expectation (average) values of series of measurements made on state such as $|\phi\rangle$ or $|R\rangle$ (happens to be 0 for $|R\rangle$).

$\hat{S}_z$ not only example of operator representing measurement of physical quantity.

Operators for other S-G magnet directions, and position, momentum, energy, and all things that one can measure.

Connection of physical quantity <—> operator in quantum theory

Very different from classical physics.

Classical physics: state = collection of quantities describing object at instant.

Quantities given numerical value.

Classical laws —> rules that connect various quantities together —> can predict future values.

Quantum mechanics: state = collection of amplitudes for object to have values of physical quantities.

Physical quantity = operator —> expectation value.

It is average value obtained from set of measurements on identical systems; none of systems necessarily have this expectation value.

Operators —> nothing by themselves; need to act on states for any information extraction.

RULE 7: Every physical variable has an associated operator $\hat{O}$.

Operators have eigenstates $|\psi\rangle$ defined by $\hat{O}\,|\psi\rangle = a\,|\psi\rangle$

where a = value of physical variable get if measured state is $|\psi\rangle$.

eigenvalue

Complete set eigenstates $\{|\psi\rangle\}$ = basis.

Operator associated with physical variable —> expectation value of series of measurements made on collection of systems in same state $|\phi\rangle$.

Given by $$\left\langle \hat{O} \right\rangle = \langle\phi|\,\hat{O}\,|\phi\rangle$$

How can we represent operators?

Think of operator as some kind of box, where we put state vector in and get another(different or same) state vector out.

or like color/hardness boxes

Similar to definition of function for space of numbers.

Some properties of operators in Quantum Mechanics:

$$\hat{Q}\left(|A\rangle + |B\rangle\right) = \hat{Q}\,|A\rangle + \hat{Q}\,|B\rangle \quad \text{(linearity)}$$
$$\hat{Q}\left(c\,|A\rangle\right) = c\hat{Q}\,|A\rangle \quad , \quad c = \textit{complex number}$$
$$\langle C|\left(\hat{Q}\,|B\rangle\right) = \langle C \mid B'\rangle = \text{ number} \equiv \langle C|\,\hat{Q}\,|B\rangle \quad \text{(matrix element)}$$
$$(\hat{Q}_1 + \hat{Q}_2)\,|A\rangle = \hat{Q}_1\,|A\rangle + \hat{Q}_2\,|A\rangle \quad \text{(linearity)}$$
$$(\hat{Q}_1\hat{Q}_2)\,|A\rangle = \hat{Q}_1(\hat{Q}_2\,|A\rangle) \text{ (order matters)}$$

Properties —> operators = LINEAR operators.

QM is understood using only linear operators —> truly amazing - simplest kind of operator that mathematicians can think of.

Nature is very efficient!!

All observables or quantities we can measure are represented by linear operators in QM.

Repeat the discussion from earlier.

Additional pass(had many discussions in between), should greatly enhance understanding and enable us to learn a few new things.

Suppose make measurements on state $|\psi\rangle$ of observable given by the operator $\hat{B}$ with eigenvalues/eigenvectors given by

$$\hat{B}\,|b_j\rangle = b_j\,|b_j\rangle \qquad j = 1, 2, 3, 4, ...$$

Then we can write

$$|\psi\rangle = \sum_k a_k\,|b_k\rangle$$

OK since eigenvectors of observable operator = complete set = basis

$$\text{-->}\quad \langle b_j \mid \psi\rangle = \langle b_j|\sum_k a_k\,|b_k\rangle = \sum_k a_k\,\langle b_j \mid b_k\rangle = \sum_k a_k\delta_{jk} = a_j$$

—> amplitude to measure b_j

Since measurement results(= eigenvalues) are values b_k, and suppose that b_k occurs n_k times where k =1 , 2 ,3 , 4,…….and where

$$\sum_k n_k = N = \text{ total number of measurements}$$

Then from the standard definition of the average value, have

$$\left\langle \hat{B} \right\rangle = \text{ average or expectation value of } \hat{B} \quad = \frac{1}{N}\sum_k n_k b_k = \sum_k \frac{n_k}{N} b_k = \sum_k b_k prob(b_k)$$

From postulates (rules) we have prob(b_k) = $|\langle b_k | \psi \rangle|^2$

—>
$$\left\langle \hat{B} \right\rangle = \sum_k b_k prob(b_k) = \sum_k b_k \left| \langle b_k \mid \psi \rangle \right|^2$$

Now $\langle b_k | \psi \rangle^* = \langle \psi | b_k \rangle$.

Therefore, $|\langle b_k | \psi \rangle|^2 = \langle \psi | b_k \rangle \langle b_k | \psi \rangle$ and

$$\left\langle \hat{B} \right\rangle = \sum_k b_k \left| \langle b_k \mid \psi \rangle \right|^2 = \sum_k b_k \langle \psi \mid b_k \rangle \langle b_k \mid \psi \rangle = \langle \psi | \left[\sum_k b_k \left| b_k \right\rangle \left\langle b_k \right| \right] |\psi \rangle$$

Now earlier definition of expectation value which was

$$\left\langle \hat{B} \right\rangle = \langle \psi | \hat{B} | \psi \rangle$$

IMPORTANT

—> can represent operator $\hat{B}$ by expression

$$\hat{B} = \sum_k b_k \left| b_k \right\rangle \left\langle b_k \right|$$

(a result we have already used several times)

IMPORTANT

A very important way represent operator, i.e., any operator is represented in terms of its eigenvalues/eigenvectors

—> called the **spectral decomposition** of operator.

Projection Operators

Operators of the form $\hat{P}_\beta = |\beta\rangle\langle\beta|$ = projection operators.

Can write any operator $\hat{B}$ as $\hat{B} = \sum_k b_k P_{b_k}$ where $P_{b_k} = |b_k\rangle\langle b_k|$

projection operator onto state $|b_k>$.

Very important property of projection operators: $\hat{I} = \sum_k |b_k\rangle\langle b_k|$ **IMPORTANT**

$$\hat{I}|\beta\rangle = \sum_k |b_k\rangle\langle b_k \mid \beta\rangle = \sum_k \langle b_k \mid \beta\rangle |b_k\rangle = |\beta\rangle$$

Clearly it is the IDENTITY operator or an operator that leaves all kets unchanged.

Some Specific Operators

$$\hat{O}_{color}\left|m\right\rangle = (+1)\left|m\right\rangle \quad , \quad \hat{O}_{color}\left|g\right\rangle = (-1)\left|g\right\rangle$$

—> $$\hat{O}_{color} = \left|m\right\rangle\left\langle m\right| - \left|g\right\rangle\left\langle g\right|$$

Does it work? We have

$$\hat{O}_{color}\left|m\right\rangle = \left(\left|m\right\rangle\left\langle m\right| - \left|g\right\rangle\left\langle g\right|\right)\left|m\right\rangle = \left|m\right\rangle$$

$$\hat{O}_{color}\left|g\right\rangle = \left(\left|m\right\rangle\left\langle m\right| - \left|g\right\rangle\left\langle g\right|\right)\left|g\right\rangle = -\left|g\right\rangle$$

as expected!

Note $|h\rangle$ state has equal amounts of magenta/green.

Therefore expectation value of color operator in hard state should = zero.

This is confirmed below.

$$\langle h| \hat{O}_{color} |h\rangle = \frac{1}{\sqrt{2}} \left(\langle m| + \langle g|\right) \hat{O}_{color} \frac{1}{\sqrt{2}} \left(|m\rangle + |g\rangle\right)$$

$$= \frac{1}{2} \left[\langle m| \hat{O}_{color} |m\rangle + \langle m| \hat{O}_{color} |g\rangle + \langle g| \hat{O}_{color} |m\rangle + \langle g| \hat{O}_{color} |g\rangle\right]$$

$$= \frac{1}{2} \left[\langle m \mid m\rangle - \langle m \mid g\rangle + \langle g \mid m\rangle - \langle g \mid g\rangle\right] \quad = \frac{1}{2} \left[1 - 0 + 0 - 1\right] = 0$$

Dirac language is very powerful!

Similarly, can represent an (UP,DOWN) S-G magnet or operator $\hat{S}_z$ by

$$\hat{S}_z = |U\rangle \langle U| - |D\rangle \langle D|$$ **IMPORTANT**

In words, the sum of projection operators for eigenvectors times eigenvalues.

Now to Figure Out How States Evolve in Time

Operator that tells us how a state evolves in time is not directly a physical variable.

= time evolution operator $\hat{U}(t)$

—-> takes a state and moves it forward in time:

$$\hat{U}(t)\,|\Psi(T)\rangle = |\Psi(T+t)\rangle$$

where $|\Psi(T)\rangle$ = state of system at time T.

For $\hat{U}(t)$ to do this accurately, it must contain all details of the behavior(interactions) of system i.e., what system is doing and how it interacts with its environment.

Example:

Consider the state $|U\rangle$ moving through space toward S-G magnet.

Electron interacts with surroundings —> disturbs its S-G orientation.

State will evolve from $|U\rangle$ to different state, $|\phi\rangle$, in a smooth and deterministic manner over time.

due to Emmy Noether

Energy of electron will determine the type of evolution, i.e., time evolution operator <—> related to energy operator for electron.

Can always describe any S-G state using $\{|U\rangle, |D\rangle\}$ basis.

$$|\Psi\rangle = a\,|U\rangle + b\,|D\rangle$$

So, what really happens when $\hat{U}(t)$ operator acts is

$$\hat{U}(t)\,|\Psi(T)\rangle = \hat{U}(t)\,[a(T)\,|U\rangle + b(T)\,|D\rangle] \;= [a(T+t)\,|U\rangle + b(T+t)\,|D\rangle]$$

(basis vectors = mathematics

coefficients = physics

so coefficients change!

$\hat{U}(t)$ changes amplitudes values at time T to new ones at time T + t.

It does not change the basis vectors!!)

Now take the extra time interval t and divide into many smaller intervals, size δt.

If 100 small pieces, then t = 100 × δt.

Time evolution operator works same way for very small time intervals as for longer ones.

Applying $\hat{U}(t)$ once takes $|\Psi(T)\rangle$ to $|\Psi(T + t)\rangle$.

Must get same answer applying $U(\delta t)$ 100 times.

Apply $U(\delta t)$ and we get

$$\hat{U}(\delta t)\,|\Psi(T)\rangle = \hat{U}(\delta t)\,[a(T)\,|U\rangle + b(T)\,|D\rangle] \;\; = [a(T+\delta t)\,|U\rangle + b(T+\delta t)\,|D\rangle]$$

For small δt , a(T + δt) not very different from a(T) (same for amplitude b).

Applying $U(\delta t)$ again gives us

$$\hat{U}(\delta t)\,|\Psi(T+\delta t)\rangle = \hat{U}(\delta t)\,[a(T+\delta t)\,|U\rangle + b(T+\delta t)\,|D\rangle] \; = [a(T+2\delta t)\,|U\rangle + b(T+2\delta t)\,|D\rangle]$$

and so forth.

State's evolution from $|\Psi(T)\rangle$ to $|\Psi(T + t)\rangle$ takes place via continuously smooth change.

(limit as δt → 0 -> DiffEQ —> Schrodinger Equation governs evolution)

study later

Evolution completely determined by physics of system, as expressed by $\hat{U}(t)$.

No randomness involved here.

Inclusion of Postulate 4 (collapse) —> smooth evolution not only way state can evolve.

States can also evolve in sharp(discontinuous) and unpredictable manner when measurement takes place

= definition of measurement.

later we will see if this statement is really necessary or whether this view of what is happening is old and outdated!!

For the moment, however, let us continue with this old line of reasoning…….

When electron in state $|\Psi\rangle$ reaches (UP,DOWN) S-G magnet, state will "change" into $|U\rangle$ with probability $|a|^2$ or into $|D\rangle$ with probability $|b|^2$.

depending on what we measure

After measurement, the state has evolved into $|U\rangle$ (i.e., $a \to 1$, $b \to 0$) or $|D\rangle$ (i.e., $a \to 0$, $b \to 1$).

During a measurement, the amplitudes a and b have not changed continuously (or so it seems) as they did with the $\hat{U}(t)$ operator.

Dramatic and unpredictable change in quantum state as a result of a measurement

—> **collapse or reduction of state**.

State collapse = radically different process from $\hat{U}(t)$ evolution,

can't be broken down into smooth progression of small steps.

Mathematically impossible for equations of QM as we currently understand them, to describe state collapse (a non-linear process cannot be described by linear operators).

Remains as add-on assumption, it cannot be predicted from within theory or so it seems.

A significant point is being made here **(much more on this later)**.

QM correctly describes the evolution of states in time(has been observed); but doesn't seem to describe the real world (measurement) unless(at this moment) we add the state collapse idea(a process that has never been observed).

World is **not** = set of evolving possibilities.

Possibilities encoded in quantum state must be linked to actual events in world.

Collapse of state <—> measurement

—> some physicists seek an instrumentalist view of quantum states.

Argue that discontinuous change in state shows state **only** represents information about system.

Before a dice thrown, information —> 1/6 probability of each face.

After throw, information state collapses since we know which face appeared.

Dice has not changed in any physical manner.

Correspondingly when quantum state collapses —> does not necessarily signal any physical difference in system being described, just a change of **knowledge** about it.

To counter this, realist —> quantum state cannot just be knowledge of system, since how, they say, can “knowledge” have direct effect on system’s behavior?

In so-called delayed choice experiments that we will discuss later we will see that our ability/ lack of ability to infer information about state of photon **does** have a direct result on outcome of experiment!!

Therefore **knowledge(information)** must be reflection of something **real** to do with the system.

Quantum states —> weird and spooky compared to classical reality we are used to, but we can still believe science is revealing real truths about world even if we find truths surprising.

SUMMARY

$\hat{U}(t)$ operator evolves state forward in time in smooth and predictable manner

$$\hat{U}(t)\,|\Psi(T)\rangle = |\Psi(T+t)\rangle$$

Exact form of $\hat{U}(t)$ operator depends on physics of system.

State collapse is sharp and unpredictable change in state as result of measurement (state not eigenstate of physical quantity measured).

Why does it seem that State Reduction is Necessary?

Reasonable to ask how this whole shebang came about.

Why are we driven to construct theory that requires this odd concept of state collapse?

Argument (from others) is:

(1) Events in microworld, (reflection or transmission of photons from half-silvered mirror) seem to be random.

—> Must represent state of microscopic objects using probabilities.

Assign number to each possibility, and number tells us probabilities of events in future.

(2) Mach-Zehnder experiment —> numbers (amplitudes) = complex numbers -> not probabilities.

Link between amplitudes and probabilities guessed (probability = $|\text{amplitude}|^2$) —> agreement with experiment.

(3) Led us to represent quantum state of system as collection of amplitudes.

(4) In certain cases, appropriate quantum state = mixture of states that have to be separate classically, i.e., many experiments only understood if the intermediate state of the photon APPEARS to follow two **different** classical paths at same time.

Without this, interference effects would not take place and actual experimental results could not be explained.

(5) Although quantum theory allows these mixed states to occur, observers seem to be protected from them

—> they are never directly observed as outcomes of experiments. i.e., photon is never observed to be on both paths at same time.

(6) Quantum state of system changes.

Prepare beam of electrons for Stern-Gerlach (S-G) experiment, find 50% exit device along each channel, indicating electrons were mixed states to start with.

However, if pass all electrons from one channel into another identical S-G device, then all emerge from same channel.

1st experiment has changed quantum state from unobservable mixed state into classical-like eigenstate that is definitely $|U\rangle$ or $|D\rangle$.

(7) Crux of problem.

Certain experiments only explained if classically forbidden mixtures allowed.

However, we live in classical world so quantum mixed states must collapse into more classical-like states that we can directly observe.

Much more later when we solve all these measurement confusions.

Finally we now explain the real experiment we started our entire discussion with!

Explaining the Double-Slit Experiment with Amplitudes

The quantum mechanical description of double-slit experiment must involve a set of amplitudes governing the passage of an electron through slits to a far away detector.

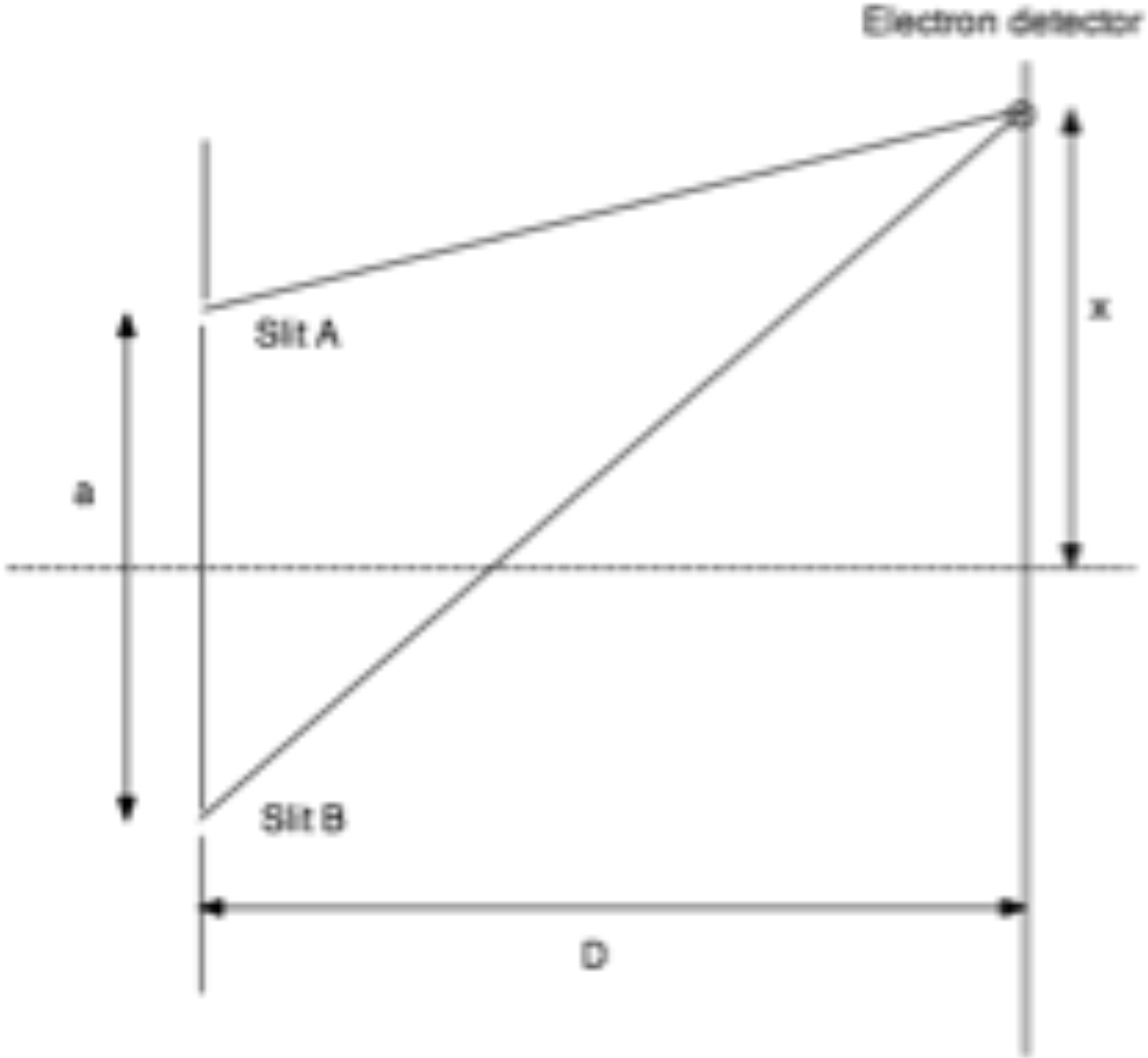
Let us define the following states

$|x\rangle$ is state corresponding to electron arriving at position x of detector (see diagram),

$|A\rangle$ corresponds to passing through slit A,

$|B\rangle$ corresponds to passing through slit B,

and $|I\rangle$ corresponds to the initial state of electron emerging from electron gun.



Then

$$\langle x \mid I \rangle$$

is the probability amplitude we want to calculate.

We assume the electron gets from source (initial state $|I\rangle$) to detector (state $|x\rangle$) via two possible intermediate states (either passing through slit A or slit B).

$$|A\rangle \qquad |B\rangle$$

Then we can write

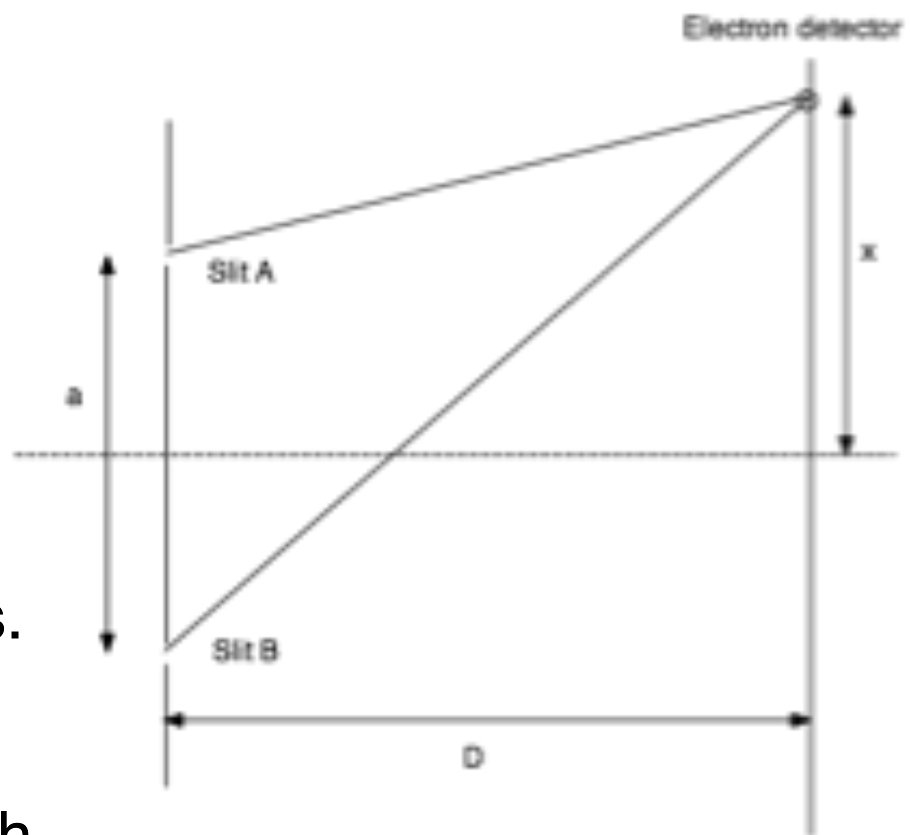
Intermediate state A **Intermediate state B**

$$\langle x \mid I\rangle = \langle x \mid A\rangle \langle A \mid I\rangle + \langle x \mid B\rangle \langle B \mid I\rangle$$

amplitude to calculate

inserted sum over intermediate states as described earlier

here **all** possibilities —> both slits included



Experimentally, we see bright and dark interference bands.

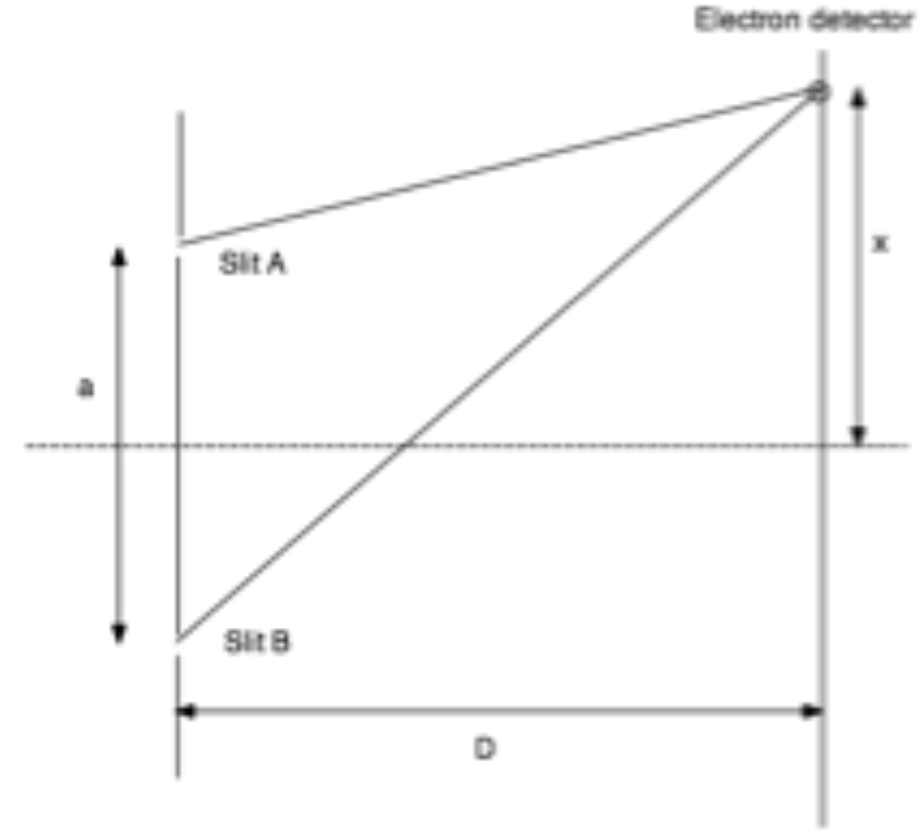
Dark band separation given by Dλ/a where λ is wavelength.

Must add together two terms in overall amplitude, two possibilities(traveling through one slit or other) which CANNOT be distinguished in the context of experiment as set up.

When combine amplitudes this way interference between terms results as we will see.

If block a slit, then one of terms will disappear

—> no interference.



Now probability that electron arrives at x is complex square of total amplitude.

$$|\langle x \mid I\rangle|^2 = [\langle x \mid A\rangle \langle A \mid I\rangle + \langle x \mid B\rangle \langle B \mid I\rangle]^* [\langle x \mid A\rangle \langle A \mid I\rangle + \langle x \mid B\rangle \langle B \mid I\rangle]$$

Expanding out:

bump would see if only A open **bump would see is only B open**

$$|\langle x \mid I\rangle|^2 = \langle x \mid A\rangle^* \langle A \mid I\rangle^* \langle x \mid A\rangle \langle A \mid I\rangle + \langle x \mid B\rangle^* \langle B \mid I\rangle^* \langle x \mid B\rangle \langle B \mid I\rangle$$

interference terms

$$+ \langle x \mid A\rangle^* \langle A \mid I\rangle^* \langle x \mid B\rangle \langle B \mid I\rangle + \langle x \mid B\rangle^* \langle B \mid I\rangle^* \langle x \mid A\rangle \langle A \mid I\rangle$$

1st two terms give that conventional (classical) sum of probabilities that one expect(bumps) for an electron that goes through one slit or other.

Final two terms is where interesting stuff resides: **the quantum interference terms**.

Digression: Phase and Physics

Let us review/reconsider complex numbers in different form

—> more useful form for our discussions.

The Complex Plane

Complex numbers are needed if mathematical objects in QM are to represent reality.

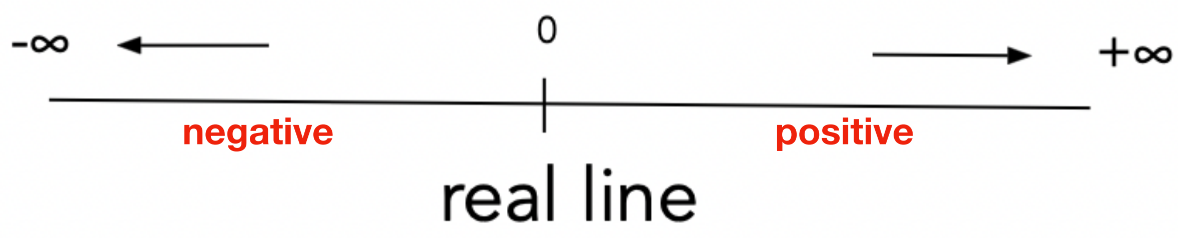
Rethink ideas already introduced, explore better **exponential** way of writing complex numbers and introduce **geometrical picture** to help us understand what is happening.

Sometimes convenient to think of complex number as denoting point on special type of graph.

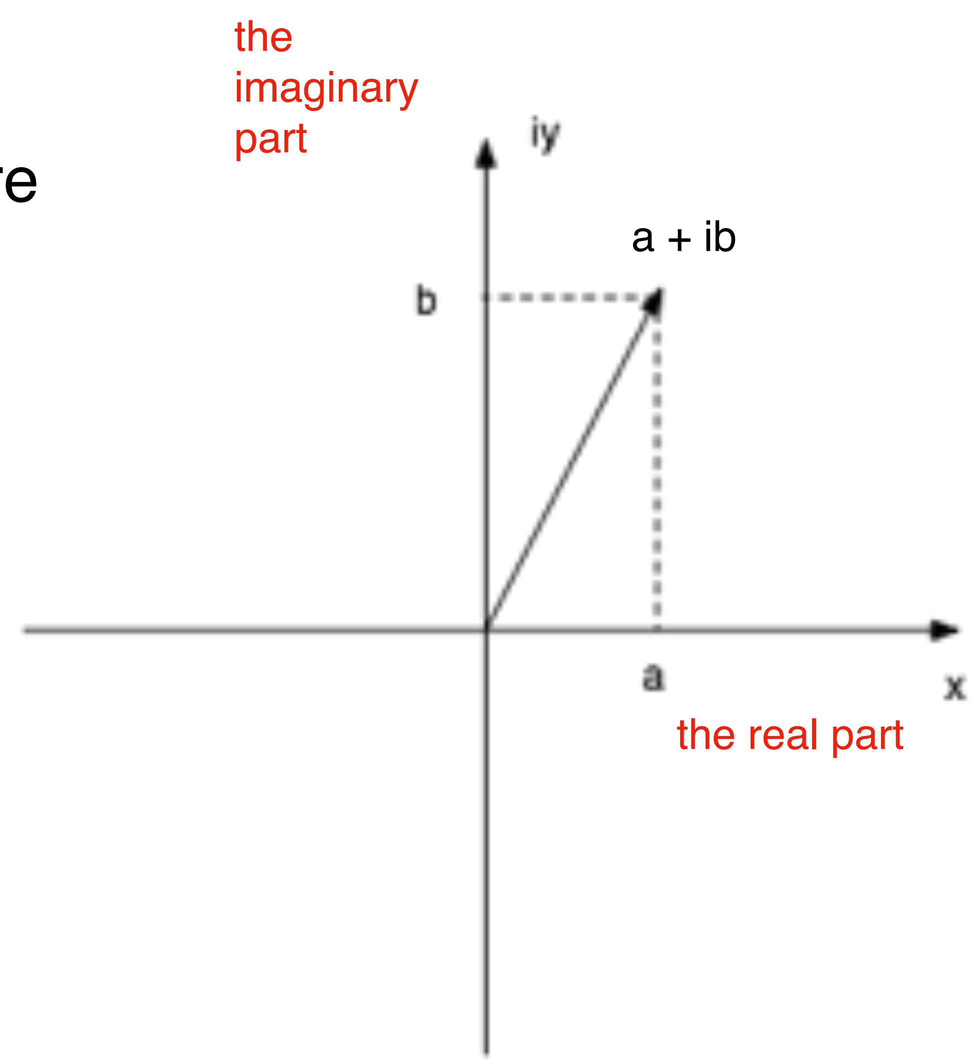
Mathematicians sometimes picture real numbers lying on line (real axis) stretching out to infinity (in two directions).

As work way along line (either direction) at some point come across the number zero.

At zero point, any points on line to right are positive real numbers and those to left are negative real numbers.



When complex numbers appeared, mathematicians thought in terms of two lines at right angles to one another.



A purely real number lies on horizontal line, and a purely imaginary number (e.g., 3i or 5i or –2.1678i) lies somewhere on vertical line running through zero point(origin).

Any two lines at right angles map out or define a 2-dimensional (2-D) region(plane)

—> in this case, the **complex plane of numbers**.

Any complex number, a + ib,

is identified with a point on the complex plane.

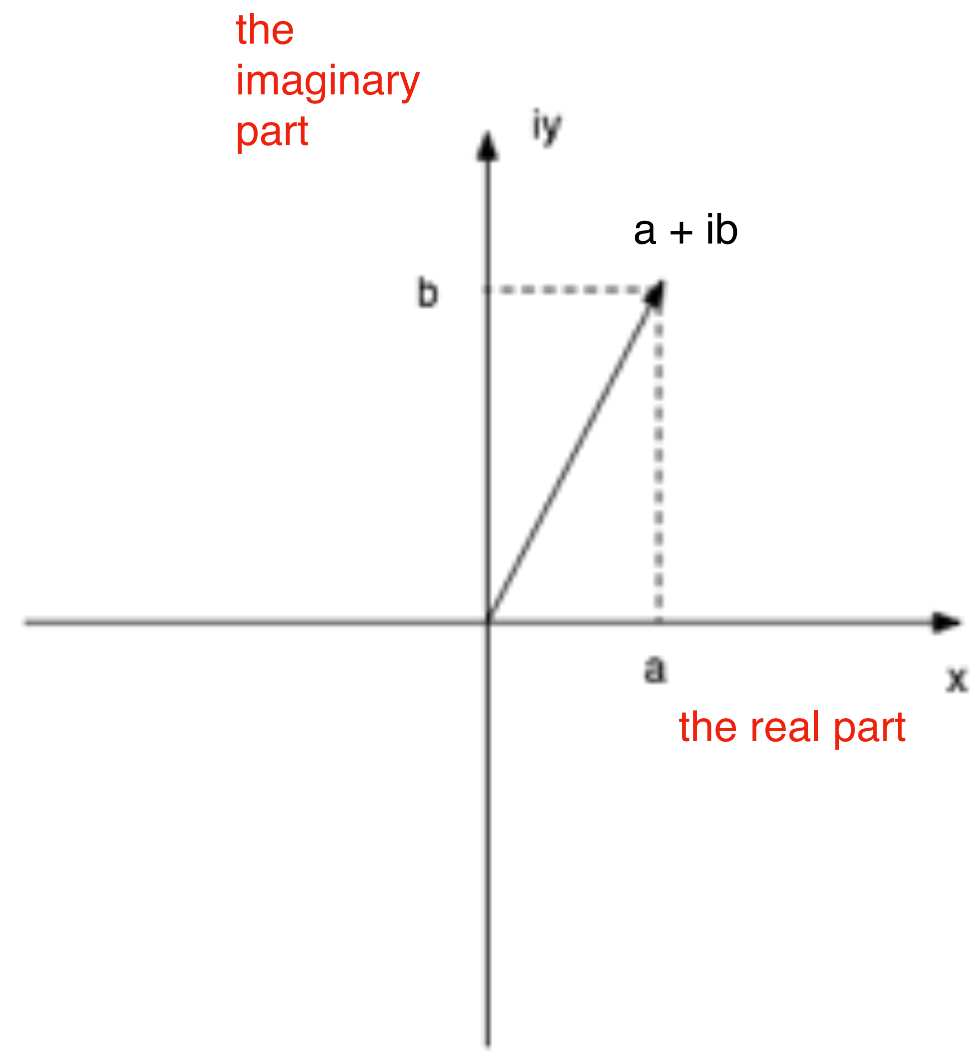
To get to point walk along real line distance a, turn left(anti-clockwise) by a right angle, and walk parallel to imaginary line distance b

—> graphical representation of the point in the complex plane.

Horizontal arrow with x next to it -> x coordinate axis.

Normal graph -> vertical arrow with y next to it

-> y coordinate axis along vertical.



Have labelled vertical line   iy for illustration (not really allowed - no such distance - but OK for us because we are using mathematics - the language of QM).

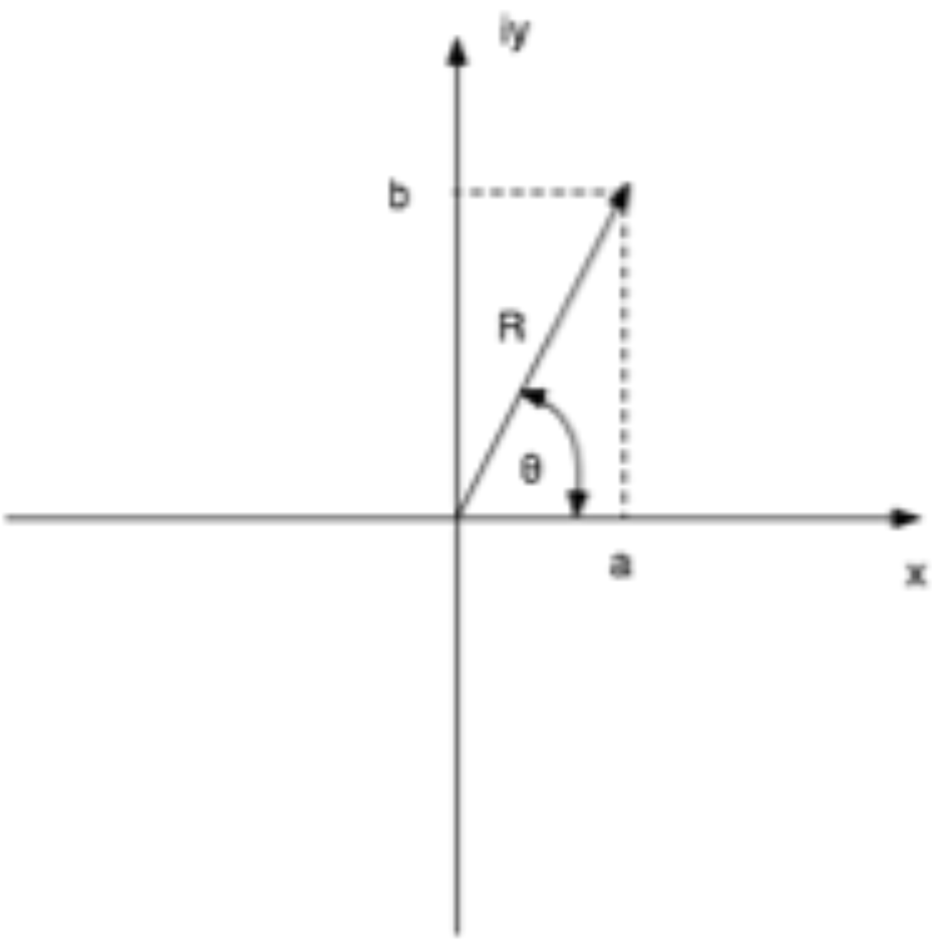
Point of the graph is to represent quantities (numbers with units) as lengths according to some scale.

Should just put y on vertical axis —> complex number is of form

$$z = (x \text{ coordinate}) + i(y \text{ coordinate})$$

Reason not done —> I wanted to **emphasize** that in figures **neither** x direction nor y direction is an actual physical direction in space.

Figure just represents a geometric way of **picturing** complex numbers.



On the plane - always more than one way of getting to particular point in 2 dimensions.

Rather than walking specific distances along two lines, can approach same point by more direct route.

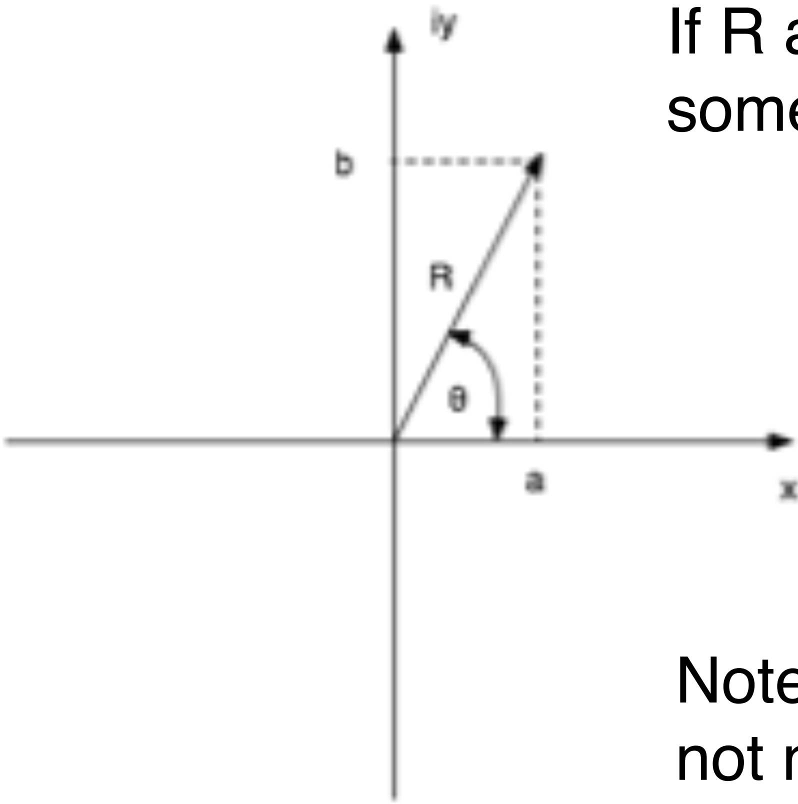
Stand on zero point(origin) looking along real line.

Turn in counterclockwise (or anticlockwise) direction

through angle θ and walk in that direction distance R as shown

—> end up at exactly same point a + ib

Can think of line of length R as hand of a clock and θ the angle that hand sweeps out (starts sweeping at 3 o'clock (+ x-axis) and goes backwards (anticlockwise)!).



If R and θ get to the same place as a and b —> then there must exist some formulas that connects two sets of numbers.

They are

$$a = R\cos\theta \ , \ b = R\sin\theta \rightarrow \frac{b}{a} = \tan\theta \ , \ R^2 = a^2 + b^2$$

Note that R is **always** positive. i.e., by convention, one does not move negative distance away from the zero point.

—> Thus, have two ways of writing same complex number as 2-tuples, either as z = a + ib = (a,b) or z = (R,θ) (derive exact formula later).

Magnitude and Phase

1st think about R ==> **magnitude** of complex number.

Complex conjugate of complex number z = a+ib is z^*= a−ib.

If multiply complex number by its conjugate —> real number.

This number is $zz^* = (a + ib)(a - ib) = a^2 + b^2 = R^2$ using Pythagorean ideas

—> mathematical way to convert complex number into real number.

Note that earlier, same procedure gave us probabilities from probability amplitudes.

Now switch ideas around.

Many complex numbers have the same R.

Draw circle around zero point(origin) of radius R;

All points on circumference of that circle —> complex numbers with same R value.

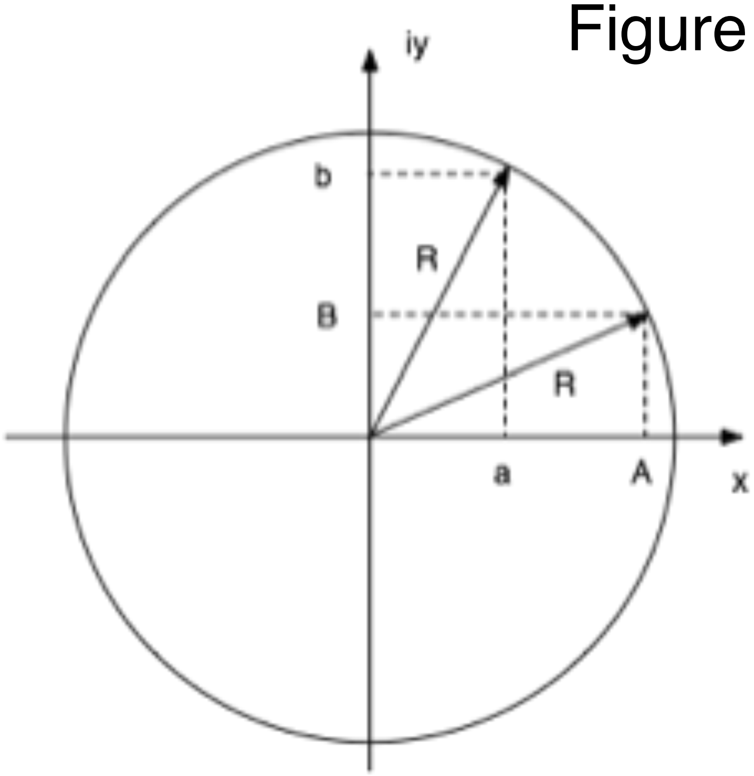


Figure illustrates idea.

Complex numbers z = a + ib and Z = A+iB have same R

—> equal magnitudes of two complex numbers.

Both lead to same absolute value.

Infinite number of such complex numbers —> same absolute value.

All complex numbers with same R differ only in value of θ —> **argument** (mathematicians) of complex number;

Physics —> call θ **the phase** of complex number.

Will see it corresponds directly to physical phase used earlier in interference experiments.

Will turn out that magnitude of complex amplitudes is related to observable probability and phase is connected to quantum interference effects.

Multiplying Complex Numbers

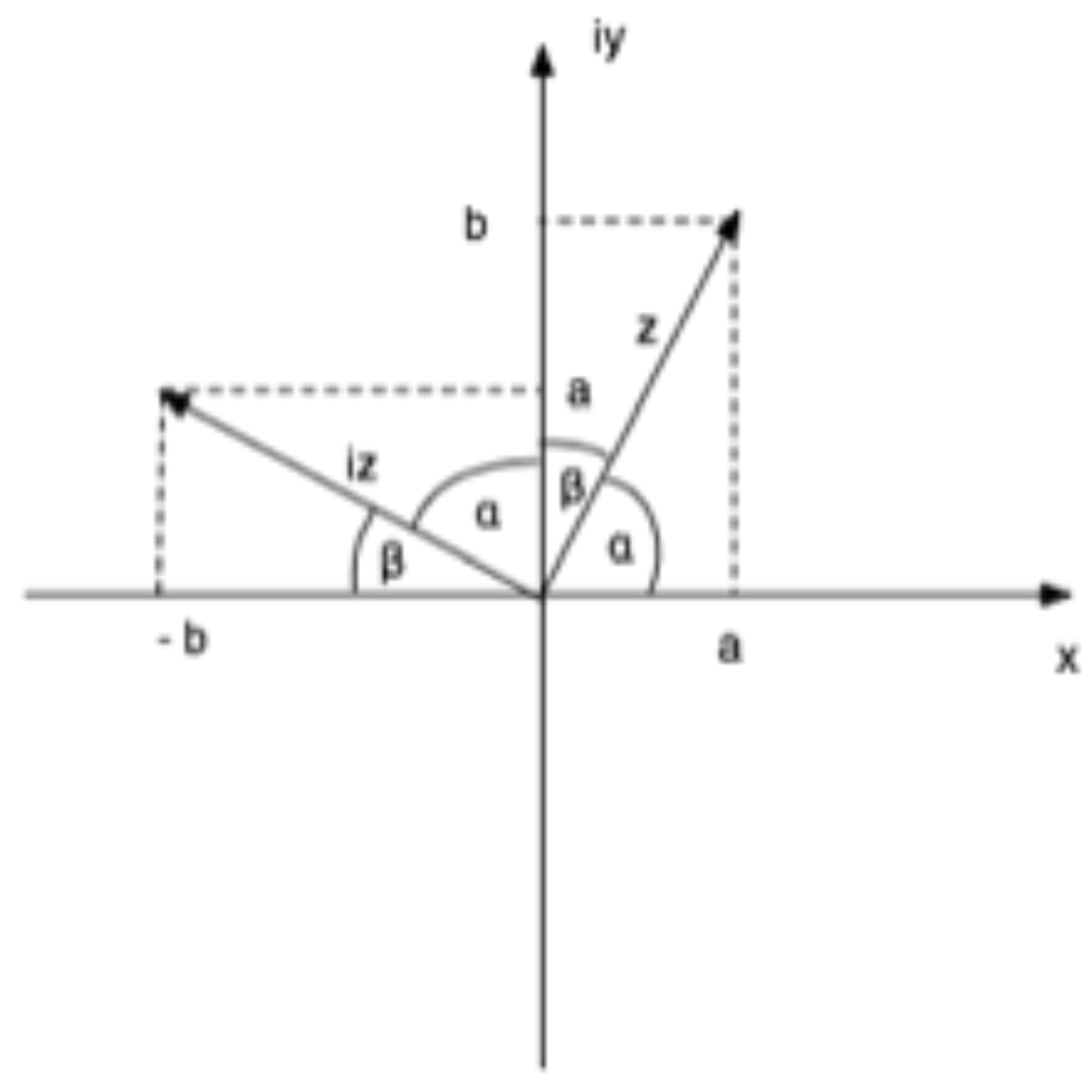
If can write complex number z = a + ib in form (R,θ),

then how would we write iz?

iz would be i(a + ib) = ia – b or –b + ia

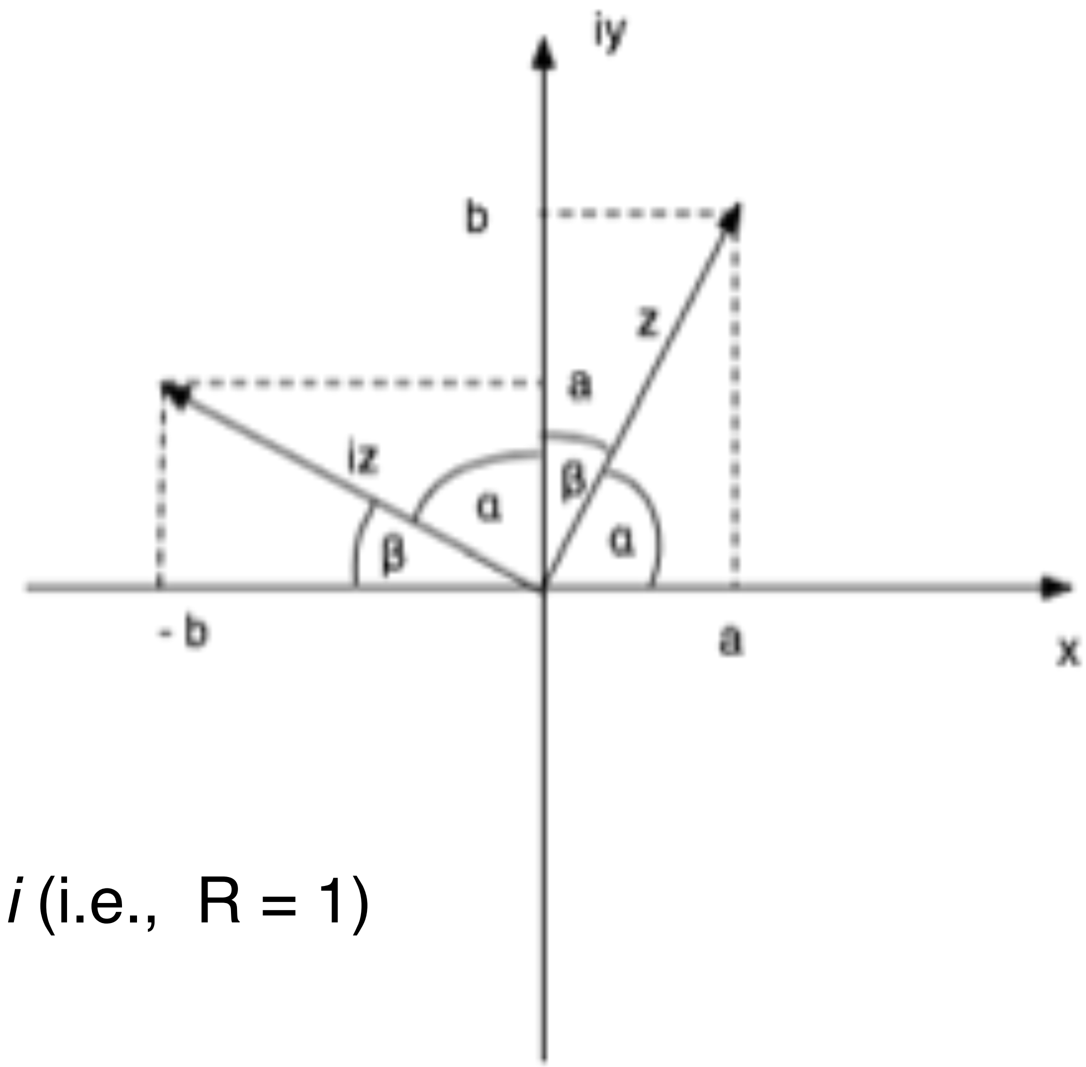
-> complex numbers z and iz have same magnitude, R.

However, have different phase, i.e., see figure.



clearly, phases of z and iz differ by 90°,

$$2\alpha + 2\beta = \pi \rightarrow \alpha + \beta = \pi/2 \rightarrow orthogonal$$



Thus, whenever multiply by i, rotate phase anticlockwise by 90°, but leave magnitude alone.

Now push these geometrical ideas further.

i is number that lies one unit along imaginary axis.

Lots of complex numbers have the same magnitude as *i* (i.e., R = 1) but different phases.

Number 1 (= 1 + 0i) for example.

What happens if we multiply by other numbers instead?

Take number z = a + ib and multiply by another number w = p + iq such that $p^2 + q^2 = 1$ —> R = 1 for w.

$$zw = (a + ib)(p + iq) = ap + iaq + ibp - bq = (ap - bq) + i(aq + bp)$$

What is magnitude of new number?

Square of magnitude = square of real part + square of imaginary part.

$$R^2 = (ap - bq)^2 + (aq + bp)^2 = a^2p^2 + b^2q^2 - 2apbq - a^2q^2 - b^2p^2 - 2aqbp$$

$$= a^2p^2 + b^2q^2 + a^2q^2 + b^2p^2 = a^2(p^2 + q^2) + b^2(p^2 + q^2) \; = a^2 + b^2$$ using $p^2 + q^2 = 1$.

Thus, multiplying any complex number z by another number complex number w, when w has magnitude 1(R=1), **doesn’t change** magnitude of z.

What about phase though?

Tougher to prove - can show, however, that phases **add**.

Easy proof later using exponential form of complex numbers and harder proof now using trigonometry (see below).

In fact, the general rule is derived as follows:

MULTIPLYING COMPLEX NUMBERS

When two complex numbers z and w are multiplied together, magnitudes multiply and **phases add**.

$$If\ z = (R_1, \theta_1)\ and\ w = (R_2, \theta_2),\ then\ zw = (R_1 R_2, \theta_1 + \theta_2)$$

Proof:(using trigonometry)

$$z = a + ib \quad , \quad w = p + iq \qquad\qquad \tan\theta_z = \frac{b}{a} \quad , \quad \tan\theta_w = \frac{q}{p}$$

$$zw = (ap - bq) + i(aq + bp)$$

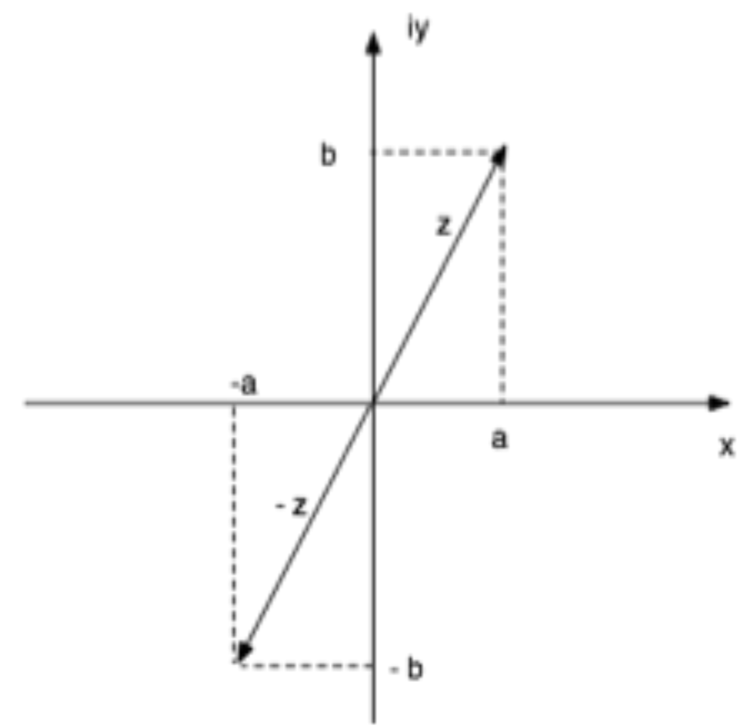
$$\tan\theta_{zw} = \frac{aq + bp}{ap - bq} = \frac{\frac{q}{p} + \frac{b}{a}}{1 - \frac{q}{p}\frac{b}{a}} = \frac{\tan\theta_z + \tan\theta_w}{1 - \tan\theta_z \tan\theta_w} = \tan(\theta_z + \theta_w) \qquad \Rightarrow \theta_{zw} = \theta_z + \theta_w$$

More Phase Stuff

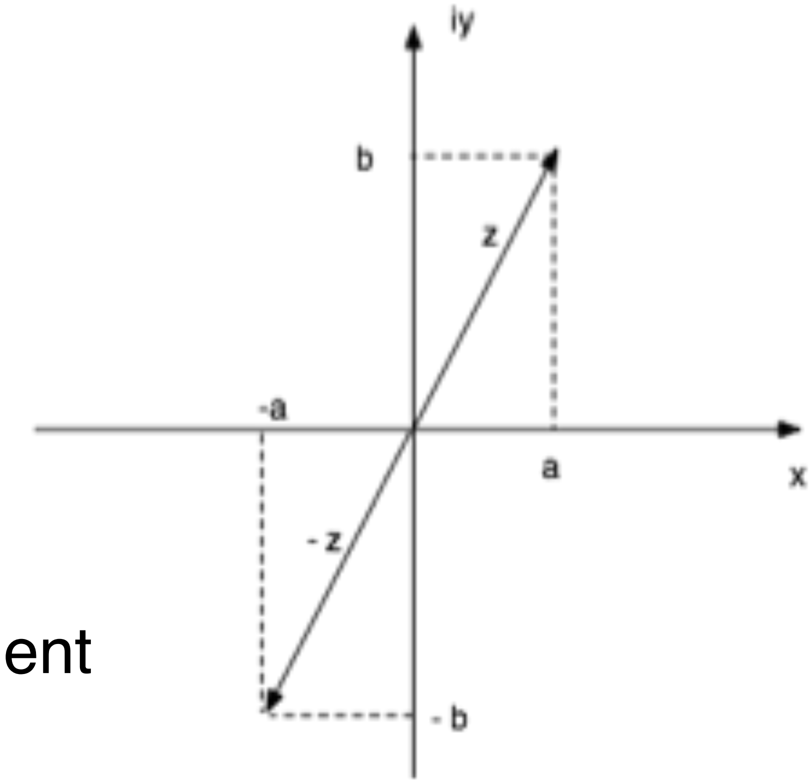
Figure shows how two complex numbers

z = a + ib   and   –z = –a – ib

are related on the complex plane.



z and –z have same R and phases different by 180° —> equivalent to multiplying z by i and again by i, as $i^2 = -1$.



Each multiplication by i shifts phase by 90°, hence 180° overall.

And Now Pulling It All Together

Complex numbers can be represented in two different, but equivalent ways

-> two numbers a and b —> $z = a+ib = (a,b)$

or

two numbers R and θ —> (R,θ)

Another way of writing complex numbers —> combines elements of both forms.

Need to review some other mathematics first (stated these relations earlier).

Some functions in mathematics represented by power series (mentioned earlier) —> real definitions of functions,

i.e., how we calculate them.

So as we mentioned earlier….

A power series representation of a function:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Special cases

$$e^{\alpha x} = 1 + \alpha + \frac{1}{2}\alpha^2 + \frac{1}{6}\alpha^3 + ... = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} x^n$$

$$\sin \alpha x = \alpha x - \frac{1}{6}(\alpha x)^3 + \frac{1}{24}(\alpha x)^5 + ... = \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n+1}}{(2n+1)!} x^{2n+1}$$

$$\cos \alpha x = 1 - \frac{1}{2}(\alpha x)^2 + \frac{1}{16}(\alpha x)^4 + ... = \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n}}{(2n)!} x^{2n}$$

Expansions are still valid if α is a complex number —> important mathematical result for QM.

Now

expand out the sum

$$e^{i\alpha x} = \sum_{n=0}^{\infty} \frac{i^n \alpha^n}{n!} x^n = 1 + i\alpha x - \frac{\alpha^2}{2!}x^2 - i\frac{\alpha^3}{3!}x^3 + \frac{\alpha^4}{4!}x^4 + i\frac{\alpha^5}{5!}x^5 - ....$$

rearrange terms

$$= \left(1 - \frac{\alpha^2}{2!}x^2 + \frac{\alpha^4}{4!}x^4 - ....\right) + \left(i\alpha x - i\frac{\alpha^3}{3!}x^3 + i\frac{\alpha^5}{5!}x^5 - ....\right)$$

recognize new series

$$= \cos\alpha x + i\sin\alpha x$$

—-> Euler relation (derived in 16th century)

Then implies that(simple algebra)

$$\sin\alpha x = \frac{e^{i\alpha x} - e^{-i\alpha x}}{2i} \quad , \quad \cos\alpha x = \frac{e^{i\alpha x} + e^{-i\alpha x}}{2}$$

Euler relation —> defines i without using square root of negative number!!

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 \qquad e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = i$$

Since

$$e^{a+b} = e^a e^b \qquad e^a = e^{a/2}e^{a/2} \rightarrow e^{a/2} = \sqrt{e^a}$$ and $$(e^a)^n = e^{na}$$

$$\sqrt{e^{i\pi}} = e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = i$$

Now from earlier we had

$$a = R\cos\theta \ , \ \ b = R\sin\theta \rightarrow \frac{b}{a} = \tan\theta \ , \ \ R^2 = a^2 + b^2$$

or

$$z = a + ib = R\cos\theta + iR\sin\theta = R(\cos\theta + i\sin\theta) = Re^{i\theta}$$

new form of complex number

Write complex conjugate z^* using exponential form.

$$z* = a - ib = R\cos\theta - iR\sin\theta = R\left[\cos(-\theta) + i\sin(-\theta)\right] = Re^{-i\theta}$$

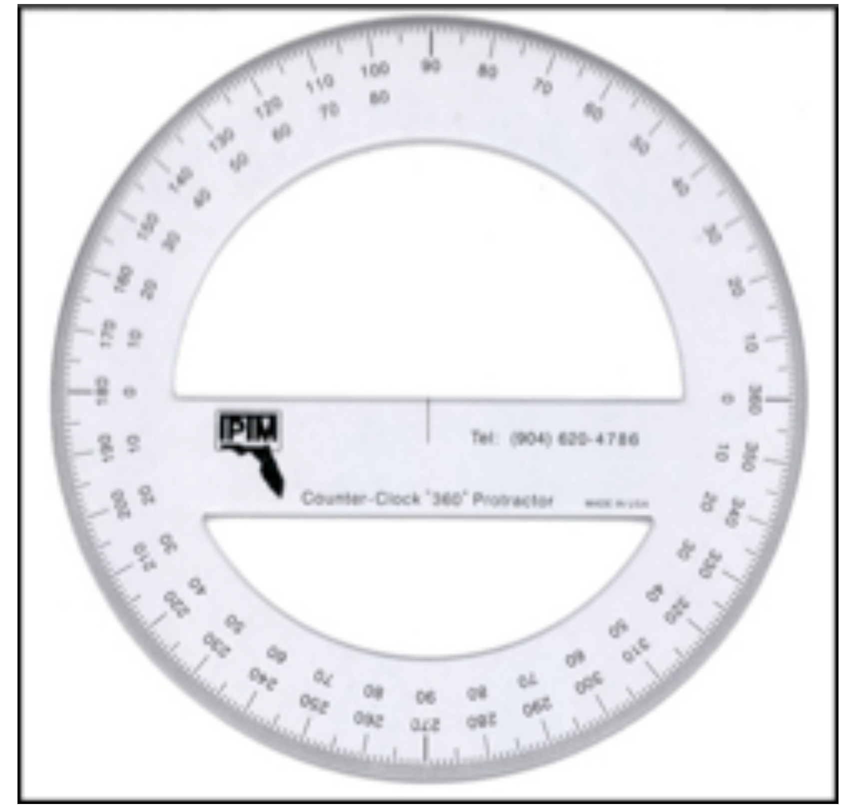
Since using cosθ/sinθ this way —> not directly related to geometry —> convenient to use different measure for θ than angle(degrees)

Mathematicians/physicists call new measure the **radian**.

Normal way measure angle —> take circle, divide circumference into 360 pieces .

Draw lines from ends of pieces to center

—> protractor.   Angle between pair of lines = 1°.



No real reason for 360 pieces.

A more interesting possibility is….

Circumference = $2\pi \times$ radius

—> imagine dividing circumference into 2π pieces.

Cannot do(not exactly) since π requires an infinite number of digits to define it.

OK since not interested in making a real protractor this way, but just finding a mathematically convenient measure of angle.

If divided circle into 2π pieces, then angle between pairs of lines = 1 radian.

Whole circle, = 360° = 2π radians. 90° = 1/4 circle = $2\pi/4$ = $\pi/2$ radians.

180° = π radians, etc.

Finally, a very useful property of the exponential form of complex numbers.

Earlier, multiplication rule

—> when multiply z_1 by z_2, magnitudes multiply and phases add.

Easier proof:

$$z_1 z_2 = \mathrm{R}_1 \ \exp(i\theta_1)\mathrm{R}_2 \ \exp(i\theta_2) = (R_1 R_2) \exp\left(i(\theta_1 + \theta_2)\right)$$

Proof now a one-liner.
That is why you keep expanding
mathematical knowledge.

Now Returning to Phase and Physics

Now we can explain electron interference pattern revealed by detectors on screen,

given the amplitudes that govern two possible routes from slits to detector.

Existence of dark bands, parts of detector where no electrons arrive

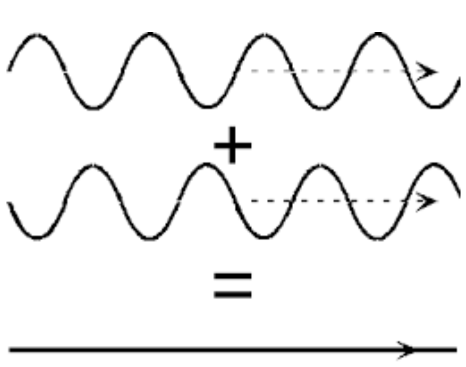
= key point we need to factor into our thinking.

When add up amplitudes for getting dark point on detector, **total** amplitude must be **zero**.

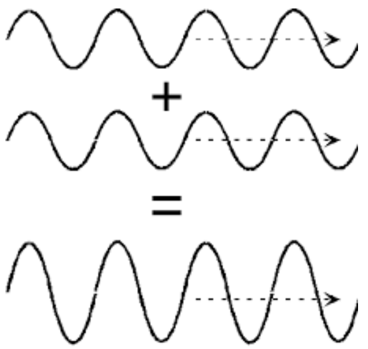
How can we be sure this happens?

If thinking of any waves interfering

—> saying path difference between two routes equivalent to multiple of half of wavelength, so that waves arrived **out of phase**.



out of phase.



in phase.

Perhaps something similar(mathematically) is happening with amplitudes.

To explain dark band at position x on detector, must have

$$\langle x \mid I\rangle = \langle x \mid A\rangle \langle A \mid I\rangle + \langle x \mid B\rangle \langle B \mid I\rangle = 0$$

Set both ⟨A|I⟩ and ⟨B|I⟩ equal same term ⟨slits|I⟩ for simplicity, I.e., if source midway between slits, these amplitudes are the ~ same.

Thus we get

$$\langle x \mid I\rangle = [\langle x \mid A\rangle + \langle x \mid B\rangle] \langle slits \mid I\rangle = 0$$

Two ways this can be true.

⟨slits | I⟩ = 0 —> electrons not getting to slits —> silly = Non-physical.

Or $\langle x \mid A \rangle + \langle x \mid B \rangle = 0$ —> related to interference.

Both true —> set up poor experiment.

If $\langle x|A \rangle + \langle x|B \rangle = 0$, then $\langle x|A \rangle = -\langle x|B \rangle$

—> two complex numbers with same magnitude, but opposite phases (like z and –z).

Suppose change position of one slit without moving other —> changes length of one path compared to other —> convert a point with destructive interference into one with constructive interference.

To do —> moved slit so that path altered by distance equal to half wavelength.

Question: does this alter phase or magnitude, or both?

Move slit even further —> path difference is $3\lambda/2$ -> destructive interference

-> two amplitudes once again, have same magnitude but opposite phases.

Unlikely simply increasing path difference —> magnitude changes.

However, can imagine alterations to path changing phase of amplitude.

Remember how phase was defined(as # wavelengths in path).

If amplitude of form $Re^{i\theta}$ and θ determined by path length, then steadily increasing path would increase angle, taking us round and round circle in complex plane.

Mathematicians don't limit angles to ≤ 360° (or 2π radians), i.e., angle of 370° —> once round circle and 10° more.

Comparing two amplitudes of this type could quite easily move us from situation where their phases are identical to out of phase and back again as path length changed.

If right, then **wavelength** associated with electron is somehow **coded** into the **amplitude phase**.

$$\lambda = \frac{h}{p}$$ from DeBroglie

So finally : An Experiment with Phase

Write two amplitudes(now just complex numbers) in the form

$$\langle x \mid A\rangle = R_1 \exp\left(i\theta_1(x,t)\right) \quad , \quad \langle x \mid B\rangle = R_2 \exp\left(i\theta_2(x,t)\right)$$

—> phase θ depends on position and time.

Put into probability calculation —> after wild algebra (either ignore it and jump to result

or if more mathematically inclined work through algebra).

$$\begin{aligned}
&|\langle x \mid I\rangle|^2 = |\langle x \mid A\rangle + \langle x \mid B\rangle|^2 \, |\langle slits \mid I\rangle|^2 \\
&= |R_1 \exp(i\theta_1(x,t)) + R_2 \exp(i\theta_2(x,t))|^2 \, |\langle slits \mid I\rangle|^2 \\
&= [R_1 \exp(i\theta_1(x,t)) + R_2 \exp(i\theta_2(x,t))]^* [R_1 \exp(i\theta_1(x,t)) + R_2 \exp(i\theta_2(x,t))] \, |\langle slits \mid I\rangle|^2 \\
&= [R_1 \exp(-i\theta_1(x,t)) + R_2 \exp(-i\theta_2(x,t))] [R_1 \exp(i\theta_1(x,t)) + R_2 \exp(i\theta_2(x,t))] \, |\langle slits \mid I\rangle|^2 \\
&= [R_1 (cos(-\theta_1(x,t)) + i \sin(-\theta_1(x,t))) + R_2 (cos(-\theta_2(x,t)) + i \sin(-\theta_2(x,t)))] \\
&\quad \times [R_1 (cos(\theta_1(x,t)) + i \sin(\theta_1(x,t))) + R_2 (cos(\theta_2(x,t)) + i \sin(\theta_2(x,t)))] \, |\langle slits \mid I\rangle|^2 \\
&= [R_1 (cos(\theta_1(x,t)) - i \sin(\theta_1(x,t))) + R_2 (cos(\theta_2(x,t)) - i \sin(\theta_2(x,t)))] \\
&\quad \times [R_1 (cos(\theta_1(x,t)) + i \sin(\theta_1(x,t))) + R_2 (cos(\theta_2(x,t)) + i \sin(\theta_2(x,t)))] \, |\langle slits \mid I\rangle|^2 \\
&= \begin{bmatrix} R_1^2 \left(cos^2(\theta_1(x,t)) + \sin^2(\theta_1(x,t))\right) + R_2^2 \left(cos^2(\theta_2(x,t)) + \sin^2(\theta_2(x,t))\right) \\ +2R_1R_2 \left(cos(\theta_1(x,t)) \, cos(\theta_2(x,t)) + \sin(\theta_1(x,t)) \sin(\theta_2(x,t))\right) \\ +iR_1R_2 \left(cos(\theta_1(x,t)) \sin(\theta_2(x,t)) + \sin(\theta_1(x,t)) \, cos(\theta_2(x,t))\right) \\ -iR_1R_2 \left(cos(\theta_1(x,t)) \sin(\theta_2(x,t)) + \sin(\theta_1(x,t)) \, cos(\theta_2(x,t))\right) \end{bmatrix} |\langle slits \mid I\rangle|^2 \\
&= \left[R_1^2 + R_2^2 + 2R_1R_2 \left(cos(\theta_1(x,t)) \, cos(\theta_2(x,t)) + \sin(\theta_1(x,t)) \sin(\theta_2(x,t))\right)\right] \, |\langle slits \mid I\rangle|^2
\end{aligned}$$

In last line:

1st two terms

-> uniform probability for electron to arrive at detector(= **classical probability result**).

If slit 2(or B) **blocked** off

—> R_1^2 (the bump) and if slit 1(or A) **blocked** off, probability —> R_2^2 (the other bump).

Last term(with cosine) -> interference pattern of light and dark bands across detector.

The Interference Term

Whole interference effect relies on path difference between two routes from slit to screen.

Size of path difference not large enough to affect intensity of electrons arriving at screen from each slit.

So two slits close together —> assume that $R_1 = R_2 = R$.

Using this —>

$$|\langle x \mid I\rangle|^2 = 2R^2\left[1 + \left(cos\left(\theta_1(x,t) - \theta_2(x,t)\right)\right)\right]|\langle slits \mid I\rangle|^2$$

1st dark band at $\theta_1(x, t) - \theta_2(x, t) = \pi$

Next dark band when $\theta_1(x, t) - \theta_2(x, t) = 3\pi$; another at 5π, etc.

So interference term guarantees series of dark bands at regular intervals across electron detector.

Theoretical formalism seems to work —> explains this particular experiment.

Spatial dependence of phase given by $$\theta = px/\hbar = 2\pi x/\lambda$$

If distance x is different for two amplitudes, then get interference pattern given by

$$A' = Ae^{2\pi i x_1/\lambda} + Ae^{2\pi i x_2/\lambda}$$

Gives intensity or brightness on screen of

$$I = |A'|^2 = |A|^2 \left| \left(e^{2\pi i x_1/\lambda} + e^{2\pi i x_2/\lambda} \right) \right|^2 \quad = 4\,|A|^2 \left(1 + \cos\left(2\pi(x_2 - x_1)/\lambda\right)\right)$$

which agrees with classical result and the experiment for the interference pattern!

So our theory of QM works for a real, basic experiment!

We now proceed as follows:

(1) First, we look at

(a) Time development and how to use QM
(b) Uncertainty Principle
(c) Interferometers; Delayed Choice; Amazing Experiments

(2) We then apply our theory to study

(a) Entanglement
(b) Einstein-Rosen Paradox
(c) Bell's Theorem

(3) Finally we look and solve at the measurement problem

(a) Importance of superposition and entanglement
(b) Schrodinger's cat
(c) What happens in a measurement
(d) Does anything collapse