

Preface

The present volume is an outgrowth of a Conference on Mathematical Tables held at Cambridge, Mass., on September 15-16, 1954, under the auspices of the National Science Foundation and the Massachusetts Institute of Technology. The purpose of the meeting was to evaluate the need for mathematical tables in the light of the availability of large scale computing machines. It was the consensus of opinion that in spite of the increasing use of the new machines the basic need for tables would continue to exist.

Numerical tables of mathematical functions are in continual demand by scientists and engineers. A greater variety of functions and higher accuracy of tabulation are now required as a result of scientific advances and, especially, of the increasing use of automatic computers. In the latter connection, the tables serve mainly for preliminary surveys of problems before programming for machine operation. For those without easy access to machines, such tables are, of course, indispensable.

Consequently, the Conference recognized that there was a pressing need for a modernized version of the classical tables of functions of Jahnke-Emde. To implement the project, the National Science Foundation requested the National Bureau of Standards to prepare such a volume and established an Ad Hoc Advisory Committee, with Professor Philip M. Morse of the Massachusetts Institute of Technology as chairman, to advise the staff of the National Bureau of Standards during the course of its preparation. In addition to the Chairman, the Committee consisted of A. Erdélyi, M. C. Gray, N. Metropolis, J. B. Rosser, H. C. Thacher, Jr., John Todd, C. B. Tompkins, and J. W. Tukey.

The primary aim has been to include a maximum of useful information within the limits of a moderately large volume, with particular attention to the needs of scientists in all fields. An attempt has been made to cover the entire field of special functions. To carry out the goal set forth by the Ad Hoc Committee, it has been necessary to supplement the tables by including the mathematical properties that are important in computation work, as well as by providing numerical methods which demonstrate the use and extension of the tables.

The Handbook was prepared under the direction of the late Milton Abramowitz, and Irene A. Stegun. Its success has depended greatly upon the cooperation of many mathematicians. Their efforts together with the cooperation of the Ad Hoc Committee are greatly appreciated. The particular contributions of these and other individuals are acknowledged at appropriate places in the text. The sponsorship of the National Science Foundation for the preparation of the material is gratefully recognized.

It is hoped that this volume will not only meet the needs of all table users but will in many cases acquaint its users with new functions.

ALLEN V. ASTIN, *Director.*

Washington, D.C.

Preface to the Ninth Printing

The enthusiastic reception accorded the "Handbook of Mathematical Functions" is little short of unprecedented in the long history of mathematical tables that began when John Napier published his tables of logarithms in 1614. Only four and one-half years after the first copy came from the press in 1964, Myron Tribus, the Assistant Secretary of Commerce for Science and Technology, presented the 100,000th copy of the Handbook to Lee A. DuBridge, then Science Advisor to the President. Today, total distribution is approaching the 150,000 mark at a scarcely diminished rate.

The success of the Handbook has not ended our interest in the subject. On the contrary, we continue our close watch over the growing and changing world of computation and to discuss with outside experts and among ourselves the various proposals for possible extension or supplementation of the formulas, methods and tables that make up the Handbook.

In keeping with previous policy, a number of errors discovered since the last printing have been corrected. Aside from this, the mathematical tables and accompanying text are unaltered. However, some noteworthy changes have been made in Chapter 2: Physical Constants and Conversion Factors, pp. 6-8. The table on page 7 has been revised to give the values of physical constants obtained in a recent reevaluation; and pages 6 and 8 have been modified to reflect changes in definition and nomenclature of physical units and in the values adopted for the acceleration due to gravity in the revised Potsdam system.

The record of continuing acceptance of the Handbook, the praise that has come from all quarters, and the fact that it is one of the most-quoted scientific publications in recent years are evidence that the hope expressed by Dr. Astin in his Preface is being amply fulfilled.

LEWIS M. BRANSCOMB, *Director*
National Bureau of Standards

November 1970

Foreword

This volume is the result of the cooperative effort of many persons and a number of organizations. The National Bureau of Standards has long been turning out mathematical tables and has had under consideration, for at least 10 years, the production of a compendium like the present one. During a Conference on Tables, called by the NBS Applied Mathematics Division on May 15, 1952, Dr. Abramowitz of that Division mentioned preliminary plans for such an undertaking, but indicated the need for technical advice and financial support.

The Mathematics Division of the National Research Council has also had an active interest in tables; since 1943 it has published the quarterly journal, "Mathematical Tables and Aids to Computation" (MTAC), editorial supervision being exercised by a Committee of the Division.

Subsequent to the NBS Conference on Tables in 1952 the attention of the National Science Foundation was drawn to the desirability of financing activity in table production. With its support a 2-day Conference on Tables was called at the Massachusetts Institute of Technology on September 15-16, 1954, to discuss the needs for tables of various kinds. Twenty-eight persons attended, representing scientists and engineers using tables as well as table producers. This conference reached consensus on several conclusions and recommendations, which were set forth in the published Report of the Conference. There was general agreement, for example, "that the advent of high-speed computing equipment changed the task of table making but definitely did not remove the need for tables". It was also agreed that "an outstanding need is for a Handbook of Tables for the Occasional Computer, with tables of usually encountered functions and a set of formulas and tables for interpolation and other techniques useful to the occasional computer". The Report suggested that the NBS undertake the production of such a Handbook and that the NSF contribute financial assistance. The Conference elected, from its participants, the following Committee: P. M. Morse (Chairman), M. Abramowitz, J. H. Curtiss, R. W. Hamming, D. H. Lehmer, C. B. Tompkins, J. W. Tukey, to help implement these and other recommendations.

The Bureau of Standards undertook to produce the recommended tables and the National Science Foundation made funds available. To provide technical guidance to the Mathematics Division of the Bureau, which carried out the work, and to provide the NSF with independent judgments on grants for the work, the Conference Committee was reconstituted as the Committee on Revision of Mathematical Tables of the Mathematics Division of the National Research Council. This, after some changes of membership, became the Committee which is signing this Foreword. The present volume is evidence that Conferences can sometimes reach conclusions and that their recommendations sometimes get acted on.

Active work was started at the Bureau in 1956. The overall plan, the selection of authors for the various chapters, and the enthusiasm required to begin the task were contributions of Dr. Abramowitz. Since his untimely death, the effort has continued under the general direction of Irene A. Stegun. The workers at the Bureau and the members of the Committee have had many discussions about content, style and layout. Though many details have had to be argued out as they came up, the basic specifications of the volume have remained the same as were outlined by the Massachusetts Institute of Technology Conference of 1954.

The Committee wishes here to register its commendation of the magnitude and quality of the task carried out by the staff of the NBS Computing Section and their expert collaborators in planning, collecting and editing these Tables, and its appreciation of the willingness with which its various suggestions were incorporated into the plans. We hope this resulting volume will be judged by its users to be a worthy memorial to the vision and industry of its chief architect, Milton Abramowitz. We regret he did not live to see its publication.

P. M. MORSE, *Chairman.*
A. ERDÉLYI
M. C. GRAY
N. C. METROPOLIS
J. B. ROSSER
H. C. THACHER, JR.
JOHN TODD
C. B. TOMPKINS
J. W. TUKEY.

Handbook of Mathematical Functions

with

Formulas, Graphs, and Mathematical Tables

Edited by Milton Abramowitz and Irene A. Stegun

1. Introduction

The present Handbook has been designed to provide scientific investigators with a comprehensive and self-contained summary of the mathematical functions that arise in physical and engineering problems. The well-known Tables of Functions by E. Jahnke and F. Emde has been invaluable to workers in these fields in its many editions¹ during the past half-century. The present volume extends the work of these authors by giving more extensive and more accurate numerical tables, and by giving larger collections of mathematical properties of the tabulated functions. The number of functions covered has also been increased.

The classification of functions and organization of the chapters in this Handbook is similar to that of *An Index of Mathematical Tables* by A. Fletcher, J. C. P. Miller, and L. Rosenhead.² In general, the chapters contain numerical tables, graphs, polynomial or rational approximations for automatic computers, and statements of the principal mathematical properties of the tabulated functions, particularly those of computa-

tional importance. Many numerical examples are given to illustrate the use of the tables and also the computation of function values which lie outside their range. At the end of the text in each chapter there is a short bibliography giving books and papers in which proofs of the mathematical properties stated in the chapter may be found. Also listed in the bibliographies are the more important numerical tables. Comprehensive lists of tables are given in the Index mentioned above, and current information on new tables is to be found in the National Research Council quarterly *Mathematics of Computation* (formerly *Mathematical Tables and Other Aids to Computation*).

The mathematical notations used in this Handbook are those commonly adopted in standard texts, particularly *Higher Transcendental Functions*, Volumes 1-3, by A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi (McGraw-Hill, 1953-55). Some alternative notations have also been listed. The introduction of new symbols has been kept to a minimum, and an effort has been made to avoid the use of conflicting notation.

2. Accuracy of the Tables

The number of significant figures given in each table has depended to some extent on the number available in existing tabulations. There has been no attempt to make it uniform throughout the Handbook, which would have been a costly and laborious undertaking. In most tables at least five significant figures have been provided, and the tabular intervals have generally been chosen to ensure that linear interpolation will yield four- or five-figure accuracy, which suffices in most physical applications. Users requiring higher

precision in their interpolates may obtain them by use of higher-order interpolation procedures, described below.

In certain tables many-figured function values are given at irregular intervals in the argument. An example is provided by Table 9.4. The purpose of these tables is to furnish "key values" for the checking of programs for automatic computers; no question of interpolation arises.

The maximum end-figure error, or "tolerance" in the tables in this Handbook is $\frac{1}{10}$ of 1 unit everywhere in the case of the elementary functions, and 1 unit in the case of the higher functions except in a few cases where it has been permitted to rise to 2 units.

¹ The most recent, the sixth, with F. Loesch added as co-author, was published in 1960 by McGraw-Hill, U.S.A., and Teubner, Germany.

² The second edition, with L. J. Comrie added as co-author, was published in two volumes in 1962 by Addison-Wesley, U.S.A., and Scientific Computing Service Ltd., Great Britain.

3. Auxiliary Functions and Arguments

One of the objects of this Handbook is to provide tables or computing methods which enable the user to evaluate the tabulated functions over complete ranges of real values of their parameters. In order to achieve this object, frequent use has been made of auxiliary functions to remove the infinite part of the original functions at their singularities, and auxiliary arguments to cope with infinite ranges. An example will make the procedure clear.

The exponential integral of positive argument is given by

$$\begin{aligned} \text{Ei}(x) &= \int_{-\infty}^x \frac{e^u}{u} du \\ &= \gamma + \ln x + \frac{x}{1 \cdot 1!} + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \dots \\ &\sim \frac{e^x}{x} \left[1 + \frac{1!}{x} + \frac{2!}{x^2} + \frac{3!}{x^3} + \dots \right] (x \rightarrow \infty) \end{aligned}$$

The logarithmic singularity precludes direct interpolation near $x=0$. The functions $\text{Ei}(x) - \ln x$ and $x^{-1}[\text{Ei}(x) - \ln x - \gamma]$, however, are well-behaved and readily interpolable in this region. Either will do as an auxiliary function; the latter was in fact selected as it yields slightly higher accuracy when $\text{Ei}(x)$ is recovered. The function $x^{-1}[\text{Ei}(x) - \ln x - \gamma]$ has been tabulated to nine decimals for the range $0 \leq x \leq \frac{1}{2}$. For $\frac{1}{2} \leq x \leq 2$, $\text{Ei}(x)$ is sufficiently well-behaved to admit direct tabulation, but for larger values of x , its exponential character predominates. A smoother and more readily interpolable function for large x is $xe^{-x}\text{Ei}(x)$; this has been tabulated for $2 \leq x \leq 10$. Finally, the range $10 \leq x \leq \infty$ is covered by use of the inverse argument x^{-1} . Twenty-one entries of $xe^{-x}\text{Ei}(x)$, corresponding to $x^{-1} = .1(-.005)0$, suffice to produce an interpolable table.

4. Interpolation

The tables in this Handbook are not provided with differences or other aids to interpolation, because it was felt that the space they require could be better employed by the tabulation of additional functions. Admittedly aids could have been given without consuming extra space by increasing the intervals of tabulation, but this would have conflicted with the requirement that linear interpolation is accurate to four or five figures.

For applications in which linear interpolation is insufficiently accurate it is intended that Lagrange's formula or Aitken's method of iterative linear interpolation³ be used. To help the user, there is a statement at the foot of most tables of the maximum error in a linear interpolate, and the number of function values needed in Lagrange's formula or Aitken's method to interpolate to full tabular accuracy.

As an example, consider the following extract from Table 5.1.

x	$xe^x E_1(x)$	x	$xe^x E_1(x)$
7.5	.89268 7854	8.0	.89823 7113
7.6	.89384 6312	8.1	.89927 7888
7.7	.89497 9666	8.2	.90029 7306
7.8	.89608 8737	8.3	.90129 6073
7.9	.89717 4302	8.4	.90227 4695

$$\left[\begin{matrix} (-6)3 \\ 5 \end{matrix} \right]$$

The numbers in the square brackets mean that the maximum error in a linear interpolate is 3×10^{-6} , and that to interpolate to the full tabular accuracy five points must be used in Lagrange's and Aitken's methods.

Let us suppose that we wish to compute the value of $xe^x E_1(x)$ for $x=7.9527$ from this table. We describe in turn the application of the methods of linear interpolation, Lagrange and Aitken, and of alternative methods based on differences and Taylor's series.

(1) Linear interpolation. The formula for this process is given by

$$f_p = (1-p)f_0 + pf_1$$

where f_0, f_1 are consecutive tabular values of the function, corresponding to arguments x_0, x_1 , respectively; p is the given fraction of the argument interval

$$p = (x - x_0)/(x_1 - x_0)$$

and f_p the required interpolate. In the present instance, we have

$$f_0 = .89717 \ 4302 \quad f_1 = .89823 \ 7113 \quad p = .527$$

The most convenient way to evaluate the formula on a desk calculating machine is to set f_0 and f_1 in turn on the keyboard, and carry out the multiplications by $1-p$ and p cumulatively; a partial check is then provided by the multiplier dial reading unity. We obtain

$$\begin{aligned} f_{.527} &= (1-.527)(.89717 \ 4302) + .527(.89823 \ 7113) \\ &= .89773 \ 4403. \end{aligned}$$

Since it is known that there is a possible error of 3×10^{-6} in the linear formula, we round off this result to .89773. The maximum possible error in this answer is composed of the error committed

³ A. C. Aitken, On interpolation by iteration of proportional parts, without the use of differences, Proc. Edinburgh Math. Soc. 3, 56-76 (1932).

by the last rounding, that is, $.4403 \times 10^{-5}$, plus 3×10^{-6} , and so certainly cannot exceed $.8 \times 10^{-5}$.

(2) Lagrange's formula. In this example, the relevant formula is the 5-point one, given by

$$f = A_{-2}(p)f_{-2} + A_{-1}(p)f_{-1} + A_0(p)f_0 + A_1(p)f_1 + A_2(p)f_2$$

Tables of the coefficients $A_k(p)$ are given in chapter 25 for the range $p=0(.01)1$. We evaluate the formula for $p=.52$, $.53$ and $.54$ in turn. Again, in each evaluation we accumulate the $A_k(p)$ in the multiplier register since their sum is unity. We now have the following subtable.

x	$xe^xE_1(x)$		
7.952	.89772 9757	10622	
7.953	.89774 0379	10620	-2
7.954	.89775 0999		

n	x_n	$y_n = xe^xE_1(x)$	$y_{0,n}$	$y_{0,1,n}$	$y_{0,1,2,n}$	$y_{0,1,2,3,n}$	$x_n - x$
0	8.0	.89823 7113					.0473
1	7.9	.89717 4302	.89773 44034				-.0527
2	8.1	.89927 7888	.89774 48264	.89773 71499			.1473
3	7.8	.89608 8737	2 90220	2394	.89773 71938		-.1527
4	8.2	.90029 7306	4 98773	1216	16	.89773 71930	.2473
5	7.7	.89497 9666	2 35221	2706	43	30	-.2527

Here

$$y_{0,n} = \frac{1}{x_n - x_0} \begin{vmatrix} y_0 & x_0 - x \\ y_n & x_n - x \end{vmatrix}$$

$$y_{0,1,n} = \frac{1}{x_n - x_1} \begin{vmatrix} y_{0,1} & x_1 - x \\ y_{0,n} & x_n - x \end{vmatrix}$$

$$y_{0,1,\dots,m-1,n} = \frac{1}{x_n - x_m} \begin{vmatrix} y_{0,1,\dots,m-1,m} & x_m - x \\ y_{0,1,\dots,m-1,n} & x_n - x \end{vmatrix}$$

If the quantities $x_n - x$ and $x_m - x$ are used as multipliers when forming the cross-product on a desk machine, their accumulation $(x_n - x) - (x_m - x)$ in the multiplier register is the divisor to be used at that stage. An extra decimal place is usually carried in the intermediate interpolates to safeguard against accumulation of rounding errors.

The order in which the tabular values are used is immaterial to some extent, but to achieve the maximum rate of convergence and at the same time minimize accumulation of rounding errors, we begin, as in this example, with the tabular argument nearest to the given argument, then take the nearest of the remaining tabular arguments, and so on.

The number of tabular values required to achieve a given precision emerges naturally in the course of the iterations. Thus in the present example six values were used, even though it was known in advance that five would suffice. The extra row confirms the convergence and provides a valuable check.

(4) Difference formulas. We use the central difference notation (chapter 25),

The numbers in the third and fourth columns are the first and second differences of the values of $xe^xE_1(x)$ (see below); the smallness of the second difference provides a check on the three interpolations. The required value is now obtained by linear interpolation:

$$f_p = .3(.89772\ 9757) + .7(.89774\ 0379)$$

$$= .89773\ 7192.$$

In cases where the correct order of the Lagrange polynomial is not known, one of the preliminary interpolations may have to be performed with polynomials of two or more different orders as a check on their adequacy.

(3) Aitken's method of iterative linear interpolation. The scheme for carrying out this process in the present example is as follows:

x_0	f_0				
		$\delta f_{1/2}$			
x_1	f_1		$\delta^2 f_1$		
		$\delta f_{3/2}$		$\delta^2 f_{3/2}$	
x_2	f_2		$\delta^2 f_2$		$\delta^4 f_2$
		$\delta f_{5/2}$		$\delta^2 f_{5/2}$	
x_3	f_3		$\delta^2 f_3$		
		$\delta f_{7/2}$			
x_4	f_4				

Here

$$\delta f_{1/2} = f_1 - f_0, \delta f_{3/2} = f_2 - f_1, \dots$$

$$\delta^2 f_1 = \delta f_{3/2} - \delta f_{1/2} = f_2 - 2f_1 + f_0$$

$$\delta^2 f_{3/2} = \delta^2 f_2 - \delta^2 f_1 = f_3 - 3f_2 + 3f_1 - f_0$$

$$\delta^4 f_2 = \delta^2 f_{5/2} - \delta^2 f_{3/2} = f_4 - 4f_3 + 6f_2 - 4f_1 + f_0$$

and so on.

In the present example the relevant part of the difference table is as follows, the differences being written in units of the last decimal place of the function, as is customary. The smallness of the high differences provides a check on the function values

x	$xe^xE_1(x)$	$\delta^2 f$	$\delta^4 f$
7.9	.89717 4302	-2 2754	-34
8.0	.89823 7113	-2 2036	-39

Applying, for example, Everett's interpolation formula

$$f_p = (1-p)f_0 + E_2(p)\delta^2 f_0 + E_4(p)\delta^4 f_0 + \dots + pf_1 + F_2(p)\delta^2 f_1 + F_4(p)\delta^4 f_1 + \dots$$

and taking the numerical values of the interpolation coefficients $E_2(p)$, $E_4(p)$, $F_2(p)$ and $F_4(p)$ from Table 25.1, we find that

$$10^9 f_{.527} = .473(89717 \ 4302) + .061196(2 \ 2754) - .012(34) \\ + .527(89823 \ 7113) + .063439(2 \ 2036) - .012(39) \\ = 89773 \ 7193.$$

We may notice in passing that Everett's formula shows that the error in a linear interpolate is approximately

$$E_2(p)\delta^2 f_0 + F_2(p)\delta^2 f_1 \approx \frac{1}{2}[E_2(p) + F_2(p)][\delta^2 f_0 + \delta^2 f_1]$$

Since the maximum value of $|E_2(p) + F_2(p)|$ in the range $0 < p < 1$ is $\frac{1}{8}$, the maximum error in a linear interpolate is approximately

$$\frac{1}{16} |\delta^2 f_0 + \delta^2 f_1|, \text{ that is, } \frac{1}{16} |f_2 - f_1 - f_0 + f_{-1}|.$$

(5) Taylor's series. In cases where the successive derivatives of the tabulated function can be computed fairly easily, Taylor's expansion

$$f(x) = f(x_0) + (x - x_0) \frac{f'(x_0)}{1!} + (x - x_0)^2 \frac{f''(x_0)}{2!} \\ + (x - x_0)^3 \frac{f'''(x_0)}{3!} + \dots$$

5. Inverse Interpolation

With linear interpolation there is no difference in principle between direct and inverse interpolation. In cases where the linear formula provides an insufficiently accurate answer, two methods are available. We may interpolate directly, for example, by Lagrange's formula to prepare a new table at a fine interval in the neighborhood of the approximate value, and then apply accurate inverse linear interpolation to the subtabulated values. Alternatively, we may use Aitken's method or even possibly the Taylor's series method, with the roles of function and argument interchanged.

It is important to realize that the accuracy of an inverse interpolate may be very different from that of a direct interpolate. This is particularly true in regions where the function is slowly varying, for example, near a maximum or minimum. The maximum precision attainable in an inverse interpolate can be estimated with the aid of the formula

$$\Delta x \approx \Delta f / \frac{df}{dx}$$

in which Δf is the maximum possible error in the function values.

Example. Given $xe^x E_1(x) = .9$, find x from the table on page X.

(i) Inverse linear interpolation. The formula for p is

$$p = (f_p - f_0) / (f_1 - f_0).$$

In the present example, we have

$$p = \frac{.9 - .89927 \ 7888}{.90029 \ 7306 - .89927 \ 7888} = \frac{72 \ 2112}{101 \ 9418} = .708357.$$

can be used. We first compute as many of the derivatives $f^{(n)}(x_0)$ as are significant, and then evaluate the series for the given value of x . An advisable check on the computed values of the derivatives is to reproduce the adjacent tabular values by evaluating the series for $x = x_{-1}$ and x_1 .

In the present example, we have

$$f(x) = xe^x E_1(x) \\ f'(x) = (1 + x^{-1})f(x) - 1 \\ f''(x) = (1 + x^{-1})f'(x) - x^{-2}f(x) \\ f'''(x) = (1 + x^{-1})f''(x) - 2x^{-2}f'(x) + 2x^{-3}f(x).$$

With $x_0 = 7.9$ and $x - x_0 = .0527$ our computations are as follows; an extra decimal has been retained in the values of the terms in the series to safeguard against accumulation of rounding errors.

k	$f^{(k)}(x_0)/k!$	$(x - x_0)^k f^{(k)}(x_0)/k!$
0	.89717 4302	.89717 4302
1	.01074 0669	.00056 6033 3
2	-.00113 7621	-.00000 3159 5
3	.00012 1987	.00000 0017 9
		.89773 7194

The desired x is therefore

$$x = x_0 + p(x_1 - x_0) = 8.1 + .708357(.1) = 8.17083 \ 57$$

To estimate the possible error in this answer, we recall that the maximum error of direct linear interpolation in this table is $\Delta f = 3 \times 10^{-6}$. An approximate value for df/dx is the ratio of the first difference to the argument interval (chapter 25), in this case .010. Hence the maximum error in x is approximately $3 \times 10^{-6} / (.010)$, that is, .0003.

(ii) Subtabulation method. To improve the approximate value of x just obtained, we interpolate directly for $p = .70, .71$ and $.72$ with the aid of Lagrange's 5-point formula,

x	$xe^x E_1(x)$	δ	δ^2
8.170	.89999 3683		
8.171	.90000 3834	1 0151	
8.172	.90001 3983	1 0149	-2

Inverse linear interpolation in the new table gives

$$p = \frac{.9 - .89999 \ 3683}{.00001 \ 0151} = .6223$$

Hence $x = 8.17062 \ 23$.

An estimate of the maximum error in this result is

$$\Delta f / \frac{df}{dx} \approx \frac{1 \times 10^{-9}}{.010} = 1 \times 10^{-7}$$

(iii) Aitken's method. This is carried out in the same manner as in direct interpolation.

n	$y_n = xe^x E_1(x)$	x_n	$x_{0,n}$	$x_{0,1,n}$	$x_{0,1,2,n}$	$x_{0,1,2,3,n}$	$y_n - y$
0	.90029 7306	8.2					.00029 7306
1	.89927 7888	8.1	8.17083 5712				-.00072 2112
2	.90129 6033	8.3	8.17023 1505	8.17061 9521			.00129 6033
3	.89823 7113	8.0	8.17113 8043	2 5948	8.17062 2244		-.00176 2887
4	.90227 4695	8.4	8.16992 9437	1 7335	415	8.17062 2318	.00227 4695
5	.89717 4302	7.9	8.17144 0382	2 8142	231	265	-.00282 5698

The estimate of the maximum error in this result is the same as in the subtabulation method. An indication of the error is also provided by the

discrepancy in the highest interpolates, in this case $x_{0,1,2,3,4}$, and $x_{0,1,2,3,5}$.

6. Bivariate Interpolation

Bivariate interpolation is generally most simply performed as a sequence of univariate interpolations. We carry out the interpolation in one direction, by one of the methods already described, for several tabular values of the second argument in the neighborhood of its given value. The interpolates are differenced as a check, and

interpolation is then carried out in the second direction.

An alternative procedure in the case of functions of a complex variable is to use the Taylor's series expansion, provided that successive derivatives of the function can be computed without much difficulty.

7. Generation of Functions from Recurrence Relations

Many of the special mathematical functions which depend on a parameter, called their index, order or degree, satisfy a linear difference equation (or recurrence relation) with respect to this parameter. Examples are furnished by the Legendre function $P_n(x)$, the Bessel function $J_n(x)$ and the exponential integral $E_n(x)$, for which we have the respective recurrence relations

$$(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0$$

$$J_{n+1} - \frac{2n}{x}J_n + J_{n-1} = 0$$

$$nE_{n+1} + xE_n = e^{-x}.$$

Particularly for automatic work, recurrence relations provide an important and powerful computing tool. If the values of $P_n(x)$ or $J_n(x)$ are known for two consecutive values of n , or $E_n(x)$ is known for one value of n , then the function may be computed for other values of n by successive applications of the relation. Since generation is carried out perforce with rounded values, it is vital to know how errors may be propagated in the recurrence process. If the errors do not grow relative to the size of the wanted function, the process is said to be stable. If, however, the relative errors grow and will eventually overwhelm the wanted function, the process is unstable.

It is important to realize that stability may depend on (i) the particular solution of the difference equation being computed; (ii) the values of x or other parameters in the difference equation;

(iii) the direction in which the recurrence is being applied. Examples are as follows.

Stability—increasing n

$$P_n(x), P_n''(x)$$

$$Q_n(x), Q_n''(x) \quad (x < 1)$$

$$Y_n(x), K_n(x)$$

$$J_{-n-\frac{1}{2}}(x), I_{-n-\frac{1}{2}}(x)$$

$$E_n(x) \quad (n < x)$$

Stability—decreasing n

$$P_n(x), P_n''(x) \quad (x < 1)$$

$$Q_n(x), Q_n''(x)$$

$$J_n(x), I_n(x)$$

$$J_{n+\frac{1}{2}}(x), I_{n+\frac{1}{2}}(x)$$

$$E_n(x) \quad (n > x)$$

$$F_n(\eta, \rho) \quad (\text{Coulomb wave function})$$

Illustrations of the generation of functions from their recurrence relations are given in the pertinent chapters. It is also shown that even in cases where the recurrence process is unstable, it may still be used when the starting values are known to sufficient accuracy.

Mention must also be made here of a refinement, due to J. C. P. Miller, which enables a recurrence process which is stable for decreasing n to be applied without any knowledge of starting values for large n . Miller's algorithm, which is well-suited to automatic work, is described in 19.28, Example 1.

8. Acknowledgments

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M. ABRAMOWITZ.

1. Mathematical Constants

DAVID S. LIEPMAN¹

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¹ National Bureau of Standards.

2. Physical Constants and Conversion Factors

A. G. McNISH ¹

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¹ National Bureau of Standards.

2. Physical Constants and Conversion Factors

The tables in this chapter supply some of the more commonly needed physical constants and conversion factors.

All scientific measurements in the fields of mechanics and heat are based upon four international arbitrarily adopted units, the magnitudes of which are fixed by four agreed on standards:

Length—the meter—fixed by the vacuum wavelength of radiation corresponding to the transition $2P_{10} - 5D_5$ of krypton 86

(1 meter = 1650763.73 λ).

Mass—the kilogram—fixed by the international kilogram at Sèvres, France.

Time—the second—fixed as 1/31,556,925.9747 of the tropical year 1900 at 12^h ephemeris time, or the duration of 9,192,631,770 cycles of the hyperfine transition frequency of cesium 133.

Temperature—the degree—fixed on a thermodynamic basis by taking the temperature for the triple point of natural water as 273.16 °K. (The Celsius scale is obtained by adding -273.15 to the Kelvin scale.)

Other units are defined in terms of them by assigning the value unity to the proportionality constant in each defining equation. The entire system, including electricity units, is called the *Système International d'Unités* (SI). Taking the 1/100 part of the meter as the unit of length and the 1/1000 part of the kilogram as the unit of mass, similarly, gives

rise to the CGS system, often used in physics and chemistry.

Table 2.1. Common Units and Conversion Factors

Quantity	SI name	CGS name	SI unit/ CGS unit
Force, F	newton	dyne	10^5
Energy, W	joule	erg	10^7
Power, P	watt		10^7

The SI unit of electric current is the ampere defined by the equation $2\Gamma_m I_1 I_2 / 4\pi = F$ giving the force in vacuo per unit length between two infinitely long parallel conductors of infinitesimal cross-section. If F is in newtons, and Γ_m has the numerical value $4\pi \times 10^{-7}$, then I_1 and I_2 are in amperes. The customary equations define the other electric and magnetic units of SI such as the volt, ohm, farad, henry, etc. The force between electric charges in a vacuum in this system is given by $Q_1 Q_2 / 4\pi \Gamma_e r^2 = F$, Γ_e having the numerical value $10^7 / 4\pi c^2$ where c is the speed of light in meters per second ($\Gamma_e = 8.854 \times 10^{-12}$).

The CGS unrationalized system is obtained by deleting 4π in the denominators in these equations and expressing F in dynes, and r in centimeters. Setting Γ_m equal to unity defines the CGS unrationalized electromagnetic system (emu), Γ_e then taking the numerical value of $1/c^2$. Setting Γ_e equal to unity defines the CGS unrationalized electrostatic system (esu), Γ_m then taking the numerical value of $1/c^2$.

Table 2.2. Names and Conversion Factors for Electric and Magnetic Units

Quantity	SI name	emu name	esu name	SI unit/ emu unit	SI unit/ esu unit
Current	ampere	abampere	statampere	10^{-1}	$\sim 3 \times 10^9$
Charge	coulomb	abcoulomb	statcoulomb	10^{-1}	$\sim 3 \times 10^9$
Potential	volt	abvolt	statvolt	10^8	$\sim (1/3) \times 10^{-2}$
Resistance	ohm	abohm	statohm	10^9	$\sim (1/9) \times 10^{-11}$
Inductance	henry	centimeter	-----	10^9	$\sim (1/9) \times 10^{-11}$
Capacitance	farad	-----	centimeter	10^{-9}	$\sim 9 \times 10^{11}$
Magnetizing force	amp. turns/ meter	oersted	-----	$4\pi \times 10^{-3*}$	$\sim 3 \times 10^{9*}$
Magnetomotive force	amp. turns	gilbert	-----	$4\pi \times 10^{-1*}$	$\sim 3/10^{6*}$
Magnetic flux	weber	maxwell	-----	10^8	$\sim (1/3) \times 10^{-2}$
Magnetic flux density	tesla	gauss	-----	10^4	$\sim (1/3) \times 10^{-6}$
Electric displacement	-----	-----	-----	10^{-5*}	$\sim 3 \times 10^{5*}$

Example: If the value assigned to a current is 100 amperes its value in abamperes is $100 \times 10^{-1} = 10$.

*Divide this number by 4π if unrationalized system is involved; other numbers are unchanged.

3. Elementary Analytical Methods

MILTON ABRAMOWITZ ¹

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¹ National Bureau of Standards. (Deceased.)

3. Elementary Analytical Methods

3.1. Binomial Theorem and Binomial Coefficients; Arithmetic and Geometric Progressions; Arithmetic, Geometric, Harmonic and Generalized Means

Binomial Theorem

3.1.1

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \dots + b^n$$

(n a positive integer)

Binomial Coefficients (see chapter 24)

3.1.2

$$\binom{n}{k} = {}_nC_k = \frac{n(n-1) \dots (n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

3.1.3

$$\binom{n}{k} = \binom{n}{n-k} = (-1)^k \binom{k-n-1}{k}$$

3.1.4

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

3.1.5

$$\binom{n}{0} = \binom{n}{n} = 1$$

3.1.6

$$1 + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$$

3.1.7

$$1 - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^n \binom{n}{n} = 0$$

Table of Binomial Coefficients $\binom{n}{k}$

3.1.8

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1											
2	1	2	1										
3	1	3	3	1									
4	1	4	6	4	1								
5	1	5	10	10	5	1							
6	1	6	15	20	15	6	1						
7	1	7	21	35	35	21	7	1					
8	1	8	28	56	70	56	28	8	1				
9	1	9	36	84	126	126	84	36	9	1			
10	1	10	45	120	210	252	210	120	45	10	1		
11	1	11	55	165	330	462	462	330	165	55	11	1	
12	1	12	66	220	495	792	924	792	495	220	66	12	1

For a more extensive table see chapter 24.

*See page 11.

3.1.9

Sum of Arithmetic Progression to n Terms

$$a + (a+d) + (a+2d) + \dots + (a+(n-1)d) = na + \frac{1}{2} n(n-1)d = \frac{n}{2} (a+l),$$

last term in series $= l = a + (n-1)d$

Sum of Geometric Progression to n Terms

3.1.10

$$s_n = a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

$$\lim_{n \rightarrow \infty} s_n = a/(1-r) \quad (-1 < r < 1)$$

Arithmetic Mean of n Quantities A

3.1.11

$$A = \frac{a_1 + a_2 + \dots + a_n}{n}$$

Geometric Mean of n Quantities G

3.1.12

$$G = (a_1 a_2 \dots a_n)^{1/n} \quad (a_k > 0, k=1, 2, \dots, n)$$

Harmonic Mean of n Quantities H

3.1.13

$$\frac{1}{H} = \frac{1}{n} \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n} \right) \quad (a_k > 0, k=1, 2, \dots, n)$$

Generalized Mean

3.1.14

$$M(t) = \left(\frac{1}{n} \sum_{k=1}^n a_k^t \right)^{1/t}$$

3.1.15

$$M(t) = 0 \quad (t < 0, \text{ some } a_k \text{ zero})$$

3.1.16

$$\lim_{t \rightarrow \infty} M(t) = \max. \quad (a_1, a_2, \dots, a_n) = \max. a$$

3.1.17

$$\lim_{t \rightarrow -\infty} M(t) = \min. \quad (a_1, a_2, \dots, a_n) = \min. a$$

3.1.18

$$\lim_{t \rightarrow 0} M(t) = G$$

3.1.19

$$M(1) = A$$

3.1.20

$$M(-1) = H$$

3.2. Inequalities

Relation Between Arithmetic, Geometric, Harmonic and Generalized Means

3.2.1

$$A \geq G \geq H, \text{ equality if and only if } a_1 = a_2 = \dots = a_n$$

3.2.2

$$\min. a < M(t) < \max. a$$

3.2.3 $\min. a < G < \max. a$

equality holds if all a_k are equal, or $t < 0$
and an a_k is zero

3.2.4 $M(t) < M(s)$ if $t < s$ unless all a_k are equal,
or $s < 0$ and an a_k is zero.

Triangle Inequalities

3.2.5 $|a_1| - |a_2| \leq |a_1 + a_2| \leq |a_1| + |a_2|$

3.2.6 $\left| \sum_{k=1}^n a_k \right| \leq \sum_{k=1}^n |a_k|$

Chebyshev's Inequality

If $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n$

$b_1 \geq b_2 \geq b_3 \geq \dots \geq b_n$

3.2.7 $n \sum_{k=1}^n a_k b_k \geq \left(\sum_{k=1}^n a_k \right) \left(\sum_{k=1}^n b_k \right)$

Hölder's Inequality for Sums

If $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, $q > 1$

3.2.8 $\sum_{k=1}^n |a_k b_k| \leq \left(\sum_{k=1}^n |a_k|^p \right)^{1/p} \left(\sum_{k=1}^n |b_k|^q \right)^{1/q}$;

equality holds if and only if $|b_k| = c|a_k|^{p-1}$ ($c = \text{constant} > 0$). If $p = q = 2$ we get

Cauchy's Inequality

3.2.9 $\left[\sum_{k=1}^n a_k b_k \right]^2 \leq \sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2$ (equality for $a_k = c b_k$,
 c constant).

Hölder's Inequality for Integrals

If $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, $q > 1$

3.2.10

$$\int_a^b |f(x)g(x)| dx \leq \left[\int_a^b |f(x)|^p dx \right]^{1/p} \left[\int_a^b |g(x)|^q dx \right]^{1/q}$$

equality holds if and only if $|g(x)| = c|f(x)|^{p-1}$
($c = \text{constant} > 0$).

If $p = q = 2$ we get

Schwarz's Inequality

3.2.11

$$\left[\int_a^b f(x)g(x) dx \right]^2 \leq \int_a^b [f(x)]^2 dx \int_a^b [g(x)]^2 dx$$

Minkowski's Inequality for Sums

If $p > 1$ and $a_k, b_k > 0$ for all k ,

3.2.12

$$\left(\sum_{k=1}^n (a_k + b_k)^p \right)^{1/p} \leq \left(\sum_{k=1}^n a_k^p \right)^{1/p} + \left(\sum_{k=1}^n b_k^p \right)^{1/p},$$

equality holds if and only if $b_k = c a_k$ ($c = \text{constant} > 0$).

Minkowski's Inequality for Integrals

If $p > 1$,

3.2.13

$$\left(\int_a^b |f(x) + g(x)|^p dx \right)^{1/p} \leq \left(\int_a^b |f(x)|^p dx \right)^{1/p} + \left(\int_a^b |g(x)|^p dx \right)^{1/p}$$

equality holds if and only if $g(x) = c f(x)$ ($c = \text{constant} > 0$).

3.3. Rules for Differentiation and Integration Derivatives

3.3.1 $\frac{d}{dx} (cu) = c \frac{du}{dx}$, c constant

3.3.2 $\frac{d}{dx} (u+v) = \frac{du}{dx} + \frac{dv}{dx}$

3.3.3 $\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

3.3.4 $\frac{d}{dx} (u/v) = \frac{v du/dx - u dv/dx}{v^2}$

3.3.5 $\frac{d}{dx} u(v) = \frac{du}{dv} \frac{dv}{dx}$

3.3.6 $\frac{d}{dx} (u^v) = u^v \left(\frac{v}{u} \frac{du}{dx} + \ln u \frac{dv}{dx} \right)$

Leibniz's Theorem for Differentiation of an Integral

3.3.7

$$\begin{aligned} \frac{d}{dc} \int_{a(c)}^{b(c)} f(x, c) dx \\ = \int_{a(c)}^{b(c)} \frac{\partial}{\partial c} f(x, c) dx + f(b, c) \frac{db}{dc} - f(a, c) \frac{da}{dc} \end{aligned}$$

Leibniz's Theorem for Differentiation of a Product**3.3.8**

$$\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n} v + \binom{n}{1} \frac{d^{n-1} u}{dx^{n-1}} \frac{dv}{dx} + \binom{n}{2} \frac{d^{n-2} u}{dx^{n-2}} \frac{d^2 v}{dx^2} + \dots + \binom{n}{r} \frac{d^{n-r} u}{dx^{n-r}} \frac{d^r v}{dx^r} + \dots + u \frac{d^n v}{dx^n}$$

3.3.9

$$\frac{dx}{dy} = 1 / \frac{dy}{dx}$$

3.3.10

$$\frac{d^2 x}{dy^2} = -\frac{d^2 y}{dx^2} \left(\frac{dy}{dx} \right)^{-3}$$

3.3.11

$$\frac{d^3 x}{dy^3} = -\left[\frac{d^3 y}{dx^3} \frac{dy}{dx} - 3 \left(\frac{d^2 y}{dx^2} \right)^2 \right] \left(\frac{dy}{dx} \right)^{-5}$$

Integration by Parts**3.3.12**

$$\int u dv = uv - \int v du$$

3.3.13

$$\int u v dx = \left(\int u dx \right) v - \int \left(\int u dx \right) \frac{dv}{dx} dx$$

Integrals of Rational Algebraic Functions

(Integration constants are omitted)

3.3.14

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} \quad (n \neq -1)$$

3.3.15

$$\int \frac{dx}{ax+b} = \frac{1}{a} \ln |ax+b|$$

The following formulas are useful for evaluating $\int \frac{P(x)dx}{(ax^2+bx+c)^n}$ where $P(x)$ is a polynomial and $n > 1$ is an integer.

3.3.16

$$\int \frac{dx}{(ax^2+bx+c)} = \frac{2}{(4ac-b^2)^{1/2}} \arctan \frac{2ax+b}{(4ac-b^2)^{1/2}} \quad (b^2-4ac < 0)$$

3.3.17

$$= \frac{1}{(b^2-4ac)^{1/2}} \ln \left| \frac{2ax+b-(b^2-4ac)^{1/2}}{2ax+b+(b^2-4ac)^{1/2}} \right| \quad (b^2-4ac > 0)$$

3.3.18

$$= \frac{-2}{2ax+b} \quad (b^2-4ac=0)$$

3.3.19

$$\int \frac{xdx}{ax^2+bx+c} = \frac{1}{2a} \ln |ax^2+bx+c| - \frac{b}{2a} \int \frac{dx}{ax^2+bx+c}$$

3.3.20

$$\int \frac{dx}{(a+bx)(c+dx)} = \frac{1}{ad-bc} \ln \left| \frac{c+dx}{a+bx} \right| \quad (ad \neq bc)$$

3.3.21

$$\int \frac{dx}{a^2+b^2x^2} = \frac{1}{ab} \arctan \frac{bx}{a}$$

3.3.22

$$\int \frac{xdx}{a^2+b^2x^2} = \frac{1}{2b^2} \ln |a^2+b^2x^2|$$

3.3.23

$$\int \frac{dx}{a^2-b^2x^2} = \frac{1}{2ab} \ln \left| \frac{a+bx}{a-bx} \right|$$

3.3.24

$$\int \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3} \arctan \frac{x}{a} + \frac{x}{2a^2(x^2+a^2)}$$

3.3.25

$$\int \frac{dx}{(x^2-a^2)^2} = \frac{-x}{2a^2(x^2-a^2)} + \frac{1}{4a^3} \ln \left| \frac{a+x}{a-x} \right|$$

Integrals of Irrational Algebraic Functions**3.3.26**

$$\int \frac{dx}{[(a+bx)(c+dx)]^{1/2}} = \frac{2}{(-bd)^{1/2}} \arctan \left[\frac{-d(a+bx)}{b(c+dx)} \right]^{1/2} \quad (bd < 0)$$

3.3.27

$$= \frac{-1}{(-bd)^{1/2}} \arcsin \left(\frac{2bdx+ad+bc}{bc-ad} \right) \quad (b > 0, d < 0)$$

3.3.28

$$= \frac{2}{(bd)^{1/2}} \ln | [bd(a+bx)]^{1/2} + b(c+dx)^{1/2} | \quad (bd > 0)$$

3.3.29

$$\int \frac{dx}{(a+bx)^{1/2}(c+dx)} = \frac{2}{[d(bc-ad)]^{1/2}} \arctan \left[\frac{d(a+bx)}{(bc-ad)} \right]^{1/2} \quad (d(ad-bc) < 0)$$

3.3.30

$$= \frac{1}{[d(ad-bc)]^{1/2}} \ln \left| \frac{d(a+bx)^{1/2} - [d(ad-bc)]^{1/2}}{d(a+bx)^{1/2} + [d(ad-bc)]^{1/2}} \right| \quad (d(ad-bc) > 0)$$

If $z^n = u_n + iv_n$, then $z^{n+1} = u_{n+1} + iv_{n+1}$ where

3.7.23 $u_{n+1} = xu_n - yv_n$; $v_{n+1} = xv_n + yu_n$
 $\mathcal{R}z^n$ and $\mathcal{I}z^n$ are called harmonic polynomials.

$$\mathbf{3.7.24} \quad \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x-iy}{x^2+y^2}$$

$$\mathbf{3.7.25} \quad \frac{1}{z^n} = \frac{\bar{z}^n}{|z|^{2n}} = (z^{-1})^n$$

Roots

$$\mathbf{3.7.26} \quad z^{\frac{1}{n}} = \sqrt[n]{z} = r^{\frac{1}{n}} e^{i\theta/n} = r^{\frac{1}{n}} \cos \frac{\theta}{n} + ir^{\frac{1}{n}} \sin \frac{\theta}{n}$$

If $-\pi < \theta \leq \pi$ this is the principal root. The other root has the opposite sign. The principal root is given by

3.7.27 $z^{\frac{1}{n}} = [\frac{1}{2}(r+x)]^{\frac{1}{n}} \pm i[\frac{1}{2}(r-x)]^{\frac{1}{n}} = u \pm iv$ where $2uv = y$ and where the ambiguous sign is taken to be the same as the sign of y .

3.7.28 $z^{1/n} = r^{1/n} e^{i\theta/n}$, (principal root if $-\pi < \theta \leq \pi$). Other roots are $r^{1/n} e^{i(\theta+2\pi k)/n}$ ($k=1, 2, 3, \dots, n-1$).

Inequalities

$$\mathbf{3.7.29} \quad \left| |z_1| - |z_2| \right| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$$

Complex Functions, Cauchy-Riemann Equations

$f(z) = f(x+iy) = u(x, y) + iv(x, y)$ where $u(x, y), v(x, y)$ are real, is *analytic* at those points $z = x+iy$ at which

$$\mathbf{3.7.30} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

If $z = re^{i\theta}$,

$$\mathbf{3.7.31} \quad \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

Laplace's Equation

The functions $u(x, y)$ and $v(x, y)$ are called harmonic functions and satisfy Laplace's equation:

Cartesian Coordinates

$$\mathbf{3.7.32} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Polar Coordinates

$$\mathbf{3.7.33} \quad r \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial \theta^2} = r \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) + \frac{\partial^2 v}{\partial \theta^2} = 0$$

3.8. Algebraic Equations

Solution of Quadratic Equations

3.8.1 Given $az^2 + bz + c = 0$,

$$z_{1,2} = -\left(\frac{b}{2a}\right) \pm \frac{1}{2a} q^{\frac{1}{2}}, \quad q = b^2 - 4ac,$$

$$z_1 + z_2 = -b/a, \quad z_1 z_2 = c/a$$

If $q > 0$, two real roots,
 $q = 0$, two equal roots,
 $q < 0$, pair of complex conjugate roots.

Solution of Cubic Equations

3.8.2 Given $z^3 + a_2 z^2 + a_1 z + a_0 = 0$, let

$$q = \frac{1}{3} a_1 - \frac{1}{9} a_2^2; \quad r = \frac{1}{6} (a_1 a_2 - 3a_0) - \frac{1}{27} a_2^3.$$

If $q^3 + r^2 > 0$, one real root and a pair of complex conjugate roots,

$q^3 + r^2 = 0$, all roots real and at least two are equal,

$q^3 + r^2 < 0$, all roots real (irreducible case).

Let

$$s_1 = [r + (q^3 + r^2)^{\frac{1}{2}}]^{\frac{1}{3}}, \quad s_2 = [r - (q^3 + r^2)^{\frac{1}{2}}]^{\frac{1}{3}}$$

then

$$z_1 = (s_1 + s_2) - \frac{a_2}{3}$$

$$z_2 = -\frac{1}{2} (s_1 + s_2) - \frac{a_2}{3} + \frac{i\sqrt{3}}{2} (s_1 - s_2)$$

$$z_3 = -\frac{1}{2} (s_1 + s_2) - \frac{a_2}{3} - \frac{i\sqrt{3}}{2} (s_1 - s_2).$$

If z_1, z_2, z_3 are the roots of the cubic equation

$$z_1 + z_2 + z_3 = -a_2$$

$$z_1 z_2 + z_1 z_3 + z_2 z_3 = a_1$$

$$z_1 z_2 z_3 = -a_0$$

Solution of Quartic Equations

3.8.3 Given $z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0$, find the real root u_1 of the cubic equation

$$u^3 - a_2 u^2 + (a_1 a_3 - 4a_0)u - (a_1^2 + a_0 a_3^2 - 4a_0 a_2) = 0$$

and determine the four roots of the quartic as solutions of the two quadratic equations

$$v^2 + \left[\frac{a_3}{2} \mp \left(\frac{a_3^2}{4} + u_1 - a_2 \right)^{\frac{1}{2}} \right] v + \frac{u_1}{2} \mp \left[\left(\frac{u_1}{2} \right)^2 - a_0 \right]^{\frac{1}{2}} = 0$$

If all roots of the cubic equation are real, use the value of u_1 which gives real coefficients in the *quadratic equation and select signs so that if

$$z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 = (z^2 + p_1 z + q_1)(z^2 + p_2 z + q_2),$$

then

$$p_1 + p_2 = a_3, p_1 p_2 + q_1 + q_2 = a_2, p_1 q_2 + p_2 q_1 = a_1, q_1 q_2 = a_0.$$

If z_1, z_2, z_3, z_4 are the roots,

$$\Sigma z_i = -a_3, \Sigma z_i z_j z_k = -a_1,$$

$$\Sigma z_i z_j = a_2, z_1 z_2 z_3 z_4 = a_0.$$

3.9. Successive Approximation Methods

General Comments

3.9.1 Let $x = x_1$ be an approximation to $x = \xi$ where $f(\xi) = 0$ and both x_1 and ξ are in the interval $a \leq x \leq b$. We define

$$x_{n+1} = x_n + c_n f(x_n) \quad (n = 1, 2, \dots).$$

Then, if $f'(x) \geq 0$ and the constants c_n are negative and bounded, the sequence x_n converges monotonically to the root ξ .

If $c_n = c = \text{constant} < 0$ and $f'(x) > 0$, then the process converges but not necessarily monotonically.

Degree of Convergence of an Approximation Process

3.9.2 Let $x_1, x_2, x_3, \dots$ be an infinite sequence of approximations to a number ξ . Then, if

$$|x_{n+1} - \xi| < A |x_n - \xi|^k, \quad (n = 1, 2, \dots)$$

where A and k are independent of n , the sequence is said to have convergence of at most the k th degree (or order or index) to ξ . If $k = 1$ and $A < 1$ the convergence is linear; if $k = 2$ the convergence is quadratic.

Regula Falsi (False Position)

3.9.3 Given $y = f(x)$ to find ξ such that $f(\xi) = 0$, choose x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ have opposite signs and compute

$$x_2 = x_1 - \frac{(x_1 - x_0)}{(f_1 - f_0)} f_1 = \frac{f_1 x_0 - f_0 x_1}{f_1 - f_0}.$$

Then continue with x_2 and either of x_0 or x_1 for which $f(x_0)$ or $f(x_1)$ is of opposite sign to $f(x_2)$.

Regula falsi is equivalent to inverse linear interpolation.

*See page 11.

Method of Iteration (Successive Substitution)

3.9.4 The iteration scheme $x_{k+1} = F(x_k)$ will converge to a zero of $x = F(x)$ if

$$(1) \quad |F'(x)| \leq q < 1 \text{ for } a \leq x \leq b,$$

$$(2) \quad a \leq x_0 \pm \frac{|F(x_0) - x_0|}{1 - q} \leq b.$$

Newton's Method of Successive Approximations

3.9.5

Newton's Rule

If $x = x_k$ is an approximation to the solution $x = \xi$ of $f(x) = 0$ then the sequence

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

will converge quadratically to $x = \xi$: (if instead of the condition (2) above),

(1) *Monotonic convergence*, $f(x_0)f''(x_0) > 0$ and $f'(x), f''(x)$ do not change sign in the interval (x_0, ξ) , or

(2) *Oscillatory convergence*, $f(x_0)f''(x_0) < 0$ and $f'(x), f''(x)$ do not change sign in the interval (x_0, x_1) , $x_0 \leq \xi \leq x_1$.

Newton's Method Applied to Real n th Roots

3.9.6 Given $x^n = N$, if x_k is an approximation $x = N^{1/n}$ then the sequence

$$x_{k+1} = \frac{1}{n} \left[\frac{N}{x_k^{n-1}} + (n-1)x_k \right]$$

will converge quadratically to x .

$$\text{If } n=2, x_{k+1} = \frac{1}{2} \left(\frac{N}{x_k} + x_k \right),$$

$$\text{If } n=3, x_{k+1} = \frac{1}{3} \left(\frac{N}{x_k^2} + 2x_k \right).$$

Aitken's δ^2 -Process for Acceleration of Sequences

3.9.7 If x_k, x_{k+1}, x_{k+2} are three successive iterates in a sequence converging with an error which is approximately in geometric progression, then

$$\bar{x}_k = x_k - \frac{(x_k - x_{k+1})^2}{\Delta^2 x_k} = \frac{x_k x_{k+2} - x_{k+1}^2}{\Delta^2 x_k},$$

$$\Delta^2 x_k = x_k - 2x_{k+1} + x_{k+2}$$

is an improved estimate of x . In fact, if $x_k = x + O(\lambda^k)$ then $\bar{x} = x + O(\lambda^k)$, $|\lambda| < 1$.

3.10. Theorems on Continued Fractions

Definitions

3.10.1

$$(1) \text{ Let } f = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots}}}$$

$$= b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \dots$$

If the number of terms is finite, f is called a terminating continued fraction. If the number of terms is infinite, f is called an infinite continued fraction and the terminating fraction

$$f_n = \frac{A_n}{B_n} = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \dots \frac{a_n}{b_n}$$

is called the n th convergent of f .

(2) If $\lim_{n \rightarrow \infty} \frac{A_n}{B_n}$ exists, the infinite continued fraction f is said to be convergent. If $a_i = 1$ and the b_i are integers there is always convergence.

Theorems

(1) If a_i and b_i are positive then $f_{2n} < f_{2n+2}$, $f_{2n-1} > f_{2n+1}$.

(2) If $f_n = \frac{A_n}{B_n}$,

$$A_n = b_n A_{n-1} + a_n A_{n-2}$$

$$B_n = b_n B_{n-1} + a_n B_{n-2}$$

where $A_{-1} = 1$, $A_0 = b_0$, $B_{-1} = 0$, $B_0 = 1$.

$$(3) \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-2} \end{bmatrix} \begin{bmatrix} b_n \\ a_n \end{bmatrix}$$

$$(4) A_n B_{n-1} - A_{n-1} B_n = (-1)^{n-1} \prod_{k=1}^n a_k$$

(5) For every $n \geq 0$,

$$f_n = b_0 + \frac{c_1 a_1}{c_1 b_1 +} \frac{c_1 c_2 a_2}{c_2 b_2 +} \frac{c_2 c_3 a_3}{c_3 b_3 +} \dots \frac{c_{n-1} c_n a_n}{c_n b_n}$$

$$(6) 1 + b_2 + b_2 b_3 + \dots + b_2 b_3 \dots b_n$$

$$= \frac{1}{1 -} \frac{b_2}{b_2 + 1 -} \frac{b_3}{b_3 + 1 -} \dots \frac{b_n}{b_n + 1}$$

$$\frac{1}{u_1} + \frac{1}{u_2} + \dots + \frac{1}{u_n} = \frac{1}{u_1 -} \frac{u_1^2}{u_1 + u_2 -} \dots \frac{u_{n-1}^2}{-u_{n-1} + u_n}$$

$$\frac{1}{a_0} \frac{x}{a_0 a_1} + \frac{x^2}{a_0 a_1 a_2} \dots + (-1)^n \frac{x^n}{a_0 a_1 a_2 \dots a_n}$$

$$= \frac{1}{a_0 +} \frac{a_0 x}{a_1 - x +} \frac{a_1 x}{a_2 - x +} \dots \frac{a_{n-1} x}{+ a_n - x}$$

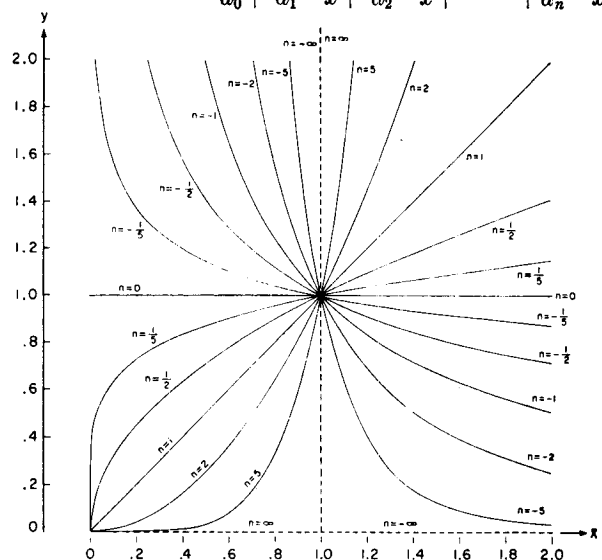


FIGURE 3.1. $y = x^n$.
 $\pm n = 0, \frac{1}{5}, \frac{1}{2}, 1, 2, 5.$

Numerical Methods

3.11. Use and Extension of the Tables

Example 1. Compute x^{19} and x^{47} for $x = 29$ using Table 3.1.

$$x^{19} = x^9 \cdot x^{10}$$

$$= (1.45071 \ 4598 \cdot 10^{13}) (4.20707 \ 2333 \cdot 10^{14})$$

$$= 6.10326 \ 1248 \cdot 10^{27}$$

$$x^{47} = (x^{24})^2 / x$$

$$= (1.25184 \ 9008 \cdot 10^{35})^2 / 29$$

$$= 5.40388 \ 2547 \cdot 10^{68}$$

Example 2. Compute $x^{-3/4}$ for $x = 9.19826$.

$$(9.19826)^{1/4} = (919.826/100)^{1/4} = (919.826)^{1/4} / 10^{1/4}$$

Linear interpolation in Table 3.1 gives $(919.826)^{1/4} \approx 5.507144$.

By Newton's method for fourth roots with $N = 919.826$,

$$\frac{1}{4} \left[\frac{919.826}{(5.507144)^3} + 3(5.507144) \right] = 5.50714 \ 3845$$

Repetition yields the same result. Thus,

$$x^{1/4} = 5.50714 \ 3845 / 10^{1/4} = 1.74151 \ 1796,$$

$$x^{-3/4} = x^1 / x = .18933 \ 05683.$$

3.12. Computing Techniques

Example 3. Solve the quadratic equation $x^2 - 18.2x + .056$ given the coefficients as $18.2 \pm .1$,

*See page 11.

.056 ± .001. From 3.8.1 the solution is

$$\begin{aligned} x &= \frac{1}{2}(18.2 \pm [(18.2)^2 - 4(.056)]^{\frac{1}{2}}) \\ &= \frac{1}{2}(18.2 \pm [331.016]^{\frac{1}{2}}) = \frac{1}{2}(18.2 \pm 18.1939) \\ &= 18.1969, .003 \end{aligned}$$

The smaller root may be obtained more accurately from

$$* \quad .056/18.1969 = .0031 \pm .0001.$$

Example 4. Compute $(-3 + .0076i)^{\frac{1}{2}}$.

From 3.7.26, $(-3 + .0076i)^{\frac{1}{2}} = u + iv$ where

$$u = \frac{y}{2v}, v = \left(\frac{r-x}{2}\right)^{\frac{1}{2}}, r = (x^2 + y^2)^{\frac{1}{2}}$$

Thus

$$r = [(-3)^2 + (.0076)^2]^{\frac{1}{2}} = (9.00005776)^{\frac{1}{2}} = 3.000009627$$

$$v = \left[\frac{3.000009627 - (-3)}{2} \right]^{\frac{1}{2}} = 1.732052196$$

$$u = \frac{y}{2v} = \frac{.0076}{2(1.732052196)} = .00219392926$$

We note that the principal square root has been computed.

Example 6. Solve the quartic equation

$$x^4 - 2.377524922x^3 + 6.073505741x^2 - 11.17938023x + 9.052655259 = 0.$$

Resolution Into Quadratic Factors

$$(x^2 + p_1x + q_1)(x^2 + p_2x + q_2)$$

by Inverse Interpolation

Starting with the trial value $q_1 = 1$ we compute successively

q_1	$q_2 = \frac{a_0}{q_1}$	$p_1 = \frac{a_1 - a_3q_1}{q_2 - q_1}$	$p_2 = a_3 - p_1$	$y(q_1) = q_1 + q_2 + p_1p_2 - a_2$
1	9.053	-1.093	-1.284	5.383
2	4.526	-2.543	.165	.032
2.2	4.115	-3.106	.729	-2.023

q_1	q_2	p_1	p_2	$y(q_1)$
2.0041	4.517067640	-2.55259257	.17506765	.00078552
2.0042	4.516842260	-2.55282851	.17530358	.00001655
2.0043	4.516616903	-2.55306447	.17553955	-.00075263

Inverse interpolation gives $q_1 = 2.004202152$, and we get finally,

q_1	q_2	p_1	p_2	$y(q_1)$
2.004202152	4.516837410	-2.55283358	.175308659	-.000000011

Example 5. Solve the cubic equation $x^3 - 18.1x - 34.8 = 0$.

To use Newton's method we first form the table of $f(x) = x^3 - 18.1x - 34.8$

x	$f(x)$
4	-43.2
5	-.3
6	72.6
7	181.5

We obtain by linear inverse interpolation:

$$x_0 = 5 + \frac{0 - (-.3)}{72.6 - (-.3)} = 5.004.$$

Using Newton's method, $f'(x) = 3x^2 - 18.1$ we get

$$\begin{aligned} x_1 &\approx x_0 - f(x_0)/f'(x_0) \\ &\approx 5.004 - \frac{(-.072159936)}{57.020048} \approx 5.00526. \end{aligned}$$

Repetition yields $x_1 = 5.005265097$. Dividing $f(x)$ by $x - 5.005265097$ gives $x^2 + 5.005265097x + 6.95267869$ the zeros of which are $-2.502632549 \pm .83036800i$.

We seek that value of q_1 for which $y(q_1) = 0$. Inverse interpolation in $y(q_1)$ gives $y(q_1) \approx 0$ for $q_1 \approx 2.003$. Then,

q_1	q_2	p_1	p_2	$y(q_1)$
2.003	4.520	-2.550	.172	.011

Inverse interpolation between $q_1 = 2.2$ and $q_1 = 2.003$ gives $q_1 = 2.0041$, and thus,

Example 10. Evaluate the integral $\int_0^\infty \frac{\sin x}{x} dx$ to 4D using the Euler transform.

$$\begin{aligned} \int_0^\infty \frac{\sin x}{x} dx &= \sum_{k=0}^\infty \int_{k\pi}^{(k+1)\pi} \frac{\sin x}{x} dx \\ &= \sum_{k=0}^\infty \int_0^\pi \frac{\sin(k\pi+t)}{k\pi+t} dt = \sum_{k=0}^\infty (-1)^k \int_0^\pi \frac{\sin t}{k\pi+t} dt. \end{aligned}$$

Evaluating the integrals in the last sum by numerical integration we get

k	$\int_0^\pi \frac{\sin t}{k\pi+t} dt$				
0	1.85194				
1	.43379				
2	.25661				
3	.18260	Δ	Δ^2	Δ^3	Δ^4
4	.14180				
5	.11593	-2587	799		
6	.09805	-1788	478	-321	
7	.08495	-1310	310	-168	153
8	.07495	-1000			

The sum to $k=3$ is 1.49216. Applying the Euler transform to the remainder we obtain

$$\begin{aligned} \frac{1}{2} (.14180) - \frac{1}{2^2} (-.02587) + \frac{1}{2^3} (.00799) \\ - \frac{1}{2^4} (-.00321) + \frac{1}{2^5} (.00153) \\ = .07090 + .00647 + .00100 + .00020 \\ + .00005 \\ = .07862 \end{aligned}$$

We obtain the value of the integral as 1.57078 as compared with 1.57080.

Example 11. Sum the series $\sum_{k=1}^\infty k^{-2} = \frac{\pi^2}{6}$ using the Euler-Maclaurin summation formula.

From 3.6.28 we have for $n=\infty$,

$$\begin{aligned} \sum_{k=1}^\infty k^{-2} &= \sum_{k=1}^{10} k^{-2} + \sum_{k=1}^\infty (k+10)^{-2} \\ &= \sum_{k=1}^{10} k^{-2} + \int_0^\infty f(k) dk - \frac{1}{2} f_0 - \frac{1}{12} f'_0 \\ &\quad + \frac{1}{720} f''_0 - \dots \end{aligned}$$

where $f(k) = (k+10)^{-2}$. Thus,

$$\begin{aligned} \sum_{k=1}^\infty k^{-2} &= 1.54976 \, 7731 + .1 \\ &\quad - .005 + .00016 \, 6667 - .00000 \, 0333 \\ &= 1.64493 \, 4065, \end{aligned}$$

as compared with $\frac{\pi^2}{6} = 1.64493 \, 4067$.

Example 12. Compute

$$\arctan x = \frac{x}{1+} \frac{x^2}{3+} \frac{4x^2}{5+} \frac{9x^2}{7+} \dots$$

to 5D for $x=.2$. Here $a_1=x$, $a_n=(n-1)^2x^2$ for $n>1$, $b_0=0$, $b_n=2n-1$, $A_{-1}=1$, $B_{-1}=0$, $A_0=0$, $B_0=1$.

For $n \geq 1$

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} A_{n-1} A_{n-2} \\ B_{n-1} B_{n-2} \end{bmatrix} \left[\begin{array}{c} 2n-1 \\ (n-1)^2 x^2 \end{array} \right] \frac{A_0}{B_0} = 0$$

$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left[\begin{array}{c} 1 \\ 2 \end{array} \right] = \begin{bmatrix} .2 \\ 1 \end{bmatrix} \frac{A_1}{B_1} = .2$$

$$\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} .2 & 0 \\ 1 & 1 \end{bmatrix} \left[\begin{array}{c} 3 \\ .04 \end{array} \right] = \begin{bmatrix} .6 \\ 3.04 \end{bmatrix} \frac{A_2}{B_2} = .197368$$

$$\begin{bmatrix} A_3 \\ B_3 \end{bmatrix} = \begin{bmatrix} .6 & .2 \\ 3.04 & 1 \end{bmatrix} \left[\begin{array}{c} 5 \\ .16 \end{array} \right] = \begin{bmatrix} 3.032 \\ 15.36 \end{bmatrix} \frac{A_3}{B_3} = .197396$$

$$\begin{bmatrix} A_4 \\ B_4 \end{bmatrix} = \begin{bmatrix} 3.032 & .6 \\ 15.36 & 3.04 \end{bmatrix} \left[\begin{array}{c} 7 \\ .36 \end{array} \right] = \begin{bmatrix} 21.440 \\ 108.6144 \end{bmatrix} \frac{A_4}{B_4} = .197396$$

Note that in carrying out the recurrence method for computing continued fractions the numerators A_n and the denominators B_n must be used as originally computed. The numerators and denominators obtained by reducing A_n/B_n to lower terms must not be used.

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4. Elementary Transcendental Functions

Logarithmic, Exponential, Circular and Hyperbolic Functions

RUTH ZUCKER¹

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¹ National Bureau of Standards.

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4. Elementary Transcendental Functions

Logarithmic, Exponential, Circular and Hyperbolic Functions

Mathematical Properties

4.1. Logarithmic Function

Integral Representation

$$4.1.1 \quad \ln z = \int_1^z \frac{dt}{t}$$

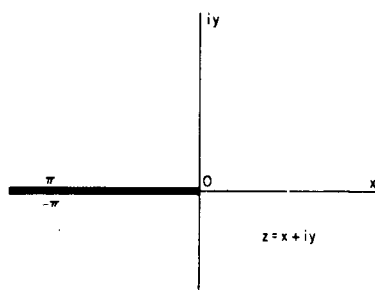


FIGURE 4.1. Branch cut for $\ln z$ and z^a .
(a not an integer or zero.)

where the path of integration does not pass through the origin or cross the negative real axis. $\ln z$ is a single-valued function, regular in the z -plane cut along the negative real axis, real when z is positive.

$$z = x + iy = re^{i\theta}.$$

$$4.1.2 \quad \ln z = \ln r + i\theta \quad (-\pi < \theta \leq \pi).$$

$$4.1.3 \quad r = (x^2 + y^2)^{\frac{1}{2}}, \quad x = r \cos \theta, \quad y = r \sin \theta,$$

$$\theta = \arctan \frac{y}{x}.$$

The general logarithmic function is the many-valued function $\text{Ln } z$ defined by

$$4.1.4 \quad \text{Ln } z = \int_1^z \frac{dt}{t}$$

where the path does not pass through the origin.

$$4.1.5 \quad \text{Ln}(re^{i\theta}) = \ln(re^{i\theta}) + 2k\pi i = \ln r + i(\theta + 2k\pi),$$

k being an arbitrary integer. $\ln z$ is said to be the *principal branch* of $\text{Ln } z$.

Logarithmic Identities

$$4.1.6 \quad \text{Ln}(z_1 z_2) = \text{Ln } z_1 + \text{Ln } z_2.$$

(i.e., every value of $\text{Ln}(z_1 z_2)$ is one of the values of $\text{Ln } z_1 + \text{Ln } z_2$.)

$$4.1.7 \quad \ln(z_1 z_2) = \ln z_1 + \ln z_2 \\ (-\pi < \arg z_1 + \arg z_2 \leq \pi)$$

$$4.1.8 \quad \text{Ln} \frac{z_1}{z_2} = \text{Ln } z_1 - \text{Ln } z_2$$

$$4.1.9 \quad \ln \frac{z_1}{z_2} = \ln z_1 - \ln z_2 \\ (-\pi < \arg z_1 - \arg z_2 \leq \pi)$$

$$4.1.10 \quad \text{Ln } z^n = n \text{Ln } z \quad (n \text{ integer})$$

$$4.1.11 \quad \ln z^n = n \ln z \\ (n \text{ integer}, \quad -\pi < n \arg z \leq \pi)$$

Special Values (see chapter 1)

$$4.1.12 \quad \ln 1 = 0$$

$$4.1.13 \quad \ln 0 = -\infty$$

$$4.1.14 \quad \ln(-1) = \pi i$$

$$4.1.15 \quad \ln(\pm i) = \pm \frac{1}{2}\pi i$$

$$4.1.16 \quad \ln e = 1, \quad e \text{ is the real number such that}$$

$$\int_1^e \frac{dt}{t} = 1$$

$$4.1.17 \quad e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = 2.71828 \, 18284 \dots$$

(see 4.2.21)

Logarithms to General Base

$$4.1.18 \quad \log_a z = \ln z / \ln a$$

$$4.1.19 \quad \log_a z = \frac{\log_b z}{\log_b a}$$

$$4.1.20 \quad \log_a b = \frac{1}{\log_b a}$$

$$4.1.21 \quad \log_e z = \ln z$$

$$4.1.22 \quad \log_{10} z = \ln z / \ln 10 = \log_{10} e \ln z \\ = (.43429 \, 44819 \dots) \ln z$$

$$4.1.23 \quad \ln z = \ln 10 \log_{10} z = (2.30258 \ 50929 \dots) \log_{10} z$$

($\log_e x = \ln x$, called natural, Napierian, or hyperbolic logarithms; $\log_{10} x$, called common or Briggs logarithms.)

Series Expansions

$$4.1.24 \quad \ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \dots$$

$$(|z| \leq 1 \text{ and } z \neq -1)$$

$$4.1.25 \quad \ln z = \left(\frac{z-1}{z}\right) + \frac{1}{2}\left(\frac{z-1}{z}\right)^2 + \frac{1}{3}\left(\frac{z-1}{z}\right)^3 + \dots$$

$$(\Re z \geq \frac{1}{2})$$

$$4.1.26 \quad \ln z = (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \dots$$

$$(|z-1| \leq 1, \quad z \neq 0)$$

$$4.1.27 \quad \ln z = 2 \left[\left(\frac{z-1}{z+1}\right) + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \frac{1}{5}\left(\frac{z-1}{z+1}\right)^5 + \dots \right]$$

$$(\Re z \geq 0, \quad z \neq 0)$$

$$4.1.28 \quad \ln\left(\frac{z+1}{z-1}\right) = 2\left(\frac{1}{z} + \frac{1}{3z^3} + \frac{1}{5z^5} + \dots\right)$$

$$(|z| \geq 1, \quad z \neq \pm 1)$$

$$4.1.29 \quad \ln(z+a) = \ln a + 2 \left[\left(\frac{z}{2a+z}\right) + \frac{1}{3}\left(\frac{z}{2a+z}\right)^3 + \frac{1}{5}\left(\frac{z}{2a+z}\right)^5 + \dots \right]$$

$$(a > 0, \quad \Re z \geq -a \neq z)$$

Limiting Values

$$4.1.30 \quad \lim_{x \rightarrow \infty} x^{-\alpha} \ln x = 0$$

$$(\alpha \text{ constant}, \quad \Re \alpha > 0)$$

$$4.1.31 \quad \lim_{x \rightarrow 0} x^\alpha \ln x = 0$$

$$(\alpha \text{ constant}, \quad \Re \alpha > 0)$$

$$4.1.32 \quad \lim_{m \rightarrow \infty} \left(\sum_{k=1}^m \frac{1}{k} - \ln m \right) = \gamma \text{ (Euler's constant)}$$

$$= .57721 \ 56649 \dots$$

$$(\text{see chapters 1, 6 and 23})$$

Inequalities

$$4.1.33 \quad \frac{x}{1+x} < \ln(1+x) < x$$

$$(x > -1, \quad x \neq 0)$$

$$4.1.34 \quad x < -\ln(1-x) < \frac{x}{1-x}$$

$$(x < 1, \quad x \neq 0)$$

$$4.1.35 \quad |\ln(1-x)| < \frac{3x}{2} \quad (0 < x \leq .5828)$$

$$4.1.36 \quad \ln x \leq x-1 \quad (x > 0)$$

$$4.1.37 \quad \ln x \leq n(x^{1/n} - 1) \text{ for any positive } n$$

$$(x > 0)$$

$$4.1.38 \quad |\ln(1+z)| \leq -\ln(1-|z|) \quad (|z| < 1)$$

Continued Fractions

$$4.1.39 \quad \ln(1+z) = \frac{z}{1+} \frac{z}{2+} \frac{z}{3+} \frac{4z}{4+} \frac{4z}{5+} \frac{9z}{6+} \dots$$

$$(z \text{ in the plane cut from } -1 \text{ to } -\infty)$$

$$4.1.40 \quad \ln\left(\frac{1+z}{1-z}\right) = \frac{2z}{1-} \frac{z^2}{3-} \frac{4z^2}{5-} \frac{9z^2}{7-} \dots$$

$$(z \text{ in the cut plane of Figure 4.7.})$$

Polynomial Approximations²

$$4.1.41 \quad \frac{1}{\sqrt{10}} \leq x \leq \sqrt{10}$$

$$\log_{10} x = a_1 t + a_3 t^3 + \epsilon(x), \quad t = (x-1)/(x+1)$$

$$|\epsilon(x)| \leq 6 \times 10^{-4}$$

$$a_1 = .86304 \quad a_3 = .36415$$

$$4.1.42 \quad \frac{1}{\sqrt{10}} \leq x \leq \sqrt{10}$$

$$\log_{10} x = a_1 t + a_3 t^3 + a_5 t^5 + a_7 t^7 + a_9 t^9 + \epsilon(x)$$

$$t = (x-1)/(x+1)$$

$$|\epsilon(x)| \leq 10^{-7}$$

$$a_1 = .86859 \ 1718 \quad a_7 = .09437 \ 6476$$

$$a_3 = .28933 \ 5524 \quad a_9 = .19133 \ 7714$$

$$a_5 = .17752 \ 2071$$

$$4.1.43 \quad 0 \leq x \leq 1$$

$$\ln(1+x) = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \epsilon(x)$$

$$|\epsilon(x)| \leq 1 \times 10^{-5}$$

$$a_1 = .99949 \ 556 \quad a_4 = -.13606 \ 275$$

$$a_2 = -.49190 \ 896 \quad a_5 = .03215 \ 845$$

$$a_3 = .28947 \ 478$$

² The approximations 4.1.41 to 4.1.44 are from C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

$$4.1.44 \quad 0 \leq x \leq 1$$

$$\ln(1+x) = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \\ + a_7x^7 + a_8x^8 + \epsilon(x)$$

$$|\epsilon(x)| \leq 3 \times 10^{-8}$$

$$\begin{array}{ll} a_1 = .99999 \ 64239 & a_5 = .16765 \ 40711 \\ a_2 = -.49987 \ 41238 & a_6 = -.09532 \ 93897 \\ a_3 = .33179 \ 90258 & a_7 = .03608 \ 84937 \\ a_4 = -.24073 \ 38084 & a_8 = -.00645 \ 35442 \end{array}$$

Approximation in Terms of Chebyshev Polynomials³

$$4.1.45 \quad 0 \leq x \leq 1$$

$$T_n^*(x) = \cos n\theta, \cos \theta = 2x - 1 \text{ (see chapter 22)}$$

$$\ln(1+x) = \sum_{n=0}^{\infty} A_n T_n^*(x)$$

n	A_n	n	A_n
0	.37645 2813	6	-.00000 8503
1	.34314 5750	7	.00000 1250
2	-.02943 7252	8	-.00000 0188
3	.00336 7089	9	.00000 0029
4	-.00043 3276	10	-.00000 0004
5	.00005 9471	11	.00000 0001

Differentiation Formulas

$$4.1.46 \quad \frac{d}{dz} \ln z = \frac{1}{z}$$

$$4.1.47 \quad \frac{d^n}{dz^n} \ln z = (-1)^{n-1} (n-1)! z^{-n}$$

Integration Formulas

$$4.1.48 \quad \int \frac{dz}{z} = \ln z$$

$$4.1.49 \quad \int \ln z \, dz = z \ln z - z$$

$$4.1.50$$

$$\int z^n \ln z \, dz = \frac{z^{n+1}}{n+1} \ln z - \frac{z^{n+1}}{(n+1)^2} \\ (n \neq -1, \ n \text{ integer})$$

$$4.1.51$$

$$\int z^n (\ln z)^m \, dz = \frac{z^{n+1} (\ln z)^m}{n+1} - \frac{m}{n+1} \int z^n (\ln z)^{m-1} \, dz \\ (n \neq -1)$$

³ The approximation 4.1.45 is from C. W. Clenshaw, Polynomial approximations to elementary functions, Math. Tables Aids Comp. 8, 143-147 (1954) (with permission).

$$4.1.52 \quad \int \frac{dz}{z \ln z} = \ln \ln z$$

$$4.1.53$$

$$\int \ln [z + (z^2 \pm 1)^{\frac{1}{2}}] \, dz = z \ln [z + (z^2 \pm 1)^{\frac{1}{2}}] - (z^2 \pm 1)^{\frac{1}{2}}$$

$$4.1.54$$

$$\int z^n \ln [z + (z^2 \pm 1)^{\frac{1}{2}}] \, dz = \frac{z^{n+1}}{n+1} \ln [z + (z^2 \pm 1)^{\frac{1}{2}}] \\ - \frac{1}{n+1} \int \frac{z^{n+1}}{(z^2 \pm 1)^{\frac{1}{2}}} \, dz \quad (n \neq -1)$$

Definite Integrals

$$4.1.55 \quad \int_0^1 \frac{\ln t}{1-t} \, dt = -\pi^2/6$$

$$4.1.56 \quad \int_0^1 \frac{\ln t}{1+t} \, dt = -\pi^2/12$$

$$4.1.57 \quad \int_0^x \frac{dt}{\ln t} = li(x) \text{ (see 5.1.3)}$$

4.2. Exponential Function

Series Expansion

$$4.2.1$$

$$e^z = \exp z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad (z = x + iy)$$

where e is the real number defined in 4.1.16

Fundamental Properties

$$4.2.2 \quad \operatorname{Ln}(\exp z) = z + 2k\pi i \quad (k \text{ any integer})$$

$$4.2.3 \quad \ln(\exp z) = z \quad (-\pi < \Im z \leq \pi)$$

$$4.2.4 \quad \exp(\ln z) = \exp(\operatorname{Ln} z) = z$$

$$4.2.5 \quad \frac{d}{dz} \exp z = \exp z$$

Definition of General Powers

$$4.2.6 \quad \text{If } N = a^z, \text{ then } z = \operatorname{Log}_a N$$

$$4.2.7 \quad a^z = \exp(z \ln a)$$

$$4.2.8 \quad \text{If } a = |a| \exp(i \arg a) \quad (-\pi < \arg a \leq \pi)$$

$$4.2.9 \quad |a^z| = |a|^x e^{-y \arg a}$$

$$4.2.10 \quad \arg(a^z) = y \ln |a| + x \arg a$$

$$4.2.11$$

$$\operatorname{Ln} a^z = z \ln a \quad \text{for one of the values of } \operatorname{Ln} a^z$$

$$4.2.12 \quad \ln a^z = x \ln a \quad (a \text{ real and positive})$$

$$4.2.13 \quad |e^z| = e^x$$

$$4.2.14 \quad \arg(e^z) = y$$

$$4.2.15 \quad a^{z_1} a^{z_2} = a^{z_1 + z_2}$$

$$4.2.16 \quad a^z b^z = (ab)^z \quad (-\pi < \arg a + \arg b \leq \pi)$$

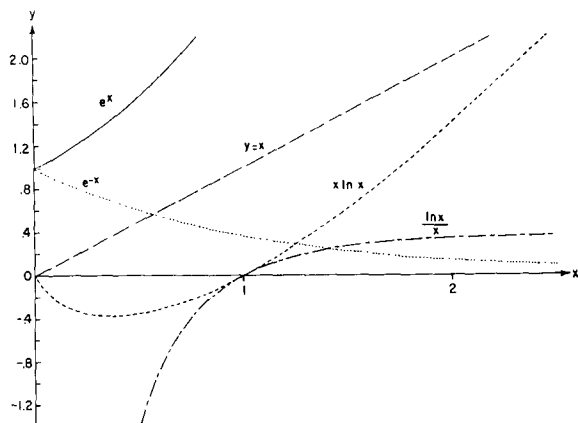


FIGURE 4.2. Logarithmic and exponential functions.

Periodic Property

$$4.2.17 \quad e^{z+2\pi ki} = e^z \quad (k \text{ any integer})$$

Exponential Identities

$$4.2.18 \quad e^{z_1} e^{z_2} = e^{z_1 + z_2}$$

$$4.2.19 \quad (e^{z_1})^{z_2} = e^{z_1 z_2} \quad (-\pi < \mathcal{I} z_1 \leq \pi)$$

The restriction $(-\pi < \mathcal{I} z_1 \leq \pi)$ can be removed if z_2 is an integer.

Limiting Values

$$4.2.20 \quad \lim_{|z| \rightarrow \infty} z^\alpha e^{-z} = 0 \quad (|\arg z| \leq \frac{1}{2}\pi - \epsilon < \frac{1}{2}\pi, \alpha \text{ constant})$$

$$4.2.21 \quad \lim_{m \rightarrow \infty} \left(1 + \frac{z}{m}\right)^m = e^z$$

Special Values (see chapter 1)

$$4.2.22 \quad e = 2.71828 \ 18284 \dots$$

$$4.2.23 \quad e^0 = 1$$

$$4.2.24 \quad e^\infty = \infty$$

$$4.2.25 \quad e^{-\infty} = 0$$

$$4.2.26 \quad e^{\pm \pi i} = -1$$

$$4.2.27 \quad e^{\pm \frac{\pi i}{2}} = \pm i$$

$$4.2.28 \quad e^{2\pi ki} = 1 \quad (k \text{ any integer})$$

Exponential Inequalities

If x is real and different from zero

$$4.2.29 \quad e^{-\frac{x}{1-x}} < 1 - x < e^{-x} \quad (x < 1)$$

$$4.2.30 \quad e^x > 1 + x$$

$$4.2.31 \quad e^x < \frac{1}{1-x} \quad (x < 1)$$

$$4.2.32 \quad \frac{x}{1+x} < (1 - e^{-x}) < x \quad (x > -1)$$

$$4.2.33 \quad x < (e^x - 1) < \frac{x}{1-x} \quad (x < 1)$$

$$4.2.34 \quad 1 + x > e^{\frac{x}{1+x}} \quad (x > -1)$$

$$4.2.35 \quad e^x > 1 + \frac{x^n}{n!} \quad (n > 0, x > 0)$$

$$4.2.36 \quad e^x > \left(1 + \frac{x}{y}\right)^y > e^{\frac{xy}{x+y}} \quad (x > 0, y > 0)$$

$$4.2.37 \quad e^{-x} < 1 - \frac{x}{2} \quad (0 < x \leq 1.5936)$$

$$4.2.38 \quad \frac{1}{4}|z| < |e^z - 1| < \frac{7}{4}|z| \quad (0 < |z| < 1)$$

$$4.2.39 \quad |e^z - 1| \leq e^{|z|} - 1 \leq |z|e^{|z|} \quad (\text{all } z)$$

Continued Fractions

$$4.2.40 \quad e^z = \frac{1}{1 - \frac{z}{1 + \frac{z}{2 - \frac{z}{3 + \frac{z}{2 - \frac{z}{5 + \frac{z}{2 - \dots}}}}}}} \quad (|z| < \infty)$$

$$= 1 + \frac{z}{1 - \frac{z}{2 + \frac{z}{3 - \frac{z}{2 + \frac{z}{5 - \frac{z}{2 + \frac{z}{7 - \dots}}}}}}} \quad (|z| < \infty)$$

$$= 1 + \frac{z}{(1 - z/2) + \frac{z^2/4 \cdot 3}{1 + \frac{z^2/4 \cdot 15}{1 + \frac{z^2/4 \cdot 35}{1 + \dots \frac{z^2/4(4n^2 - 1)}{1 + \dots}}}} \quad (|z| < \infty)$$

$$4.2.41 \quad e^z - e_{n-1}(z) = \frac{z^n}{n!} - \frac{n!z}{(n+1)!} + \frac{z}{(n+2)!} - \frac{(n+1)z}{(n+3)!} + \frac{2z}{(n+4)!} - \frac{(n+2)z}{(n+5)!} + \frac{3z}{(n+6)!} - \dots \quad (|z| < \infty)$$

(For $e_n(z)$ see 6.5.11)

4.2.42

$$e^{2a \arctan \frac{1}{z}} = 1 + \frac{2a}{z-a} \frac{a^2+1}{3z+5} \frac{a^2+4}{5z+7} \frac{a^2+9}{7z+\dots}$$

(z in the cut plane of Figure 4.4.)

Polynomial Approximations⁴

$$4.2.43 \quad 0 \leq x \leq \ln 2 = .693 \dots$$

$$e^{-x} = 1 + a_1 x + a_2 x^2 + \epsilon(x)$$

$$|\epsilon(x)| \leq 3 \times 10^{-5}$$

$$a_1 = -.9664 \quad a_2 = .3536$$

$$4.2.44 \quad 0 \leq x \leq \ln 2$$

$$e^{-x} = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \epsilon(x)$$

$$|\epsilon(x)| \leq 3 \times 10^{-11}$$

$$a_1 = -.99986 \ 84 \quad a_3 = -.15953 \ 32$$

$$a_2 = .49829 \ 26 \quad a_4 = .02936 \ 41$$

$$4.2.45 \quad 0 \leq x \leq \ln 2$$

$$e^{-x} = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \epsilon(x)$$

$$|\epsilon(x)| \leq 2 \times 10^{-10}$$

$$a_1 = -.99999 \ 99995 \quad a_5 = -.00830 \ 13598$$

$$a_2 = .49999 \ 99206 \quad a_6 = .00132 \ 98820$$

$$a_3 = -.16666 \ 53019 \quad a_7 = -.00014 \ 13161$$

$$a_4 = .04165 \ 73475$$

$$4.2.46^5 \quad 0 \leq x \leq 1$$

$$10^x = (1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4)^2 + \epsilon(x)$$

$$|\epsilon(x)| \leq 7 \times 10^{-4}$$

$$a_1 = 1.14991 \ 96 \quad a_3 = .20800 \ 30$$

$$a_2 = .67743 \ 23 \quad a_4 = .12680 \ 89$$

$$4.2.47 \quad 0 \leq x \leq 1$$

$$10^x = (1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7)^2 + \epsilon(x)$$

$$|\epsilon(x)| < 5 \times 10^{-8}$$

$$a_1 = 1.15129 \ 277603 \quad a_5 = .01742 \ 111988$$

$$a_2 = .66273 \ 088429 \quad a_6 = .00255 \ 491796$$

$$a_3 = .25439 \ 357484 \quad a_7 = .00093 \ 264267$$

$$a_4 = .07295 \ 173666$$

⁴ The approximations 4.2.43 to 4.2.45 are from B. Carlson, M. Goldstein, Rational approximation of functions, Los Alamos Scientific Laboratory LA-1943, Los Alamos, N. Mex., 1955 (with permission).

⁵ The approximations 4.2.46 to 4.2.47 are from C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

Approximations in Terms of Chebyshev Polynomials⁶

$$4.2.48 \quad 0 \leq x \leq 1$$

$$T_n^*(x) = \cos n\theta, \quad \cos \theta = 2x - 1 \quad (\text{see chapter 22})$$

$$e^x = \sum_{n=0}^{\infty} A_n T_n^*(x) \quad e^{-x} = \sum_{n=0}^{\infty} A_n T_n^*(x)$$

n	A_n	n	A_n
0	1.75338 7654	0	.64503 5270
1	.85039 1654	1	-.31284 1606
2	.10520 8694	2	.03870 4116
3	.00872 2105	3	-.00320 8683
4	.00054 3437	4	.00019 9919
5	.00002 7115	5	-.00000 9975
6	.00000 1128	6	.00000 0415
7	.00000 0040	7	-.00000 0015
8	.00000 0001		

Differentiation Formulas

$$4.2.49 \quad \frac{d}{dz} e^z = e^z$$

$$4.2.50 \quad \frac{d^n}{dz^n} e^{az} = a^n e^{az}$$

$$4.2.51 \quad \frac{d}{dz} a^z = a^z \ln a$$

$$4.2.52 \quad \frac{d}{dz} z^a = a z^{a-1}$$

$$4.2.53 \quad \frac{d}{dz} z^z = (1 + \ln z) z^z$$

Integration Formulas

$$4.2.54 \quad \int e^{az} dz = e^{az}/a$$

$$4.2.55 \quad \int z^n e^{az} dz = \frac{e^{az}}{a^{n+1}} [(az)^n - n(az)^{n-1} + n(n-1)(az)^{n-2} + \dots + (-1)^{n-1} n! (az) + (-1)^n n!] \quad (n \geq 0)$$

$$4.2.56 \quad \int \frac{e^{az}}{z^n} dz = -\frac{e^{az}}{(n-1)z^{n-1}} + \frac{a}{n-1} \int \frac{e^{az}}{z^{n-1}} dz \quad (n > 1)$$

(See chapters 5, 7 and 29 for other integrals involving exponential functions.)

4.3. Circular Functions

Definitions

$$4.3.1 \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad (z = x + iy)$$

$$4.3.2 \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

⁶ The approximations 4.2.48 are from C. W. Clenshaw, Polynomial approximations to elementary functions, Math. Tables Aids Comp. 8, 143-147 (1954) (with permission).

$$4.3.3 \quad \tan z = \frac{\sin z}{\cos z}$$

$$4.3.4 \quad \csc z = \frac{1}{\sin z}$$

$$4.3.5 \quad \sec z = \frac{1}{\cos z}$$

$$4.3.6 \quad \cot z = \frac{1}{\tan z}$$

Periodic Properties

$$4.3.7 \quad \sin(z + 2k\pi) = \sin z \quad (k \text{ any integer})$$

$$4.3.8 \quad \cos(z + 2k\pi) = \cos z$$

$$4.3.9 \quad \tan(z + k\pi) = \tan z$$

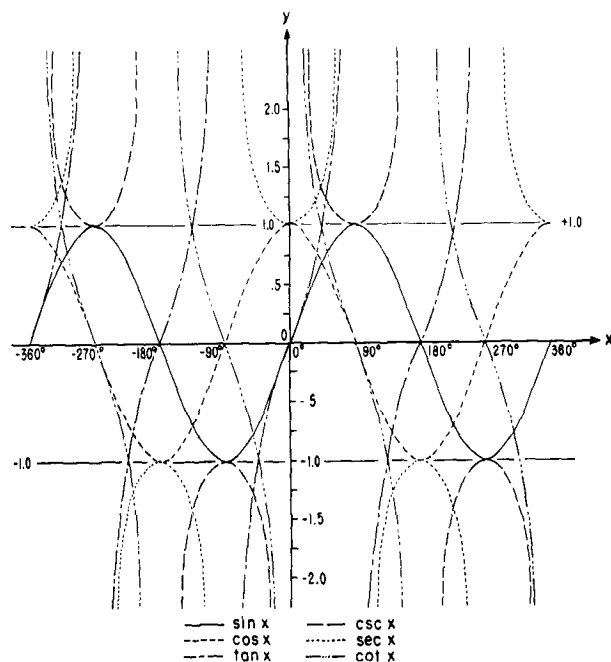


FIGURE 4.3. Circular functions.

Relations Between Circular Functions

$$4.3.10 \quad \sin^2 z + \cos^2 z = 1$$

$$4.3.11 \quad \sec^2 z - \tan^2 z = 1$$

$$4.3.12 \quad \csc^2 z - \cot^2 z = 1$$

Negative Angle Formulas

$$4.3.13 \quad \sin(-z) = -\sin z$$

$$4.3.14 \quad \cos(-z) = \cos z$$

$$4.3.15 \quad \tan(-z) = -\tan z$$

Addition Formulas

$$4.3.16 \quad \sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

$$4.3.17 \quad \cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$

$$4.3.18 \quad \tan(z_1 + z_2) = \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$$

$$4.3.19 \quad \cot(z_1 + z_2) = \frac{\cot z_1 \cot z_2 - 1}{\cot z_2 + \cot z_1}$$

Half-Angle Formulas

$$4.3.20 \quad \sin \frac{z}{2} = \pm \left(\frac{1 - \cos z}{2} \right)^{\frac{1}{2}}$$

$$4.3.21 \quad \cos \frac{z}{2} = \pm \left(\frac{1 + \cos z}{2} \right)^{\frac{1}{2}}$$

$$4.3.22 \quad \tan \frac{z}{2} = \pm \left(\frac{1 - \cos z}{1 + \cos z} \right)^{\frac{1}{2}} = \frac{1 - \cos z}{\sin z} = \frac{\sin z}{1 + \cos z}$$

The ambiguity in sign may be resolved with the aid of a diagram.

Transformation of Trigonometric Integrals

If $\tan \frac{u}{2} = z$ then

$$4.3.23 \quad \sin u = \frac{2z}{1+z^2}, \quad \cos u = \frac{1-z^2}{1+z^2}, \quad du = \frac{2}{1+z^2} dz$$

Multiple-Angle Formulas

$$4.3.24 \quad \sin 2z = 2 \sin z \cos z = \frac{2 \tan z}{1 + \tan^2 z}$$

$$4.3.25 \quad \begin{aligned} \cos 2z &= 2 \cos^2 z - 1 = 1 - 2 \sin^2 z \\ &= \cos^2 z - \sin^2 z = \frac{1 - \tan^2 z}{1 + \tan^2 z} \end{aligned}$$

$$4.3.26 \quad \tan 2z = \frac{2 \tan z}{1 - \tan^2 z} = \frac{2 \cot z}{\cot^2 z - 1} = \frac{2}{\cot z - \tan z}$$

$$4.3.27 \quad \sin 3z = 3 \sin z - 4 \sin^3 z$$

$$4.3.28 \quad \cos 3z = -3 \cos z + 4 \cos^3 z$$

$$4.3.29 \quad \sin 4z = 8 \cos^3 z \sin z - 4 \cos z \sin z$$

$$4.3.30 \quad \cos 4z = 8 \cos^4 z - 8 \cos^2 z + 1$$

Products of Sines and Cosines

$$4.3.31 \quad 2 \sin z_1 \sin z_2 = \cos(z_1 - z_2) - \cos(z_1 + z_2)$$

$$4.3.32 \quad 2 \cos z_1 \cos z_2 = \cos(z_1 - z_2) + \cos(z_1 + z_2)$$

$$4.3.33 \quad 2 \sin z_1 \cos z_2 = \sin(z_1 - z_2) + \sin(z_1 + z_2)$$

Addition and Subtraction of Two Circular Functions

4.3.34

$$\sin z_1 + \sin z_2 = 2 \sin \left(\frac{z_1 + z_2}{2} \right) \cos \left(\frac{z_1 - z_2}{2} \right)$$

4.3.35

$$\sin z_1 - \sin z_2 = 2 \cos \left(\frac{z_1 + z_2}{2} \right) \sin \left(\frac{z_1 - z_2}{2} \right)$$

4.3.36

$$\cos z_1 + \cos z_2 = 2 \cos \left(\frac{z_1 + z_2}{2} \right) \cos \left(\frac{z_1 - z_2}{2} \right)$$

4.3.37

$$\cos z_1 - \cos z_2 = -2 \sin \left(\frac{z_1 + z_2}{2} \right) \sin \left(\frac{z_1 - z_2}{2} \right)$$

4.3.38

$$\tan z_1 \pm \tan z_2 = \frac{\sin (z_1 \pm z_2)}{\cos z_1 \cos z_2}$$

4.3.39

$$\cot z_1 \pm \cot z_2 = \frac{\sin (z_2 \pm z_1)}{\sin z_1 \sin z_2}$$

Relations Between Squares of Sines and Cosines

4.3.40

$$\sin^2 z_1 - \sin^2 z_2 = \sin (z_1 + z_2) \sin (z_1 - z_2)$$

4.3.41

$$\cos^2 z_1 - \cos^2 z_2 = -\sin (z_1 + z_2) \sin (z_1 - z_2)$$

4.3.42

$$\cos^2 z_1 - \sin^2 z_2 = \cos (z_1 + z_2) \cos (z_1 - z_2)$$

4.3.43

Signs of the Circular Functions
in the Four Quadrants

Quadrant	sin csc	cos sec	tan cot
I	+	+	+
II	+	-	-
III	-	-	+
IV	-	+	-

4.3.44

Functions of Angles in Any Quadrant in Terms of Angles in the First Quadrant. ($0 \leq \theta \leq \frac{\pi}{2}$, k any integer)

	$-\theta$	$\frac{\pi}{2} \pm \theta$	$\pi \pm \theta$	$\frac{3\pi}{2} \pm \theta$	$2k\pi \pm \theta$
sin.....	$-\sin \theta$	$\cos \theta$	$\mp \sin \theta$	$-\cos \theta$	$\pm \sin \theta$
cos.....	$\cos \theta$	$\mp \sin \theta$	$-\cos \theta$	$\pm \sin \theta$	$\mp \cos \theta$
tan.....	$-\tan \theta$	$\mp \cot \theta$	$\pm \tan \theta$	$\mp \cot \theta$	$\pm \tan \theta$
csc.....	$-\csc \theta$	$+\sec \theta$	$\mp \csc \theta$	$-\sec \theta$	$\pm \csc \theta$
sec.....	$\sec \theta$	$\mp \csc \theta$	$-\sec \theta$	$\pm \csc \theta$	$\mp \sec \theta$
cot.....	$-\cot \theta$	$\mp \tan \theta$	$\pm \cot \theta$	$\mp \tan \theta$	$\pm \cot \theta$

4.3.45

Relations Between Circular (or Inverse Circular) Functions

	$\sin x = a$	$\cos x = a$	$\tan x = a$	$\csc x = a$	$\sec x = a$	$\cot x = a$
sin x.....	a	$(1-a^2)^{\frac{1}{2}}$	$a(1+a^2)^{-\frac{1}{2}}$	a^{-1}	$a^{-1}(a^2-1)^{\frac{1}{2}}$	$(1+a^2)^{-\frac{1}{2}}$
cos x.....	$(1-a^2)^{\frac{1}{2}}$	a	$(1+a^2)^{-\frac{1}{2}}$	$a^{-1}(a^2-1)^{\frac{1}{2}}$	a^{-1}	$a(1+a^2)^{-\frac{1}{2}}$
tan x.....	$a(1-a^2)^{-\frac{1}{2}}$	$a^{-1}(1-a^2)^{\frac{1}{2}}$	a	$(a^2-1)^{-\frac{1}{2}}$	$(a^2-1)^{\frac{1}{2}}$	a^{-1}
csc x.....	a^{-1}	$(1-a^2)^{-\frac{1}{2}}$	$a^{-1}(1+a^2)^{\frac{1}{2}}$	a	$a(a^2-1)^{-\frac{1}{2}}$	$(1+a^2)^{\frac{1}{2}}$
sec x.....	$(1-a^2)^{-\frac{1}{2}}$	a^{-1}	$(1+a^2)^{\frac{1}{2}}$	$a(a^2-1)^{-\frac{1}{2}}$	a	$a^{-1}(1+a^2)^{\frac{1}{2}}$
cot x.....	$a^{-1}(1-a^2)^{\frac{1}{2}}$	$a(1-a^2)^{-\frac{1}{2}}$	a^{-1}	$(a^2-1)^{\frac{1}{2}}$	$(a^2-1)^{-\frac{1}{2}}$	a

$(0 \leq x \leq \frac{\pi}{2})$ Illustration: If $\sin x = a$, $\cot x = a^{-1}(1-a^2)^{\frac{1}{2}}$

$$\operatorname{arcsec} a = \operatorname{arccot} (a^2-1)^{-\frac{1}{2}}$$

4.3.46 Circular Functions for Certain Angles

	0 0°	$\pi/12$ 15°	$\pi/6$ 30°	$\pi/4$ 45°	$\pi/3$ 60°
sin	0	$\frac{\sqrt{2}}{4}(\sqrt{3}-1)$	1/2	$\sqrt{2}/2$	$\sqrt{3}/2$
cos	1	$\frac{\sqrt{2}}{4}(\sqrt{3}+1)$	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2
tan	0	$2-\sqrt{3}$	$\sqrt{3}/3$	1	$\sqrt{3}$
csc	∞	$\sqrt{2}(\sqrt{3}+1)$	2	$\sqrt{2}$	$2\sqrt{3}/3$
sec	1	$\sqrt{2}(\sqrt{3}-1)$	$2\sqrt{3}/3$	$\sqrt{2}$	2
cot	∞	$2+\sqrt{3}$	$\sqrt{3}$	1	$\sqrt{3}/3$

	$5\pi/12$ 75°	$\pi/2$ 90°	$7\pi/12$ 105°	$2\pi/3$ 120°
sin	$\frac{\sqrt{2}}{4}(\sqrt{3}+1)$	1	$\frac{\sqrt{2}}{4}(\sqrt{3}+1)$	$\sqrt{3}/2$
cos	$\frac{\sqrt{2}}{4}(\sqrt{3}-1)$	0	$-\frac{\sqrt{2}}{4}(\sqrt{3}-1)$	-1/2
tan	$2+\sqrt{3}$	∞	$-(2+\sqrt{3})$	$-\sqrt{3}$
csc	$\sqrt{2}(\sqrt{3}-1)$	1	$\sqrt{2}(\sqrt{3}-1)$	$2\sqrt{3}/3$
sec	$\sqrt{2}(\sqrt{3}+1)$	∞	$-\sqrt{2}(\sqrt{3}+1)$	-2
cot	$2-\sqrt{3}$	0	$-(2-\sqrt{3})$	$-\sqrt{3}/3$

	$3\pi/4$ 135°	$5\pi/6$ 150°	$11\pi/12$ 165°	π 180°
sin	$\sqrt{2}/2$	1/2	$\frac{\sqrt{2}}{4}(\sqrt{3}-1)$	0
cos	$-\sqrt{2}/2$	$-\sqrt{3}/2$	$-\frac{\sqrt{2}}{4}(\sqrt{3}+1)$	-1
tan	-1	$-\sqrt{3}/3$	$-(2-\sqrt{3})$	0
csc	$\sqrt{2}$	2	$\sqrt{2}(\sqrt{3}+1)$	∞
sec	$-\sqrt{2}$	$-2\sqrt{3}/3$	$-\sqrt{2}(\sqrt{3}-1)$	-1
cot	-1	$-\sqrt{3}$	$-(2+\sqrt{3})$	∞

Euler's Formula

$$4.3.47 \quad e^z = e^{x+iy} = e^x (\cos y + i \sin y)$$

De Moivre's Theorem

$$4.3.48 \quad (\cos z + i \sin z)^v = \cos vz + i \sin vz$$

($-\pi < \Re z \leq \pi$ unless v is an integer)

Relation to Hyperbolic Functions (see 4.5.7 to 4.5.12)

$$4.3.49 \quad \sin z = -i \sinh iz$$

$$4.3.50 \quad \cos z = \cosh iz$$

$$4.3.51 \quad \tan z = -i \tanh iz$$

$$4.3.52 \quad \csc z = i \operatorname{csch} iz$$

$$4.3.53 \quad \sec z = \operatorname{sech} iz$$

$$4.3.54 \quad \cot z = i \coth iz$$

Circular Functions in Terms of Real and Imaginary Parts

$$4.3.55 \quad \sin z = \sin x \cosh y + i \cos x \sinh y$$

$$4.3.56 \quad \cos z = \cos x \cosh y - i \sin x \sinh y$$

$$4.3.57 \quad \tan z = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$$

$$4.3.58 \quad \cot z = \frac{\sin 2x - i \sinh 2y}{\cosh 2y - \cos 2x}$$

Modulus and Phase (Argument) of Circular Functions

$$4.3.59 \quad |\sin z| = (\sin^2 x + \sinh^2 y)^{\frac{1}{2}}$$

$$= [\tfrac{1}{2} (\cosh 2y - \cos 2x)]^{\frac{1}{2}}$$

$$4.3.60 \quad \arg \sin z = \arctan (\cot x \tanh y)$$

$$4.3.61 \quad |\cos z| = (\cos^2 x + \sinh^2 y)^{\frac{1}{2}}$$

$$= [\tfrac{1}{2} (\cosh 2y + \cos 2x)]^{\frac{1}{2}}$$

$$4.3.62 \quad \arg \cos z = -\arctan (\tan x \tanh y)$$

$$4.3.63 \quad |\tan z| = \left(\frac{\cosh 2y - \cos 2x}{\cosh 2y + \cos 2x} \right)^{\frac{1}{2}}$$

$$4.3.64 \quad \arg \tan z = \arctan \left(\frac{\sinh 2y}{\sin 2x} \right)$$

Series Expansions

$$4.3.65$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \quad (|z| < \infty)$$

$$4.3.66$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \quad (|z| < \infty)$$

4.3.67

$$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + \dots$$

$$+ \frac{(-1)^{n-1} 2^{2n} (2^{2n}-1) B_{2n}}{(2n)!} z^{2n-1} + \dots \quad \left(|z| < \frac{\pi}{2}\right)$$

4.3.68

$$\csc z = \frac{1}{z} + \frac{z}{6} + \frac{7}{360} z^3 + \frac{31}{15120} z^5 + \dots$$

$$+ \frac{(-1)^{n-1} 2(2^{2n-1}-1) B_{2n}}{(2n)!} z^{2n-1} + \dots \quad (|z| < \pi)$$

4.3.69

$$\sec z = 1 + \frac{z^2}{2} + \frac{5z^4}{24} + \frac{61z^6}{720} + \dots$$

$$+ \frac{(-1)^n E_{2n}}{(2n)!} z^{2n} + \dots \quad \left(|z| < \frac{\pi}{2}\right)$$

4.3.70

$$\cot z = \frac{1}{z} - \frac{z}{3} + \frac{z^3}{45} - \frac{2z^5}{945} + \dots$$

$$- \frac{(-1)^{n-1} 2^{2n} B_{2n}}{(2n)!} z^{2n-1} + \dots \quad (|z| < \pi)$$

4.3.71

$$\ln \frac{\sin z}{z} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} B_{2n}}{n(2n)!} z^{2n} \quad (|z| < \pi)$$

4.3.72

$$\ln \cos z = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} (2^{2n}-1) B_{2n}}{n(2n)!} z^{2n} \quad (|z| < \frac{1}{2}\pi)$$

4.3.73

$$\ln \frac{\tan z}{z} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n-1}-1) B_{2n}}{n(2n)!} z^{2n}$$

$$(|z| < \frac{1}{2}\pi)$$

where B_n and E_n are the Bernoulli and Euler numbers (see chapter 23).

Limiting Values

$$4.3.74 \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$4.3.75 \quad \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

$$4.3.76 \quad \lim_{n \rightarrow \infty} n \sin \frac{x}{n} = x$$

$$4.3.77 \quad \lim_{n \rightarrow \infty} n \tan \frac{x}{n} = x$$

$$4.3.78 \quad \lim_{n \rightarrow \infty} \cos \frac{x}{n} = 1$$

Inequalities

$$4.3.79 \quad \frac{\sin x}{x} > \frac{2}{\pi} \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

$$4.3.80 \quad \sin x \leq x \leq \tan x \quad \left(0 \leq x \leq \frac{\pi}{2}\right)$$

$$4.3.81 \quad \cos x \leq \frac{\sin x}{x} \leq 1 \quad (0 \leq x \leq \pi)$$

$$4.3.82 \quad \pi < \frac{\sin \pi x}{x(1-x)} \leq 4 \quad (0 < x < 1)$$

$$4.3.83 \quad |\sinh y| \leq |\sin z| \leq \cosh y$$

$$4.3.84 \quad |\sinh y| \leq |\cos z| \leq \cosh y$$

$$4.3.85 \quad |\csc z| \leq \operatorname{csch} |y|$$

$$4.3.86 \quad |\cos z| \leq \cosh |z|$$

$$4.3.87 \quad |\sin z| \leq \sinh |z|$$

$$4.3.88 \quad |\cos z| < 2, \quad |\sin z| \leq \frac{6}{5} |z| \quad (|z| < 1)$$

Infinite Products

$$4.3.89 \quad \sin z = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2 \pi^2}\right)$$

$$4.3.90 \quad \cos z = \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{(2k-1)^2 \pi^2}\right)$$

Expansion in Partial Fractions

$$4.3.91 \quad \cot z = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{1}{z^2 - k^2 \pi^2}$$

$$(z \neq 0, \pm \pi, \pm 2\pi, \dots)$$

$$4.3.92 \quad \csc^2 z = \sum_{k=-\infty}^{\infty} \frac{1}{(z - k\pi)^2}$$

$$(z \neq 0, \pm \pi, \pm 2\pi, \dots)$$

$$4.3.93 \quad \csc z = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{(-1)^k}{z^2 - k^2 \pi^2}$$

$$(z \neq 0, \pm \pi, \pm 2\pi, \dots)$$

Continued Fractions

$$4.3.94 \quad \tan z = \frac{z}{1 - \frac{z^2}{3 - \frac{z^2}{5 - \frac{z^2}{7 - \dots}}}} \quad \left(z \neq \frac{\pi}{2} \pm n\pi\right)$$

4.3.95

$$\tan az = \frac{a \tan z}{1 + \frac{(1-a^2) \tan^2 z}{3 + \frac{(4-a^2) \tan^2 z}{5 + \frac{(9-a^2) \tan^2 z}{7 + \dots}}}} \quad \left(-\frac{\pi}{2} < \Re z < \frac{\pi}{2}, \quad az \neq \frac{\pi}{2} \pm n\pi\right)$$

Polynomial Approximations ⁷

4.3.96

$$0 \leq x \leq \frac{\pi}{2}$$

$$\frac{\sin x}{x} = 1 + a_2 x^2 + a_4 x^4 + \epsilon(x)$$

$$|\epsilon(x)| \leq 2 \times 10^{-4}$$

$$a_2 = -.16605$$

$$a_4 = .00761$$

4.3.97

$$0 \leq x \leq \frac{\pi}{2}$$

$$\frac{\sin x}{x} = 1 + a_2 x^2 + a_4 x^4 + a_6 x^6 + a_8 x^8 + a_{10} x^{10} + \epsilon(x)$$

$$|\epsilon(x)| \leq 2 \times 10^{-9}$$

$$a_2 = -.16666 \ 66664$$

$$a_8 = .00000 \ 27526$$

$$a_4 = .00833 \ 33315$$

$$a_{10} = -.00000 \ 00239$$

$$a_6 = -.00019 \ 84090$$

4.3.98

$$0 \leq x \leq \frac{\pi}{2}$$

$$\cos x = 1 + a_2 x^2 + a_4 x^4 + \epsilon(x)$$

$$|\epsilon(x)| \leq 9 \times 10^{-4}$$

$$a_2 = -.49670$$

$$a_4 = .03705$$

4.3.99

$$0 \leq x \leq \frac{\pi}{2}$$

$$\cos x = 1 + a_2 x^2 + a_4 x^4 + a_6 x^6 + a_8 x^8 + a_{10} x^{10} + \epsilon(x)$$

$$|\epsilon(x)| \leq 2 \times 10^{-9}$$

$$a_2 = -.49999 \ 99963$$

$$a_8 = .00002 \ 47609$$

$$a_4 = .04166 \ 66418$$

$$a_{10} = -.00000 \ 02605$$

$$a_6 = -.00138 \ 88397$$

4.3.100

$$0 \leq x \leq \frac{\pi}{4}$$

$$\frac{\tan x}{x} = 1 + a_2 x^2 + a_4 x^4 + \epsilon(x)$$

$$|\epsilon(x)| \leq 1 \times 10^{-3}$$

$$a_2 = .31755$$

$$a_4 = .20330$$

4.3.101

$$0 \leq x \leq \frac{\pi}{4}$$

$$\frac{\tan x}{x} = 1 + a_2 x^2 + a_4 x^4 + a_6 x^6 + a_8 x^8 + a_{10} x^{10} + a_{12} x^{12} + \epsilon(x)$$

$$|\epsilon(x)| \leq 2 \times 10^{-8}$$

$$a_2 = .33333 \ 14036$$

$$a_8 = .02456 \ 50893$$

$$a_4 = .13339 \ 23995$$

$$a_{10} = .00290 \ 05250$$

$$a_6 = .05337 \ 40603$$

$$a_{12} = .00951 \ 68091$$

4.3.102

$$0 \leq x \leq \frac{\pi}{4}$$

$$* \quad x \cot x = 1 + a_2 x^2 + a_4 x^4 + \epsilon(x)$$

$$|\epsilon(x)| \leq 3 \times 10^{-5}$$

$$a_2 = -.332867$$

$$a_4 = -.024369$$

4.3.103

$$0 \leq x \leq \frac{\pi}{4}$$

$$x \cot x = 1 + a_2 x^2 + a_4 x^4 + a_6 x^6 + a_8 x^8 + a_{10} x^{10} + \epsilon(x)$$

$$|\epsilon(x)| \leq 4 \times 10^{-10}$$

$$a_2 = -.33333 \ 33410$$

$$a_8 = -.00020 \ 78504$$

$$a_4 = -.02222 \ 20287$$

$$a_{10} = -.00002 \ 62619$$

$$a_6 = -.00211 \ 77168$$

Approximations in Terms of Chebyshev Polynomials ⁸

4.3.104

$$-1 \leq x \leq 1$$

$$T_n^*(x) = \cos n\theta, \cos \theta = 2x - 1 \quad (\text{see chapter 22})$$

$$\sin \frac{1}{2}\pi x = x \sum_{n=0}^{\infty} A_n T_n^*(x^2) \quad \cos \frac{1}{2}\pi x = \sum_{n=0}^{\infty} A_n T_n^*(x^2)$$

n	A_n	n	A_n
0	1.27627 8962	0	.47200 1216
1	-.28526 1569	1	-.49940 3258
2	.00911 8016	2	.02799 2080
3	-.00013 6587	3	-.00059 6695
4	.00000 1185	4	.00000 6704
5	-.00000 0007	5	-.00000 0047

⁷ The approximations 4.3.96 to 4.3.103 are from B. Carlson, M. Goldstein, Rational approximation of functions, Los Alamos Scientific Laboratory LA-1943, Los Alamos, N. Mex., 1955 (with permission).

⁸ The approximations 4.3.104 are from C. W. Clenshaw, Polynomial approximations to elementary functions, Math. Tables Aids Comp. 8, 143-147 (1954) (with permission).

*See page II.

Differentiation Formulas

$$4.3.105 \quad \frac{d}{dz} \sin z = \cos z$$

$$4.3.106 \quad \frac{d}{dz} \cos z = -\sin z$$

$$4.3.107 \quad \frac{d}{dz} \tan z = \sec^2 z$$

$$4.3.108 \quad \frac{d}{dz} \csc z = -\csc z \cot z$$

$$4.3.109 \quad \frac{d}{dz} \sec z = \sec z \tan z$$

$$4.3.110 \quad \frac{d}{dz} \cot z = -\csc^2 z$$

$$4.3.111 \quad \frac{d^n}{dz^n} \sin z = \sin\left(z + \frac{1}{2}n\pi\right)$$

$$4.3.112 \quad \frac{d^n}{dz^n} \cos z = \cos\left(z + \frac{1}{2}n\pi\right)$$

Integration Formulas

$$4.3.113 \quad \int \sin z \, dz = -\cos z$$

$$4.3.114 \quad \int \cos z \, dz = \sin z$$

$$4.3.115 \quad \int \tan z \, dz = -\ln \cos z = \ln \sec z$$

$$4.3.116 \quad \int \csc z \, dz = \ln \tan \frac{z}{2} = \ln (\csc z - \cot z) = \frac{1}{2} \ln \frac{1 - \cos z}{1 + \cos z}$$

$$4.3.117 \quad \int \sec z \, dz = \ln (\sec z + \tan z) = \ln \tan \left(\frac{\pi}{4} + \frac{z}{2}\right) = \text{gd}^{-1}(z)$$

= Inverse Gudermannian Function

$$\text{gd } z = 2 \arctan e^z - \frac{\pi}{2}$$

$$4.3.118 \quad \int \cot z \, dz = \ln \sin z = -\ln \csc z$$

$$4.3.119 \quad \int z^n \sin z \, dz = -z^n \cos z + n \int z^{n-1} \cos z \, dz$$

$$4.3.120 \quad \int \frac{\sin z}{z^n} \, dz = \frac{-\sin z}{(n-1)z^{n-1}} + \frac{1}{n-1} \int \frac{\cos z}{z^{n-1}} \, dz \quad (n > 1)$$

$$4.3.121 \quad \int \frac{z}{\sin^2 z} \, dz = -z \cot z - \ln \sin z$$

4.3.122

$$\int \frac{z \, dz}{\sin^n z} = \frac{-z \cos z}{(n-1) \sin^{n-1} z} - \frac{1}{(n-1)(n-2) \sin^{n-2} z} + \frac{(n-2)}{(n-1)} \int \frac{z \, dz}{\sin^{n-2} z} \quad (n > 2)$$

4.3.123

$$\int z^n \cos z \, dz = z^n \sin z - n \int z^{n-1} \sin z \, dz$$

4.3.124

$$\int \frac{\cos z}{z^n} \, dz = -\frac{\cos z}{(n-1)z^{n-1}} - \frac{1}{n-1} \int \frac{\sin z}{z^{n-1}} \, dz \quad (n > 1)$$

$$4.3.125 \quad \int \frac{z}{\cos^2 z} \, dz = z \tan z + \ln \cos z$$

4.3.126

$$\int \frac{z \, dz}{\cos^n z} = \frac{z \sin z}{(n-1) \cos^{n-1} z} - \frac{1}{(n-1)(n-2) \cos^{n-2} z} + \frac{(n-2)}{(n-1)} \int \frac{z \, dz}{\cos^{n-2} z} \quad (n > 2)$$

4.3.127

$$\begin{aligned} \int \sin^m z \cos^n z \, dz &= \frac{\sin^{m+1} z \cos^{n-1} z}{m+n} \\ &+ \frac{(n-1)}{(m+n)} \int \sin^m z \cos^{n-2} z \, dz \\ &= -\frac{\sin^{m-1} z \cos^{n+1} z}{m+n} \\ &+ \frac{(m-1)}{(m+n)} \int \sin^{m-2} z \cos^n z \, dz \quad (m \neq -n) \end{aligned}$$

4.3.128

$$\begin{aligned} \int \frac{dz}{\sin^m z \cos^n z} &= \frac{1}{(n-1) \sin^{m-1} z \cos^{n-1} z} \\ &+ \frac{m+n-2}{n-1} \int \frac{dz}{\sin^m z \cos^{n-2} z} \quad (n > 1) \\ &= \frac{-1}{(m-1) \sin^{m-1} z \cos^{n-1} z} \\ &+ \frac{m+n-2}{m-1} \int \frac{dz}{\sin^{m-2} z \cos^n z} \quad (m > 1) \end{aligned}$$

$$4.3.129 \quad \int \tan^n z \, dz = \frac{\tan^{n-1} z}{n-1} - \int \tan^{n-2} z \, dz \quad (n \neq 1)$$

$$4.3.130 \quad \int \cot^n z \, dz = -\frac{\cot^{n-1} z}{n-1} - \int \cot^{n-2} z \, dz \quad (n \neq 1)$$

4.3.131

$$\int \frac{dz}{a+b \sin z} = \frac{2}{(a^2-b^2)^{\frac{1}{2}}} \arctan \frac{a \tan \left(\frac{z}{2}\right) + b}{(a^2-b^2)^{\frac{1}{2}}} \quad (a^2 > b^2)$$

$$= \frac{1}{(b^2-a^2)^{\frac{1}{2}}} \ln \left[\frac{a \tan \left(\frac{z}{2}\right) + b - (b^2-a^2)^{\frac{1}{2}}}{a \tan \left(\frac{z}{2}\right) + b + (b^2-a^2)^{\frac{1}{2}}} \right] \quad (b^2 > a^2)$$

4.3.132

$$\int \frac{dz}{1 \pm \sin z} = \mp \tan \left(\frac{\pi}{4} \mp \frac{z}{2} \right)$$

4.3.133

$$\int \frac{dz}{a+b \cos z} = \frac{2}{(a^2-b^2)^{\frac{1}{2}}} \arctan \frac{(a-b) \tan \frac{z}{2}}{(a^2-b^2)^{\frac{1}{2}}} \quad (a^2 > b^2)$$

$$= \frac{1}{(b^2-a^2)^{\frac{1}{2}}} \ln \left[\frac{(b-a) \tan \frac{z}{2} + (b^2-a^2)^{\frac{1}{2}}}{(b-a) \tan \frac{z}{2} - (b^2-a^2)^{\frac{1}{2}}} \right] \quad (b^2 > a^2)$$

4.3.134

$$\int \frac{dz}{1+\cos z} = \tan \frac{z}{2}$$

4.3.135

$$\int \frac{dz}{1-\cos z} = -\cot \frac{z}{2}$$

4.3.136

$$\int e^{az} \sin bz \, dz = \frac{e^{az}}{a^2+b^2} (a \sin bz - b \cos bz)$$

4.3.137

$$\int e^{az} \cos bz \, dz = \frac{e^{az}}{a^2+b^2} (a \cos bz + b \sin bz)$$

4.3.138

$$\int e^{az} \sin^n bz \, dz = \frac{e^{az} \sin^{n-1} bz}{a^2+n^2b^2} (a \sin bz - nb \cos bz)$$

$$+ \frac{n(n-1)b^2}{a^2+n^2b^2} \int e^{az} \sin^{n-2} bz \, dz$$

4.3.139

$$\int e^{az} \cos^n bz \, dz = \frac{e^{az} \cos^{n-1} bz}{a^2+n^2b^2} (a \cos bz + nb \sin bz)$$

$$+ \frac{n(n-1)b^2}{a^2+n^2b^2} \int e^{az} \cos^{n-2} bz \, dz$$

Definite Integrals

4.3.140

$$\int_0^\pi \sin mt \sin nt \, dt = 0$$

($m \neq n$, m and n integers)

$$\int_0^\pi \cos mt \cos nt \, dt = 0$$

$$4.3.141 \quad \int_0^\pi \sin^2 nt \, dt = \int_0^\pi \cos^2 nt \, dt = \frac{\pi}{2}$$

(n an integer, $n \neq 0$)

$$4.3.142 \quad \int_0^\infty \frac{\sin mt}{t} \, dt = \frac{\pi}{2} \quad (m > 0)$$

$$= 0 \quad (m = 0)$$

$$= -\frac{\pi}{2} \quad (m < 0)$$

$$4.3.143 \quad \int_0^\infty \frac{\cos at - \cos bt}{t} \, dt = \ln(b/a)$$

$$4.3.144 \quad \int_0^\infty \sin t^2 \, dt = \int_0^\infty \cos t^2 \, dt = \frac{1}{2} \sqrt{\frac{\pi}{2}}$$

$$4.3.145 \quad \int_0^{\pi/2} \ln \sin t \, dt = \int_0^{\pi/2} \ln \cos t \, dt = -\frac{\pi}{2} \ln 2$$

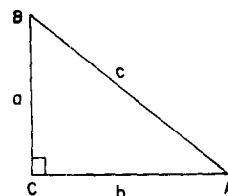
$$4.3.146 \quad \int_0^\infty \frac{\cos mt}{1+t^2} \, dt = \frac{\pi}{2} e^{-m}$$

(See chapters 5 and 7 for other integrals involving circular functions.)

(See [5.3] for Fourier transforms.)

4.3.147

Formulas for Solution of Plane Right Triangles



If A , B and C are the vertices (C the right angle), and a , b and c the sides opposite respectively,

$$\sin A = \frac{a}{c} = \frac{1}{\csc A}$$

$$\cos A = \frac{b}{c} = \frac{1}{\sec A}$$

$$\tan A = \frac{a}{b} = \frac{1}{\cot A}$$

$$\text{versine } A = \text{vers } A = 1 - \cos A$$

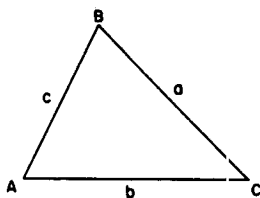
$$\text{coversine } A = \text{covers } A = 1 - \sin A$$

$$\text{haversine } A = \text{hav } A = \frac{1}{2} \text{vers } A$$

$$\text{exsecant } A = \text{exsec } A = \sec A - 1$$

4.3.148

Formulas for Solution of Plane Triangles



In a triangle with angles A , B and C and sides opposite a , b and c respectively,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\cos A = \frac{c^2 + b^2 - a^2}{2bc}$$

$$a = b \cos C + c \cos B$$

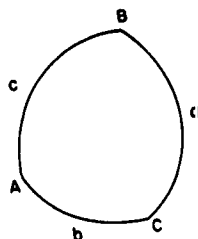
$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$$

$$\text{area} = \frac{bc \sin A}{2} = [s(s-a)(s-b)(s-c)]^{\frac{1}{2}}$$

$$s = \frac{1}{2}(a+b+c)$$

4.3.149

Formulas for Solution of Spherical Triangles



If A , B and C are the three angles and a , b and c the opposite sides,

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$= \frac{\cos b \cos (c \pm \theta)}{\cos \theta}$$

where $\tan \theta = \tan b \cos A$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

4.4. Inverse Circular Functions

Definitions

4.4.1

$$\arcsin z = \int_0^z \frac{dt}{(1-t^2)^{\frac{1}{2}}} \quad (z=x+iy)$$

4.4.2

$$\arccos z = \int_z^1 \frac{dt}{(1-t^2)^{\frac{1}{2}}} = \frac{\pi}{2} - \arcsin z$$

4.4.3

$$\arctan z = \int_0^z \frac{dt}{1+t^2}$$

The path of integration must not cross the real axis in the case of 4.4.1 and 4.4.2 and the imaginary axis in the case of 4.4.3 except possibly inside the unit circle. Each function is single-valued and regular in the z -plane cut along the real axis from $-\infty$ to -1 and $+1$ to $+\infty$ in the case of 4.4.1 and 4.4.2 and along the imaginary axis from i to $i\infty$ and $-i$ to $-i\infty$ in the case of 4.4.3.

Inverse circular functions are also written $\arcsin z = \sin^{-1} z$, $\arccos z = \cos^{-1} z$, $\arctan z = \tan^{-1} z$,

When $-1 \leq x \leq 1$, $\arcsin x$ and $\arccos x$ are real and

$$4.4.4 \quad -\frac{1}{2}\pi \leq \arcsin x \leq \frac{1}{2}\pi, \quad 0 < \arccos x \leq \pi$$

$$4.4.5 \quad \arctan z + \operatorname{arccot} z = \pm \frac{1}{2}\pi \quad \begin{matrix} Rz \geq 0 \\ Rz < 0 \end{matrix}$$

$$4.4.6 \quad \operatorname{arccsc} z = \arcsin 1/z$$

$$4.4.7 \quad \operatorname{arcsec} z = \arccos 1/z$$

$$4.4.8 \quad \operatorname{arccot} z = \arctan 1/z$$

$$4.4.9 \quad \operatorname{arcsec} z + \operatorname{arccsc} z = \frac{1}{2}\pi$$

(see 4.3.45)

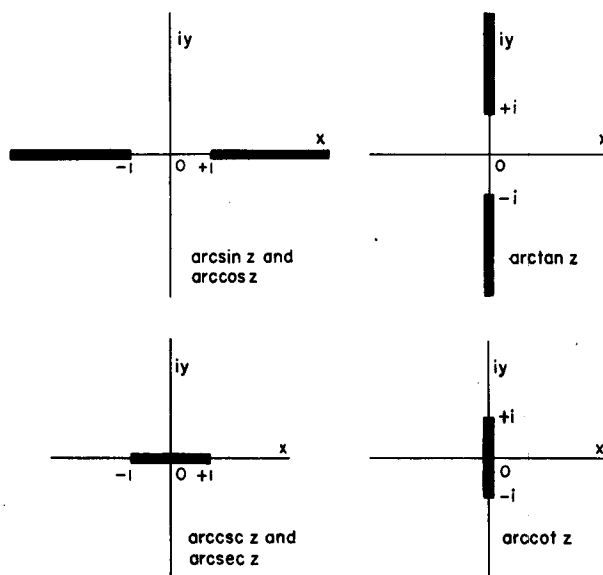


FIGURE 4.4. Branch cuts for inverse circular functions.

Fundamental Property

The general solutions of the equations

$$\sin t = z$$

$$\cos t = z$$

$$\tan t = z$$

are respectively

$$4.4.10 \quad t = \text{Arcsin } z = (-1)^k \arcsin z + k\pi$$

$$4.4.11 \quad t = \text{Arccos } z = \pm \arccos z + 2k\pi$$

$$4.4.12 \quad t = \text{Arctan } z = \arctan z + k\pi \quad (z^2 \neq -1)$$

where k is an arbitrary integer.

$$4.4.13 \quad \begin{array}{ccc} \text{Interval containing principal value} & & \\ y & x \text{ positive} & x \text{ negative} \\ & \text{or zero} & \end{array}$$

$$\arcsin x \text{ and } \arctan x \quad 0 \leq y \leq \pi/2 \quad -\pi/2 \leq y < 0$$

$$\arccos x \text{ and } \text{arcsec } x \quad 0 \leq y \leq \pi/2 \quad \pi/2 < y \leq \pi$$

$$\text{arccot } x \text{ and } \text{arccsc } x \quad 0 \leq y \leq \pi/2 \quad -\pi/2 \leq y < 0$$

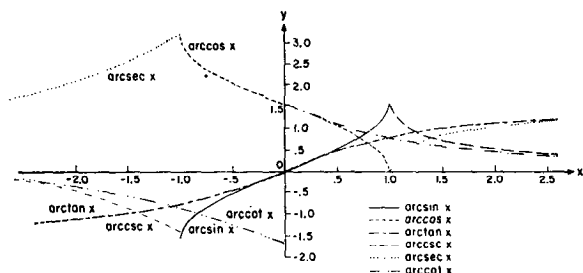


FIGURE 4.5. Inverse circular functions.

Functions of Negative Arguments

$$4.4.14 \quad \arcsin(-z) = -\arcsin z$$

$$4.4.15 \quad \arccos(-z) = \pi - \arccos z$$

$$4.4.16 \quad \arctan(-z) = -\arctan z$$

$$4.4.17 \quad \text{arccsc}(-z) = -\text{arccsc } z$$

$$4.4.18 \quad \text{arcsec}(-z) = \pi - \text{arcsec } z$$

$$4.4.19 \quad \text{arccot}(-z) = -\text{arccot } z$$

Relation to Inverse Hyperbolic Functions (see 4.6.14 to 4.6.19)

$$4.4.20 \quad \text{Arcsin } z = -i \text{Arcsinh } iz$$

$$4.4.21 \quad \text{Arccos } z = \pm i \text{Arccosh } z$$

$$4.4.22 \quad \text{Arctan } z = -i \text{Arctanh } iz \quad (z^2 \neq -1)$$

$$4.4.23 \quad \text{Arccsc } z = i \text{Arccsch } iz$$

$$4.4.24 \quad \text{Arcsec } z = \pm i \text{Arcsech } z$$

$$4.4.25 \quad \text{Arccot } z = i \text{Arccoth } iz$$

Logarithmic Representations

$$4.4.26 \quad \text{Arcsin } x = -i \text{Ln}[(1-x^2)^{-1/2} + ix] \quad (x^2 \leq 1)$$

$$4.4.27 \quad \text{Arccos } x = -i \text{Ln}[x + i(1-x^2)^{1/2}] \quad (x^2 \leq 1)$$

$$4.4.28 \quad \text{Arctan } x = \frac{i}{2} \text{Ln} \frac{1-ix}{1+ix} = \frac{i}{2} \text{Ln} \frac{i+x}{i-x} \quad (x \text{ real})$$

$$4.4.29 \quad \text{Arccsc } x = -i \text{Ln} \left[\frac{(x^2-1)^{1/2} + i}{x} \right] \quad (x^2 \geq 1)$$

$$4.4.30 \quad \text{Arcsec } x = -i \text{Ln} \left[\frac{1+i(x^2-1)^{1/2}}{x} \right] \quad (x^2 \geq 1)$$

$$4.4.31 \quad \text{Arccot } x = \frac{i}{2} \text{Ln} \left(\frac{ix+1}{ix-1} \right) = \frac{i}{2} \text{Ln} \left(\frac{x-i}{x+i} \right) \quad (x \text{ real})$$

Addition and Subtraction of Two Inverse Circular Functions**4.4.32**

$$\begin{aligned} \text{Arcsin } z_1 \pm \text{Arcsin } z_2 \\ = \text{Arcsin} [z_1(1-z_2^2)^{1/2} \pm z_2(1-z_1^2)^{1/2}] \end{aligned}$$

4.4.33

$$\begin{aligned} \text{Arccos } z_1 \pm \text{Arccos } z_2 \\ = \text{Arccos} \{ z_1 z_2 \mp [(1-z_1^2)(1-z_2^2)]^{1/2} \} \end{aligned}$$

4.4.34

$$\text{Arctan } z_1 \pm \text{Arctan } z_2 = \text{Arctan} \left(\frac{z_1 \pm z_2}{1 \mp z_1 z_2} \right)$$

4.4.35

$$\begin{aligned} \text{Arcsin } z_1 \pm \text{Arccos } z_2 \\ = \text{Arcsin} \{ z_1 z_2 \pm [(1-z_1^2)(1-z_2^2)]^{1/2} \} \\ = \text{Arccos} [z_2(1-z_1^2)^{1/2} \mp z_1(1-z_2^2)^{1/2}] \end{aligned}$$

4.4.36

$$\begin{aligned} \text{Arctan } z_1 \pm \text{Arccot } z_2 \\ = \text{Arctan} \left(\frac{z_1 z_2 \pm 1}{z_2 \mp z_1} \right) = \text{Arccot} \left(\frac{z_2 \mp z_1}{z_1 z_2 \pm 1} \right) \end{aligned}$$

Inverse Circular Functions in Terms of Real and Imaginary Parts**4.4.37**

$$\begin{aligned} \text{Arcsin } z = k\pi + (-1)^k \arcsin \beta \\ + (-1)^k i \ln [\alpha + (\alpha^2 - 1)^{1/2}] \end{aligned}$$

4.4.38

$$\text{Arccos } z = 2k\pi \pm \{ \arccos \beta - i \ln [\alpha + (\alpha^2 - 1)^{1/2}] \}$$

4.4.39

$$\operatorname{Arctan} z = k\pi + \frac{1}{2} \arctan \left(\frac{2x}{1-x^2-y^2} \right) + \frac{i}{4} \ln \left[\frac{x^2 + (y+1)^2}{x^2 + (y-1)^2} \right] \quad (z^2 \neq -1)$$

where k is an integer or zero and

$$\alpha = \frac{1}{2} [(x+1)^2 + y^2]^{\frac{1}{2}} + \frac{1}{2} [(x-1)^2 + y^2]^{\frac{1}{2}}$$

$$\beta = \frac{1}{2} [(x+1)^2 + y^2]^{\frac{1}{2}} - \frac{1}{2} [(x-1)^2 + y^2]^{\frac{1}{2}}$$

Series Expansions

4.4.40

$$\arcsin z = z + \frac{z^3}{2 \cdot 3} + \frac{1 \cdot 3 z^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 z^7}{2 \cdot 4 \cdot 6 \cdot 7} + \dots \quad (|z| < 1)$$

4.4.41

$$\arcsin (1-z) = \frac{\pi}{2} - (2z)^{\frac{1}{2}} \left[1 + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots (2k-1)}{2^{2k} (2k+1) k!} z^k \right] \quad (|z| < 2)$$

4.4.42

$$\begin{aligned} \arctan z &= z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \dots \quad (|z| \leq 1 \text{ and } z^2 \neq -1) \\ &= \frac{\pi}{2} - \frac{1}{z} + \frac{1}{3z^3} - \frac{1}{5z^5} + \dots \quad (|z| > 1 \text{ and } z^2 \neq -1) \\ &= \frac{z}{1+z^2} \left[1 + \frac{2}{3} \frac{z^2}{1+z^2} + \frac{2 \cdot 4}{3 \cdot 5} \left(\frac{z^2}{1+z^2} \right)^2 + \dots \right] \quad (z^2 \neq -1) \end{aligned}$$

Continued Fractions

$$4.4.43 \quad \arctan z = \frac{z}{1 + \frac{z^2}{3 + \frac{4z^2}{5 + \frac{9z^2}{7 + \frac{16z^2}{9 + \dots}}}}} \quad (z \text{ in the cut plane of Figure 4.4.})$$

$$4.4.44 \quad \frac{\arcsin z}{\sqrt{1-z^2}} = \frac{z}{1 - \frac{1 \cdot 2z^2}{3 - \frac{1 \cdot 2z^2}{5 - \frac{3 \cdot 4z^2}{7 - \frac{3 \cdot 4z^2}{9 - \dots}}}}} \quad (z \text{ in the cut plane of Figure 4.4.})$$

Polynomial Approximations⁹

4.4.45

$$0 \leq x \leq 1$$

$$\arcsin x = \frac{\pi}{2} - (1-x)^{\frac{1}{2}} (a_0 + a_1 x + a_2 x^2 + a_3 x^3) + \epsilon(x)$$

$$|\epsilon(x)| \leq 5 \times 10^{-5}$$

$$a_0 = 1.57072 \ 88$$

$$a_2 = .07426 \ 10$$

$$a_1 = -.21211 \ 44$$

$$a_3 = -.01872 \ 93$$

4.4.46

$$0 \leq x \leq 1$$

$$\arcsin x = \frac{\pi}{2} - (1-x)^{\frac{1}{2}} (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7) + \epsilon(x)$$

$$|\epsilon(x)| \leq 2 \times 10^{-8}$$

$$a_0 = 1.57079 \ 63050$$

$$a_4 = .03089 \ 18810$$

$$a_1 = -.21459 \ 88016$$

$$a_5 = -.01708 \ 81256$$

$$a_2 = .08897 \ 89874$$

$$a_6 = .00667 \ 00901$$

$$a_3 = -.05017 \ 43046$$

$$a_7 = -.00126 \ 24911$$

4.4.47

$$-1 \leq x \leq 1$$

$$\arctan x = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + a_9 x^9 + \epsilon(x)$$

$$|\epsilon(x)| \leq 10^{-5}$$

$$a_1 = .99986 \ 60$$

$$a_7 = -.08513 \ 30$$

$$a_3 = -.33029 \ 95$$

$$a_9 = .02083 \ 51$$

$$a_5 = .18014 \ 10$$

4.4.48¹⁰

$$-1 \leq x \leq 1$$

$$\arctan x = \frac{x}{1 + .28x^2} + \epsilon(x)$$

$$|\epsilon(x)| \leq 5 \times 10^{-3}$$

4.4.49¹¹

$$0 \leq x \leq 1$$

$$\frac{\arctan x}{x} = 1 + \sum_{k=1}^8 a_{2k} x^{2k} + \epsilon(x)$$

$$|\epsilon(x)| \leq 2 \times 10^{-8}$$

$$a_2 = -.33333 \ 14528$$

$$a_{10} = -.07528 \ 96400$$

$$a_4 = .19993 \ 55085$$

$$a_{12} = .04290 \ 96138$$

$$a_6 = -.14208 \ 89944$$

$$a_{14} = -.01616 \ 57367$$

$$a_8 = .10656 \ 26393$$

$$a_{16} = .00286 \ 62257$$

¹⁰ The approximation 4.4.48 is from C. Hastings, Jr., Note 143, Math. Tables Aids Comp. 6, 68 (1953) (with permission).

¹¹ The approximation 4.4.49 is from B. Carlson, M. Goldstein, Rational approximation of functions, Los Alamos Scientific Laboratory LA-1943, Los Alamos, N. Mex., 1955 (with permission).

⁹ The approximations 4.4.45 to 4.4.47 are from C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

Approximations in Terms of Chebyshev Polynomials ¹²

$$4.4.50 \quad -1 \leq x \leq 1$$

$$T_n^*(x) = \cos n\theta, \quad \cos \theta = 2x - 1 \quad (\text{see chapter 22})$$

$$\arctan x = x \sum_{n=0}^{\infty} A_n T_n^*(x^2)$$

n	A_n	n	A_n
0	.88137 3587	6	.00000 3821
1	-.10589 2925	7	-.00000 0570
2	.01113 5843	8	.00000 0086
3	-.00138 1195	9	-.00000 0013
4	.00018 5743	10	.00000 0002
5	-.00002 6215		

For $x > 1$, use $\arctan x = \frac{1}{2}\pi - \arctan(1/x)$

$$4.4.51 \quad -\frac{1}{2}\sqrt{2} \leq x \leq \frac{1}{2}\sqrt{2}$$

$$\arcsin x = x \sum_{n=0}^{\infty} A_n T_n^*(2x^2)$$

$$0 \leq x \leq \frac{1}{2}\sqrt{2}$$

$$\arccos x = \frac{1}{2}\pi - x \sum_{n=0}^{\infty} A_n T_n^*(2x^2)$$

n	A_n	n	A_n
0	1.05123 1959	5	.00000 5881
1	.05494 6487	6	.00000 0777
2	.00408 0631	7	.00000 0107
3	.00040 7890	8	.00000 0015
4	.00004 6985	9	.00000 0002

For $\frac{1}{2}\sqrt{2} \leq x \leq 1$, use $\arcsin x = \arccos(1-x^2)^{\frac{1}{2}}$, $\arccos x = \arcsin(1-x^2)^{\frac{1}{2}}$.

Differentiation Formulas

$$4.4.52 \quad \frac{d}{dz} \arcsin z = (1-z^2)^{-\frac{1}{2}}$$

$$4.4.53 \quad \frac{d}{dz} \arccos z = -(1-z^2)^{-\frac{1}{2}}$$

$$4.4.54 \quad \frac{d}{dz} \arctan z = \frac{1}{1+z^2}$$

$$4.4.55 \quad \frac{d}{dz} \operatorname{arccot} z = \frac{-1}{1+z^2}$$

$$4.4.56 \quad \frac{d}{dz} \operatorname{arcsec} z = \frac{1}{z(z^2-1)^{\frac{1}{2}}}$$

¹² The approximations 4.4.50 to 4.4.51 are from C. W. Clenshaw, Polynomial approximations to elementary functions, Math. Tables Aids Comp. 8, 143-147 (1954) (with permission).

$$4.4.57 \quad \frac{d}{dz} \operatorname{arccsc} z = -\frac{1}{z(z^2-1)^{\frac{1}{2}}}$$

Integration Formulas

$$4.4.58 \quad \int \arcsin z \, dz = z \arcsin z + (1-z^2)^{\frac{1}{2}}$$

$$4.4.59 \quad \int \arccos z \, dz = z \arccos z - (1-z^2)^{\frac{1}{2}}$$

$$4.4.60 \quad \int \arctan z \, dz = z \arctan z - \frac{1}{2} \ln(1+z^2)$$

$$4.4.61 \quad \int \operatorname{arccsc} z \, dz = z \operatorname{arccsc} z \pm \ln[z + (z^2-1)^{\frac{1}{2}}]$$

$$\begin{cases} 0 < \operatorname{arccsc} z < \frac{\pi}{2} \\ -\frac{\pi}{2} < \operatorname{arccsc} z < 0 \end{cases}$$

$$4.4.62 \quad \int \operatorname{arcsec} z \, dz = z \operatorname{arcsec} z \mp \ln[z + (z^2-1)^{\frac{1}{2}}]$$

$$\begin{cases} 0 < \operatorname{arcsec} z < \frac{\pi}{2} \\ \frac{\pi}{2} < \operatorname{arcsec} z < \pi \end{cases}$$

$$4.4.63 \quad \int \operatorname{arccot} z \, dz = z \operatorname{arccot} z + \frac{1}{2} \ln(1+z^2)$$

$$4.4.64 \quad \int z \arcsin z \, dz = \left(\frac{z^2}{2} - \frac{1}{4}\right) \arcsin z + \frac{z}{4} (1-z^2)^{\frac{1}{2}}$$

$$4.4.65 \quad \int z^n \arcsin z \, dz = \frac{z^{n+1}}{n+1} \arcsin z - \frac{1}{n+1} \int \frac{z^{n+1}}{(1-z^2)^{\frac{1}{2}}} dz \quad (n \neq -1)$$

$$4.4.66 \quad \int z \arccos z \, dz = \left(\frac{z^2}{2} - \frac{1}{4}\right) \arccos z - \frac{z}{4} (1-z^2)^{\frac{1}{2}}$$

$$4.4.67 \quad \int z^n \arccos z \, dz = \frac{z^{n+1}}{n+1} \arccos z + \frac{1}{n+1} \int \frac{z^{n+1}}{(1-z^2)^{\frac{1}{2}}} dz \quad (n \neq -1)$$

$$4.4.68 \quad \int z \arctan z \, dz = \frac{1}{2} (1+z^2) \arctan z - \frac{z}{2}$$

4.4.69

$$\int z^n \arctan z \, dz = \frac{z^{n+1}}{n+1} \arctan z - \frac{1}{n+1} \int \frac{z^{n+1}}{1+z^2} dz$$

($n \neq -1$)

4.4.70

$$\int z \operatorname{arccot} z \, dz = \frac{1}{2} (1+z^2) \operatorname{arccot} z + \frac{z}{2}$$

4.4.71

$$\int z^n \operatorname{arccot} z \, dz = \frac{z^{n+1}}{n+1} \operatorname{arccot} z + \frac{1}{n+1} \int \frac{z^{n+1}}{1+z^2} dz$$

($n \neq -1$)

4.5. Hyperbolic Functions

Definitions

$$4.5.1 \quad \sinh z = \frac{e^z - e^{-z}}{2} \quad (z = x + iy)$$

$$4.5.2 \quad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$4.5.3 \quad \tanh z = \sinh z / \cosh z$$

$$4.5.4 \quad \operatorname{csch} z = 1 / \sinh z$$

$$4.5.5 \quad \operatorname{sech} z = 1 / \cosh z$$

$$4.5.6 \quad \operatorname{coth} z = 1 / \tanh z$$

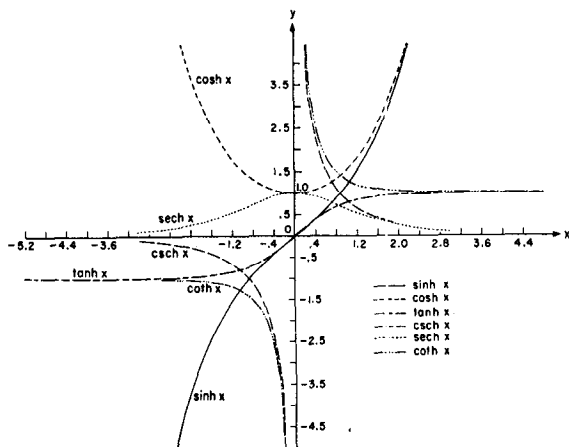


FIGURE 4.6. Hyperbolic functions.

Relation to Circular Functions (see 4.3.49 to 4.3.54)

Hyperbolic formulas can be derived from trigonometric identities by replacing z by iz

$$4.5.7 \quad \sinh z = -i \sin iz$$

$$4.5.8 \quad \cosh z = \cos iz$$

$$4.5.9 \quad \tanh z = -i \tan iz$$

$$4.5.10 \quad \operatorname{csch} z = i \csc iz$$

$$4.5.11 \quad \operatorname{sech} z = \sec iz$$

$$4.5.12 \quad \operatorname{coth} z = i \cot iz$$

Periodic Properties

$$4.5.13 \quad \sinh(z + 2k\pi i) = \sinh z$$

(k any integer)

$$4.5.14 \quad \cosh(z + 2k\pi i) = \cosh z$$

$$4.5.15 \quad \tanh(z + k\pi i) = \tanh z$$

Relations Between Hyperbolic Functions

$$4.5.16 \quad \cosh^2 z - \sinh^2 z = 1$$

$$4.5.17 \quad \tanh^2 z + \operatorname{sech}^2 z = 1$$

$$4.5.18 \quad \operatorname{coth}^2 z - \operatorname{csch}^2 z = 1$$

$$4.5.19 \quad \cosh z + \sinh z = e^z$$

$$4.5.20 \quad \cosh z - \sinh z = e^{-z}$$

Negative Angle Formulas

$$4.5.21 \quad \sinh(-z) = -\sinh z$$

$$4.5.22 \quad \cosh(-z) = \cosh z$$

$$4.5.23 \quad \tanh(-z) = -\tanh z$$

Addition Formulas

$$4.5.24 \quad \sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$$

$$4.5.25 \quad \cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$

$$4.5.26 \quad \tanh(z_1 + z_2) = (\tanh z_1 + \tanh z_2) / (1 + \tanh z_1 \tanh z_2)$$

$$4.5.27 \quad \operatorname{coth}(z_1 + z_2) = (\operatorname{coth} z_1 \operatorname{coth} z_2 + 1) / (\operatorname{coth} z_2 + \operatorname{coth} z_1)$$

Half-Angle Formulas

$$4.5.28 \quad \sinh \frac{z}{2} = \left(\frac{\cosh z - 1}{2} \right)^{\frac{1}{2}}$$

4.5.29

$$\cosh \frac{z}{2} = \left(\frac{\cosh z + 1}{2} \right)^{\frac{1}{2}}$$

4.5.30

$$\tanh \frac{z}{2} = \left(\frac{\cosh z - 1}{\cosh z + 1} \right)^{\frac{1}{2}} = \frac{\cosh z - 1}{\sinh z} = \frac{\sinh z}{\cosh z + 1}$$

Multiple-Angle Formulas

$$4.5.31 \quad \sinh 2z = 2 \sinh z \cosh z = \frac{2 \tanh z}{1 - \tanh^2 z}$$

$$4.5.32 \quad \cosh 2z = 2 \cosh^2 z - 1 = 2 \sinh^2 z + 1 \\ = \cosh^2 z + \sinh^2 z$$

$$4.5.33 \quad \tanh 2z = \frac{2 \tanh z}{1 + \tanh^2 z}$$

$$4.5.34 \quad \sinh 3z = 3 \sinh z + 4 \sinh^3 z$$

$$4.5.35 \quad \cosh 3z = -3 \cosh z + 4 \cosh^3 z$$

$$4.5.36 \quad \sinh 4z = 4 \sinh^3 z \cosh z + 4 \cosh^3 z \sinh z$$

$$4.5.37 \quad \cosh 4z = \cosh^4 z + 6 \sinh^2 z \cosh^2 z + \sinh^4 z$$

Products of Hyperbolic Sines and Cosines

$$4.5.38 \quad 2 \sinh z_1 \sinh z_2 = \cosh (z_1 + z_2) \\ - \cosh (z_1 - z_2)$$

$$4.5.39 \quad 2 \cosh z_1 \cosh z_2 = \cosh (z_1 + z_2) \\ + \cosh (z_1 - z_2)$$

$$4.5.40 \quad 2 \sinh z_1 \cosh z_2 = \sinh (z_1 + z_2) \\ + \sinh (z_1 - z_2)$$

Addition and Subtraction of Two Hyperbolic Functions

4.5.41

$$\sinh z_1 + \sinh z_2 = 2 \sinh \left(\frac{z_1 + z_2}{2} \right) \cosh \left(\frac{z_1 - z_2}{2} \right)$$

4.5.42

$$\sinh z_1 - \sinh z_2 = 2 \cosh \left(\frac{z_1 + z_2}{2} \right) \sinh \left(\frac{z_1 - z_2}{2} \right)$$

4.5.43

$$\cosh z_1 + \cosh z_2 = 2 \cosh \left(\frac{z_1 + z_2}{2} \right) \cosh \left(\frac{z_1 - z_2}{2} \right)$$

4.5.44

$$\cosh z_1 - \cosh z_2 = 2 \sinh \left(\frac{z_1 + z_2}{2} \right) \sinh \left(\frac{z_1 - z_2}{2} \right)$$

4.5.45

$$\tanh z_1 + \tanh z_2 = \frac{\sinh (z_1 + z_2)}{\cosh z_1 \cosh z_2}$$

4.5.46

$$\coth z_1 + \coth z_2 = \frac{\sinh (z_1 + z_2)}{\sinh z_1 \sinh z_2}$$

Relations Between Squares of Hyperbolic Sines and Cosines

4.5.47

$$\sinh^2 z_1 - \sinh^2 z_2 = \sinh (z_1 + z_2) \sinh (z_1 - z_2) \\ = \cosh^2 z_1 - \cosh^2 z_2$$

4.5.48

$$\sinh^2 z_1 + \cosh^2 z_2 = \cosh (z_1 + z_2) \cosh (z_1 - z_2) \\ = \cosh^2 z_1 + \sinh^2 z_2$$

Hyperbolic Functions in Terms of Real and Imaginary Parts

$$(z = x + iy)$$

$$4.5.49 \quad \sinh z = \sinh x \cosh y + i \cosh x \sinh y$$

$$4.5.50 \quad \cosh z = \cosh x \cosh y + i \sinh x \sinh y$$

$$4.5.51 \quad \tanh z = \frac{\sinh 2x + i \sin 2y}{\cosh 2x + \cos 2y}$$

$$4.5.52 \quad \coth z = \frac{\sinh 2x - i \sin 2y}{\cosh 2x - \cos 2y}$$

De Moivre's Theorem

$$4.5.53 \quad (\cosh z + \sinh z)^n = \cosh nz + \sinh nz$$

Modulus and Phase (Argument) of Hyperbolic Functions

$$4.5.54 \quad |\sinh z| = (\sinh^2 x + \sin^2 y)^{\frac{1}{2}} \\ = [\tfrac{1}{2}(\cosh 2x - \cos 2y)]^{\frac{1}{2}}$$

$$4.5.55 \quad \arg \sinh z = \arctan (\coth x \tan y)$$

$$4.5.56 \quad |\cosh z| = (\sinh^2 x + \cos^2 y)^{\frac{1}{2}} \\ = [\tfrac{1}{2}(\cosh 2x + \cos 2y)]^{\frac{1}{2}}$$

$$4.5.57 \quad \arg \cosh z = \arctan (\tanh x \tan y)$$

$$4.5.58 \quad |\tanh z| = \left(\frac{\cosh 2x - \cos 2y}{\cosh 2x + \cos 2y} \right)^{\frac{1}{2}}$$

$$4.5.59 \quad \arg \tanh z = \arctan \left(\frac{\sin 2y}{\sinh 2x} \right)$$

4.5.60

Relations Between Hyperbolic (or Inverse Hyperbolic) Functions

	$\sinh x=a$	$\cosh x=a$	$\tanh x=a$	$\operatorname{csch} x=a$	$\operatorname{sech} x=a$	$\coth x=a$
$\sinh x$ -----	a	$(a^2-1)^{\frac{1}{2}}$	$a(1-a^2)^{-\frac{1}{2}}$	a^{-1}	$a^{-1}(1-a^2)^{\frac{1}{2}}$	$(a^2-1)^{-\frac{1}{2}}$
$\cosh x$ -----	$(1+a^2)^{\frac{1}{2}}$	a	$(1-a^2)^{-\frac{1}{2}}$	$a^{-1}(1+a^2)^{\frac{1}{2}}$	a^{-1}	$a(a^2-1)^{-\frac{1}{2}}$
$\tanh x$ -----	$a(1+a^2)^{-\frac{1}{2}}$	$a^{-1}(a^2-1)^{\frac{1}{2}}$	a	$(1+a^2)^{-\frac{1}{2}}$	$(1-a^2)^{\frac{1}{2}}$	a^{-1}
$\operatorname{csch} x$ -----	a^{-1}	$(a^2-1)^{-\frac{1}{2}}$	$a^{-1}(1-a^2)^{\frac{1}{2}}$	a	$a(1-a^2)^{-\frac{1}{2}}$	$(a^2-1)^{\frac{1}{2}}$
$\operatorname{sech} x$ -----	$(1+a^2)^{-\frac{1}{2}}$	a^{-1}	$(1-a^2)^{\frac{1}{2}}$	$a(1+a^2)^{-\frac{1}{2}}$	a	$a^{-1}(a^2-1)^{\frac{1}{2}}$
$\coth x$ -----	$a^{-1}(a^2+1)^{\frac{1}{2}}$	$a(a^2-1)^{-\frac{1}{2}}$	a^{-1}	$(1+a^2)^{\frac{1}{2}}$	$(1-a^2)^{-\frac{1}{2}}$	a

Illustration: If $\sinh x=a$, $\coth x=a^{-1}(a^2+1)^{\frac{1}{2}}$ $\operatorname{arcsech} a=\operatorname{arccoth} (1-a^2)^{-\frac{1}{2}}$

4.5.61 Special Values of the Hyperbolic Functions

z	0	$\frac{\pi}{2}i$	πi	$\frac{3\pi}{2}i$	∞
$\sinh z$ -----	0	i	0	$-i$	∞
$\cosh z$ -----	1	0	-1	0	∞
$\tanh z$ -----	0	∞i	0	$-\infty i$	1
$\operatorname{csch} z$ -----	∞	$-i$	∞	i	0
$\operatorname{sech} z$ -----	1	∞	-1	∞	0
$\coth z$ -----	∞	0	∞	0	1

Series Expansions

$$4.5.62 \quad \sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \dots \quad (|z| < \infty)$$

$$4.5.63 \quad \cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \dots \quad (|z| < \infty)$$

$$4.5.64 \quad \tanh z = z - \frac{z^3}{3} + \frac{2}{15}z^5 - \frac{17}{315}z^7 + \dots + \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}z^{2n-1} + \dots$$

$$\left(|z| < \frac{\pi}{2}\right)$$

4.5.65

$$\operatorname{csch} z = \frac{1}{z} - \frac{z}{6} + \frac{7}{360}z^3 - \frac{31}{15120}z^5 + \dots - \frac{2(2^{2n-1}-1)B_{2n}}{(2n)!}z^{2n-1} + \dots$$

$$(|z| < \pi)$$

4.5.66

$$\operatorname{sech} z = 1 - \frac{z^2}{2} + \frac{5}{24}z^4 - \frac{61}{720}z^6 + \dots + \frac{E_{2n}}{(2n)!}z^{2n} + \dots$$

$$\left(|z| < \frac{\pi}{2}\right)$$

4.5.67

$$\coth z = \frac{1}{z} + \frac{z}{3} - \frac{z^3}{45} + \frac{2}{945}z^5 - \dots + \frac{2^{2n}B_{2n}}{(2n)!}z^{2n-1} + \dots$$

$$(|z| < \pi)$$

where B_n and E_n are the n th Bernoulli and Euler numbers, see chapter 23.

Infinite Products

$$4.5.68 \quad \sinh z = z \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{k^2\pi^2}\right)$$

$$4.5.69 \quad \cosh z = \prod_{k=1}^{\infty} \left[1 + \frac{4z^2}{(2k-1)^2\pi^2}\right]$$

Continued Fraction

$$4.5.70 \quad \tanh z = \frac{z}{1 + \frac{z^2}{3 + \frac{z^2}{5 + \frac{z^2}{7 + \dots}}}}$$

$$\left(z \neq \frac{\pi}{2}i \pm n\pi i\right)$$

Differentiation Formulas

$$4.5.71 \quad \frac{d}{dz} \sinh z = \cosh z$$

$$4.5.72 \quad \frac{d}{dz} \cosh z = \sinh z$$

$$4.5.73 \quad \frac{d}{dz} \tanh z = \operatorname{sech}^2 z$$

$$4.5.74 \quad \frac{d}{dz} \operatorname{csch} z = -\operatorname{csch} z \coth z$$

*See page 11.

$$4.5.75 \quad \frac{d}{dz} \operatorname{sech} z = -\operatorname{sech} z \tanh z$$

$$4.5.76 \quad \frac{d}{dz} \coth z = -\operatorname{csch}^2 z$$

Integration Formulas

$$4.5.77 \quad \int \sinh z \, dz = \cosh z$$

$$4.5.78 \quad \int \cosh z \, dz = \sinh z$$

$$4.5.79 \quad \int \tanh z \, dz = \ln \cosh z$$

$$4.5.80 \quad \int \operatorname{csch} z \, dz = \ln \tanh \frac{z}{2}$$

$$4.5.81 \quad \int \operatorname{sech} z \, dz = \arctan (\sinh z)$$

$$4.5.82 \quad \int \coth z \, dz = \ln \sinh z$$

$$4.5.83 \quad \int z^n \sinh z \, dz = z^n \cosh z - n \int z^{n-1} \cosh z \, dz$$

$$4.5.84 \quad \int z^n \cosh z \, dz = z^n \sinh z - n \int z^{n-1} \sinh z \, dz$$

$$4.5.85 \quad \int \sinh^m z \cosh^n z \, dz = \frac{1}{m+n} \sinh^{m+1} z \cosh^{n-1} z \\ + \frac{n-1}{m+n} \int \sinh^m z \cosh^{n-2} z \, dz \\ = \frac{1}{m+n} \sinh^{m-1} z \cosh^{n+1} z \\ - \frac{m-1}{m+n} \int \sinh^{m-2} z \cosh^n z \, dz \quad (m+n \neq 0)$$

$$4.5.86 \quad \int \frac{dz}{\sinh^m z \cosh^n z} = \frac{-1}{m-1} \frac{1}{\sinh^{m-1} z \cosh^{n-1} z} \\ - \frac{m+n-2}{m-1} \int \frac{dz}{\sinh^{m-2} z \cosh^n z} \quad (m \neq 1) \\ = \frac{1}{n-1} \frac{1}{\sinh^{m-1} z \cosh^{n-1} z} \\ + \frac{m+n-2}{n-1} \int \frac{dz}{\sinh^m z \cosh^{n-2} z} \quad (n \neq 1)$$

4.5.87

$$\int \tanh^n z \, dz = -\frac{\tanh^{n-1} z}{n-1} + \int \tanh^{n-2} z \, dz \quad (n \neq 1)$$

4.5.88

$$\int \coth^n z \, dz = -\frac{\coth^{n-1} z}{n-1} + \int \coth^{n-2} z \, dz \quad (n \neq 1)$$

(See chapters 5 and 7 for other integrals involving hyperbolic functions.)

4.6. Inverse Hyperbolic Functions

Definitions

$$4.6.1 \quad \operatorname{arsinh} z = \int_0^z \frac{dt}{(1+t^2)^{\frac{1}{2}}} \quad (z=x+iy)$$

$$4.6.2 \quad \operatorname{arcosh} z = \int_1^z \frac{dt}{(t^2-1)^{\frac{1}{2}}}$$

$$4.6.3 \quad \operatorname{artanh} z = \int_0^z \frac{dt}{1-t^2}$$

The paths of integration must not cross the following cuts.

4.6.1 imaginary axis from $-i\infty$ to $-i$ and i to $i\infty$

4.6.2 real axis from $-\infty$ to $+1$

4.6.3 real axis from $-\infty$ to -1 and $+1$ to $+\infty$

Inverse hyperbolic functions are also written $\sinh^{-1} z$, $\operatorname{arsinh} z$, $\mathcal{A}r \sinh z$, etc.

$$4.6.4 \quad \operatorname{arcsch} z = \operatorname{arsinh} 1/z$$

$$4.6.5 \quad \operatorname{arcsech} z = \operatorname{arcosh} 1/z$$

$$4.6.6 \quad \operatorname{arcoth} z = \operatorname{artanh} 1/z$$

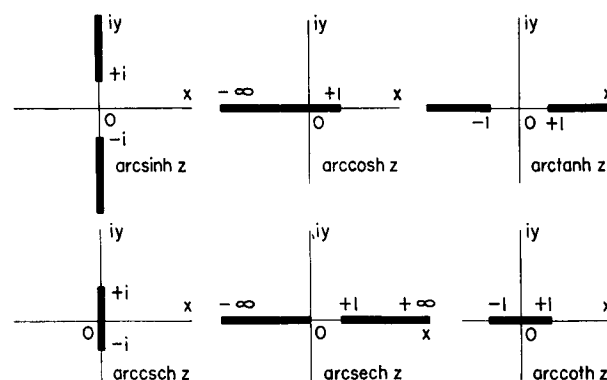


FIGURE 4.7. Branch cuts for inverse hyperbolic functions.

$$4.6.7 \quad \operatorname{arctanh} z = \operatorname{arccoth} z \pm \frac{1}{2}\pi i$$

(see 4.5.60) (according as $\Im z \geq 0$)

Fundamental Property

The general solutions of the equations

$$z = \sinh t$$

$$z = \cosh t$$

$$z = \tanh t$$

are respectively

$$4.6.8 \quad t = \operatorname{Arcsinh} z = (-1)^k \operatorname{arsinh} z + k\pi i$$

$$4.6.9 \quad t = \operatorname{Arccosh} z = \pm \operatorname{arccosh} z + 2k\pi i$$

$$4.6.10 \quad t = \operatorname{Arctanh} z = \operatorname{arctanh} z + k\pi i$$

(k , integer)

Functions of Negative Arguments

$$4.6.11 \quad \operatorname{arsinh} (-z) = -\operatorname{arsinh} z$$

$$*4.6.12 \quad \operatorname{arccosh} (-z) = \pi i - \operatorname{arccosh} z$$

$$4.6.13 \quad \operatorname{arctanh} (-z) = -\operatorname{arctanh} z$$

Relation to Inverse Circular Functions (see 4.4.20 to 4.4.25)

Hyperbolic identities can be derived from trigonometric identities by replacing z by iz .

$$4.6.14 \quad \operatorname{Arcsinh} z = -i \operatorname{Arcsin} iz$$

$$4.6.15 \quad \operatorname{Arccosh} z = \pm i \operatorname{Arccos} z$$

$$4.6.16 \quad \operatorname{Arctanh} z = -i \operatorname{Arctan} iz$$

$$4.6.17 \quad \operatorname{Arccsch} z = i \operatorname{Arccsc} iz$$

$$4.6.18 \quad \operatorname{Arcsech} z = \pm i \operatorname{Arcsec} z$$

$$4.6.19 \quad \operatorname{Arcoth} z = i \operatorname{Arccot} iz$$

Logarithmic Representations

$$4.6.20 \quad \operatorname{arsinh} x = \ln [x + (x^2 + 1)^{\frac{1}{2}}]$$

$$4.6.21 \quad \operatorname{arccosh} x = \ln [x + (x^2 - 1)^{\frac{1}{2}}] \quad (x \geq 1)$$

$$4.6.22 \quad \operatorname{arctanh} x = \frac{1}{2} \ln \frac{1+x}{1-x} \quad (0 \leq x^2 < 1)$$

$$4.6.23 \quad \operatorname{arcsch} x = \ln \left[\frac{1}{x} + \left(\frac{1}{x^2} + 1 \right)^{\frac{1}{2}} \right] \quad (x \neq 0)$$

$$4.6.24 \quad \operatorname{arcsech} x = \ln \left[\frac{1}{x} + \left(\frac{1}{x^2} - 1 \right)^{\frac{1}{2}} \right] \quad (0 < x \leq 1)$$

$$4.6.25 \quad \operatorname{arccoth} x = \frac{1}{2} \ln \frac{x+1}{x-1} \quad (x^2 > 1)$$

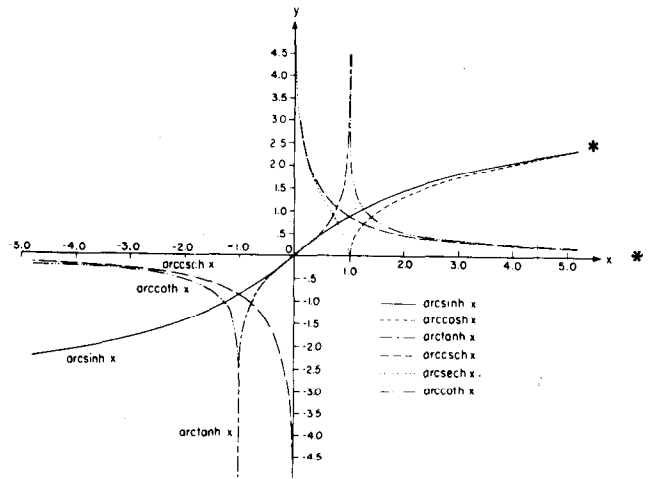


FIGURE 4.8. Inverse hyperbolic functions.

Addition and Subtraction of Two Inverse Hyperbolic Functions**4.6.26**

$$\operatorname{Arcsinh} z_1 \pm \operatorname{Arcsinh} z_2$$

$$= \operatorname{Arcsinh} [z_1(1+z_2^2)^{\frac{1}{2}} \pm z_2(1+z_1^2)^{\frac{1}{2}}]$$

4.6.27

$$\operatorname{Arccosh} z_1 \pm \operatorname{Arccosh} z_2$$

$$= \operatorname{Arccosh} \{z_1 z_2 \pm [(z_1^2 - 1)(z_2^2 - 1)]^{\frac{1}{2}}\}$$

4.6.28

$$\operatorname{Arctanh} z_1 \pm \operatorname{Arctanh} z_2 = \operatorname{Arctanh} \left(\frac{z_1 \pm z_2}{1 \pm z_1 z_2} \right)$$

4.6.29

$$\operatorname{Arcsinh} z_1 \pm \operatorname{Arccosh} z_2$$

$$= \operatorname{Arcsinh} \{z_1 z_2 \pm [(1+z_1^2)(z_2^2-1)]^{\frac{1}{2}}\}$$

$$= \operatorname{Arccosh} [z_2(1+z_1^2)^{\frac{1}{2}} \pm z_1(z_2^2-1)^{\frac{1}{2}}]$$

4.6.30

$$\operatorname{Arctanh} z_1 \pm \operatorname{Arcoth} z_2 = \operatorname{Arctanh} \left(\frac{z_1 z_2 \pm 1}{z_2 \pm z_1} \right)$$

$$= \operatorname{Arcoth} \left(\frac{z_2 \pm z_1}{z_1 z_2 \pm 1} \right)$$

Series Expansions

4.6.31

$$\begin{aligned} \operatorname{arsinh} z &= z - \frac{1}{2 \cdot 3} z^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} z^5 \\ &\quad - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} z^7 + \dots \\ &\quad (|z| < 1) \\ &= \ln 2z + \frac{1}{2 \cdot 2z^2} - \frac{1 \cdot 3}{2 \cdot 4 \cdot 4z^4} \\ &\quad + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 6z^6} - \dots \\ &\quad (|z| > 1) \end{aligned}$$

4.6.32

$$\begin{aligned} \operatorname{arcosh} z &= \ln 2z - \frac{1}{2 \cdot 2z^2} - \frac{1 \cdot 3}{2 \cdot 4 \cdot 4z^4} \\ &\quad - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 6z^6} - \dots \\ &\quad (|z| > 1) \end{aligned}$$

$$4.6.33 \quad \operatorname{artanh} z = z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \dots \quad (|z| < 1)$$

$$4.6.34 \quad \operatorname{arcoth} z = \frac{1}{z} + \frac{1}{3z^3} + \frac{1}{5z^5} + \frac{1}{7z^7} + \dots \quad (|z| > 1)$$

Continued Fractions

$$4.6.35 \quad \operatorname{artanh} z = \frac{z}{1 - \frac{z^2}{3 - \frac{4z^2}{5 - \frac{9z^2}{7 - \dots}}}}$$

(z in the cut plane of Figure 4.7.)

4.6.36

$$\frac{\operatorname{arsinh} z}{\sqrt{1+z^2}} = \frac{z}{1+} \frac{1 \cdot 2z^2}{3+} \frac{1 \cdot 2z^2}{5+} \frac{3 \cdot 4z^2}{7+} \frac{3 \cdot 4z^2}{9+} \dots$$

Differentiation Formulas

$$4.6.37 \quad \frac{d}{dz} \operatorname{arsinh} z = (1+z^2)^{-1/2}$$

$$4.6.38 \quad \frac{d}{dz} \operatorname{arcosh} z = (z^2-1)^{-1/2}$$

$$4.6.39 \quad \frac{d}{dz} \operatorname{artanh} z = (1-z^2)^{-1}$$

$$4.6.40 \quad \frac{d}{dz} \operatorname{arcsch} z = \mp \frac{1}{z(1+z^2)^{1/2}} \quad (\text{according as } \Re z \geq 0)$$

$$4.6.41 \quad \frac{d}{dz} \operatorname{arcsech} z = \mp \frac{1}{z(1-z^2)^{1/2}}$$

$$4.6.42 \quad \frac{d}{dz} \operatorname{arcoth} z = (1-z^2)^{-1}$$

Integration Formulas

$$4.6.43 \quad \int \operatorname{arsinh} z \, dz = z \operatorname{arsinh} z - (1+z^2)^{1/2}$$

$$4.6.44 \quad \int \operatorname{arcosh} z \, dz = z \operatorname{arcosh} z - (z^2-1)^{1/2}$$

$$4.6.45 \quad \int \operatorname{artanh} z \, dz = z \operatorname{artanh} z + \frac{1}{2} \ln(1-z^2)$$

$$4.6.46 \quad \int \operatorname{arcsch} z \, dz = z \operatorname{arcsch} z \pm \operatorname{arsinh} z \quad * \\ (\text{according as } \Re z \geq 0)$$

$$4.6.47 \quad \int \operatorname{arcsech} z \, dz = z \operatorname{arcsech} z \pm \operatorname{arcsin} z \quad *$$

$$4.6.48 \quad \int \operatorname{arcoth} z \, dz = z \operatorname{arcoth} z + \frac{1}{2} \ln(z^2-1)$$

$$4.6.49 \quad \int z \operatorname{arsinh} z \, dz = \frac{2z^2+1}{4} \operatorname{arsinh} z - \frac{z}{4} (z^2+1)^{1/2}$$

$$4.6.50 \quad \int z^n \operatorname{arsinh} z \, dz = \frac{z^{n+1}}{n+1} \operatorname{arsinh} z - \frac{1}{n+1} \int \frac{z^{n+1}}{(1+z^2)^{1/2}} dz \\ (n \neq -1)$$

$$4.6.51 \quad \int z \operatorname{arcosh} z \, dz = \frac{2z^2-1}{4} \operatorname{arcosh} z - \frac{z}{4} (z^2-1)^{1/2}$$

$$4.6.52 \quad \int z^n \operatorname{arcosh} z \, dz = \frac{z^{n+1}}{n+1} \operatorname{arcosh} z - \frac{1}{n+1} \int \frac{z^{n+1}}{(z^2-1)^{1/2}} dz \\ (n \neq -1)$$

$$4.6.53 \quad \int z \operatorname{artanh} z \, dz = \frac{z^2-1}{2} \operatorname{artanh} z + \frac{z}{2}$$

$$4.6.54 \quad \int z^n \operatorname{artanh} z \, dz = \frac{z^{n+1}}{n+1} \operatorname{artanh} z - \frac{1}{n+1} \int \frac{z^{n+1}}{1-z^2} dz \\ (n \neq -1)$$

$$4.6.55 \quad \int z \operatorname{arcsch} z \, dz = \frac{z^2}{2} \operatorname{arcsch} z \pm \frac{1}{2} (1+z^2)^{1/2} \quad * \\ (\text{according as } \Re z \geq 0)$$

$$4.6.56 \quad \int z^n \operatorname{arcsech} z \, dz = \frac{z^{n+1}}{n+1} \operatorname{arcsech} z \pm \frac{1}{n+1} \int \frac{z^n}{(z^2+1)^{1/2}} dz \quad * \\ (n \neq -1)$$

*See page II.

$$4.6.57 \quad \int z \operatorname{arcsech} z \, dz = \frac{z^2}{2} \operatorname{arcsech} z \mp \frac{1}{2} (1-z^2)^{\frac{1}{2}} \\ \text{(according as } \Re z \geq 0 \text{)}$$

4.6.58

$$\int z^n \operatorname{arcsech} z \, dz = \frac{z^{n+1}}{n+1} \operatorname{arcsech} z \pm \frac{1}{n+1} \int \frac{z^n}{(1-z^2)^{\frac{1}{2}}} dz \\ (n \neq -1)$$

4.6.59

$$\int z \operatorname{arccoth} z \, dz = \frac{z^2-1}{2} \operatorname{arccoth} z + \frac{z}{2}$$

4.6.60

$$\int z^n \operatorname{arccoth} z \, dz = \frac{z^{n+1}}{n+1} \operatorname{arccoth} z + \frac{1}{n+1} \int \frac{z^{n+1}}{z^2-1} dz \\ (n \neq -1)$$

Numerical Methods

4.7. Use and Extension of the Tables

NOTE: In the examples given it is assumed that the arguments are exact.

Example 1. Computation of Common Logarithms.

To compute common logarithms, the number must be expressed in the form $x \cdot 10^q$, ($1 \leq x < 10$, $-\infty \leq q \leq \infty$). The common logarithm of $x \cdot 10^q$ consists of an integral part which is called the characteristic and a decimal part which is called the mantissa. Table 4.1 gives the common logarithm of x .

x	$x \cdot 10^q$	$\log_{10} x \cdot 10^q$
.009836	$9.836 \cdot 10^{-3}$	$\bar{3}.99281 \, 85 = (-2.00718 \, 15)$
.09836	$9.836 \cdot 10^{-2}$	$\bar{2}.99281 \, 85 = (-1.00718 \, 15)$
.9836	$9.836 \cdot 10^{-1}$	$\bar{1}.99281 \, 85 = (-0.00718 \, 15)$
9.836	$9.836 \cdot 10^0$	0.99281 85
98.36	$9.836 \cdot 10^1$	1.99281 85
983.6	$9.836 \cdot 10^2$	2.99281 85

Interpolation in Table 4.1 between 983 and 984 gives .99281 85 as the mantissa of 9836.

Note that $\bar{3}.99281 \, 85 = -3 + .99281 \, 85$. When q is negative the common logarithm can be expressed in the alternative forms

$$\log_{10} (.009836) = \bar{3}.99281 \, 85 = 7.99281 \, 85 - 10 \\ = -2.00718 \, 15.$$

The last form is convenient for conversion from common logarithms to natural logarithms.

The inverse of $\log_{10} x$ is called the antilogarithm of x , and is written $\operatorname{antilog} x$ or $\log^{-1} x$. The logarithm of the reciprocal of a number is called the cologarithm, written colog .

Example 2.

Compute $x^{-3/4}$ for $x=9.19826$ to 10D using the Table of Common Logarithms.

From Table 4.1, four-point Lagrangian interpolation gives $\log_{10} (9.19826) = .96370 \, 56812$. Then, $-\frac{3}{4} \log_{10} (x) = -.72277 \, 92609 = 9.27722 \, 07391 - 10$.

Linear inverse interpolation in Table 4.1 yields $\operatorname{antilog} (\bar{1}.27722) = .18933$. For 10 place accuracy subtabulation with 4-point Lagrangian interpolants produces the table

N	$\log_{10} N$	Δ	Δ^2
.18933	.27721 94350	2 29379	
.18934	.27724 23729	2 29366	-13
.18935	.27726 53095		

By linear inverse interpolation

$$x^{-3/4} = .18933 \, 05685.$$

Example 3.

Convert $\log_{10} x$ to $\ln x$ for $x=.009836$.

Using 4.1.23 and Table 4.1, $\ln (.009836) = \ln 10 \log_{10} (.009836) = 2.30258 \, 5093 (-2.00718 \, 15) = -4.62170 \, 62$.

Example 4.

Compute $\ln x$ for $x=.00278$ to 6D.

Using 4.1.7, 4.1.11 and Table 4.2, $\ln (.00278) = \ln (.278 \cdot 10^{-2}) = \ln (.278) - 2 \ln 10 = -5.885304$.

Linear interpolation between $x=.002$ and $x=.003$ would give $\ln(.00278) = -5.898$. To obtain 5 decimal place accuracy with linear interpolation it is necessary that $x > .175$.

Example 5.

Compute $\ln x$ for $x=1131.718$ to 8D.

Using 4.1.7, 4.1.11 and Table 4.2

$$\ln 1131.718 = \ln \left(\frac{1131.718}{1131} \cdot 1131 \right) \\ = \ln \frac{1131.718}{1131} + \ln 1.131 + \ln 10^3 \\ = \ln (1.00063 \, 4836) + \ln 1.131 + 3 \ln 10.$$

*See page 11.

Then from 4.1.24

$$\begin{aligned}\ln 1131.718 &= (.00063 \ 4836) - \frac{1}{2}(.00063 \ 4836)^2 \\ &+ \ln 1.131 + 3 \ln 10 = .00063 \ 4836 - .00000 \ 0202 \\ &+ .12310 \ 2197 + 6.90775 \ 5279 = 7.03149 \ 211.\end{aligned}$$

Example 6.

Compute $\ln x$ working with 16D for
 $x = 1.38967 \ 12458 \ 179231$.

Since $\frac{x}{1.389} = 1.00048 \ 32583 \ 282384 = 1 + a$, using
 4.1.24 and Table 4.2 we compute successively

$$\begin{aligned}a &= .00048 \ 32583 \ 282384 \\ -\frac{a^2}{2} &= -\quad \quad \quad 1167 \ 693059 \\ \frac{a^3}{3} &= \quad \quad \quad 376199 \\ -\frac{a^4}{4} &= -\quad \quad \quad 136 \\ \ln(1+a) &= .00048 \ 31415 \ 965388 \\ \ln 1.389 &= .32858 \ 40637 \ 722067 \\ \ln x &= .32906 \ 72053 \ 687455.\end{aligned}$$

Example 7.

Compute the principal value of $\ln(\pm 2 \pm 3i)$.
 From 4.1.2, 4.1.3 and Tables 4.2 and 4.14.

$$\begin{aligned}\ln(2+3i) &= \frac{1}{2} \ln(2^2+3^2) + i \arctan \frac{3}{2} \\ &= 1.282475 + i(.982794) \\ \ln(-2+3i) &= \frac{1}{2} \ln 13 + i\left(\pi - \arctan \frac{3}{2}\right) \\ &= 1.282475 + i(2.158799) \\ \ln(-2-3i) &= \frac{1}{2} \ln 13 + i\left(-\pi + \arctan \frac{3}{2}\right) \\ &= 1.282475 - i(2.158799) \\ \ln(2-3i) &= \frac{1}{2} \ln 13 + i\left(-\arctan \frac{3}{2}\right) \\ &= 1.282475 - i(.982794).\end{aligned}$$

Example 8.

Compute $(.227)^{.69}$ to 7D.
 Using 4.2.7 and Tables 4.2 and 4.4,

$$\begin{aligned}(.227)^{.69} &= e^{.69 \ln(.227)} = e^{.69(-1.48280 \ 5282)} \\ &= e^{-1.02313 \ 5631} = .35946 \ 60.\end{aligned}$$

Example 9.

Compute $e^{4.99728 \ 69}$ to 7S.
 Using 4.2.18 and Table 4.4,

$$e^{4.99728 \ 69} = e^{4.9} e^{.09728 \ 69}.$$

Linear interpolation gives $e^{.09728 \ 69} = 1.10217 \ 67$
 with an error of 1×10^{-7} ,

$$e^{4.99728 \ 69} = (134.28978)(1.10217 \ 67) = 148.0111.$$

Example 10.

Compute e^x to 18D for

$$x = .86725 \ 13489 \ 24685 \ 12693.$$

Let $a = x - .867$. Using 4.2.1, compute successively

$$\begin{aligned}1.00000 \ 00000 \ 00000 \ 00000 \\ a = .00025 \ 13489 \ 24685 \ 12693 \\ \frac{a^2}{2!} = \quad \quad \quad 315 \ 88140 \ 97019 \\ \frac{a^3}{3!} = \quad \quad \quad 2646 \ 54842 \\ \frac{a^4}{4!} = \quad \quad \quad 16630\end{aligned}$$

$$e^a = 1.00025 \ 13805 \ 15472 \ 81184$$

$$e^{.867} = 2.37976 \ 08513 \ 29496 \ 863 \text{ from Table 4.4}$$

$$e^x e^{.867} = e^x = 2.38035 \ 90768 \ 39006 \ 089.$$

Example 11.

Compute $e^{.648}$ to 7S.

Let $n = \frac{x}{\ln 10}$ and $d =$ the decimal part of $\frac{x}{\ln 10}$.

Then

$$\begin{aligned}\exp x &= \exp\left(\frac{x}{\ln 10} \ln 10\right) = \exp[(n+d) \ln 10] \\ &= \exp(\ln 10^n) \exp(d \ln 10) \\ &= 10^n \exp(d \ln 10)\end{aligned}$$

From Table 4.4

$$e^{.648} = \exp\left(\frac{.648}{\ln 10} \ln 10\right) = \exp(281.42282 \ 42 \ln 10)$$

$$= 10^{281} \exp(.42282 \ 42 \ln 10) = 10^{281} \exp .97358 \ 87$$

$$= 10^{281} (2.647428) = (281) 2.647428.$$

Example 12.

Compute e^{-x} for $x = .75$ using the expansion in Chebyshev polynomials.

Following the procedure in [4.3] we have from 4.2.48

$$e^{-x} = \sum_{k=0}^7 A_k T_k^*(x)$$

* where $T_k^*(x)$ are the Chebyshev polynomials defined in chapter 22. Assuming $b_8 = b_9 = 0$ we generate b_k , $k=7, 6, 5, \dots, 0$ from the recurrence relation

$$b_k = (4x-2)b_{k+1} - b_{k+2} + A_k$$

k	b_k
7	-.00000 0015
6	.00000 0400
5	-.00000 9560
4	.00018 9959
3	-.00300 9164
2	.03550 4993
1	-.27432 7449
0	.33520 2828

since $f(x) = b_0 - (2x-1)b_1$,

$$e^{-.75} = .33520 \ 2828 - (.5)(-.27432 \ 7449) \\ = .47236 \ 6553.$$

Example 13.

Express $38^\circ 42' 32''$ in radians to 6D.

$$1^\circ = .01745 \ 32925 \ 19943 \ 29577 \ r \\ 1' = .00029 \ 08882 \ 08665 \ 72159 \ 62 \ r \\ 1'' = .00000 \ 48481 \ 36811 \ 09535 \ 9936 \ r$$

Therefore

$$38^\circ = .66322 \ 51 \ r \\ 42' = .01221 \ 73 \ r \\ 32'' = .00015 \ 51 \ r \\ 38^\circ 42' 32'' = .675598 \ r.$$

Example 14.

Express $x = 1.6789$ radians in degrees, minutes and seconds to the nearest tenth of a second.

From Table 1.1 giving the mathematical constants we have

$$1 \ r = \frac{180^\circ}{\pi} = 57.29577 \ 95130^\circ \dots \\ 1.6789 \ r = 96.19388^\circ \\ .19388^\circ \times 60 = 11.633' \\ .633' \times 60 = 38.0'' \\ 1.6789 \ r = 96^\circ 11' 38.0''.$$

Example 15.

Compute $\sin x$ and $\cos x$ for $x = 2.317$ to 7D. From 4.3.44 and Table 4.6

$$\sin(2.317) = \sin(\pi - 2.317) = \sin(.82459 \ 2654) \\ = .73427 \ 12$$

$$\cos(2.317) = \cos(\pi - 2.317) = -\cos(.82459 \ 2654) \\ = -.67885 \ 60.$$

Linear interpolation for $x = .82459 \ 2654$ gives an error of 9×10^{-8} .

Example 16.

Compute $\sin x$ for $x = 12.867$ to 8D. From 4.3.16 and Tables 4.6 and 4.8

$$\sin(12.867) = \sin 12 \cos .867 + \cos 12 \sin .867 \\ = .29612 \ 142.$$

The method of reduction to an angle in the first quadrant which was given in Example 15 may also be used.

Example 17.

Compute $\sin x$ to 19D for

$$x = .86725 \ 13489 \ 24685 \ 12693.$$

Let $\alpha = .867$, $\beta = x - \alpha$. From 4.3.16 and Table 4.6

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ \sin \alpha = .76239 \ 10208 \ 07866 \ 22598 \\ \cos \alpha = .64711 \ 66288 \ 94312 \ 75010$$

With the series expansions for $\sin \beta$ and $\cos \beta$ we compute successively

$$\begin{array}{r} 1.00000 \ 00000 \ 00000 \ 00000 \\ -\frac{\beta^2}{2!} = - \quad \quad \quad 315 \ 88140 \ 97019 \\ \frac{\beta^4}{4!} = \quad \quad \quad \quad \quad \quad 16630 \\ \hline \cos \beta = .99999 \ 99684 \ 11859 \ 19611 \\ \beta = .00025 \ 13489 \ 24685 \ 12693 \\ -\frac{\beta^3}{3!} = - \quad \quad \quad 2646 \ 54842 \\ \frac{\beta^5}{5!} = \quad \quad \quad \quad \quad \quad 1 \\ \hline \sin \beta = .00025 \ 13489 \ 22038 \ 57852 \\ \sin \alpha \cos \beta = .76239 \ 09967 \ 25351 \ 31308 \\ \cos \alpha \sin \beta = .00016 \ 26520 \ 67105 \ 82436 \\ \hline \sin x = .76255 \ 36487 \ 92457 \ 1374 \end{array}$$

This procedure is equivalent to interpolation with Taylor's formula 3.6.4.

Example 18.

In the plane triangle ABC , $a=123$, $B=29^\circ 16'$, $c=321$; find A , b .

$$b^2 = a^2 + c^2 - 2ac \cos B = (123)^2 + (321)^2 - 2(123)(321) \cos 29^\circ 16'$$

$$b = 221.99934 \ 00$$

$$\sin A = \frac{a \sin B}{b} = \frac{(123)(.48887 \ 50196)}{221.99934 \ 00} = .27086 \ 39918$$

$$A = 15^\circ 42' 56.469''.$$

Example 19.

In the plane triangle ABC , $a=4$, $b=7$, $c=9$, find A , B , and C .

$$\cos A = \frac{c^2 + b^2 - a^2}{2bc} = \frac{81 + 49 - 16}{2 \cdot 7 \cdot 9} = \frac{114}{126} = .90476 \ 1905$$

$$A = .43997 \ 5954 = 25^\circ 12' 31.6''$$

$$\sin A = .42591 \ 7709$$

$$\sin B = \frac{7(.42591 \ 7709)}{4}, \quad B = .84106 \ 8670 \\ = 48^\circ 11' 22.9''$$

$$\sin C = \frac{9(.42591 \ 7709)}{4}, \quad C = 1.86054 \ 803 \\ = 106^\circ 36' 5.6''$$

where the supplementary angle must be chosen for C . As a check we get $A+B+C=180^\circ 00' .1''$.

Example 20.

Compute $\cot x$ for $x=.4589$ to 6D.

Since $x < .5$, using Table 4.9 with interpolation in $(x^{-1} - \cot x)$, we find $\frac{1}{.4589} - \cot(.4589) = .155159$. Therefore $\cot(.4589) = 2.179124 - .155159 = 2.023965$.

Example 21.

Compute $\arcsin x$ for $x=.99511$.

For $x > .95$, using Table 4.14 with interpolation in the auxiliary function $f(x)$ we find

$$\arcsin x = \frac{\pi}{2} - [2(1-x)]^{\frac{1}{2}} f(x)$$

$$\arcsin(.99511) = \frac{\pi}{2} - [2(.00489)]^{\frac{1}{2}} f(.99511) \\ = 1.57079 \ 6327 - (.09889 \ 388252) \\ \quad (1.00040 \ 7951) \\ = 1.47186 \ 2100.$$

Example 22.

Compute $\arctan 20$ and $\operatorname{arccot} 20$ to 9D.

Using 4.4.5, 4.4.8, and Table 4.14 *

$$\arctan 20 = \frac{\pi}{2} - \arctan 1/20 = 1.52083 \ 7931$$

$$\operatorname{arccot} 20 = \frac{\pi}{2} - \arctan 20 = \arctan .05 = .04995 \ 8396.$$

Example 23.

Express $z=3+9i$ in polar form.

$$z = x + iy = re^{i\theta}, \text{ where } r = (x^2 + y^2)^{\frac{1}{2}},$$

$$\theta = \arctan \frac{y}{x} + 2\pi k, \quad k \text{ is an integer. For } k=0,$$

$$r = (3^2 + 9^2)^{\frac{1}{2}} = \sqrt{90} = 9.486833$$

$$\theta = \arctan 9/3 = \arctan 3 = 1.24904 \ 58.$$

$$\text{Thus } 3+9i = 9.486833 \exp(1.24904 \ 58i).$$

Example 24.

Compute $\arctan x$ for $x=1/3$ to 12D.

From 4.4.34 and 4.4.42 we have

$$\begin{aligned} \arctan x &= \arctan(x_0 + h) \\ &= \arctan x_0 + \arctan \frac{h}{1 + x_0 h + x_0^2} \\ &= \arctan x_0 + \left(\frac{h}{1 + x_0 h + x_0^2} \right) - \frac{1}{3} \left(\frac{h}{1 + x_0 h + x_0^2} \right)^3 + \dots \end{aligned}$$

We have

$$\begin{aligned} x &= \frac{1}{3} = .33333 \ 33333 \ 33 \text{ so that } h = .00033 \ 33333 \ 33 \\ \text{and, from Table 4.14, } \arctan x_0 &= \arctan .333 \\ &= .32145 \ 05244 \ 03. \quad \text{Since } \frac{h}{1 + x_0 h + x_0^2} = \\ &.00030 \ 00300 \ 03 \text{ we get} \end{aligned}$$

$$\begin{aligned} \arctan x &= .32145 \ 05244 \ 03 + .00030 \ 00300 \ 03 \\ &\quad - .00000 \ 00000 \ 09 \\ &= .32175 \ 05543 \ 97. \end{aligned}$$

If x is given in the form b/a it is convenient to use 4.4.34 in the form

$$\arctan \frac{b}{a} = \arctan x_0 + \arctan \frac{b - ax_0}{a + bx_0}$$

In the present example we get

$$\arctan \frac{1}{3} = \arctan .333 + \arctan \frac{1}{3333}$$

*See page 11.

Example 25.

Compute $\operatorname{arcsec} 2.8$ to 5D.
Using 4.3.45 and Table 4.14

$$\begin{aligned}\operatorname{arcsec} z &= \arcsin \frac{(z^2 - 1)^{\frac{1}{2}}}{z} \\ \operatorname{arcsec} 2.8 &= \arcsin \frac{[(2.8)^2 - 1]^{\frac{1}{2}}}{2.8} \\ &= \arcsin .93404\ 97735 \\ &= 1.20559\end{aligned}$$

or using 4.3.45 and Table 4.14

$$\begin{aligned}\operatorname{arcsec} z &= \arctan (z^2 - 1)^{\frac{1}{2}} \\ \operatorname{arcsec} 2.8 &= \arctan 2.61533\ 9366 \\ &= \frac{\pi}{2} - \arctan .38235\ 95564, \\ &\quad \text{from 4.4.3 and 4.4.8} \\ &= 1.570796 - .365207 \\ &= 1.20559.\end{aligned}$$

Example 26.

Compute $\operatorname{arctanh} x$ for $x = .96035$ to 6D.
From 4.6.22 and Table 4.2

$$\begin{aligned}\operatorname{arctanh} .96035 &= \frac{1}{2} \ln \frac{1 + .96035}{1 - .96035} = \frac{1}{2} \ln \frac{1.96035}{.03965} \\ &= \frac{1}{2} \ln 49.44136\ 191 \\ &= \frac{1}{2} (3.90078\ 7359) = 1.950394.\end{aligned}$$

Example 27.

Compute $\operatorname{arccosh} x$ for $x = 1.5368$ to 6D.
Using Table 4.17

$$\begin{aligned}\frac{\operatorname{arccosh} x}{(x^2 - 1)^{\frac{1}{2}}} &= \frac{\operatorname{arccosh} 1.5368}{[(1.5368)^2 - 1]^{\frac{1}{2}}} = .852346 \\ \operatorname{arccosh} 1.5368 &= (.852346)(1.361754)^{\frac{1}{2}} \\ &= (.852346)(1.166942) \\ &= .994638.\end{aligned}$$

Example 28.

Compute $\operatorname{arccosh} x$ for $x = 31.2$ to 5D.
Using Tables 4.2 and 4.17 with $1/x = 1/31.2$
 $= .03205\ 128205$

$$\begin{aligned}\operatorname{arccosh} 31.2 - \ln 31.2 &= .692886 \\ \operatorname{arccosh} 31.2 &= .692886 + 3.440418 = 4.13330.\end{aligned}$$

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5. Exponential Integral and Related Functions

WALTER GAUTSCHI¹ AND WILLIAM F. CAHILL²

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$\text{Si}(x)$, $\text{Ci}(x)$, 10D; $\text{Ei}(x)$, $E_1(x)$, 9D; $x=.5(.01)2$	
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5. Exponential Integral and Related Functions

Mathematical Properties

5.1. Exponential Integral

Definitions

$$5.1.1 \quad E_1(z) = \int_z^{\infty} \frac{e^{-t}}{t} dt \quad (|\arg z| < \pi)$$

$$5.1.2 \quad \text{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt = \int_{-\infty}^x \frac{e^t}{t} dt \quad (x > 0)$$

$$5.1.3 \quad \text{li}(x) = \int_0^x \frac{dt}{\ln t} = \text{Ei}(\ln x) \quad (x > 1)$$

5.1.4

$$E_n(z) = \int_1^{\infty} \frac{e^{-zt}}{t^n} dt \quad (n=0, 1, 2, \dots; \Re z > 0)$$

5.1.5

$$\alpha_n(z) = \int_1^{\infty} t^n e^{-zt} dt \quad (n=0, 1, 2, \dots; \Re z > 0)$$

$$5.1.6 \quad \beta_n(z) = \int_{-1}^1 t^n e^{-zt} dt \quad (n=0, 1, 2, \dots)$$

In 5.1.1 it is assumed that the path of integration excludes the origin and does not cross the negative real axis.

Analytic continuation of the functions in 5.1.1, 5.1.2, and 5.1.4 for $n > 0$ yields multi-valued functions with branch points at $z=0$ and $z=\infty$.³ They are single-valued functions in the z -plane cut along the negative real axis.⁴ The function $\text{li}(z)$, the logarithmic integral, has an additional branch point at $z=1$.

Interrelations

5.1.7

$$E_1(-x \pm i0) = -\text{Ei}(x) \mp i\pi,$$

$$-\text{Ei}(x) = \frac{1}{2}[E_1(-x+i0) + E_1(-x-i0)] \quad (x > 0)$$

³ Some authors [5.14], [5.16] use the entire function $\int_0^z (1-e^{-t})dt/t$ as the basic function and denote it by $\text{Ein}(z)$. We have $\text{Ein}(z) = E_1(z) + \ln z + \gamma$.

⁴ Various authors define the integral $\int_{-\infty}^z (e^t/t)dt$ in the z -plane cut along the positive real axis and denote it also by $\text{Ei}(z)$. For $z=x>0$ additional notations such as $\overline{\text{Ei}}(x)$ (e.g., in [5.10], [5.25]), $E^*(x)$ (in [5.2]), $\text{Ei}^*(x)$ (in [5.6]) are then used to designate the principal value of the integral. Correspondingly, $E_1(x)$ is often denoted by $-\text{Ei}(-x)$.

Explicit Expressions for $\alpha_n(z)$ and $\beta_n(z)$

$$5.1.8 \quad \alpha_n(z) = n! z^{-n-1} e^{-z} \left(1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!}\right)$$

5.1.9

$$\beta_n(z) = n! z^{-n-1} \{ e^z [1 - z + \frac{z^2}{2!} - \dots + (-1)^n \frac{z^n}{n!}] - e^{-z} (1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!}) \}$$

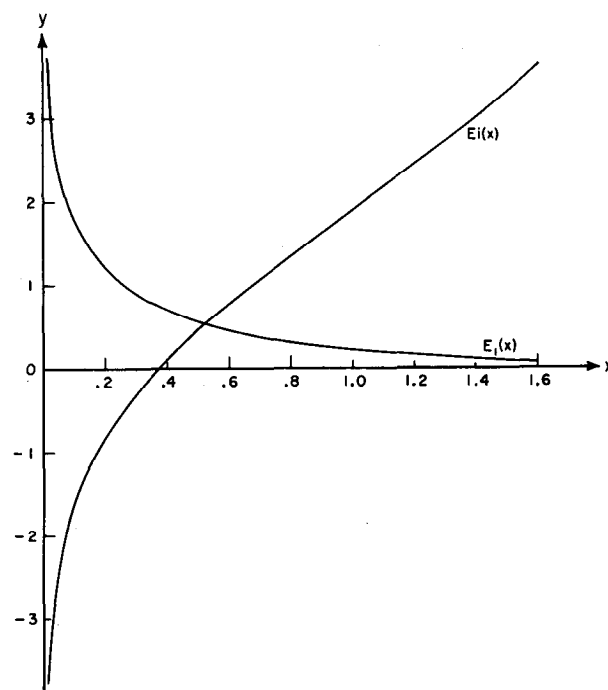


FIGURE 5.1. $y = \text{Ei}(x)$ and $y = E_1(x)$.

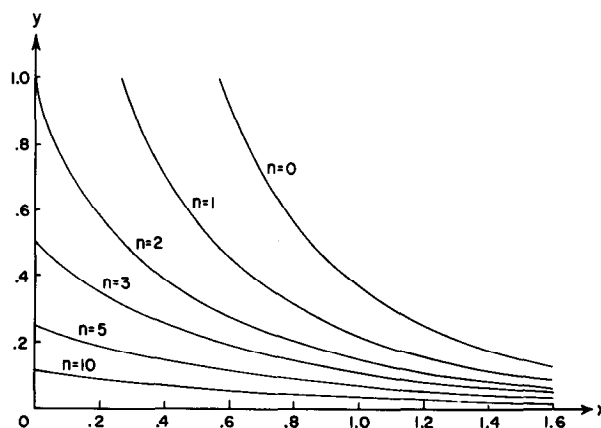
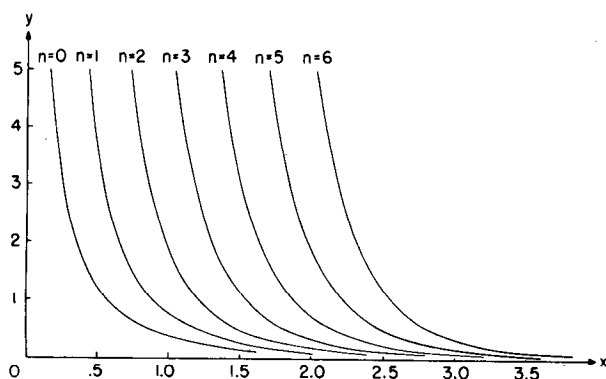
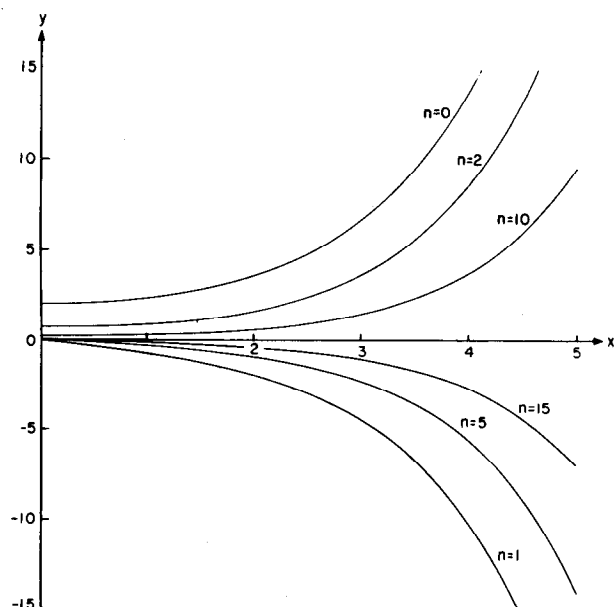


FIGURE 5.2. $y = E_n(x)$
 $n=0, 1, 2, 3, 5, 10$


 FIGURE 5.3. $y = \alpha_n(x)$
 $n = 0(1)6$

 FIGURE 5.4. $y = \beta_n(x)$
 $n = 0, 1, 2, 5, 10, 15$

Series Expansions

$$5.1.10 \quad \text{Ei}(x) = \gamma + \ln x + \sum_{n=1}^{\infty} \frac{x^n}{nn!} \quad (x > 0)$$

5.1.11

$$E_1(z) = -\gamma - \ln z - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{nn!} \quad (|\arg z| < \pi)$$

5.1.12

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!} [-\ln z + \psi(n)] - \sum_{\substack{m=0 \\ m \neq n-1}}^{\infty} \frac{(-z)^m}{(m-n+1)m!} \quad (|\arg z| < \pi)$$

$$\psi(1) = -\gamma, \quad \psi(n) = -\gamma + \sum_{m=1}^{n-1} \frac{1}{m} \quad (n > 1)$$

 $\gamma = .57721\ 56649 \dots$ is Euler's constant.

Symmetry Relation

$$5.1.13 \quad E_n(\bar{z}) = \overline{E_n(z)}$$

Recurrence Relations

5.1.14

$$E_{n+1}(z) = \frac{1}{n} [e^{-z} - zE_n(z)] \quad (n = 1, 2, 3, \dots)$$

$$5.1.15 \quad z\alpha_n(z) = e^{-z} + n\alpha_{n-1}(z) \quad (n = 1, 2, 3, \dots)$$

5.1.16

$$z\beta_n(z) = (-1)^n e^z - e^{-z} + n\beta_{n-1}(z) \quad (n = 1, 2, 3, \dots)$$

Inequalities [5.8], [5.4]

5.1.17

$$\frac{n-1}{n} E_n(x) < E_{n+1}(x) < E_n(x) \quad (x > 0; n = 1, 2, 3, \dots)$$

5.1.18

$$E_n^2(x) < E_{n-1}(x)E_{n+1}(x) \quad (x > 0; n = 1, 2, 3, \dots)$$

5.1.19

$$\frac{1}{x+n} < e^x E_n(x) \leq \frac{1}{x+n-1} \quad (x > 0; n = 1, 2, 3, \dots)$$

5.1.20

$$\frac{1}{2} \ln \left(1 + \frac{2}{x} \right) < e^x E_1(x) < \ln \left(1 + \frac{1}{x} \right) \quad (x > 0)$$

5.1.21

$$\frac{d}{dx} \left[\frac{E_n(x)}{E_{n-1}(x)} \right] > 0 \quad (x > 0; n = 1, 2, 3, \dots)$$

Continued Fraction

5.1.22

$$E_n(z) = e^{-z} \left(\frac{1}{z+1} + \frac{n}{z+1} + \frac{1}{z+1} + \frac{n+1}{z+1} + \frac{2}{z+1} \dots \right) \quad (|\arg z| < \pi)$$

Special Values

$$5.1.23 \quad E_n(0) = \frac{1}{n-1} \quad (n > 1)$$

$$5.1.24 \quad E_0(z) = \frac{e^{-z}}{z}$$

$$5.1.25 \quad \alpha_0(z) = \frac{e^{-z}}{z}, \quad \beta_0(z) = \frac{2}{z} \sinh z$$

Derivatives

$$5.1.26 \quad \frac{dE_n(z)}{dz} = -E_{n-1}(z) \quad (n=1, 2, 3, \dots)$$

5.1.27

$$\frac{d^n}{dz^n} [e^z E_1(z)] = \frac{d^{n-1}}{dz^{n-1}} [e^z E_1(z)] + \frac{(-1)^n (n-1)!}{z^n} \quad (n=1, 2, 3, \dots)$$

Definite and Indefinite Integrals

(For more extensive tables of integrals see [5.3], [5.6], [5.11], [5.12], [5.13]. For integrals involving $E_n(x)$ see [5.9].)

$$5.1.28 \quad \int_0^\infty \frac{e^{-at}}{b+t} dt = e^{ab} E_1(ab)$$

5.1.29

$$\int_0^\infty \frac{e^{iat}}{b+t} dt = e^{-iab} E_1(-iab) \quad (a>0, b>0)$$

5.1.30

$$\int_0^\infty \frac{t-ib}{t^2+b^2} e^{iat} dt = e^{ab} E_1(ab) \quad (a>0, b>0)$$

5.1.31

$$\int_0^\infty \frac{t+ib}{t^2+b^2} e^{iat} dt = e^{-ab} (-\text{Ei}(ab) + i\pi) \quad (a>0, b>0)$$

5.1.32

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt = \ln \frac{b}{a}$$

5.1.33

$$\int_0^\infty E_1^2(t) dt = 2 \ln 2$$

5.1.34

$$\int_0^\infty e^{-at} E_n(t) dt = \frac{(-1)^{n-1}}{a^n} \left[\ln(1+a) + \sum_{k=1}^{n-1} \frac{(-1)^k a^k}{k} \right] \quad (a>-1)$$

5.1.35

$$\int_0^1 \frac{e^{at} \sin bt}{t} dt = \pi - \arctan \frac{b}{a} + \mathcal{J} E_1(-a+ib) \quad (a>0, b>0)$$

5.1.36

$$\int_0^1 \frac{e^{-at} \sin bt}{t} dt = \arctan \frac{b}{a} + \mathcal{J} E_1(a+ib) \quad (a>0, b \text{ real})$$

5.1.37

$$\int_0^1 \frac{e^{at}(1-\cos bt)}{t} dt = \frac{1}{2} \ln \left(1 + \frac{b^2}{a^2} \right) + \text{Ei}(a) + \mathcal{R} E_1(-a+ib) \quad (a>0, b \text{ real})$$

5.1.38

$$\int_0^1 \frac{e^{-at}(1-\cos bt)}{t} dt = \frac{1}{2} \ln \left(1 + \frac{b^2}{a^2} \right) - E_1(a) + \mathcal{R} E_1(a+ib) \quad (a>0, b \text{ real})$$

5.1.39

$$\int_0^z \frac{1-e^{-t}}{t} dt = E_1(z) + \ln z + \gamma$$

5.1.40

$$\int_0^x \frac{e^t-1}{t} dt = \text{Ei}(x) - \ln x - \gamma \quad (x>0)$$

5.1.41

$$\int \frac{e^{ix}}{a^2+x^2} dx = \frac{i}{2a} [e^{-a} E_1(-a-ix) - e^a E_1(a-ix)] + \text{const.}$$

5.1.42

$$\int \frac{x e^{ix}}{a^2+x^2} dx = -\frac{1}{2} [e^{-a} E_1(-a-ix) + e^a E_1(a-ix)] + \text{const.}$$

5.1.43

$$\int \frac{e^x}{a^2+x^2} dx = -\frac{1}{a} \mathcal{J}(e^{ia} E_1(-x+ia)) + \text{const.} \quad (a>0)$$

5.1.44

$$\int \frac{x e^x}{a^2+x^2} dx = -\mathcal{R}(e^{ia} E_1(-x+ia)) + \text{const.} \quad (a>0)$$

Relation to Incomplete Gamma Function (see 6.5)

$$5.1.45 \quad E_n(z) = z^{n-1} \Gamma(1-n, z)$$

$$5.1.46 \quad \alpha_n(z) = z^{-n-1} \Gamma(n+1, z)$$

$$5.1.47 \quad \beta_n(z) = z^{-n-1} [\Gamma(n+1, -z) - \Gamma(n+1, z)]$$

Relation to Spherical Bessel Functions (see 10.2)

$$5.1.48 \quad \alpha_0(z) = \sqrt{\frac{2}{\pi z}} K_{\frac{1}{2}}(z), \quad \beta_0(z) = \sqrt{\frac{2\pi}{z}} I_{\frac{1}{2}}(z)$$

$$5.1.49 \quad \alpha_1(z) = \sqrt{\frac{2}{\pi z}} K_{\frac{3}{2}}(z), \quad \beta_1(z) = -\sqrt{\frac{2\pi}{z}} I_{\frac{3}{2}}(z)$$

Number-Theoretic Significance of $\text{li}(x)$

(Assuming Riemann's hypothesis that all non-real zeros of $\zeta(z)$ have a real part of $\frac{1}{2}$)

$$5.1.50 \quad \text{li}(x) - \pi(x) = O(\sqrt{x} \ln x) \quad (x \rightarrow \infty)$$

$\pi(x)$ is the number of primes less than or equal to x .

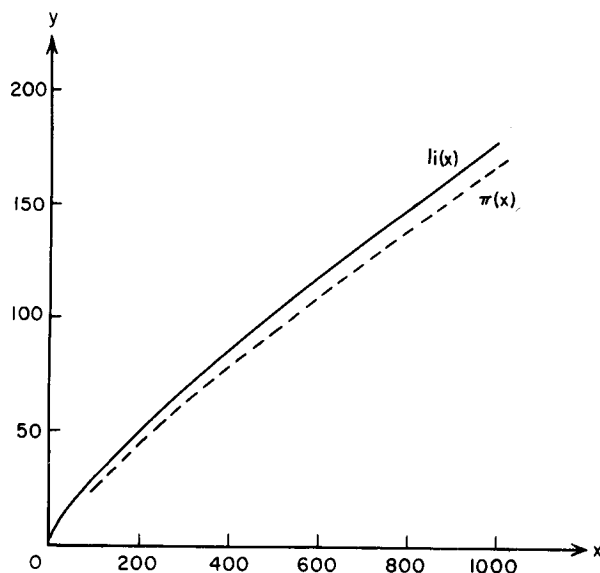


FIGURE 5.5. $y = \text{li}(x)$ and $y = \pi(x)$

Asymptotic Expansion

5.1.51

$$E_n(z) \sim \frac{e^{-z}}{z} \left\{ 1 - \frac{n}{z} + \frac{n(n+1)}{z^2} - \frac{n(n+1)(n+2)}{z^3} + \dots \right\} \quad (|\arg z| < \frac{3}{2}\pi)$$

Representation of $E_n(x)$ for Large n

5.1.52

$$E_n(x) = \frac{e^{-x}}{x+n} \left\{ 1 + \frac{n}{(x+n)^2} + \frac{n(n-2x)}{(x+n)^4} + \frac{n(6x^2 - 8nx + n^2)}{(x+n)^6} + R(n, x) \right\}$$

$$- .36n^{-4} \leq R(n, x) \leq \left(1 + \frac{1}{x+n-1} \right) n^{-4} \quad (x > 0)$$

Polynomial and Rational Approximations⁵

5.1.53

$$0 \leq x \leq 1$$

$$E_1(x) + \ln x = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \epsilon(x)$$

$$|\epsilon(x)| < 2 \times 10^{-7}$$

⁵ The approximation 5.1.53 is from E. E. Allen, Note 169, MTAC 8, 240 (1954); approximations 5.1.54 and 5.1.56 are from C. Hastings, Jr., Approximations for digital computers, Princeton Univ. Press, Princeton, N.J., 1955; approximation 5.1.55 is from C. Hastings, Jr., Note 143, MTAC 7, 68 (1953) (with permission).

$$\begin{array}{ll} a_0 = -.57721 \ 566 & a_3 = .05519 \ 968 \\ a_1 = .99999 \ 193 & a_4 = -.00976 \ 004 \\ a_2 = -.24991 \ 055 & a_5 = .00107 \ 857 \end{array}$$

5.1.54

$$1 \leq x < \infty$$

$$xe^xE_1(x) = \frac{x^2 + a_1x + a_2}{x^2 + b_1x + b_2} + \epsilon(x)$$

$$|\epsilon(x)| < 5 \times 10^{-5}$$

$$\begin{array}{ll} a_1 = 2.334733 & b_1 = 3.330657 \\ a_2 = .250621 & b_2 = 1.681534 \end{array}$$

5.1.55

$$10 \leq x < \infty$$

$$xe^xE_1(x) = \frac{x^2 + a_1x + a_2}{x^2 + b_1x + b_2} + \epsilon(x)$$

$$|\epsilon(x)| < 10^{-7}$$

$$\begin{array}{ll} a_1 = 4.03640 & b_1 = 5.03637 \\ a_2 = 1.15198 & b_2 = 4.19160 \end{array}$$

5.1.56

$$1 \leq x < \infty$$

$$xe^xE_1(x) = \frac{x^4 + a_1x^3 + a_2x^2 + a_3x + a_4}{x^4 + b_1x^3 + b_2x^2 + b_3x + b_4} + \epsilon(x)$$

$$|\epsilon(x)| < 2 \times 10^{-8}$$

$$\begin{array}{ll} a_1 = 8.57332 \ 87401 & b_1 = 9.57332 \ 23454 \\ a_2 = 18.05901 \ 69730 & b_2 = 25.63295 \ 61486 \\ a_3 = 8.63476 \ 08925 & b_3 = 21.09965 \ 30827 \\ a_4 = .26777 \ 37343 & b_4 = 3.95849 \ 69228 \end{array}$$

5.2. Sine and Cosine Integrals

Definitions

$$5.2.1 \quad \text{Si}(z) = \int_0^z \frac{\sin t}{t} dt$$

5.2.2⁶

$$\text{Ci}(z) = \gamma + \ln z + \int_0^z \frac{\cos t - 1}{t} dt \quad (|\arg z| < \pi)$$

5.2.3⁷

$$\text{Shi}(z) = \int_0^z \frac{\sinh t}{t} dt$$

5.2.4⁷

$$\text{Chi}(z) = \gamma + \ln z + \int_0^z \frac{\cosh t - 1}{t} dt \quad (|\arg z| < \pi)$$

⁶ Some authors [5.14], [5.16] use the entire function $\int_0^z (1 - \cos t) dt/t$ as the basic function and denote it by $\text{Cin}(z)$. We have

$$\text{Cin}(z) = -\text{Ci}(z) + \ln z + \gamma.$$

⁷ The notations $\text{Sih}(z) = \int_0^z \sinh t dt/t$, $\text{Cinh}(z) = \int_0^z (\cosh t - 1) dt/t$ have also been proposed [5.14.]

$$5.2.5 \quad \text{si}(z) = \text{Si}(z) - \frac{\pi}{2}$$

Auxiliary Functions

$$5.2.6 \quad f(z) = \text{Ci}(z) \sin z - \text{si}(z) \cos z$$

$$5.2.7 \quad g(z) = -\text{Ci}(z) \cos z - \text{si}(z) \sin z$$

Sine and Cosine Integrals in Terms of Auxiliary Functions

$$5.2.8 \quad \text{Si}(z) = \frac{\pi}{2} - f(z) \cos z - g(z) \sin z$$

$$5.2.9 \quad \text{Ci}(z) = f(z) \sin z - g(z) \cos z$$

Integral Representations

$$5.2.10 \quad \text{si}(z) = - \int_0^{\frac{\pi}{2}} e^{-z \cos t} \cos(z \sin t) dt$$

$$5.2.11 \quad \text{Ci}(z) + E_1(z) = \int_0^{\frac{\pi}{2}} e^{-z \cos t} \sin(z \sin t) dt$$

$$5.2.12 \quad f(z) = \int_0^{\infty} \frac{\sin t}{t+z} dt = \int_0^{\infty} \frac{e^{-zt}}{t^2+1} dt \quad (\Re z > 0)$$

$$5.2.13 \quad g(z) = \int_0^{\infty} \frac{\cos t}{t+z} dt = \int_0^{\infty} \frac{te^{-zt}}{t^2+1} dt \quad (\Re z > 0)$$

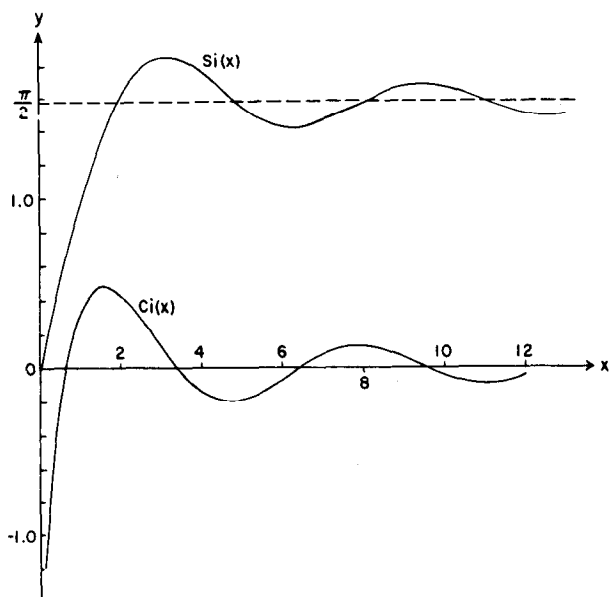


FIGURE 5.6. $y = \text{Si}(x)$ and $y = \text{Ci}(x)$

Series Expansions

$$5.2.14 \quad \text{Si}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)(2n+1)!}$$

$$5.2.15 \quad \text{Si}(z) = \pi \sum_{n=0}^{\infty} J_{n+\frac{1}{2}}^2\left(\frac{z}{2}\right)$$

$$5.2.16 \quad \text{Ci}(z) = \gamma + \ln z + \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{2n(2n)!}$$

$$5.2.17 \quad \text{Shi}(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)(2n+1)!}$$

$$5.2.18 \quad \text{Chi}(z) = \gamma + \ln z + \sum_{n=1}^{\infty} \frac{z^{2n}}{2n(2n)!}$$

Symmetry Relations

$$5.2.19 \quad \text{Si}(-z) = -\text{Si}(z), \quad \text{Si}(\bar{z}) = \overline{\text{Si}(z)}$$

$$5.2.20$$

$$\text{Ci}(-z) = \text{Ci}(z) - i\pi \quad (0 < \arg z < \pi)$$

$$\text{Ci}(\bar{z}) = \overline{\text{Ci}(z)}$$

Relation to Exponential Integral

$$5.2.21$$

$$\text{Si}(z) = \frac{1}{2i} [E_1(iz) - E_1(-iz)] + \frac{\pi}{2} \quad (|\arg z| < \frac{\pi}{2})$$

$$5.2.22 \quad \text{Si}(ix) = \frac{i}{2} [\text{Ei}(x) + E_1(x)] \quad (x > 0)$$

$$5.2.23$$

$$\text{Ci}(z) = -\frac{1}{2} [E_1(iz) + E_1(-iz)] \quad (|\arg z| < \frac{\pi}{2})$$

$$5.2.24 \quad \text{Ci}(ix) = \frac{1}{2} [\text{Ei}(x) - E_1(x)] + i\frac{\pi}{2} \quad (x > 0)$$

Value at Infinity

$$5.2.25 \quad \lim_{x \rightarrow \infty} \text{Si}(x) = \frac{\pi}{2}$$

Integrals

(For more extensive tables of integrals see [5.3], [5.6], [5.11], [5.12], [5.13].)

$$5.2.26 \quad \int_z^{\infty} \frac{\sin t}{t} dt = -\text{si}(z) \quad (|\arg z| < \pi)$$

$$5.2.27 \quad \int_z^{\infty} \frac{\cos t}{t} dt = -\text{Ci}(z) \quad (|\arg z| < \pi)$$

$$5.2.28 \quad \int_0^{\infty} e^{-at} \text{Ci}(t) dt = -\frac{1}{2a} \ln(1+a^2) \quad (\Re a > 0)^*$$

$$5.2.29 \quad \int_0^{\infty} e^{-at} \text{si}(t) dt = -\frac{1}{a} \arctan a \quad (\Re a > 0)$$

$$5.2.30 \quad \int_0^{\infty} \cos t \text{Ci}(t) dt = \int_0^{\infty} \sin t \text{si}(t) dt = -\frac{\pi}{4}$$

*See page 11.

$$5.2.31 \quad \int_0^{\infty} \text{Ci}^2(t) dt = \int_0^{\infty} \text{si}^2(t) dt = \frac{\pi}{2}$$

$$5.2.32^* \quad \int_0^{\infty} \text{Ci}(t) \text{si}(t) dt = \ln 2$$

5.2.33

$$\int_0^1 \frac{(1-e^{-at}) \cos bt}{t} dt = \frac{1}{2} \ln \left(1 + \frac{a^2}{b^2} \right) + \text{Ci}(b) \\ + \mathcal{J} E_1(a+ib) \quad (a \text{ real}, b > 0)$$

Asymptotic Expansions

5.2.34

$$f(z) \sim \frac{1}{z} \left(1 - \frac{2!}{z^2} + \frac{4!}{z^4} - \frac{6!}{z^6} + \dots \right) \quad (|\arg z| < \pi)$$

5.2.35

$$g(z) \sim \frac{1}{z^2} \left(1 - \frac{3!}{z^2} + \frac{5!}{z^4} - \frac{7!}{z^6} + \dots \right) \quad (|\arg z| < \pi)$$

Rational Approximations⁸

5.2.36

$$1 \leq x < \infty$$

$$f(x) = \frac{1}{x} \left(\frac{x^4 + a_1 x^2 + a_2}{x^4 + b_1 x^2 + b_2} \right) + \epsilon(x)$$

$$|\epsilon(x)| < 2 \times 10^{-4}$$

$$a_1 = 7.241163 \quad b_1 = 9.068580$$

$$a_2 = 2.463936 \quad b_2 = 7.157433$$

5.2.37

$$1 \leq x < \infty$$

$$g(x) = \frac{1}{x^2} \left(\frac{x^4 + a_1 x^2 + a_2}{x^4 + b_1 x^2 + b_2} \right) + \epsilon(x)$$

$$|\epsilon(x)| < 10^{-4}$$

$$a_1 = 7.547478 \quad b_1 = 12.723684 \quad *$$

$$a_2 = 1.564072 \quad b_2 = 15.723606 \quad *$$

5.2.38

$$1 \leq x < \infty$$

$$f(x) = \frac{1}{x} \left(\frac{x^8 + a_1 x^6 + a_2 x^4 + a_3 x^2 + a_4}{x^8 + b_1 x^6 + b_2 x^4 + b_3 x^2 + b_4} \right) + \epsilon(x)$$

$$|\epsilon(x)| < 5 \times 10^{-7}$$

$$a_1 = 38.027264 \quad b_1 = 40.021433$$

$$a_2 = 265.187033 \quad b_2 = 322.624911$$

$$a_3 = 335.677320 \quad b_3 = 570.236280$$

$$a_4 = 38.102495 \quad b_4 = 157.105423$$

5.2.39

$$1 \leq x < \infty$$

$$g(x) = \frac{1}{x^2} \left(\frac{x^8 + a_1 x^6 + a_2 x^4 + a_3 x^2 + a_4}{x^8 + b_1 x^6 + b_2 x^4 + b_3 x^2 + b_4} \right) + \epsilon(x)$$

$$|\epsilon(x)| < 3 \times 10^{-7}$$

$$a_1 = 42.242855 \quad b_1 = 48.196927$$

$$a_2 = 302.757865 \quad b_2 = 482.485984$$

$$a_3 = 352.018498 \quad b_3 = 1114.978885$$

$$a_4 = 21.821899 \quad b_4 = 449.690326$$

Numerical Methods

5.3. Use and Extension of the Tables

Example 1. Compute $\text{Ci}(.25)$ to 5D.

From **Tables 5.1** and **4.2** we have

$$\frac{\text{Ci}(.25) - \ln(.25) - \gamma}{(.25)^2} = -.249350,$$

$$\text{Ci}(.25) = (.25)^2 (-.249350) + (-1.38629) \\ + .577216 = -.82466.$$

Example 2. Compute $\text{Ei}(8)$ to 5S.

From **Table 5.1** we have $xe^{-x}\text{Ei}(x) = 1.18185$ for $x=8$. From **Table 4.4**, $e^8 = 2.98096 \times 10^3$. Thus $\text{Ei}(8) = 440.38$.

^{*}See page II.

⁸ From C. Hastings, Jr., *Approximations for digital computers*, Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

Example 3. Compute $\text{Si}(20)$ to 5D.

Since $1/20 = .05$ from **Table 5.2** we find $f(20) = .049757$, $g(20) = .002464$. From **Table 4.8**, $\sin 20 = .912945$, $\cos 20 = .408082$. Using **5.2.8**

$$\text{Si}(20) = \frac{\pi}{2} - f(20) \cos 20 - g(20) \sin 20 \\ = 1.570796 - .022555 = 1.54824.$$

Example 4. Compute $E_n(x)$, $n=1(1)N$, to 5S for $x=1.275$, $N=10$.

If x is less than about five, the recurrence relation **5.1.14** can be used in increasing order of n without serious loss of accuracy.

By quadratic interpolation in **Table 5.1** we get $E_1(1.275) = .1408099$, and from **Table 4.4**, $e^{-1.275} = .2794310$. The recurrence formula **5.1.14** then yields

n	$E_n(1.275)$	$E_n(1.275)$
1	.1408099	6 .0430168
2	.0998984	7 .0374307
3	.0760303	8 .0331009
4	.0608307	9 .0296534
5	.0504679	10 .0268469

Interpolating directly in Table 5.4 for $n=10$ we get $E_{10}(1.275)=.0268470$ as a check.

Example 5. Compute $E_n(x)$, $n=1(1)N$, to 5S for $x=10$, $N=10$.

If, as in this example, x is appreciably larger than five and $N \leq x$, then the recurrence relation 5.1.14 may be safely used in decreasing order of n ([5.5]). From Table 5.5 for $x^{-1}=.1$ we get $(x+10)e^x E_{10}(x)=1.02436$ so that $E_{10}(10)=2.32529 \times 10^{-6}$. Using this as the initial value we obtain column (2).

n	$10^6 E_n(10)$ (1)	$10^6 E_n(10)$ (2)
1	.41570	.41570
2	.38300	.38302
3	.35500	.35488
4	.33000	.33041
5	.31000	.30898
6	.28800	.29005
7	.27667	.27325
8	.25333	.25822
9	.25084	.24472
10	.22573	.23253

From Table 5.2 we get $xe^x E_1(x)=.915633$ so that $E_1(10)=4.15697 \times 10^{-6}$ as a check. Forward recurrence starting with $E_1(10)=4.1570 \times 10^{-6}$ yields the values in column (1). The underlined figures are in error.

Example 6. Compute $E_n(x)$, $n=1(1)N$, to 5S for $x=12.3$, $N=20$.

If N is appreciably larger than x , and x appreciably larger than five, then the recurrence relation 5.1.14 should be used in the backward direction to generate $E_n(x)$ for $n < n_0$, and in the forward direction to generate $E_n(x)$ for $n > n_0$, where $n_0 = \langle x \rangle$.

From 5.1.52, with $n_0=12$, $x=12.3$, we have

$$E_{n_0}(x) = \frac{e^{-12.3}}{24.3} (1 + .02032 - .00043 - .00001) = 1.91038 \times 10^{-7}.$$

Using the recurrence relation 5.1.14, as indicated, we get

n	$10^6 E_n(12.3)$	$10^6 E_n(12.3)$	n
12	.191038	.191038	12
11	.199213	.183498	13
10	.208098	.176516	14
9	.217793	.170042	15
8	.228406	.164015	16
7	.240073	.158397	17
6	.252951	.153144	18
5	.267234	.148226	19
4	.283155	.143608	20
3	.300998		
2	.321117		
1	.343953		

From Tables 5.2 and 5.5 we find $E_1(12.3)=.343953 \times 10^{-6}$, $E_{20}(12.3)=.143609 \times 10^{-6}$ as a check.

Example 7. Compute $\alpha_n(2)$ to 6S for $n=1(1)5$.

The recurrence formula 5.1.15 can be used for all $x > 0$ in increasing order of n without loss of accuracy. From 5.1.25 we have $\alpha_0(2) = \frac{1}{2}e^{-2} = .0676676$, so we get

n	$\alpha_n(2)$
0	.0676676
1	.101501
2	.169169
3	.321421
4	.710510
5	1.84394

Independent calculation with 5.1.8 yields the same result for $\alpha_5(2)$.

The functions $\alpha_0(x)$ and $\alpha_1(x)$ can be obtained from Table 10.8 using 5.1.48, 5.1.49.

Example 8. Compute $\beta_n(x)$, $n=0(1)N$ to 6S for $x=1$, $N=5$.

Use the recurrence relation 5.1.16 in increasing order of n if

$$x > .368N + .184 \ln N + .821$$

and in decreasing order of n otherwise [5.5].

From 5.1.9 with $n=5$ we get $\beta_5(1)=-.324297$ correctly rounded to 6D. Using the recurrence formula 5.1.16 in decreasing order of n and carrying 9D we get the values in column (2).

n	$\beta_n(1)$ (1)	$\beta_n(1)$ (2)
0	2.35040 2	2.35040 2389
1	-.73575 9269	-.73575 8880
2	.87888 3849	.87888 4629
3	-.44950 9722	-.44950 7383
4	.55236 3499	.55237 2854
5	-.32434 3774	-.32429 7

Using forward recurrence instead, starting with

$\beta_0(1)=2 \sinh 1=2.350402$ and again carrying 9D, we obtain column (1). The underlined figures are in error. The above shows that three significant figures are lost in forward recurrence, whereas about three significant figures are gained in backward recurrence!

An alternative procedure is to start with an arbitrary value for n sufficiently large (see also [5.1]). To illustrate, starting with the value zero at $n=11$ we get

n	$\beta_n(1)$	n	$\beta_n(1)$
11	0.	5	-.324297
10	.280560	4	.552373
9	-.206984	3	-.449507
8	.319908	2	.878885
7	-.253812	1	-.735759
6	.404621	0	2.350402

The functions $\beta_0(x)$ and $\beta_1(x)$ can be obtained from Table 10.8 using 5.1.48, 5.1.49.

Example 9. Compute $E_1(z)$ for $z=3.2578+6.8943i$.

From Table 5.6 we have for $z_0=x_0+iy_0=3+7i$

$$z_0 e^{z_0} E_1(z_0) = .934958 + .095598i,$$

$$e^{z_0} E_1(z_0) = .059898 - .107895i.$$

From Taylor's formula with $f(z) = e^z E_1(z)$ we have

$$f(z) = f(z_0 + \Delta z) = f(z_0) + \frac{f'(z_0)}{1!} \Delta z + \frac{f''(z_0)}{2!} (\Delta z)^2 + \dots$$

with $\Delta z = z - z_0 = .2578 - .1057i$. Thus with 5.1.27 we get

k	$f^{(k)}(z_0)/k!$	$(\Delta z)^k f^{(k)}(z_0)/k!$
0	.059898 - .107895i	.059898 - .107895i
1	.008174 + .012795i	.003460 + .002435i
2	-.001859 + .000155i	-.000094 + .000110i
3	.000088 - .000212i	-.000003 - .000004i

$$f(z) = .063261 - .105354i$$

$$e^{-z} = .031510 - .022075i$$

$$E_1(z) = -.000332 - .004716i$$

Repeating the calculation with $z_0=3+6i$ and $\Delta z=.2578+.8943i$ we get the same result.

An alternative procedure is to perform bivariate interpolation in the real and imaginary parts of $ze^z E_1(z)$.

Example 10. Compute $E_1(z)$ for $z=-4.2+12.7i$.

Using the formula at the bottom of Table 5.6

$$\begin{aligned} e^z E_1(z) &\approx \frac{.711093}{-3.784225 + 12.7i} \\ &\quad + \frac{.278518}{-1.90572 + 12.7i} + \frac{.010389}{2.0900 + 12.7i} \\ &= -.0184106 - .0736698i \\ E_1(z) &\approx -1.87133 - 4.70540i. \end{aligned}$$

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Tables

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- $E_1\left(\frac{\Delta E}{kT}\right)$, $1-\frac{\Delta E}{kT} \exp\left(\frac{\Delta E}{kT}\right) E_1\left(\frac{\Delta E}{kT}\right)$; $\Delta E=.2(.2)2$, $T=25(25)1000$, $T=150(10)390$, 3-4S; x^{-1} , $\exp(-x^{-1})$, $x \exp(-x^{-1})$, $E_1(x^{-1})$, $\int_0^x \exp(-t^{-1}) dt$, $x^{-1} \exp(x^{-1}) E_1(x^{-1})$, $1-x^{-1} \exp(x^{-1}) E_1(x^{-1})$; $x=.01(.0001).1$, 5-6S.
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6. Gamma Function and Related Functions

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¹ National Bureau of Standards.

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6. Gamma Function and Related Functions

Mathematical Properties

6.1. Gamma (Factorial) Function

Euler's Integral

$$\begin{aligned} 6.1.1 \quad \Gamma(z) &= \int_0^{\infty} t^{z-1} e^{-t} dt \quad (\Re z > 0) \\ &= k^z \int_0^{\infty} t^{z-1} e^{-kt} dt \quad (\Re z > 0, \Re k > 0) \end{aligned}$$

Euler's Formula

$$6.1.2 \quad \Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{z(z+1) \dots (z+n)} \quad (z \neq 0, -1, -2, \dots)$$

Euler's Infinite Product

$$\begin{aligned} 6.1.3 \quad \frac{1}{\Gamma(z)} &= z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n} \right) e^{-z/n} \right] \quad (|z| < \infty) \\ \gamma &= \lim_{m \rightarrow \infty} \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{m} - \ln m \right] \\ &= .57721 \ 56649 \dots \end{aligned}$$

γ is known as Euler's constant and is given to 25 decimal places in chapter 1. $\Gamma(z)$ is single valued and analytic over the entire complex plane, save for the points $z = -n$ ($n = 0, 1, 2, \dots$) where it possesses simple poles with residue $(-1)^n/n!$. Its reciprocal $1/\Gamma(z)$ is an entire function possessing simple zeros at the points $z = -n$ ($n = 0, 1, 2, \dots$).

Hankel's Contour Integral

$$6.1.4 \quad \frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_C (-t)^{-z} e^{-t} dt \quad (|z| < \infty)$$

The path of integration C starts at $+\infty$ on the real axis, circles the origin in the counterclockwise direction and returns to the starting point.

Factorial and Π Notations

$$6.1.5 \quad \Pi(z) = z! = \Gamma(z+1)$$

Integer Values

$$6.1.6 \quad \Gamma(n+1) = 1 \cdot 2 \cdot 3 \dots (n-1)n = n!$$

6.1.7

$$\lim_{z \rightarrow n} \frac{1}{\Gamma(-z)} = 0 = \frac{1}{(-n-1)!} \quad (n = 0, 1, 2, \dots)$$

Fractional Values

$$6.1.8 \quad \Gamma\left(\frac{1}{2}\right) = 2 \int_0^{\infty} e^{-t^2} dt = \pi^{\frac{1}{2}} = 1.77245 \ 38509 \dots = \left(-\frac{1}{2}\right)!$$

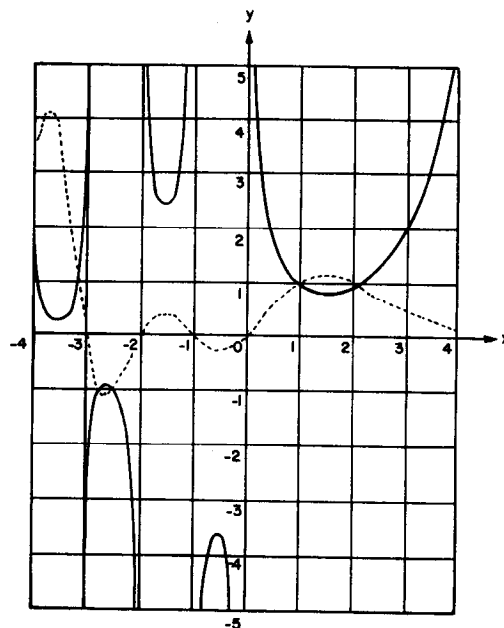


FIGURE 6.1. Gamma function. *

—, $y = \Gamma(x)$, - - - , $y = 1/\Gamma(x)$

$$6.1.9 \quad \Gamma(3/2) = \frac{1}{2} \pi^{\frac{1}{2}} = .88622 \ 69254 \dots = \left(\frac{1}{2}\right)!$$

$$\begin{aligned} 6.1.10 \quad \Gamma\left(n + \frac{1}{4}\right) &= \frac{1 \cdot 5 \cdot 9 \cdot 13 \dots (4n-3)}{4^n} \Gamma\left(\frac{1}{4}\right) \\ \Gamma\left(\frac{1}{4}\right) &= 3.62560 \ 99082 \dots \end{aligned}$$

$$\begin{aligned} 6.1.11 \quad \Gamma\left(n + \frac{1}{3}\right) &= \frac{1 \cdot 4 \cdot 7 \cdot 10 \dots (3n-2)}{3^n} \Gamma\left(\frac{1}{3}\right) \\ \Gamma\left(\frac{1}{3}\right) &= 2.67893 \ 85347 \dots \end{aligned}$$

$$6.1.12 \quad \Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots (2n-1)}{2^n} \Gamma\left(\frac{1}{2}\right)$$

$$\begin{aligned} 6.1.13 \quad \Gamma\left(n + \frac{2}{3}\right) &= \frac{2 \cdot 5 \cdot 8 \cdot 11 \dots (3n-1)}{3^n} \Gamma\left(\frac{2}{3}\right) \\ \Gamma\left(\frac{2}{3}\right) &= 1.35411 \ 79394 \dots \end{aligned}$$

$$\begin{aligned} 6.1.14 \quad \Gamma\left(n + \frac{3}{4}\right) &= \frac{3 \cdot 7 \cdot 11 \cdot 15 \dots (4n-1)}{4^n} \Gamma\left(\frac{3}{4}\right) \\ \Gamma\left(\frac{3}{4}\right) &= 1.22541 \ 67024 \dots \end{aligned}$$

*See page II.

Recurrence Formulas

$$6.1.15 \quad \Gamma(z+1) = z\Gamma(z) = z! = z(z-1)!$$

$$6.1.16$$

$$\begin{aligned} \Gamma(n+z) &= (n-1+z)(n-2+z) \dots (1+z)\Gamma(1+z) \\ &= (n-1+z)! \\ &= (n-1+z)(n-2+z) \dots (1+z)z! \end{aligned}$$

Reflection Formula

$$6.1.17 \quad \Gamma(z)\Gamma(1-z) = -z\Gamma(-z)\Gamma(z) = \pi \csc \pi z$$

$$= \int_0^{\infty} \frac{t^{z-1}}{1+t} dt \quad (0 < \Re z < 1)$$

Duplication Formula

$$6.1.18 \quad \Gamma(2z) = (2\pi)^{-\frac{1}{2}} 2^{2z-\frac{1}{2}} \Gamma(z) \Gamma(z+\frac{1}{2})$$

Triplcation Formula

$$6.1.19 \quad \Gamma(3z) = (2\pi)^{-1} 3^{3z-\frac{1}{2}} \Gamma(z) \Gamma(z+\frac{1}{3}) \Gamma(z+\frac{2}{3})$$

Gauss' Multiplication Formula

$$6.1.20 \quad \Gamma(nz) = (2\pi)^{\frac{1}{2}(1-n)} n^{nz-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right)$$

Binomial Coefficient

$$6.1.21 \quad \binom{z}{w} = \frac{z!}{w!(z-w)!} = \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}$$

Pochhammer's Symbol

$$6.1.22$$

$$(z)_0 = 1,$$

$$(z)_n = z(z+1)(z+2) \dots (z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)}$$

Gamma Function in the Complex Plane

$$6.1.23 \quad \Gamma(\bar{z}) = \overline{\Gamma(z)}; \ln \Gamma(\bar{z}) = \overline{\ln \Gamma(z)}$$

$$6.1.24 \quad \arg \Gamma(z+1) = \arg \Gamma(z) + \arctan \frac{y}{x}$$

$$6.1.25 \quad \left| \frac{\Gamma(x+iy)}{\Gamma(x)} \right|^2 = \prod_{n=0}^{\infty} \left[1 + \frac{y^2}{(x+n)^2} \right]^{-1}$$

$$6.1.26 \quad |\Gamma(x+iy)| \leq |\Gamma(x)|$$

$$6.1.27$$

$$\arg \Gamma(x+iy) = \mathcal{N}(x) + \sum_{n=0}^{\infty} \left(\frac{y}{x+n} - \arctan \frac{y}{x+n} \right)$$

$$(x+iy \neq 0, -1, -2, \dots)$$

where

$$\psi(z) = \Gamma'(z)/\Gamma(z)$$

$$6.1.28$$

$$\Gamma(1+iy) = iy \Gamma(iy)$$

$$6.1.29 \quad \Gamma(iy)\Gamma(-iy) = |\Gamma(iy)|^2 = \frac{\pi}{y \sinh \pi y}$$

$$6.1.30 \quad \Gamma(\tfrac{1}{2}+iy)\Gamma(\tfrac{1}{2}-iy) = |\Gamma(\tfrac{1}{2}+iy)|^2 = \frac{\pi}{\cosh \pi y}$$

$$6.1.31 \quad \Gamma(1+iy)\Gamma(1-iy) = |\Gamma(1+iy)|^2 = \frac{\pi y}{\sinh \pi y}$$

$$6.1.32 \quad \Gamma(\tfrac{1}{4}+iy)\Gamma(\tfrac{3}{4}-iy) = \frac{\pi\sqrt{2}}{\cosh \pi y + i \sinh \pi y}$$

Power Series

$$6.1.33$$

$$\begin{aligned} \ln \Gamma(1+z) &= -\ln(1+z) + z(1-\gamma) \\ &+ \sum_{n=2}^{\infty} (-1)^n [\zeta(n)-1] z^n/n \quad (|z| < 2) \end{aligned}$$

$\zeta(n)$ is the Riemann Zeta Function (see chapter 23).

Series Expansion² for $1/\Gamma(z)$

$$6.1.34 \quad \frac{1}{\Gamma(z)} = \sum_{k=1}^{\infty} c_k z^k \quad (|z| < \infty)$$

k	c_k
1	1.00000 00000 000000
2	0.57721 56649 015329
3	-0.65587 80715 202538
4	-0.04200 26350 340952
5	0.16653 86113 822915
6	-0.04219 77345 555443
7	-0.00962 19715 278770
8	0.00721 89432 466630
9	-0.00116 51675 918591
10	-0.00021 52416 741149
11	0.00012 80502 823882
12	-0.00002 01348 547807
13	-0.00000 12504 934821
14	0.00000 11330 272320
15	-0.00000 02056 338417
16	0.00000 00061 160950
17	0.00000 00050 020075
18	-0.00000 00011 812746
19	0.00000 00001 043427
20	0.00000 00000 077823
21	-0.00000 00000 036968
22	0.00000 00000 005100
23	-0.00000 00000 000206
24	-0.00000 00000 000054
25	0.00000 00000 000014
26	0.00000 00000 000001

² The coefficients c_k are from H. T. Davis, Tables of higher mathematical functions, 2 vols., Principia Press, Bloomington, Ind., 1933, 1935 (with permission); with corrections due to H. E. Salzer.

Polynomial Approximations³

6.1.35 $0 \leq x \leq 1$

$$\Gamma(x+1) = x! = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \epsilon(x)$$

$$|\epsilon(x)| \leq 5 \times 10^{-5}$$

$$a_1 = -.57486 \ 46 \quad a_4 = .42455 \ 49$$

$$a_2 = .95123 \ 63 \quad a_5 = -.10106 \ 78$$

$$a_3 = -.69985 \ 88$$

6.1.36 $0 \leq x \leq 1$

$$\Gamma(x+1) = x! = 1 + b_1x + b_2x^2 + \dots + b_8x^8 + \epsilon(x)$$

$$|\epsilon(x)| \leq 3 \times 10^{-7}$$

$$b_1 = -.57719 \ 1652 \quad b_5 = -.75670 \ 4078$$

$$b_2 = .98820 \ 5891 \quad b_6 = .48219 \ 9394$$

$$b_3 = -.89705 \ 6937 \quad b_7 = -.19352 \ 7818$$

$$b_4 = .91820 \ 6857 \quad b_8 = .03586 \ 8343$$

Stirling's Formula

6.1.37

$$\Gamma(z) \sim e^{-z} z^{z-\frac{1}{2}} (2\pi)^{\frac{1}{2}} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \dots \right] \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

$$+ \frac{163879}{209018880z^5} - \dots]$$

6.1.38

$$x! = \sqrt{2\pi} x^{x+\frac{1}{2}} \exp\left(-x + \frac{\theta}{12x}\right) \quad (x > 0, 0 < \theta < 1)$$

Asymptotic Formulas

6.1.39

$$\Gamma(az+b) \sim \sqrt{2\pi} e^{-az} (az)^{az+b-\frac{1}{2}} \quad (|\arg z| < \pi, a > 0)$$

6.1.40

$$\ln \Gamma(z) \sim (z - \frac{1}{2}) \ln z - z + \frac{1}{2} \ln(2\pi) + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)z^{2m-1}} \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

For B_n see chapter 23

6.1.41

$$\ln \Gamma(z) \sim (z - \frac{1}{2}) \ln z - z + \frac{1}{2} \ln(2\pi) + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \frac{1}{1680z^7} + \dots \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

³ From C. Hastings, Jr., Approximations for digital computers, Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

Error Term for Asymptotic Expansion

6.1.42

If

$$R_n(z) = \ln \Gamma(z) - (z - \frac{1}{2}) \ln z + z - \frac{1}{2} \ln(2\pi) - \sum_{m=1}^n \frac{B_{2m}}{2m(2m-1)z^{2m-1}}$$

then

$$|R_n(z)| \leq \frac{|B_{2n+2}| K(z)}{(2n+1)(2n+2)|z|^{2n+1}}$$

where

$$K(z) = \text{upper bound } |z^2/(u^2+z^2)|_{u \geq 0}$$

For z real and positive, R_n is less in absolute value than the first term neglected and has the same sign.

6.1.43

$$\mathcal{R} \ln \Gamma(iy) = \mathcal{R} \ln \Gamma(-iy)$$

$$= \frac{1}{2} \ln \left(\frac{\pi}{y \sinh \pi y} \right)$$

$$\sim \frac{1}{2} \ln(2\pi) - \frac{1}{2} \pi y - \frac{1}{2} \ln y, \quad (y \rightarrow +\infty)$$

6.1.44

$$\mathcal{I} \ln \Gamma(iy) = \arg \Gamma(iy) = -\arg \Gamma(-iy)$$

$$= -\mathcal{I} \ln \Gamma(-iy)$$

$$\sim y \ln y - y - \frac{1}{4} \pi - \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_{2n}}{(2n-1)(2n)y^{2n-1}} \quad (y \rightarrow +\infty)$$

$$6.1.45 \quad \lim_{|y| \rightarrow \infty} (2\pi)^{-\frac{1}{2}} |\Gamma(x+iy)| e^{\frac{1}{2}\pi|y|} |y|^{\frac{1}{2}-x} = 1$$

$$6.1.46 \quad \lim_{n \rightarrow \infty} n^{b-a} \frac{\Gamma(n+a)}{\Gamma(n+b)} = 1$$

6.1.47

$$z^{b-a} \frac{\Gamma(z+a)}{\Gamma(z+b)} \sim 1 + \frac{(a-b)(a+b-1)}{2z} + \frac{1}{12} \binom{a-b}{2} \left(3(a+b-1)^2 - a+b-1 \right) \frac{1}{z^2} + \dots$$

as $z \rightarrow \infty$ along any curve joining $z=0$ and $z=\infty$, providing $z \neq -a, -a-1, \dots; z \neq -b, -b-1, \dots$

Continued Fraction

6.1.48

$$\ln \Gamma(z) + z - (z - \frac{1}{2}) \ln z - \frac{1}{2} \ln(2\pi) \\ = \frac{a_0}{z} + \frac{a_1}{z+1} + \frac{a_2}{z+2} + \frac{a_3}{z+3} + \frac{a_4}{z+4} + \frac{a_5}{z+5} + \dots \quad (\Re z > 0)$$

$$a_0 = \frac{1}{12}, a_1 = \frac{1}{30}, a_2 = \frac{53}{210}, a_3 = \frac{195}{371},$$

$$a_4 = \frac{22999}{22737}, a_5 = \frac{29944523}{19733142}, a_6 = \frac{109535241009}{48264275462}$$

Wallis' Formula⁴

6.1.49

$$\frac{2}{\pi} \int_0^{\pi/2} (\sin x)^{2n} dx = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)} \\ = \frac{(2n)!}{2^{2n} (n!)^2} = \frac{1}{2^{2n}} \binom{2n}{n} = \frac{\Gamma(n+\frac{1}{2})}{\pi^{\frac{1}{2}} \Gamma(n+1)} \\ \sim \frac{1}{\pi^{\frac{1}{2}} n^{\frac{1}{2}}} \left[1 - \frac{1}{8n} + \frac{1}{128n^2} - \dots \right] \quad (n \rightarrow \infty)$$

Some Definite Integrals

6.1.50

$$\ln \Gamma(z) = \int_0^\infty \left[(z-1) e^{-t} - \frac{e^{-t} - e^{-zt}}{1 - e^{-t}} \right] \frac{dt}{t} \quad (\Re z > 0) \\ = (z - \frac{1}{2}) \ln z - z + \frac{1}{2} \ln 2\pi \\ + 2 \int_0^\infty \frac{\arctan(t/z)}{e^{2\pi t} - 1} dt \quad (\Re z > 0)$$

6.2. Beta Function

6.2.1

$$B(z, w) = \int_0^1 t^{z-1} (1-t)^{w-1} dt = \int_0^\infty \frac{t^{z-1}}{(1+t)^{z+w}} dt \\ = 2 \int_0^{\pi/2} (\sin t)^{2z-1} (\cos t)^{2w-1} dt \quad (\Re z > 0, \Re w > 0)$$

$$6.2.2 \quad B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = B(w, z)$$

6.3. Psi (Digamma) Function⁵

$$6.3.1 \quad \psi(z) = d[\ln \Gamma(z)]/dz = \Gamma'(z)/\Gamma(z)$$

⁴ Some authors employ the special double factorial notation as follows:

$$(2n)!! = 2 \cdot 4 \cdot 6 \dots (2n) = 2^n n!$$

$$(2n-1)!! = 1 \cdot 3 \cdot 5 \dots (2n-1) = \pi^{-\frac{1}{2}} 2^n \Gamma(n + \frac{1}{2})$$

⁵ Some authors write $\psi(z) = \frac{d}{dz} \ln \Gamma(z+1)$ and similarly for the polygamma functions.

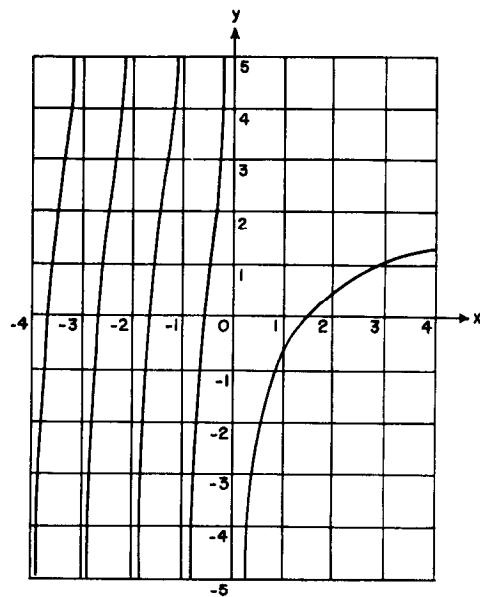


FIGURE 6.2. Psi function.

$$y = \psi(x) = d \ln \Gamma(x)/dx$$

Integer Values

$$6.3.2 \quad \psi(1) = -\gamma, \psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1} \quad (n \geq 2)$$

Fractional Values

6.3.3

$$\psi(\frac{1}{2}) = -\gamma - 2 \ln 2 = -1.96351 00260 21423 \dots$$

6.3.4

$$\psi(n + \frac{1}{2}) = -\gamma - 2 \ln 2 + 2 \left(1 + \frac{1}{3} + \dots + \frac{1}{2n-1} \right) \quad (n \geq 1)$$

Recurrence Formulas

$$6.3.5 \quad \psi(z+1) = \psi(z) + \frac{1}{z}$$

6.3.6

$$\psi(n+z) = \frac{1}{(n-1)+z} + \frac{1}{(n-2)+z} + \dots \\ + \frac{1}{2+z} + \frac{1}{1+z} + \psi(1+z)$$

Reflection Formula

$$6.3.7 \quad \psi(1-z) = \psi(z) + \pi \cot \pi z$$

Duplication Formula

$$6.3.8 \quad \psi(2z) = \frac{1}{2}\psi(z) + \frac{1}{2}\psi(z + \frac{1}{2}) + \ln 2$$

Psi Function in the Complex Plane

$$6.3.9 \quad \psi(\bar{z}) = \overline{\psi(z)}$$

$$6.3.10$$

$$\Re \psi(iy) = \Re \psi(-iy) = \Re \psi(1+iy) = \Re \psi(1-iy)$$

$$6.3.11 \quad \Im \psi(iy) = \frac{1}{2}y^{-1} + \frac{1}{2}\pi \coth \pi y$$

$$6.3.12 \quad \Im \psi(\frac{1}{2} + iy) = \frac{1}{2}\pi \tanh \pi y$$

$$6.3.13 \quad \Im \psi(1+iy) = -\frac{1}{2y} + \frac{1}{2}\pi \coth \pi y$$

$$= y \sum_{n=1}^{\infty} (n^2 + y^2)^{-1}$$

Series Expansions

$$6.3.14 \quad \psi(1+z) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^{n-1} \quad (|z| < 1)$$

$$6.3.15$$

$$\psi(1+z) = \frac{1}{2}z^{-1} - \frac{1}{2}\pi \cot \pi z - (1-z^2)^{-1} + 1 - \gamma$$

$$- \sum_{n=1}^{\infty} [\zeta(2n+1) - 1] z^{2n} \quad (|z| < 2)$$

$$6.3.16$$

$$\psi(1+z) = -\gamma + \sum_{n=1}^{\infty} \frac{z}{n(n+z)} \quad (z \neq -1, -2, -3, \dots)$$

$$6.3.17$$

$$\Re \psi(1+iy) = 1 - \gamma - \frac{1}{1+y^2}$$

$$+ \sum_{n=1}^{\infty} (-1)^{n+1} [\zeta(2n+1) - 1] y^{2n}$$

$$(|y| < 2)$$

$$= -\gamma + y^2 \sum_{n=1}^{\infty} n^{-1} (n^2 + y^2)^{-1}$$

$$(-\infty < y < \infty)$$

Asymptotic Formulas

$$6.3.18$$

$$\psi(z) \sim \ln z - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}}$$

$$= \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \dots$$

$$(z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

$$6.3.19$$

$$\Re \psi(1+iy) \sim \ln y + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_{2n}}{2ny^{2n}}$$

$$= \ln y + \frac{1}{12y^2} + \frac{1}{120y^4} + \frac{1}{252y^6} + \dots$$

$$(y \rightarrow \infty)$$

Extrema^a of $\Gamma(x)$ — Zeros of $\psi(x)$

$$\Gamma'(x_n) = \psi(x_n) = 0$$

n	x_n	$\Gamma(x_n)$
0	+1.462	+0.886
1	-0.504	-3.545
2	-1.573	+2.302
3	-2.611	-0.888
4	-3.635	+0.245
5	-4.653	-0.053
6	-5.667	+0.009
7	-6.678	-0.001

17 379
821

$$x_0 = 1.46163 \quad 21449 \quad 68362 \approx 1 + \frac{6}{13}$$

$$\Gamma(x_0) = .88560 \quad 31944 \quad 10889$$

$$6.3.20 \quad x_n = -n + (\ln n)^{-1} + o[(\ln n)^{-2}]$$

Definite Integrals

$$6.3.21$$

$$\psi(z) = \int_0^{\infty} \left[\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} \right] dt \quad (\Re z > 0)$$

$$= \int_0^{\infty} \left[e^{-t} - \frac{1}{(1+t)^z} \right] \frac{dt}{t}$$

$$= \ln z - \frac{1}{2z} - 2 \int_0^{\infty} \frac{t dt}{(t^2 + z^2)(e^{2\pi t} - 1)}$$

$$\left(|\arg z| < \frac{\pi}{2} \right)$$

$$6.3.22$$

$$\psi(z) + \gamma = \int_0^{\infty} \frac{e^{-t} - e^{-zt}}{1-e^{-t}} dt = \int_0^1 \frac{1-t^{z-1}}{1-t} dt$$

$$\gamma = \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{te^t} \right) dt$$

$$= \int_0^{\infty} \left(\frac{1}{1+t} - e^{-t} \right) \frac{dt}{t}$$

^a From W. Sibagaki, Theory and applications of the gamma function, Iwanami Syoten, Tokyo, Japan, 1952 (with permission).

6.4. Polygamma Functions⁷

6.4.1

$$\psi^{(n)}(z) = \frac{d^n}{dz^n} \psi(z) = \frac{d^{n+1}}{dz^{n+1}} \ln \Gamma(z) \quad (n=1, 2, 3, \dots)$$

$$* \quad = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-zt}}{1 - e^{-t}} dt \quad (\Re z > 0)$$

$\psi^{(n)}(z)$, ($n=0, 1, \dots$), is a single valued analytic function over the entire complex plane save at the points $z = -m$ ($m=0, 1, 2, \dots$) where it possesses poles of order $(n+1)$.

Integer Values

6.4.2*

$$\psi^{(n)}(1) = (-1)^{n+1} n! \zeta(n+1) \quad (n=1, 2, 3, \dots)$$

6.4.3

$$\psi^{(m)}(n+1) = (-1)^m m! \left[-\zeta(m+1) + 1 + \frac{1}{2^{m+1}} + \dots + \frac{1}{n^{m+1}} \right]$$

Fractional Values

6.4.4

$$\psi^{(n)}\left(\frac{1}{2}\right) = (-1)^{n+1} n! (2^{n+1} - 1) \zeta(n+1) \quad (n=1, 2, \dots)$$

$$6.4.5 \quad \psi'(n + \frac{1}{2}) = \frac{1}{2} \pi^2 - 4 \sum_{k=1}^n (2k-1)^{-2}$$

Recurrence Formula

$$6.4.6 \quad \psi^{(n)}(z+1) = \psi^{(n)}(z) + (-1)^n n! z^{-n-1}$$

Reflection Formula

6.4.7

$$\psi^{(n)}(1-z) + (-1)^{n+1} \psi^{(n)}(z) = (-1)^n \pi \frac{d^n}{dz^n} \cot \pi z$$

Multiplication Formula

6.4.8

$$* \quad \psi^{(n)}(mz) = \delta \ln m + \frac{1}{m^{n+1}} \sum_{k=0}^{m-1} \psi^{(n)}\left(z + \frac{k}{m}\right)$$

$$\delta = 1, \quad n=0$$

$$\delta = 0, \quad n > 0$$

⁷ ψ' is known as the trigamma function. ψ'' , $\psi^{(3)}$, $\psi^{(4)}$ are the tetra-, penta-, and hexagramma functions respectively. Some authors write $\psi(z) = d[\ln \Gamma(z+1)]/dz$, and similarly for the polygamma functions.

* See page II.

Series Expansions

6.4.9

$$\psi^{(n)}(1+z) = (-1)^{n+1} \left[n! \zeta(n+1) - \frac{(n+1)!}{1!} \zeta(n+2)z + \frac{(n+2)!}{2!} \zeta(n+3)z^2 - \dots \right] \quad (|z| < 1)$$

6.4.10

$$\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} (z+k)^{-n-1} \quad (z \neq 0, -1, -2, \dots)$$

Asymptotic Formulas

6.4.11

$$\psi^{(n)}(z) \sim (-1)^{n-1} \left[\frac{(n-1)!}{z^n} + \frac{n!}{2z^{n+1}} + \sum_{k=1}^{\infty} B_{2k} \frac{(2k+n-1)!}{(2k)! z^{2k+n}} \right] \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

6.4.12

$$\psi'(z) \sim \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} - \frac{1}{30z^5} + \frac{1}{42z^7} - \frac{1}{30z^9} + \dots \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

6.4.13

$$\psi''(z) \sim -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{2z^4} + \frac{1}{6z^6} - \frac{1}{6z^8} + \frac{3}{10z^{10}} - \frac{5}{6z^{12}} + \dots \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

6.4.14

$$\psi^{(3)}(z) \sim \frac{2}{z^3} + \frac{3}{z^4} + \frac{2}{z^5} - \frac{1}{z^7} + \frac{4}{3z^9} - \frac{3}{z^{11}} + \frac{10}{z^{13}} - \dots \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

6.5. Incomplete Gamma Function
(see also 26.4)

6.5.1

$$P(a, x) = \frac{1}{\Gamma(a)} \int_0^x e^{-t} t^{a-1} dt \quad (\Re a > 0)$$

6.5.2

$$\gamma(a, x) = P(a, x) \Gamma(a) = \int_0^x e^{-t} t^{a-1} dt \quad (\Re a > 0)$$

6.5.3

$$\Gamma(a, x) = \Gamma(a) - \gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt$$

6.5.4

$$\gamma^*(a, x) = x^{-a} P(a, x) = \frac{x^{-a}}{\Gamma(a)} \gamma(a, x)$$

γ^* is a single valued analytic function of a and x possessing no finite singularities.

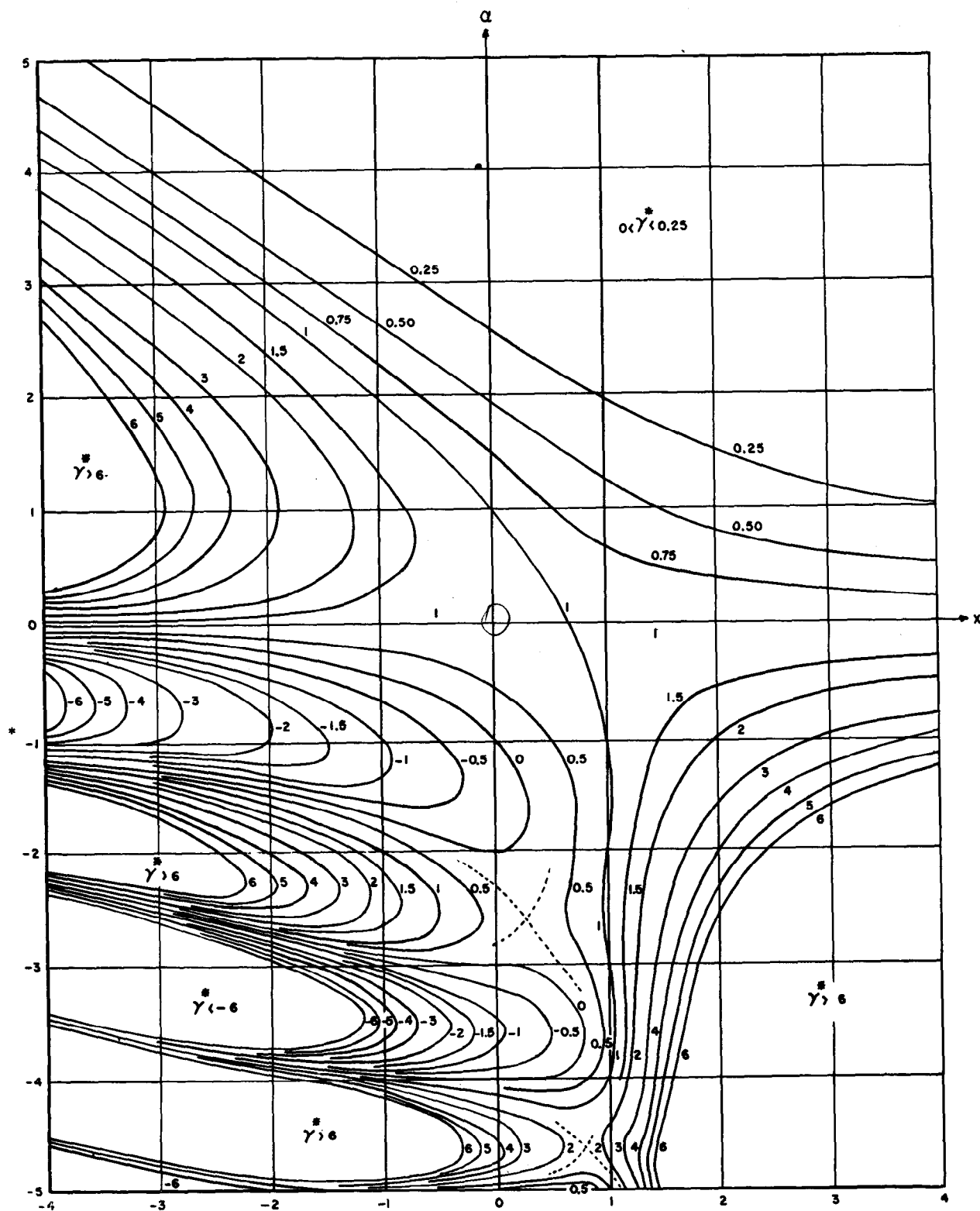


FIGURE 6.3. Incomplete gamma function.

$$\gamma^*(a, x) = \frac{x^{-a}}{\Gamma(a)} \int_0^x e^{-t} t^{a-1} dt$$

From F. G. Tricomi, Sulla funzione gamma incompleta, Annali di Matematica, IV, 33, 1950 (with permission).

*See page II.

6.5.5

Probability Integral of the χ^2 -Distribution

$$P(\chi^2|\nu) = \frac{1}{2^{\frac{1}{2}\nu}\Gamma\left(\frac{\nu}{2}\right)} \int_0^{\chi^2} t^{\frac{1}{2}\nu-1} e^{-\frac{t}{2}} dt$$

6.5.6

(Pearson's Form of the Incomplete Gamma Function)

$$I(u, p) = \frac{1}{\Gamma(p+1)} \int_0^{u\sqrt{p+1}} e^{-t} t^p dt \\ = P(p+1, u\sqrt{p+1})$$

$$6.5.7 \quad C(x, a) = \int_x^\infty t^{a-1} \cos t \, dt \quad (\Re a < 1)$$

$$6.5.8 \quad S(x, a) = \int_x^\infty t^{a-1} \sin t \, dt \quad (\Re a < 1)$$

6.5.9

$$E_n(x) = \int_1^\infty e^{-xt} t^{-n} dt = x^{n-1} \Gamma(1-n, x)$$

6.5.10

$$\alpha_n(x) = \int_1^\infty e^{-xt} t^n dt = x^{-n-1} \Gamma(1+n, x)$$

6.5.11

$$e_n(x) = \sum_{j=0}^n \frac{x^j}{j!}$$

Incomplete Gamma Function as a Confluent Hypergeometric Function (see chapter 13)

$$6.5.12 \quad \gamma(a, x) = a^{-1} x^a e^{-x} M(1, 1+a, x) \\ = a^{-1} x^a M(a, 1+a, -x)$$

Special Values

6.5.13

$$P(n, x) = 1 - \left(1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!}\right) e^{-x} \\ = 1 - e_{n-1}(x) e^{-x}$$

For relation to the Poisson distribution, see 26.4.

$$6.5.14 \quad \gamma^*(-n, x) = x^n$$

$$6.5.15 \quad \Gamma(0, x) = \int_x^\infty e^{-t} t^{-1} dt = E_1(x)$$

$$6.5.16 \quad \gamma\left(\frac{1}{2}, x^2\right) = 2 \int_0^x e^{-t^2} dt = \sqrt{\pi} \operatorname{erf} x$$

$$6.5.17 \quad \Gamma\left(\frac{1}{2}, x^2\right) = 2 \int_x^\infty e^{-t^2} dt = \sqrt{\pi} \operatorname{erfc} x$$

$$6.5.18 \quad \frac{1}{2}\sqrt{\pi} x \gamma^*\left(\frac{1}{2}, -x^2\right) = \int_0^x e^{t^2} dt$$

$$6.5.19 \quad \Gamma(-n, x) = \frac{(-1)^n}{n!} \left[E_1(x) - e^{-x} \sum_{j=0}^{n-1} \frac{(-1)^j j!}{x^{j+1}} \right]$$

$$6.5.20 \quad \Gamma(a, ix) = e^{\frac{1}{2}\pi i a} [C(x, a) - iS(x, a)]$$

Recurrence Formulas

$$6.5.21 \quad P(a+1, x) = P(a, x) - \frac{x^a e^{-x}}{\Gamma(a+1)}$$

$$6.5.22 \quad \gamma(a+1, x) = a\gamma(a, x) - x^a e^{-x}$$

$$6.5.23 \quad \gamma^*(a-1, x) = x\gamma^*(a, x) + \frac{e^{-x}}{\Gamma(a)}$$

Derivatives and Differential Equations

6.5.24

$$\left(\frac{\partial \gamma^*}{\partial \alpha}\right)_{\alpha=0} = - \int_x^\infty \frac{e^{-t} dt}{t} - \ln x = -E_1(x) - \ln x$$

$$6.5.25 \quad \frac{\partial \gamma(a, x)}{\partial x} = - \frac{\partial \Gamma(a, x)}{\partial x} = x^{a-1} e^{-x}$$

6.5.26

$$\frac{\partial^n}{\partial x^n} [x^{-a} \Gamma(a, x)] = (-1)^n x^{-a-n} \Gamma(a+n, x) \\ (n=0, 1, 2, \dots)$$

6.5.27

$$\frac{\partial^n}{\partial x^n} [e^x x^a \gamma^*(a, x)] = e^x x^{a-n} \gamma^*(a-n, x) \\ (n=0, 1, 2, \dots)$$

$$6.5.28 \quad x \frac{\partial^2 \gamma^*}{\partial x^2} + (a+1+x) \frac{\partial \gamma^*}{\partial x} + a\gamma^* = 0$$

Series Developments

6.5.29

$$\gamma^*(a, z) = e^{-z} \sum_{n=0}^\infty \frac{z^n}{\Gamma(a+n+1)} = \frac{1}{\Gamma(a)} \sum_{n=0}^\infty \frac{(-z)^n}{(a+n)n!} \\ (|z| < \infty)$$

6.5.30

$$\gamma(a, x+y) - \gamma(a, x) = e^{-x} x^{a-1} \sum_{n=0}^{\infty} \frac{(a-1)(a-2) \cdots (a-n)}{x^n} [1 - e^{-y} e_n(y)]$$

$$(|y| < |x|)$$

Continued Fraction

6.5.31

$$\Gamma(a, x) = e^{-x} x^a \left(\frac{1}{x+} \frac{1-a}{1+} \frac{1}{x+} \frac{2-a}{1+} \frac{2}{x+} \cdots \right)$$

$$(x > 0, |a| < \infty)$$

Asymptotic Expansions

6.5.32

$$\Gamma(a, z) \sim z^{a-1} e^{-z} \left[1 + \frac{a-1}{z} + \frac{(a-1)(a-2)}{z^2} + \cdots \right]$$

$$(z \rightarrow \infty \text{ in } |\arg z| < \frac{3\pi}{2})$$

Suppose $R_n(a, z) = u_{n+1}(a, z) + \cdots$ is the remainder after n terms in this series. Then if a, z are real, we have for $n > a - 2$

$$|R_n(a, z)| \leq |u_{n+1}(a, z)|$$

and $\text{sign } R_n(a, z) = \text{sign } u_{n+1}(a, z)$.

$$6.5.33 \quad \gamma(a, z) \sim \sum_{n=0}^{\infty} \frac{(-1)^n z^{a+n}}{(a+n)n!} \quad (a \rightarrow +\infty)$$

$$6.5.34 \quad \lim_{n \rightarrow \infty} \frac{e_n(\alpha n)}{e^{\alpha n}} = \begin{cases} 0 & \text{for } \alpha > 1 \\ \frac{1}{2} & \text{for } \alpha = 1 \\ 1 & \text{for } 0 \leq \alpha < 1 \end{cases}$$

6.5.35

$$\Gamma(z+1, z) \sim e^{-z} z^z \left(\sqrt{\frac{\pi}{2}} z^{\frac{1}{2}} + \frac{2}{3} + \frac{\sqrt{2\pi}}{24} \frac{1}{z^{\frac{1}{2}}} + \cdots \right)$$

$$(z \rightarrow \infty \text{ in } |\arg z| < \frac{1}{2}\pi)$$

Numerical Methods

6.7. Use and Extension of the Tables

Example 1. Compute $\Gamma(6.38)$ to 8S. Using the recurrence relation 6.1.16 and Table 6.1 we have,

$$\Gamma(6.38) = [(5.38)(4.38)(3.38)(2.38)(1.38)]\Gamma(1.38)$$

$$= 232.43671.$$

Example 2. Compute $\ln \Gamma(56.38)$, using Table 6.4 and linear interpolation in f_2 . We have

$$\ln \Gamma(56.38) = (56.38 - \frac{1}{2}) \ln (56.38) - (56.38)$$

$$+ f_2(56.38)$$

Definite Integrals

6.5.36

$$\int_0^{\infty} e^{-at} \Gamma(b, ct) dt = \frac{\Gamma(b)}{a} \left[1 - \frac{c^b}{(a+c)^b} \right]$$

$$(\Re(a+c) > 0, \Re b > -1)$$

6.5.37

$$\int_0^{\infty} t^{a-1} \Gamma(b, t) dt = \frac{\Gamma(a+b)}{a}$$

$$(\Re(a+b) > 0, \Re a > 0)$$

6.6. Incomplete Beta Function

$$6.6.1 \quad B_x(a, b) = \int_0^x t^{a-1} (1-t)^{b-1} dt$$

$$6.6.2 \quad I_x(a, b) = B_x(a, b) / B(a, b)$$

For statistical applications, see 26.5.

Symmetry

$$6.6.3 \quad I_x(a, b) = 1 - I_{1-x}(b, a)$$

Relation to Binomial Expansion

$$6.6.4 \quad I_p(a, n-a+1) = \sum_{j=a}^n \binom{n}{j} p^j (1-p)^{n-j}$$

For binomial distribution, see 26.1.

Recurrence Formulas

$$6.6.5 \quad I_x(a, b) = x I_x(a-1, b) + (1-x) I_x(a, b-1)$$

$$6.6.6 \quad (a+b-ax) I_x(a, b) = a(1-x) I_x(a+1, b-1) + b I_x(a, b+1)$$

$$6.6.7 \quad (a+b) I_x(a, b) = a I_x(a+1, b) + b I_x(a, b+1)$$

Relation to Hypergeometric Function

$$6.6.8 \quad B_x(a, b) = a^{-1} x^a F(a, 1-b; a+1; x)$$

The error of linear interpolation in the table of the function f_2 is smaller than 10^{-7} in this region. Hence, $f_2(56.38) = .92041 \ 67$ and $\ln \Gamma(56.38) = 169.85497 \ 42$.

Direct interpolation in Table 6.4 of $\log_{10} \Gamma(n)$ eliminates the necessity of employing logarithms. However, the error of linear interpolation is .002 so that $\log_{10} \Gamma(n)$ is obtained with a relative error of 10^{-5} .

*See page 11.

Example 3. Compute $\psi(6.38)$ to 8S. Using the recurrence relation 6.3.6 and Table 6.1.

$$\begin{aligned}\psi(6.38) &= \frac{1}{5.38} + \frac{1}{4.38} + \frac{1}{3.38} + \frac{1}{2.38} + \frac{1}{1.38} + \psi(1.38) \\ &= 1.77275\ 59.\end{aligned}$$

Example 4. Compute $\psi(56.38)$. Using Table 6.3 we have $\psi(56.38) = \ln 56.38 - f_3(56.38)$.

The error of linear interpolation in the table of the function f_3 is smaller than 8×10^{-7} in this region. Hence, $f_3(56.38) = .00889\ 53$ and $\psi(56.38) = 4.023219$.

Example 5. Compute $\ln \Gamma(1-i)$. From the reflection principle 6.1.23 and Table 6.7, $\ln \Gamma(1-i) = \ln \Gamma(1+i) = -.6509 + .3016i$.

Example 6. Compute $\ln \Gamma(\frac{1}{2} + \frac{1}{2}i)$. Taking the logarithm of the recurrence relation 6.1.15 we have,

$$\begin{aligned}\ln \Gamma(\tfrac{1}{2} + \tfrac{1}{2}i) &= \ln \Gamma(\tfrac{3}{2} + \tfrac{1}{2}i) - \ln(\tfrac{1}{2} + \tfrac{1}{2}i) \\ &= -.23419 + .03467i \\ &\quad - (\tfrac{1}{2} \ln \tfrac{1}{2} + i \arctan 1) \\ &= .11239 - .75073i\end{aligned}$$

The logarithms of complex numbers are found from 4.1.2.

Example 7. Compute $\ln \Gamma(3+7i)$ using the duplication formula 6.1.18. Taking the logarithm of 6.1.18, we have

$$\begin{aligned}-\tfrac{1}{2} \ln 2\pi &= -.91894 \\ (\tfrac{3}{2} + 7i) \ln 2 &= 1.73287 + 4.85203i \\ \ln \Gamma(\tfrac{3}{2} + \tfrac{7}{2}i) &= -3.31598 + 2.32553i \\ \ln \Gamma(2 + \tfrac{7}{2}i) &= -2.66047 + 2.93869i \\ \ln \Gamma(3+7i) &= -5.16252 + 10.11625i\end{aligned}$$

Example 8. Compute $\ln \Gamma(3+7i)$ to 5D using the asymptotic formula 6.1.41. We have

$$\ln(3+7i) = 2.03022\ 15 + 1.16590\ 45i.$$

Then,

$$\begin{aligned}(2.5+7i) \ln(3+7i) &= -3.0857779 + 17.1263119i \\ -(3+7i) &= -3.0000000 - 7.0000000i \\ \tfrac{1}{2} \ln(2\pi) &= .9189385 \\ [12(3+7i)]^{-1} &= .0043103 - .0100575i \\ -[360(3+7i)^3]^{-1} &= .0000059 - .0000022i\end{aligned}$$

$$\ln \Gamma(3+7i) = -5.16252 + 10.11625i$$

6.8. Summation of Rational Series by Means of Polygamma Functions

An infinite series whose general term is a rational function of the index may always be reduced to a finite series of psi and polygamma functions. The method will be illustrated by writing the explicit formula when the denominator contains a triple root.

Let the general term of an infinite series have the form

$$u_n = \frac{p(n)}{d_1(n)d_2(n)d_3(n)}$$

where

$$d_1(n) = (n + \alpha_1)(n + \alpha_2) \dots (n + \alpha_m)$$

$$d_2(n) = (n + \beta_1)^2(n + \beta_2)^2 \dots (n + \beta_r)^2$$

$$d_3(n) = (n + \gamma_1)^3(n + \gamma_2)^3 \dots (n + \gamma_s)^3$$

where $p(n)$ is a polynomial of degree $m + 2r + 3s - 2$ at most and where the constants α_i , β_i , and γ_i are distinct. Expand u_n in partial fractions as follows

$$\begin{aligned}u_n &= \sum_{k=1}^m \frac{a_k}{(n + \alpha_k)} + \sum_{k=1}^r \frac{b_{1k}}{(n + \beta_k)} + \frac{b_{2k}}{(n + \beta_k)^2} \\ &\quad + \sum_{k=1}^s \frac{c_{1k}}{(n + \gamma_k)} + \frac{c_{2k}}{(n + \gamma_k)^2} + \frac{c_{3k}}{(n + \gamma_k)^3} \\ &\quad \sum_{k=1}^m a_k + \sum_{k=1}^r b_{1k} + \sum_{k=1}^s c_{1k} = 0.\end{aligned}$$

Then, we may express $\sum_{n=1}^{\infty} u_n$ in terms of the constants appearing in this partial fraction expansion as follows

$$\begin{aligned}\sum_{n=1}^{\infty} u_n &= -\sum_{j=1}^m a_j \psi(1 + \alpha_j) \\ &\quad - \sum_{j=1}^r b_{1j} \psi(1 + \beta_j) + \sum_{j=1}^r b_{2j} \psi'(1 + \beta_j) \\ &\quad - \sum_{j=1}^s c_{1j} \psi(1 + \gamma_j) + \sum_{j=1}^s c_{2j} \psi'(1 + \gamma_j) \\ &\quad - \sum_{j=1}^s \frac{c_{3j}}{2!} \psi''(1 + \gamma_j).\end{aligned}$$

Higher order repetitions in the denominator are handled similarly. If the denominator contains

only simple or double roots, omit the corresponding lines.

Example 9. Find

$$s = \sum_{n=1}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)}.$$

Since

$$\frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\frac{1}{3}}{n+1} - \frac{1}{n+\frac{1}{2}} + \frac{\frac{2}{3}}{n+\frac{1}{4}},$$

we have

$$\alpha_1=1, \alpha_2=\frac{1}{2}, \alpha_3=\frac{1}{4}, a_1=\frac{1}{3}, a_2=-1, a_3=\frac{2}{3}.$$

Thus,

$$s = -\frac{1}{3}\psi(2) + \psi(1\frac{1}{2}) - \frac{2}{3}\psi(1\frac{1}{4}) = .047198.$$

Example 10.

$$\text{Find } s = \sum_{n=1}^{\infty} \frac{1}{n^2(8n+1)^2}.$$

$$\text{Since } \frac{1}{n^2(8n+1)^2} = -\frac{16}{n} + \frac{16}{n+\frac{1}{8}} + \frac{1}{n^2} + \frac{1}{(n+\frac{1}{8})^2},$$

we have,

$$\beta_1=0, \beta_2=\frac{1}{8}, b_{11}=-16, b_{12}=16, b_{21}=1, b_{22}=1.$$

Therefore

$$s = 16\psi(1) - 16\psi(1\frac{1}{8}) + \psi'(1) + \psi'(1\frac{1}{8}) = .013499.$$

Example 11.

$$\text{Evaluate } s = \sum_{n=1}^{\infty} \frac{1}{(n^2+1)(n^2+4)} \quad (\text{see also 6.3.13}).$$

$$\text{We have, } \frac{1}{(n^2+1)(n^2+4)} = \frac{i}{6} \left(\frac{1}{n+i} - \frac{1}{n-i} \right) - \frac{i}{12} \left(\frac{1}{n+2i} - \frac{1}{n-2i} \right).$$

$$\text{Hence, } a_1 = \frac{i}{6}, a_2 = -\frac{i}{6}, a_3 = -\frac{i}{12}, a_4 = \frac{i}{12},$$

$$\alpha_1=i, \alpha_2=-i, \alpha_3=2i, \alpha_4=-2i,$$

and therefore

$$s = \frac{-i}{6} [\psi(1+i) - \psi(1-i)] + \frac{i}{12} [\psi(1+2i) - \psi(1-2i)].$$

By 6.3.9, this reduces to

$$s = \frac{1}{3} \mathcal{J} \psi(1+i) - \frac{1}{6} \mathcal{J} \psi(1+2i).$$

From Table 6.8, $s = .13876$.

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For references to tabular material on the incomplete gamma and incomplete beta functions, see the references in chapter 26.

Voir aussi M.O.C.

7. Error Function and Fresnel Integrals

WALTER GAUTSCHI¹

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7. Error Function and Fresnel Integrals

Mathematical Properties

7.1. Error Function

Definitions

$$7.1.1 \quad \operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

$$7.1.2 \quad \operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt = 1 - \operatorname{erf} z$$

$$7.1.3 \quad w(z) = e^{-z^2} \left(1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{t^2} dt \right) = e^{-z^2} \operatorname{erfc}(-iz)$$

In 7.1.2 the path of integration is subject to the restriction $\arg t \rightarrow \alpha$ with $|\alpha| < \frac{\pi}{4}$ as $t \rightarrow \infty$ along the path. ($\alpha = \frac{\pi}{4}$ is permissible if $\Re t^2$ remains bounded to the left.)

Integral Representation

$$7.1.4 \quad w(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{e^{-t^2} dt}{z-t} = \frac{2iz}{\pi} \int_0^{\infty} \frac{e^{-t^2} dt}{z^2 - t^2} \quad (\Im z > 0)$$

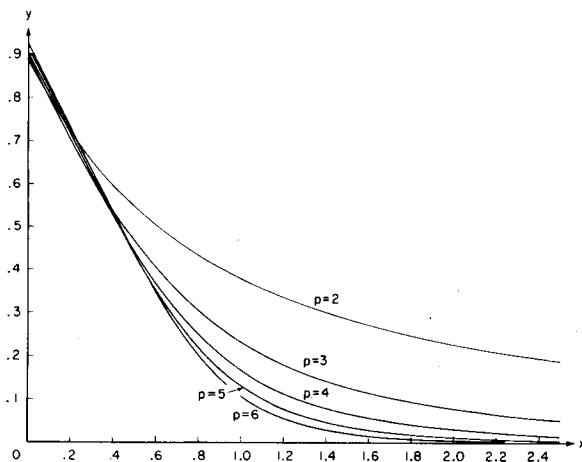


FIGURE 7.1. $y = e^{x^2} \int_x^\infty e^{-t^2} dt$.
 $p=2(1)6$

Series Expansions

$$7.1.5 \quad \operatorname{erf} z = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}$$

$$7.1.6 \quad = \frac{2}{\sqrt{\pi}} e^{-z^2} \sum_{n=0}^{\infty} \frac{2^n}{1 \cdot 3 \cdots (2n+1)} z^{2n+1}$$

$$7.1.7 \quad = \sqrt{2} \sum_{n=0}^{\infty} (-1)^n [I_{2n+1/2}(z^2) - I_{2n+3/2}(z^2)]$$

$$7.1.8 \quad w(z) = \sum_{n=0}^{\infty} \frac{(iz)^n}{\Gamma\left(\frac{n}{2} + 1\right)}$$

For $I_{n-1/2}(x)$, see chapter 10.

Symmetry Relations

$$7.1.9 \quad \operatorname{erf}(-z) = -\operatorname{erf} z$$

$$7.1.10 \quad \operatorname{erf} \bar{z} = \overline{\operatorname{erf} z}$$

$$7.1.11 \quad w(-z) = 2e^{-z^2} - w(z)$$

$$7.1.12 \quad w(\bar{z}) = \overline{w(-z)}$$

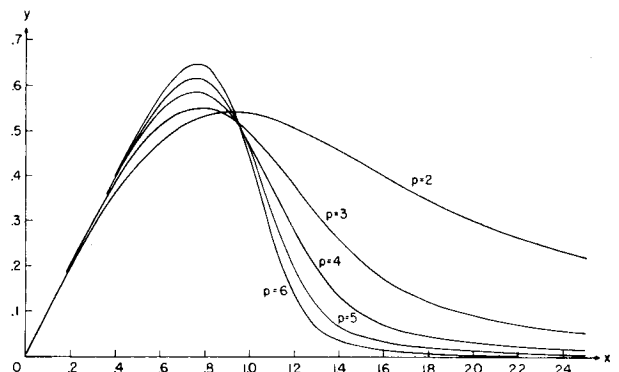
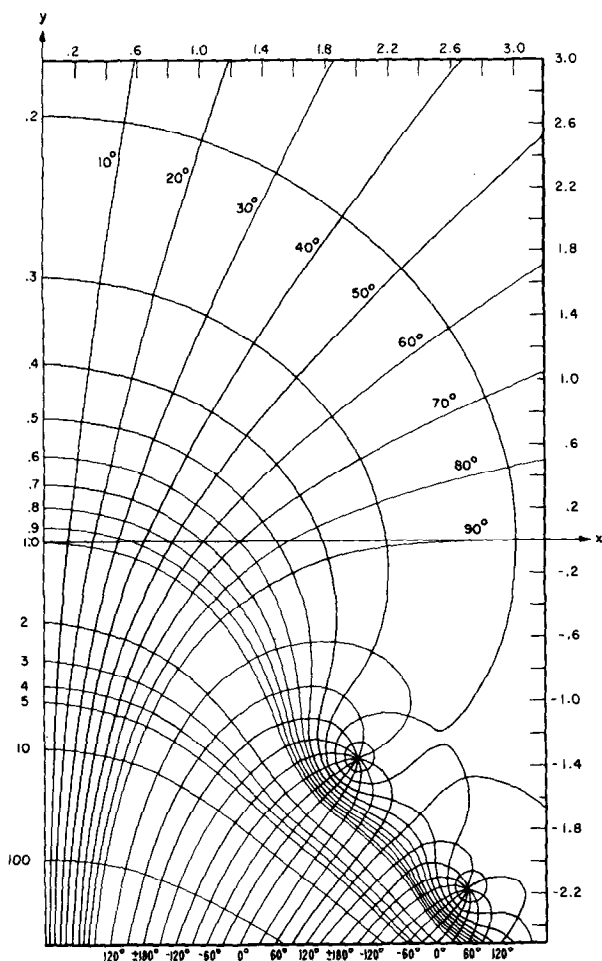


FIGURE 7.2. $y = e^{-x^2} \int_0^x e^{t^2} dt$.
 $p=2(1)6$

FIGURE 7.3. Altitude Chart of $w(z)$.**Inequalities [7.11], [7.17]**

7.1.13

$$\frac{1}{x + \sqrt{x^2 + 2}} < e^{x^2} \int_x^\infty e^{-t^2} dt \leq \frac{1}{x + \sqrt{x^2 + \frac{4}{\pi}}} \quad (x \geq 0)$$

(For other inequalities see [7.2].)

Continued Fractions

7.1.14

$$2e^{z^2} \int_z^\infty e^{-t^2} dt = \frac{1}{z + \frac{1/2}{z + \frac{1}{z + \frac{3/2}{z + \frac{2}{z + \dots}}}}} \quad (\Re z > 0)$$

7.1.15

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty \frac{e^{-t^2} dt}{z - t} = \frac{1}{z - \frac{1/2}{z - \frac{1}{z - \frac{3/2}{z - \frac{2}{z - \dots}}}}} = \frac{1}{\sqrt{\pi}} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{H_k^{(n)}}{z - x_k^{(n)}} \quad (\Im z \neq 0)$$

$x_k^{(n)}$ and $H_k^{(n)}$ are the zeros and weight factors of the Hermite polynomials. For numerical values see chapter 25.

Value at Infinity

$$7.1.16 \quad \operatorname{erf} z \rightarrow 1 \quad \left(z \rightarrow \infty \text{ in } |\arg z| < \frac{\pi}{4} \right)$$

Maximum and Inflection Points for Dawson's Integral [7.31]

$$F(x) = e^{-x^2} \int_0^x e^{t^2} dt$$

$$7.1.17 \quad F(.92413 \ 88730 \dots) = .54104 \ 42246 \dots$$

$$7.1.18 \quad F(1.50197 \ 52682 \dots) = .42768 \ 66160 \dots$$

Derivatives

7.1.19

$$\frac{d^{n+1}}{dz^{n+1}} \operatorname{erf} z = (-1)^n \frac{2}{\sqrt{\pi}} H_n(z) e^{-z^2} \quad (n=0, 1, 2, \dots)$$

7.1.20

$$w^{(n+2)}(z) + 2zw^{(n+1)}(z) + 2(n+1)w^{(n)}(z) = 0 \quad (n=0, 1, 2, \dots)$$

$$w^{(0)}(z) = w(z), \quad w'(z) = -2zw(z) + \frac{2i}{\sqrt{\pi}}$$

(For the Hermite polynomials $H_n(z)$ see chapter 22.)

Relation to Confluent Hypergeometric Function (see chapter 13)

7.1.21

$$\operatorname{erf} z = \frac{2z}{\sqrt{\pi}} M\left(\frac{1}{2}, \frac{3}{2}, -z^2\right) = \frac{2z}{\sqrt{\pi}} e^{-z^2} M\left(1, \frac{3}{2}, z^2\right)$$

The Normal Distribution Function With Mean m and Standard Deviation σ (see chapter 26)

$$7.1.22 \quad \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{(t-m)^2}{2\sigma^2}} dt = \frac{1}{2} \left(1 + \operatorname{erf} \left(\frac{x-m}{\sigma\sqrt{2}} \right) \right)$$

Asymptotic Expansion

7.1.23

$$\sqrt{\pi} z e^{z^2} \operatorname{erfc} z \sim 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1 \cdot 3 \dots (2m-1)}{(2z^2)^m} \quad \left(z \rightarrow \infty, |\arg z| < \frac{3\pi}{4} \right)$$

If $R_n(z)$ is the remainder after n terms then

7.1.24

$$R_n(z) = (-1)^n \frac{1 \cdot 3 \cdots (2n-1)}{(2z^2)^n} \theta,$$

$$\theta = \int_0^\infty e^{-t} \left(1 + \frac{t}{z^2}\right)^{-n-\frac{1}{2}} dt \quad \left(|\arg z| < \frac{\pi}{2}\right)$$

$$|\theta| < 1 \quad \left(|\arg z| < \frac{\pi}{4}\right)$$

For x real, $R_n(x)$ is less in absolute value than the first neglected term and of the same sign.

Rational Approximations² ($0 \leq x < \infty$)

7.1.25

$$\operatorname{erf} x = 1 - (a_1 t + a_2 t^2 + a_3 t^3) e^{-x^2} + \epsilon(x), \quad t = \frac{1}{1+px}$$

$$|\epsilon(x)| \leq 2.5 \times 10^{-5}$$

$$p = .47047 \quad a_1 = .34802 \ 42 \quad a_2 = -.09587 \ 98$$

$$a_3 = .74785 \ 56$$

7.1.26

$$\operatorname{erf} x = 1 - (a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5) e^{-x^2} + \epsilon(x),$$

$$t = \frac{1}{1+px}$$

$$|\epsilon(x)| \leq 1.5 \times 10^{-7}$$

$$p = .32759 \ 11 \quad a_1 = .25482 \ 9592$$

$$a_2 = -.28449 \ 6736 \quad a_3 = 1.42141 \ 3741$$

$$a_4 = -1.45315 \ 2027 \quad a_5 = 1.06140 \ 5429$$

7.1.27

$$\operatorname{erf} x = 1 - \frac{1}{[1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4]^4} + \epsilon(x)$$

$$|\epsilon(x)| \leq 5 \times 10^{-4}$$

$$a_1 = .278393 \quad a_2 = .230389$$

$$a_3 = .000972 \quad a_4 = .078108$$

7.1.28

$$\operatorname{erf} x = 1 - \frac{1}{[1 + a_1 x + a_2 x^2 + \cdots + a_6 x^6]^6} + \epsilon(x)$$

$$|\epsilon(x)| \leq 3 \times 10^{-7}$$

$$a_1 = .07052 \ 30784 \quad a_2 = .04228 \ 20123$$

$$a_3 = .00927 \ 05272 \quad a_4 = .00015 \ 20143$$

$$a_5 = .00027 \ 65672 \quad a_6 = .00004 \ 30638$$

² Approximations 7.1.25-7.1.28 are from C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N. J., 1955 (with permission).

Infinite Series Approximation for Complex Error Function [7.19]

7.1.29

$$\operatorname{erf}(x+iy) = \operatorname{erf} x + \frac{e^{-x^2}}{2\pi x} [(1 - \cos 2xy) + i \sin 2xy]$$

$$+ \frac{2}{\pi} e^{-x^2} \sum_{n=1}^{\infty} \frac{e^{-4n^2}}{n^2 + 4x^2} [f_n(x, y) + i g_n(x, y)] + \epsilon(x, y)$$

where

$$f_n(x, y) = 2x - 2x \cosh ny \cos 2xy + n \sinh ny \sin 2xy$$

$$g_n(x, y) = 2x \cosh ny \sin 2xy + n \sinh ny \cos 2xy$$

$$|\epsilon(x, y)| \approx 10^{-16} |\operatorname{erf}(x+iy)|$$

7.2. Repeated Integrals of the Error Function

Definition

7.2.1

$$i^n \operatorname{erfc} z = \int_z^\infty i^{n-1} \operatorname{erfc} t \, dt \quad (n=0, 1, 2, \dots)$$

$$i^{-1} \operatorname{erfc} z = \frac{2}{\sqrt{\pi}} e^{-z^2}, \quad i^0 \operatorname{erfc} z = \operatorname{erfc} z$$

Differential Equation

$$7.2.2 \quad \frac{d^2 y}{dz^2} + 2z \frac{dy}{dz} - 2ny = 0$$

$$y = A i^n \operatorname{erfc} z + B i^n \operatorname{erfc}(-z)$$

(A and B are constants.)

Expression as a Single Integral

$$7.2.3 \quad i^n \operatorname{erfc} z = \frac{2}{\sqrt{\pi}} \int_z^\infty \frac{(t-z)^n}{n!} e^{-t^2} dt$$

Power Series³

$$7.2.4 \quad i^n \operatorname{erfc} z = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{2^{n-k} k! \Gamma\left(1 + \frac{n-k}{2}\right)}$$

Recurrence Relations

7.2.5

$$i^n \operatorname{erfc} z = -\frac{z}{n} i^{n-1} \operatorname{erfc} z + \frac{1}{2n} i^{n-2} \operatorname{erfc} z$$

$$(n=1, 2, 3, \dots)$$

7.2.6

$$2(n+1)(n+2) i^{n+2} \operatorname{erfc} z$$

$$= (2n+1+2z^2) i^n \operatorname{erfc} z - \frac{1}{2} i^{n-2} \operatorname{erfc} z$$

$$(n=1, 2, 3, \dots)$$

³ The terms in this series corresponding to $k=n+2, n+4, n+6, \dots$ are understood to be zero.

Value at Zero

7.2.7

$$i^n \operatorname{erfc} 0 = \frac{1}{2^n \Gamma\left(\frac{n}{2} + 1\right)} \quad (n = -1, 0, 1, 2, \dots)$$

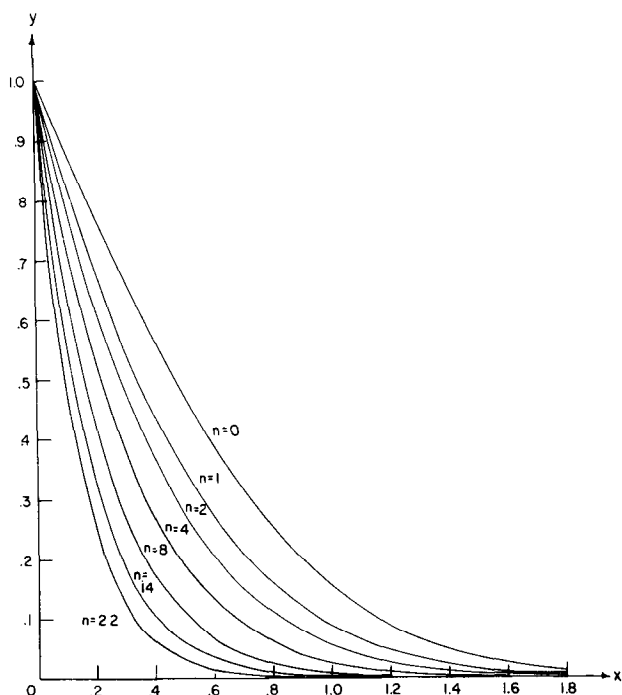


FIGURE 7.4. Repeated Integrals of the Error Function.

$$y = 2^n \Gamma\left(\frac{n}{2} + 1\right) i^n \operatorname{erfc} z$$

$n = 0, 1, 2, 4, 8, 14, 22$

Derivatives

$$7.2.8 \quad \frac{d}{dz} i^n \operatorname{erfc} z = -i^{n-1} \operatorname{erfc} z \quad (n = 0, 1, 2, \dots)$$

7.2.9

$$\frac{d^n}{dz^n} (e^{z^2} \operatorname{erfc} z) = (-1)^n 2^n n! e^{z^2} i^n \operatorname{erfc} z$$

$(n = 0, 1, 2, \dots)$

Relation to $Hh_n(z)$ (see 19.14)

$$7.2.10 \quad i^n \operatorname{erfc} z = \frac{1}{(2^{n-1} \pi)^{1/2}} Hh_n(\sqrt{2} z)$$

Relation to Hermite Polynomials (see chapter 22)

$$7.2.11 \quad (-1)^n i^n \operatorname{erfc} z + i^n \operatorname{erfc} (-z) = \frac{i^{-n}}{2^{n-1} n!} H_n(iz)$$

Relation to the Confluent Hypergeometric Function
(see chapter 13)

7.2.12

$$i^n \operatorname{erfc} z = e^{-z^2} \left[\frac{1}{2^n \Gamma\left(\frac{n}{2} + 1\right)} M\left(\frac{n+1}{2}, \frac{1}{2}, z^2\right) - \frac{z}{2^{n-1} \Gamma\left(\frac{n+1}{2}\right)} M\left(\frac{n}{2} + 1, \frac{3}{2}, z^2\right) \right]$$

Relation to Parabolic Cylinder Functions (see chapter 19)

$$7.2.13 \quad i^n \operatorname{erfc} z = \frac{e^{-1/2 z^2}}{(2^{n-1} \pi)^{1/2}} D_{-n-1}(z\sqrt{2})$$

Asymptotic Expansion

7.2.14

$$i^n \operatorname{erfc} z \sim \frac{2}{\sqrt{\pi}} \frac{e^{-z^2}}{(2z)^{n+1}} \sum_{m=0}^{\infty} \frac{(-1)^m (2m+n)!}{n! m! (2z)^{2m}}$$

$(z \rightarrow \infty, |\arg z| < \frac{3\pi}{4})$

7.3. Fresnel Integrals

Definition

$$7.3.1 \quad C(z) = \int_0^z \cos\left(\frac{\pi}{2} t^2\right) dt$$

$$7.3.2 \quad S(z) = \int_0^z \sin\left(\frac{\pi}{2} t^2\right) dt$$

The following functions are also in use:

7.3.3

$$C_1(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos t^2 dt, \quad C_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}} dt$$

7.3.4

$$S_1(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin t^2 dt, \quad S_2(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}} dt$$

Auxiliary Functions

7.3.5

$$f(z) = \left[\frac{1}{2} - S(z) \right] \cos\left(\frac{\pi}{2} z^2\right) - \left[\frac{1}{2} - C(z) \right] \sin\left(\frac{\pi}{2} z^2\right)$$

7.3.6

$$g(z) = \left[\frac{1}{2} - C(z) \right] \cos\left(\frac{\pi}{2} z^2\right) + \left[\frac{1}{2} - S(z) \right] \sin\left(\frac{\pi}{2} z^2\right)$$

Interrelations

$$7.3.7 \quad C(x) = C_1\left(x\sqrt{\frac{\pi}{2}}\right) = C_2\left(\frac{\pi}{2} x^2\right)$$

$$7.3.8 \quad S(x) = S_1 \left(x \sqrt{\frac{\pi}{2}} \right) = S_2 \left(\frac{\pi}{2} x^2 \right)$$

$$7.3.9 \quad C(z) = \frac{1}{2} + f(z) \sin \left(\frac{\pi}{2} z^2 \right) - g(z) \cos \left(\frac{\pi}{2} z^2 \right)$$

$$7.3.10 \quad S(z) = \frac{1}{2} - f(z) \cos \left(\frac{\pi}{2} z^2 \right) - g(z) \sin \left(\frac{\pi}{2} z^2 \right)$$

Series Expansions

$$7.3.11 \quad C(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n}}{(2n)!(4n+1)} z^{4n+1}$$

7.3.12

$$C(z) = \cos \left(\frac{\pi}{2} z^2 \right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{1 \cdot 3 \cdots (4n+1)} z^{4n+1} \\ + \sin \left(\frac{\pi}{2} z^2 \right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{1 \cdot 3 \cdots (4n+3)} z^{4n+3}$$

$$7.3.13 \quad S(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n+1}}{(2n+1)!(4n+3)} z^{4n+3}$$

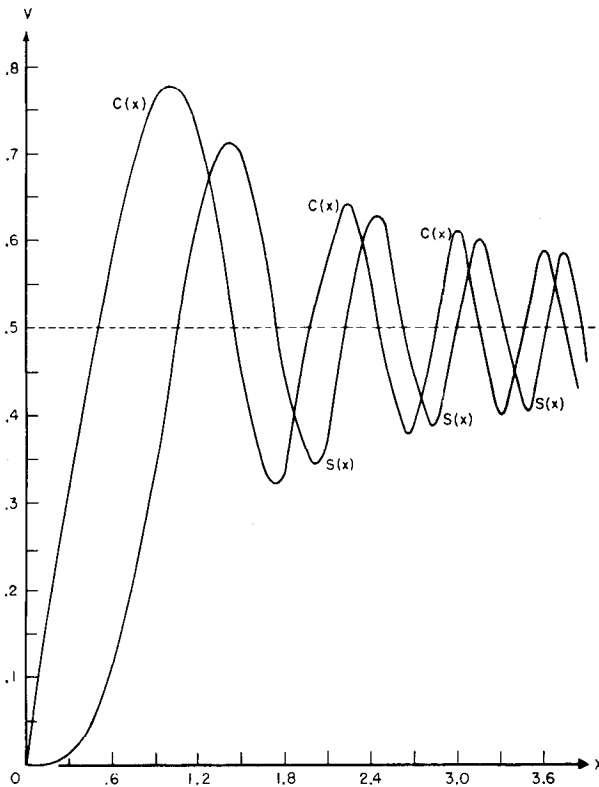


FIGURE 7.5. Fresnel Integrals.
 $y = C(x)$, $y = S(x)$

7.3.14

$$S(z) = -\cos \left(\frac{\pi}{2} z^2 \right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{1 \cdot 3 \cdots (4n+3)} z^{4n+3} \\ + \sin \left(\frac{\pi}{2} z^2 \right) \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{1 \cdot 3 \cdots (4n+1)} z^{4n+1}$$

$$7.3.15 \quad C_2(z) = J_{1/2}(z) + J_{5/2}(z) + J_{9/2}(z) + \cdots$$

$$7.3.16 \quad S_2(z) = J_{3/2}(z) + J_{7/2}(z) + J_{11/2}(z) + \cdots$$

For Bessel functions $J_{n+1/2}(z)$ see chapter 10.

Symmetry Relations

$$7.3.17 \quad C(-z) = -C(z), \quad S(-z) = -S(z)$$

$$7.3.18 \quad C(iz) = iC(z), \quad S(iz) = -iS(z)$$

$$7.3.19 \quad C(\bar{z}) = \overline{C(z)}, \quad S(\bar{z}) = \overline{S(z)}$$

Value at Infinity

$$7.3.20 \quad C(x) \rightarrow \frac{1}{2}, \quad S(x) \rightarrow \frac{1}{2} \quad (x \rightarrow \infty)$$

Derivatives

$$7.3.21 \quad \frac{df(x)}{dx} = -\pi x g(x), \quad \frac{dg(x)}{dx} = \pi x f(x) - 1$$

Relation to Error Function (see 7.1.1, 7.1.3)

7.3.22

$$C(z) + iS(z) = \frac{1+i}{2} \operatorname{erf} \left[\frac{\sqrt{\pi}}{2} (1-i)z \right] \\ = \frac{1+i}{2} \left\{ 1 - e^{-\frac{\pi}{2} z^2} w \left[\frac{\sqrt{\pi}}{2} (1+i)z \right] \right\}$$

$$7.3.23 \quad g(x) = \mathcal{R} \left\{ \frac{1+i}{2} w \left[\frac{\sqrt{\pi}}{2} (1+i)x \right] \right\}$$

$$7.3.24 \quad f(x) = \mathcal{I} \left\{ \frac{1+i}{2} w \left[\frac{\sqrt{\pi}}{2} (1+i)x \right] \right\}$$

Relation to Confluent Hypergeometric Function (see chapter 13)

7.3.25

$$C(z) + iS(z) = z M \left(\frac{1}{2}, \frac{3}{2}, i \frac{\pi}{2} z^2 \right) \\ = z e^{i \frac{\pi}{2} z^2} M \left(1, \frac{3}{2}, -i \frac{\pi}{2} z^2 \right)$$

Relation to Spherical Bessel Functions (see chapter 10)

$$7.3.26 \quad C_2(z) = \frac{1}{2} \int_0^z J_{-1/2}(t) dt, \quad S_2(z) = \frac{1}{2} \int_0^z J_{1/2}(t) dt$$

Asymptotic Expansions

7.3.27

$$\pi z f(z) \sim 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1 \cdot 3 \cdots (4m-1)}{(\pi z^2)^{2m}} \quad \left(z \rightarrow \infty, |\arg z| < \frac{\pi}{2} \right)$$

7.3.28

$$\pi z g(z) \sim \sum_{m=0}^{\infty} (-1)^m \frac{1 \cdot 3 \cdots (4m+1)}{(\pi z^2)^{2m+1}} \quad \left(z \rightarrow \infty, |\arg z| < \frac{\pi}{2} \right)$$

If $R_n^{(f)}(z)$, $R_n^{(g)}(z)$ are the remainders after n terms in 7.3.27, 7.3.28, respectively, then

7.3.29

$$R_n^{(f)}(z) = (-1)^n \frac{1 \cdot 3 \cdots (4n-1)}{(\pi z^2)^{2n}} \theta^{(f)},$$

$$\theta^{(f)} = \frac{1}{\Gamma(2n + \frac{1}{2})} \int_0^{\infty} \frac{e^{-t} t^{2n-1}}{1 + \left(\frac{2t}{\pi z^2}\right)^2} dt \quad \left(|\arg z| < \frac{\pi}{4} \right)$$

7.3.30

$$R_n^{(g)}(z) = (-1)^n \frac{1 \cdot 3 \cdots (4n+1)}{(\pi z^2)^{2n}} \theta^{(g)},$$

$$\theta^{(g)} = \frac{1}{\Gamma(2n + \frac{3}{2})} \int_0^{\infty} \frac{e^{-t} t^{2n+1}}{1 + \left(\frac{2t}{\pi z^2}\right)^2} dt \quad \left(|\arg z| < \frac{\pi}{4} \right)$$

$$7.3.31 \quad |\theta^{(f)}| < 1, |\theta^{(g)}| < 1 \quad \left(|\arg z| \leq \frac{\pi}{8} \right)$$

For x real, $R_n^{(f)}(x)$ and $R_n^{(g)}(x)$ are less in absolute value than the first neglected term and of the same sign.

Rational Approximations⁴ ($0 \leq x \leq \infty$)

7.3.32

$$f(x) = \frac{1 + .926x}{2 + 1.792x + 3.104x^2} + \epsilon(x) \quad |\epsilon(x)| \leq 2 \times 10^{-3}$$

7.3.33

$$g(x) = \frac{1}{2 + 4.142x + 3.492x^2 + 6.670x^3} + \epsilon(x)$$

$$|\epsilon(x)| \leq 2 \times 10^{-3}$$

(For more accurate approximations see [7.1].)

7.4. Definite and Indefinite Integrals

For a more extensive list of integrals see [7.5], [7.8], [7.15].

$$7.4.1 \quad \int_0^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

⁴ Approximations 7.3.32, 7.3.33 are based on those given in C. Hastings, Jr., Approximations for calculating Fresnel integrals, Approximation Newsletter, April 1956, Note 10. [See also MTAC 10, 173, 1956.]

7.4.2

$$\int_0^{\infty} e^{-(at^2 + 2bt + c)} dt = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{\frac{b^2 - ac}{a}} \operatorname{erfc} \frac{b}{\sqrt{a}} \quad (\Re a > 0)$$

7.4.3

$$\int_0^{\infty} e^{-at^2 - \frac{b}{t^2}} dt = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-2\sqrt{ab}} \quad (\Re a > 0, \Re b > 0)$$

7.4.4

$$\int_0^{\infty} t^{2n} e^{-at^2} dt = \frac{1 \cdot 3 \cdots (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}$$

$$= \frac{\Gamma(n + \frac{1}{2})}{2a^{n+\frac{1}{2}}} \quad (\Re a > 0; n = 0, 1, 2, \dots)$$

7.4.5

$$\int_0^{\infty} t^{2n+1} e^{-at^2} dt = \frac{n!}{2a^{n+1}} \quad (\Re a > 0; n = 0, 1, 2, \dots)$$

7.4.6

$$\int_0^{\infty} e^{-at^2} \cos(2xt) dt = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-\frac{x^2}{a}} \quad (\Re a > 0)$$

7.4.7

$$\int_0^{\infty} e^{-at^2} \sin(2xt) dt = \frac{1}{\sqrt{a}} e^{-x^2/a} \int_0^{x/\sqrt{a}} e^{t^2} dt$$

$$(\Re a > 0)$$

7.4.8

$$\int_0^{\infty} \frac{e^{-at} dt}{\sqrt{t+z^2}} = \sqrt{\frac{\pi}{a}} e^{az^2} \operatorname{erfc} \sqrt{a} z \quad (\Re a > 0, \Re z > 0)$$

7.4.9

$$\int_0^{\infty} \frac{e^{-at} dt}{\sqrt{t(t+z)}} = \frac{\pi}{\sqrt{z}} e^{az} \operatorname{erfc} \sqrt{a} z$$

$$(\Re a > 0, z \neq 0, |\arg z| < \pi)$$

7.4.10

$$\int_0^{\infty} \frac{e^{-at^2} dt}{t+x} = e^{-ax^2} \left[\sqrt{\pi} \int_0^{\sqrt{ax}} e^{t^2} dt - \frac{1}{2} \operatorname{Ei}(ax^2) \right]$$

$$(a > 0, x > 0)$$

7.4.11

$$\int_0^{\infty} \frac{e^{-at^2} dt}{t^2 + x^2} = \frac{\pi}{2x} e^{ax^2} \operatorname{erfc} \sqrt{a} x \quad (a > 0, x > 0)$$

$$7.4.12 \quad \int_0^1 \frac{e^{-at^2} dt}{t^2 + 1} = \frac{\pi}{4} e^a [1 - (\operatorname{erf} \sqrt{a})^2] \quad (a > 0)$$

7.4.13

$$\int_{-\infty}^{\infty} \frac{ye^{-t^2} dt}{(x-t)^2 + y^2} = \pi \mathcal{R}w(x+iy) \quad (x \text{ real}, y > 0)$$

7.4.14

$$\int_{-\infty}^{\infty} \frac{(x-t)e^{-t^2} dt}{(x-t)^2 + y^2} = \pi \mathcal{J} w(x+iy) \quad (x \text{ real}, y > 0)$$

7.4.15

$$\int_0^{\infty} \frac{[t^2 - (x^2 - y^2)]e^{-t^2} dt}{t^4 - 2(x^2 - y^2)t^2 + (x^2 + y^2)^2} = \frac{\pi}{2} \mathcal{R} \frac{w(x+iy)}{y-ix} \\ (x \text{ real}, y > 0)$$

7.4.16

$$\int_0^{\infty} \frac{2xye^{-t^2} dt}{t^4 - 2(x^2 - y^2)t^2 + (x^2 + y^2)^2} = \frac{\pi}{2} \mathcal{J} \frac{w(x+iy)}{y-ix} \\ (x \text{ real}, y > 0)$$

7.4.17

$$\int_0^{\infty} e^{-at} \operatorname{erf} bt \, dt = \frac{1}{a} e^{\frac{a^2}{4b^2}} \operatorname{erfc} \frac{a}{2b} \\ (\mathcal{R}a > 0, |\arg b| < \frac{\pi}{4})$$

7.4.18

$$\int_0^{\infty} \sin(2at) \operatorname{erfc} bt \, dt = \frac{1}{2a} [1 - e^{-(a/b)^2}] (a > 0, \mathcal{R}b > 0)$$

7.4.19

$$\int_0^{\infty} e^{-at} \operatorname{erf} \sqrt{bt} \, dt = \frac{1}{a} \sqrt{\frac{b}{a+b}} \quad (\mathcal{R}(a+b) > 0)$$

7.4.20

$$\int_0^{\infty} e^{-at} \operatorname{erfc} \sqrt{\frac{b}{t}} \, dt = \frac{1}{a} e^{-2\sqrt{ab}} \quad (\mathcal{R}a > 0, \mathcal{R}b > 0)$$

7.4.21

$$\int_0^{\infty} e^{(a-b)t} \operatorname{erfc} \left(\sqrt{at} + \sqrt{\frac{c}{t}} \right) dt = \frac{e^{-2(\sqrt{ac} + \sqrt{bc})}}{\sqrt{b}(\sqrt{a} + \sqrt{b})} \\ (\mathcal{R}b > 0, \mathcal{R}c > 0)$$

7.4.22

$$\int_0^{\infty} e^{-at} \cos(t^2) dt = \sqrt{\frac{\pi}{2}} \left\{ \left[\frac{1}{2} - S\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right) \right] \cos\left(\frac{a^2}{4}\right) - \left[\frac{1}{2} - C\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right) \right] \sin\left(\frac{a^2}{4}\right) \right\} \quad (\mathcal{R}a > 0)$$

7.4.23

$$\int_0^{\infty} e^{-at} \sin(t^2) dt = \sqrt{\frac{\pi}{2}} \left\{ \left[\frac{1}{2} - C\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right) \right] \cos\left(\frac{a^2}{4}\right) + \left[\frac{1}{2} - S\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right) \right] \sin\left(\frac{a^2}{4}\right) \right\} \quad (\mathcal{R}a > 0)$$

7.4.24

$$\int_0^{\infty} e^{-at} \frac{\sin(t^2)}{t} dt = \frac{\pi}{2} \left[\frac{1}{2} - C\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right) \right]^2 + \frac{\pi}{2} \left[\frac{1}{2} - S\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right) \right]^2 \quad (\mathcal{R}a > 0)$$

7.4.25

$$\int_0^{\infty} \frac{e^{-at}\sqrt{t}}{t^2 + b^2} dt = \pi \sqrt{\frac{2}{b}} \left\{ \left[\frac{1}{2} - C\left(\sqrt{\frac{2ab}{\pi}}\right) \right] \cos(ab) + \left[\frac{1}{2} - S\left(\sqrt{\frac{2ab}{\pi}}\right) \right] \sin(ab) \right\} \quad (\mathcal{R}a > 0, \mathcal{R}b > 0)$$

7.4.26

$$\int_0^{\infty} \frac{e^{-at} dt}{\sqrt{t}(t^2 + b^2)} = \frac{\pi}{b} \sqrt{\frac{2}{b}} \left\{ \left[\frac{1}{2} - S\left(\sqrt{\frac{2ab}{\pi}}\right) \right] \cos(ab) - \left[\frac{1}{2} - C\left(\sqrt{\frac{2ab}{\pi}}\right) \right] \sin(ab) \right\} \quad (\mathcal{R}a > 0, \mathcal{R}b > 0)$$

7.4.27

$$\int_0^{\infty} e^{-at} C(t) dt = \frac{1}{a} \left\{ \left[\frac{1}{2} - S\left(\frac{a}{\pi}\right) \right] \cos\left(\frac{a^2}{2\pi}\right) - \left[\frac{1}{2} - C\left(\frac{a}{\pi}\right) \right] \sin\left(\frac{a^2}{2\pi}\right) \right\} \quad (\mathcal{R}a > 0)$$

7.4.28

$$\int_0^{\infty} e^{-at} S(t) dt = \frac{1}{a} \left\{ \left[\frac{1}{2} - C\left(\frac{a}{\pi}\right) \right] \cos\left(\frac{a^2}{2\pi}\right) + \left[\frac{1}{2} - S\left(\frac{a}{\pi}\right) \right] \sin\left(\frac{a^2}{2\pi}\right) \right\} \quad (\mathcal{R}a > 0)$$

7.4.29

$$\int_0^{\infty} e^{-at} C\left(\sqrt{\frac{2t}{\pi}}\right) dt = \frac{1}{2a(\sqrt{a^2+1}-a)^{\frac{1}{2}}\sqrt{a^2+1}} \quad (\mathcal{R}a > 0)$$

7.4.30

$$\int_0^{\infty} e^{-at} S\left(\sqrt{\frac{2t}{\pi}}\right) dt = \frac{1}{2a(\sqrt{a^2+1}+a)^{\frac{1}{2}}\sqrt{a^2+1}} \quad (\mathcal{R}a > 0)$$

$$7.4.31 \quad \int_0^{\infty} \left\{ \left[\frac{1}{2} - C(t) \right]^2 + \left[\frac{1}{2} - S(t) \right]^2 \right\} dt = \frac{1}{\pi}$$

7.4.32

$$\int e^{-(ax^2+2bx+c)} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{\frac{b^2-ac}{a}} \operatorname{erf}\left(\sqrt{a}x + \frac{b}{\sqrt{a}}\right) + \text{const.} \quad (a \neq 0)$$

7.4.33

$$\int e^{-ax^2 - \frac{b^2}{x^2}} dx = \frac{\sqrt{\pi}}{4a} \left[e^{2ab} \operatorname{erf} \left(ax + \frac{b}{x} \right) + e^{-2ab} \operatorname{erf} \left(ax - \frac{b}{x} \right) \right] + \text{const.} \quad (a \neq 0)$$

7.4.34

$$\int e^{-ax^2 + \frac{b^2}{x^2}} dx = -\frac{\sqrt{\pi}}{4a} e^{-a^2 x^2 + \frac{b^2}{x^2}} \left[w \left(\frac{b}{x} + iax \right) + w \left(-\frac{b}{x} + iax \right) \right] + \text{const.} \quad (a \neq 0)$$

$$7.4.35 \quad \int \operatorname{erf} x dx = x \operatorname{erf} x + \frac{1}{\sqrt{\pi}} e^{-x^2} + \text{const.}$$

7.4.36

$$\int e^{ax} \operatorname{erf} bx dx = \frac{1}{a} \left[e^{ax} \operatorname{erf} bx - e^{\frac{a^2}{4b^2}} \operatorname{erf} \left(bx - \frac{a}{2b} \right) \right] + \text{const.} \quad (a \neq 0)$$

7.4.37

$$\int e^{ax} \operatorname{erf} \sqrt{\frac{b}{x}} dx = \frac{1}{a} \left\{ e^{ax} \operatorname{erf} \sqrt{\frac{b}{x}} + \frac{1}{2} e^{ax - \frac{b}{x}} \left[w \left(\sqrt{ax} + i \sqrt{\frac{b}{x}} \right) + w \left(-\sqrt{ax} + i \sqrt{\frac{b}{x}} \right) \right] \right\} + \text{const.} \quad (a \neq 0)$$

7.4.38

$$\int \cos(ax^2 + 2bx + c) dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos \left(\frac{b^2 - ac}{a} \right) C \left[\sqrt{\frac{2}{a\pi}} (ax + b) \right] + \sin \left(\frac{b^2 - ac}{a} \right) S \left[\sqrt{\frac{2}{a\pi}} (ax + b) \right] \right\} + \text{const.}$$

7.4.39

$$\int \sin(ax^2 + 2bx + c) dx = \sqrt{\frac{\pi}{2a}} \left\{ \cos \left(\frac{b^2 - ac}{a} \right) S \left[\sqrt{\frac{2}{a\pi}} (ax + b) \right] - \sin \left(\frac{b^2 - ac}{a} \right) C \left[\sqrt{\frac{2}{a\pi}} (ax + b) \right] \right\} + \text{const.}$$

$$7.4.40 \quad \int C(x) dx = xC(x) - \frac{1}{\pi} \sin \left(\frac{\pi}{2} x^2 \right) + \text{const.}$$

$$7.4.41 \quad \int S(x) dx = xS(x) + \frac{1}{\pi} \cos \left(\frac{\pi}{2} x^2 \right) + \text{const.}$$

Numerical Methods

7.5. Use and Extension of the Tables

Example 1. Compute $\operatorname{erf} .745$ and $e^{-(.745)^2}$ using Taylor's series.

With the aid of Taylor's theorem and 7.1.19 it can be shown that

$$\begin{aligned} \operatorname{erf} (x_0 + ph) &= \operatorname{erf} x_0 \\ &+ \frac{2}{\sqrt{\pi}} e^{-x_0^2} ph \left[1 - phx_0 + \frac{1}{3} p^2 h^2 (2x_0^2 - 1) \right] + \epsilon \\ e^{-(x_0 + ph)^2} &= e^{-x_0^2} \left[1 - 2phx_0 + p^2 h^2 (2x_0^2 - 1) \right. \\ &\quad \left. - \frac{2}{3} p^3 h^3 x_0 (2x_0^2 - 3) \right] + \eta \end{aligned}$$

where $|\epsilon| < 1.2 \times 10^{-10}$, $|\eta| < 3.2 \times 10^{-10}$ if $h = 10^{-2}$, $|p| \leq \frac{1}{2}$. With $x_0 = .74$, $p = .5$ and using Table 7.1

$$\begin{aligned} \operatorname{erf} .745 &= .70467 \ 80779 + (.5)(.00652 \ 58247) \times \\ &\quad [1 - (.005)(.74) + (.00000 \ 83333)(.0952)] \\ &= .70792 \ 8920 \end{aligned}$$

$$\begin{aligned} e^{-(.745)^2} &= \frac{\sqrt{\pi}}{2} (.65258 \ 24665) [1 - .0074 \\ &\quad + (.000025)(.0952) + (.00000 \ 00833)(.74)(1.9048)] \\ &= .57405 \ 7910. \end{aligned}$$

As a check the computation was repeated with $x_0 = .75$, $p = -.5$.

Example 2. Compute $\operatorname{erfc} x$ to 5S for $x = 4.8$. We have $1/x^2 = .0434028$. With Table 7.2 and linear interpolation in Table 7.3, we obtain

$$\begin{aligned} \operatorname{erfc} 4.8 &= \frac{1}{4.8} (1.11253)(10^{-10})(.552669) \frac{\sqrt{\pi}}{2} \\ &= (1.1352)10^{-11}. \end{aligned}$$

Example 3. Compute $e^{-x^2} \int_0^x e^{t^2} dt$ to 5S for $x=6.5$.

With $1/x^2=.0236686$ and linear interpolation in **Table 7.5**

$$e^{-(6.5)^2} \int_0^{6.5} e^{t^2} dt = (.506143)/(6.5) = .077868.$$

Example 4. Compute $i^2 \operatorname{erfc} 1.72$ using the recurrence relation and **Table 7.1**.

By **7.2.1**, using **Table 7.1**,

$$i^{-1} \operatorname{erfc} 1.72 = .05856 \quad 50.$$

Using the recurrence relation **7.2.5** and **Table 7.1**

$$i \operatorname{erfc} 1.72 = -(1.72)(.01499 \quad 72) + (.5)(.05856 \quad 50) \\ = .0034873$$

$$i^2 \operatorname{erfc} 1.72 = -(86)(.0034873) + (.25)(.01499 \quad 72) \\ = .0007502.$$

Note the loss of two significant digits.

Example 5. Compute $i^k \operatorname{erfc} 1.72$ for $k=1, 2, 3$ by backward recurrence.

Let the sequence $w_\mu^m(x)$ ($\mu=m, m-1, \dots, 1, 0, -1$) be generated by backward use of the recurrence relation **7.2.5** starting with $w_{m+2}^m=0$, $w_{m+1}^m=1$. Then, for any fixed k , (see [7.7]),

$$\lim_{m \rightarrow \infty} \frac{w_k^m(x)}{w_{-1}^m(x)} = \frac{\sqrt{\pi}}{2} e^{x^2} i^k \operatorname{erfc} x \quad (x > 0).$$

With $x=1.72$, $m=15$ we obtain

μ	$w_\mu^{15}(1.72)$	μ	$w_\mu^{15}(1.72)$	μ	$w_\mu^{15}(1.72)$	μ	$w_\mu^{15}(1.72)$
17	0	12	(3) 2.1011	7	(7) 2.5879	2	(11) 1.2920
16	1	11	(4) 1.3831	6	(8) 1.5569	1	(11) 6.0064
15	3.44	10	(4) 9.8005	5	(8) 8.9787	0	(12) 2.5330
14	(1) 4.3834	9	(5) 6.4143	4	(9) 4.9570	-1	(13) 1.0087
13	(2) 2.5399	8	(6) 4.1666	3	(10) 2.6031		

From **Table 7.1** we have $\frac{2}{\sqrt{\pi}} e^{-(1.72)^2} = .058565$.

Thus,

$$i \operatorname{erfc} 1.72 \approx (.058565)(6.0064 \times 10^{11})/1.0087 \times 10^{13} \\ = 3.4873 \times 10^{-3}$$

$$i^2 \operatorname{erfc} 1.72 \approx (.058565)(1.2920 \times 10^{11})/1.0087 \times 10^{13} \\ = 7.5013 \times 10^{-4}$$

$$i^3 \operatorname{erfc} 1.72 \approx (.058565)(2.6031 \times 10^{10})/1.0087 \times 10^{13} \\ = 1.5114 \times 10^{-4}.$$

Example 6. Compute $C(8.65)$ using **Table 7.8**.

With $x=8.65$, $1/x=.115607$ we have from **Table 7.8** by linear interpolation

$$f(8.65) = .036797, \quad g(8.65) = .000159.$$

From **Table 4.6**

$$\sin\left(\frac{\pi}{2} x^2\right) = -.961382, \quad \cos\left(\frac{\pi}{2} x^2\right) = -.275218.$$

Using **7.3.9**

$$C(8.65) = .5 + (.036797)(-.961382) \\ - (.000159)(-.275218) = .46467.$$

Example 7. Compute $S_1(1.1)$ to 10D.

Using **7.3.8** and **7.3.10** we obtain by 6-pt interpolation in **Table 7.8**

$$S_1(1.1) = S\left(1.1 \sqrt{\frac{2}{\pi}}\right) \\ = S(.87767 \quad 30169) = .31865 \quad 57172.$$

Example 8. Compute $S_2(5.24)$ to 6D.

Enter **Table 7.7** in the column headed by u . Using Aitken's scheme of interpolation

u	$S_2(u)$					
5.20310 58	.43280 06	.03689 42				
5.31808 80	.41573 97	-.07808 80	.42732 63			
5.08938 01	.45093 88	.15061 99	691 63	.42718 63		
5.43432 70	.39999 44	-.19432 70	756 60	6 52	.42717 71	
4.97691 11	.46990 94	.26308 89	674 79	9 39	61	.42717 67

$$S_2(5.24) = .427177$$

Example 9. Compute $S_2(5.24)$ using Taylor's series and **Table 7.8**.

Using **7.3.21** we can write Taylor's series for $f_2(u)$

$$= f\left(\sqrt{\frac{2u}{\pi}}\right) \text{ and } g_2(u) = g\left(\sqrt{\frac{2u}{\pi}}\right) \text{ in the form}$$

$$f_2(u) = c_0 + c_1(u - u_0) + \frac{c_2}{2!}(u - u_0)^2 + \frac{c_3}{3!}(u - u_0)^3 + \dots,$$

$$g_2(u) = -\left[c_1 + c_2(u - u_0) + \frac{c_3}{2!}(u - u_0)^2 + \frac{c_4}{3!}(u - u_0)^3 + \dots \right],$$

where

$$\begin{aligned} c_0 &= f_2(u_0), \quad c_1 = -g_2(u_0), \\ c_{k+2} &= -c_k + (-1)^k \frac{1 \cdot 3 \dots (2k-1)}{\sqrt{2\pi} u_0 (2u_0)^k} \\ &\quad (k=0, 1, 2, \dots). \end{aligned}$$

Consulting **Table 7.8** we chose $u_0 = 1/185638 = 5.386819$, thus having $u - u_0 = 5.24 - 5.386819 = -.146819$. From **Table 7.8**

$$f_2(u_0) = .168270, \quad g_2(u_0) = .014483.$$

Hence, applying the series above,

$$f_2(5.24) = .170436, \quad g_2(5.24) = .015030.$$

Using the 4th formula at the bottom of **Table 7.8**

$$\begin{aligned} S_2(5.24) &= .5 - (.170436)(.503471) \\ &\quad - (.015030)(-.864012) = .42718. \end{aligned}$$

Example 10. Compute $S_2(2)$ using **7.3.16**.

Generating the values of $J_{n+1/2}(2)$ as described in chapter 10 we find

$$\begin{aligned} S_2(2) &= J_{3/2}(2) + J_{7/2}(2) + J_{11/2}(2) + J_{15/2}(2) + \dots \\ &= .49129 + .06852 + .00297 + .00006 = .56284. \end{aligned}$$

Example 11. Compute $\int_1^\infty \frac{Y_0(t)}{t} dt$ by numerical integration using **Tables 9.1** and **7.8**. [$Y_0(t)$ is the Bessel function of the second kind defined in **9.1.16**.]

We decompose the integral into three parts

$$\begin{aligned} \int_1^\infty Y_0(t) \frac{dt}{t} &= \int_1^{10} Y_0(t) \frac{dt}{t} + \int_{10}^\infty [Y_0(t) - \tilde{Y}_0(t)] \frac{dt}{t} \\ &\quad + \int_{10}^\infty \tilde{Y}_0(t) \frac{dt}{t} \end{aligned}$$

where

$$\begin{aligned} \tilde{Y}_0(t) &= \left(1 - \frac{9}{128t^2}\right) \frac{\sin\left(t - \frac{\pi}{4}\right)}{\sqrt{\frac{1}{2}\pi t}} \\ &\quad - \left(1 - \frac{75}{128t^2}\right) \frac{\cos\left(t - \frac{\pi}{4}\right)}{8t\sqrt{\frac{1}{2}\pi t}} \end{aligned}$$

represents the first two terms of the asymptotic expansion **9.2.2**.

By numerical integration, using **Table 9.1**,

$$\int_1^{10} Y_0(t) \frac{dt}{t} = .41826 \ 00.$$

Using the fact that the remainder terms of the asymptotic expansion are less in absolute value than the first neglected terms, we can estimate

$$\begin{aligned} \left| \int_{10}^\infty [Y_0(t) - \tilde{Y}_0(t)] \frac{dt}{t} \right| &\leq \sqrt{\frac{2}{\pi}} \int_{10}^\infty \left[\frac{3^2 \cdot 5^2 \cdot 7^2}{2^{12} \cdot 4!} t^{-11/2} \right. \\ &\quad \left. + \frac{3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2}{2^{15} \cdot 5!} t^{-13/2} \right] dt = 7.33 \times 10^{-7}. \end{aligned}$$

Finally,

$$\begin{aligned} \int_{10}^\infty \tilde{Y}_0(t) \frac{dt}{t} &= \frac{14659}{6720} \sqrt{2} [1 - C_2(10) - S_2(10)] \\ &\quad - \frac{5953819 \cos 10 - \sin 10}{2688000 \sqrt{10\pi}} \\ &\quad - \frac{23107 \cos 10 + \sin 10}{2150400 \sqrt{10\pi}} = -.02298 \ 78, \end{aligned}$$

using **Tables 7.8** and **4.8**. Hence

$$\int_1^\infty Y_0(t) \frac{dt}{t} = .41826 \ 00 - .02298 \ 78 = .39527 \ 22.$$

The answer correct to 8D is .39527 290 (**Table 11.2**).

Example 12. Compute $w(.44 + .67i)$ using bivariate linear interpolation.

By linear interpolation in **Table 7.9** along the x -direction at $y = .6$ and $y = .7$

$$\begin{aligned} w(.44 + .6i) &\approx .6(.522246 + .167880i) + .4(.498591 \\ &\quad + .202666i) = .512784 + .181794i \end{aligned}$$

$$\begin{aligned} w(.44 + .7i) &\approx .6(.487556 + .147975i) + .4(.467521 \\ &\quad + .179123i) = .479542 + .160434i. \end{aligned}$$

By linear interpolation along the y -direction at $x = .44$

$$\begin{aligned} w(.44 + .67i) &\approx .3(.512784 + .181794i) + .7(.479542 \\ &\quad + .160434i) = .489515 + .166842i. \end{aligned}$$

The correct answer is .489557 + .166889i.

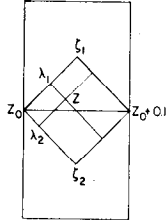
Example 13. Compute $\mathcal{R}w(z)$ for $z = .44 + .61i$.

Bivariate linear interpolation, as described in **Example 12**, is most accurate if z lies near the center or along a diagonal of one of the squares of the tabular grid [7.6]. It is not as accurate for z near the midpoint of a side of a square, as in this example. However, we may introduce an auxil-

iary square (see diagram) which contains z close to its center. Bivariate linear interpolation can then be applied within this auxiliary square.

The values of $w(z)$ needed at $z=\zeta_1$, and $z=\zeta_2$ are easily approximated by the average of the four neighboring tabular values. Furthermore the parts to be used are given by

$$\frac{|z_0 - \lambda_1|}{|z_0 - \zeta_1|} = p_1 + p_2, \quad \frac{|z_0 - \lambda_2|}{|z_0 - \zeta_2|} = p_1 - p_2$$



where $z = z_0 + .1(p_1 + ip_2)$. Thus, with $z_0 = .4 + .6i$, $\zeta_1 = .45 + .65i$, $\zeta_2 = .45 + .55i$, $p_1 = .4$, $p_2 = .1$, we get from Table 7.9

$$\Re w(\zeta_1) \approx \frac{1}{4}(.522246 + .498591 + .487556 + .467521) = .493979$$

$$\Re w(\zeta_2) \approx \frac{1}{4}(.522246 + .498591 + .561252 + .533157) = .528812$$

$$\Re w(z) \approx [1 - (.4 + .1)]\{[1 - (.4 - .1)].522246 + (.4 - .1).528812\} + (.4 + .1) \times \{[1 - (.4 - .1)].493979 + (.4 - .1).498591\} = .509789.$$

The correct answer is .509756. Straightforward bivariate interpolation gives .509460.

Example 14. Compute $\mathcal{J}w(.39 + .61i)$ to 6D using Taylor's series.

Let $z = .39 + .61i$, $z_0 = .4 + .6i$. From 7.1.20, and using Table 7.9, we have

$$w(z_0) = .522246 + .167880i$$

$$w'(z_0) = -.21634 + .36738i, \quad z - z_0 = (-1 + i)10^{-2}$$

$$\frac{1}{2}w''(z_0) = -.215 - .185i, \quad (z - z_0)^2 = -2i \times 10^{-4}$$

$$\mathcal{J}w(z) = .167880 - .0021634 - .0036738 + .0000430 = .162086.$$

Example 15. Compute $w(.4 - 1.3i)$.

From 7.1.11, 7.1.12

$$w(.4 - 1.3i) = \overline{w(-.4 - 1.3i)} = 2e^{-(.4 - 1.3i)^2} - \overline{w(.4 + 1.3i)}.$$

Using Tables 7.9, 4.4 and 4.6

$$w(.4 - 1.3i) = 4.33342 + 8.04201i.$$

Example 16. Compute $w(7 + 2i)$.

Using the second formula at the end of Table 7.9

$$w(7 + 2i) = (-2 + 7i) \left(\frac{.5124242}{44.72474 + 28i} + \frac{.05176536}{42.27525 + 28i} \right) = .021853 + .075010i.$$

Example 17. Compute $\operatorname{erf}(2 + i)$.

From 7.1.3, 7.1.12 we have

$$\operatorname{erf} z = 1 - e^{-z^2} w(iz) = 1 - e^{y^2 - x^2} (\cos 2xy - i \sin 2xy) \overline{w(y + ix)} \quad (z = x + iy).$$

Using Tables 7.9, 4.4, 4.6

$$\operatorname{erf}(2 + i) = 1 - e^{-3} (\cos 4 - i \sin 4) \overline{w(1 + 2i)} = 1.003606 - .0112590i.$$

Example 18. Compute $S_1\left(\left(\frac{1}{2} + i\right)\sqrt{2}\right)$.

From 7.3.22, 7.3.8, 7.3.18 we have

$$S_1(z) = \frac{1}{2} - \frac{1-i}{4} e^{iz^2} w \left[(1+i) \frac{z}{\sqrt{2}} \right] - \frac{1+i}{4} e^{-iz^2} w \left[(i-1) \frac{z}{\sqrt{2}} \right].$$

Setting $z = \left(\frac{1}{2} + i\right)\sqrt{2}$ and making use of 7.1.11, 7.1.12, and Table 7.9

$$S_1\left(\left(\frac{1}{2} + i\right)\sqrt{2}\right) = -\frac{i}{2} - \frac{1-i}{4} e^{-2} \left(\cos \frac{3}{2} - i \sin \frac{3}{2} \right) \overline{w\left(\frac{1}{2} + \frac{3}{2}i\right)} + \frac{1+i}{4} e^2 \left(\cos \frac{3}{2} + i \sin \frac{3}{2} \right) w\left(\frac{3}{2} + \frac{1}{2}i\right) = -.990734 - .681619i.$$

Example 19. Compute $\int_0^\infty e^{-(1/4)t^2 - 3t} \cos(2t) dt$ using Table 7.9.

Setting $b = y + ix$, $c = 0$ in 7.4.2 and using 7.1.3, 7.1.12 we find

$$\int_0^\infty e^{-at^2 - 2yt} \cos(2xt) dt = \frac{1}{2} \sqrt{\frac{\pi}{a}} \Re w\left(\frac{x + iy}{\sqrt{a}}\right) \quad (a > 0, x, y \text{ real}).$$

Hence from Table 7.9

$$\int_0^\infty e^{-(1/4)t^2 - 3t} \cos(2t) dt = \sqrt{\pi} \Re w(2 + 3i) = .231761.$$

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 $z=x$; $x=0(.001)2(.01)10$; 5D;
 $z=\rho e^{i\theta}$; $\theta=2.5^\circ(2.5^\circ)30^\circ(1.25^\circ)35^\circ(.625^\circ)40^\circ$;
 $\rho=\rho_0(.001)\rho_0'(.01)\rho_0''(.0002)5$, $0 \leq \rho_0 \leq \rho_0' \leq \rho_0'' \leq 5$, 5D;
 $z=iy$; $y=0(.001)3(.0002)5$, 5S.
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 $\theta=45^\circ(.3125^\circ)48.75^\circ(.625^\circ)55^\circ(1.25^\circ)65^\circ(2.5^\circ)90^\circ$;
 $\rho=\rho_0(.001)\rho_0'(.01)\rho_0''$, $0 \leq \rho_0 < \rho_0' \leq \rho_0'' \leq 5$, 5D;
 $z=x$; $x=0(.001)10$, 5S.
- [7.30] J. Kaye, A table of the first eleven repeated integrals of the error function, *J. Math. Phys.* **34**, 119-125 (1955). $i\operatorname{erfc} x$, $x=0(.01).2(.05)1(.1)3$, $n=-1(1)11$, 6D.
- [7.31] B. Lohmander and S. Rittsten, Table of the function $y=e^{-x^2} \int_0^x e^{t^2} dt$, *Kungl. Fysiogr. Sällsk. i Lund Förh.* **28**, 45-52 (1958). $x=0(.01)3(.02)5$, $x^{-1}=0(.005).2$, 10D; $x=.5(.5)10$, 20D. Contains also 20D values for maximum and inflection points.

- [7.32] W. Lash Miller and A. R. Gordon, Numerical evaluation of infinite series and integrals which arise in certain problems of linear heat flow, electrochemical diffusion, etc., *J. Phys. Chem.* **35**, 2785-2884 (1931).

$$F(x) = e^{-x^2} \int_0^x e^{t^2} dt; x=0(.01)1.99, \quad 6D;$$

$$x=2(.01)4(.05)7.5(.1)10(.2)12, \quad 8S.$$

- [7.33] National Bureau of Standards, Tables of the error function and its derivative, *Applied Math. Series* 41, 2d ed. (U.S. Government Printing Office, Washington, D.C., 1954).

$$(2/\sqrt{\pi})e^{-x^2}, \operatorname{erf} x, x=0(.0001)1(.001)5.6, \quad 15D;$$

$$(2/\sqrt{\pi})e^{-x^2}, \operatorname{erfc} x, x=4(.01)10, \quad 8S.$$

- [7.34] T. Pearcey, Table of the Fresnel integral (Cambridge Univ. Press, London, England, 1956).

$$C\left(\sqrt{\frac{2x}{\pi}}\right), S\left(\sqrt{\frac{2x}{\pi}}\right), x=0(.01)50, \quad 6-7D.$$

- [7.35] *Tablitsy integralov Frenelya* (Izdat. Akad. Nauk SSSR., Moscow, U.S.S.R., 1953). $C(x), S(x), x=0(.001)25, 7D$; $S(x), x=0(.001) .58, 7S$; $C(x), x=0(.001) .101, 7S$.

- [7.36] A. van Wijngaarden and W. L. Scheen, Table of Fresnel integrals, *Verh. Nederl. Akad. Wetensch., Afd. Natuurk. Sec. I*, 19, No. 4, 1-26 (1949). $C(x), S(x), x=0(.01)20, 5D$. (Also contains numerical values of the coefficients in Taylor and asymptotic expansions.)

8. Legendre Functions

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¹ National Bureau of Standards.

8. Legendre Functions

Mathematical Properties

Notation

The conventions used are $z=x+iy$, x , y real, and in particular, x always means a real number in the interval $-1 \leq x \leq +1$ with $\cos \theta = x$ where θ is likewise a real number; n and m are positive integers or zero; ν and μ are unrestricted except where otherwise indicated.

Other notations are:

$$P^n(x) \text{ for } \frac{n!P_n(x)}{(2n-1)!!}$$

$$P_{nm}(x) \text{ for } (-1)^m P_n^m(x)$$

$$T_n^m(x) \text{ for } (-1)^m P_n^m(x)$$

$$\overline{P}_n^m(x) \text{ for } (-1)^m \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}} P_n^m(x)$$

$$\mathfrak{P}_\nu^\mu(z) \text{ for } P_\nu^\mu(z), \mathfrak{Q}_\nu^\mu(z) \text{ for } Q_\nu^\mu(z) \quad (\Re z > 1)$$

$$\mathfrak{Q}_\nu^\mu(z) \text{ for } e^{i\mu\pi} Q_\nu^\mu(z)$$

$$Q_\nu^\mu(z) \text{ for } \frac{\sin(\nu+\mu)\pi}{\sin \nu\pi} Q_\nu^\mu(z)$$

Various other definitions of the functions occur as well as mixing of definitions.

8.1. Differential Equation

8.1.1

$$(1-z^2) \frac{d^2 w}{dz^2} - 2z \frac{dw}{dz} + [\nu(\nu+1) - \frac{\mu^2}{1-z^2}] w = 0$$

Solutions

(Degree ν and order μ with singularities at $z=\pm 1$, ∞ as ordinary branch points— μ , ν arbitrary complex constants.)

$P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ —Associated Legendre Functions (Spherical Harmonics) of the First and Second Kinds²

$$|\arg(z \pm 1)| < \pi, \quad |\arg z| < \pi$$

$$(z^2-1)^{\frac{1}{2}\mu} = (z-1)^{\frac{1}{2}\mu} (z+1)^{\frac{1}{2}\mu}$$

(For $P_\nu^\mu(z)$, $\mu=0$, Legendre polynomials, see chapter 22.)

8.1.2

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left[\frac{z+1}{z-1} \right]^{\frac{1}{2}\mu} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right) \quad (|1-z| < 2)$$

(For $F(a, b; c; z)$ see chapter 15.)

$$8.1.3 \quad Q_\nu^\mu(z) = e^{i\mu\pi} 2^{-\nu-1} \pi^{-\frac{1}{2}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} z^{-\nu-\mu-1} (z^2-1)^{\frac{1}{2}\mu} F\left(1+\frac{\nu}{2}+\frac{\mu}{2}, \frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right) \quad (|z| > 1)$$

Alternate Forms

(Additional forms may be obtained by means of the transformation formulas of the hypergeometric function, see [8.1].)

$$8.1.4 \quad P_\nu^\mu(z) = 2^\nu \pi^{\frac{1}{2}} (z^2-1)^{-\frac{1}{2}\mu} \left\{ \frac{F\left(-\frac{\nu}{2}-\frac{\mu}{2}, \frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}; \frac{1}{2}; z^2\right)}{\Gamma\left(\frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2}\right) \Gamma\left(1+\frac{\nu}{2}-\frac{\mu}{2}\right)} - 2z \frac{F\left(\frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2}, 1+\frac{\nu}{2}-\frac{\mu}{2}; \frac{3}{2}; z^2\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}\right) \Gamma\left(-\frac{\nu}{2}-\frac{\mu}{2}\right)} \right\} \quad (|z^2| < 1)$$

$$8.1.5 \quad P_\nu^\mu(z) = \frac{2^{-\nu-1} \pi^{-\frac{1}{2}} \Gamma(-\frac{1}{2}-\nu) z^{-\nu+\mu-1}}{(z^2-1)^{\mu/2} \Gamma(-\nu-\mu)} F\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}, 1+\frac{\nu}{2}-\frac{\mu}{2}; \nu+\frac{3}{2}; z^{-2}\right) + \frac{2^\nu \Gamma(\frac{1}{2}+\nu) z^{\nu+\mu}}{(z^2-1)^{\mu/2} \Gamma(1+\nu-\mu)} F\left(-\frac{\nu}{2}-\frac{\mu}{2}, \frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2}; \frac{1}{2}-\nu; z^{-2}\right) \quad (|z^{-2}| < 1)$$

$$8.1.6 \quad e^{-i\mu\pi} Q_\nu^\mu(z) = \frac{\Gamma(1+\nu+\mu) \Gamma(-\mu) (z-1)^{\frac{1}{2}\mu} (z+1)^{-\frac{1}{2}\mu}}{2\Gamma(1+\nu-\mu)} F\left(-\nu, 1+\nu; 1+\mu; \frac{1-z}{2}\right) + \frac{1}{2} \Gamma(\mu) (z+1)^{\frac{1}{2}\mu} (z-1)^{-\frac{1}{2}\mu} F\left(-\nu, 1+\nu; 1-\mu; \frac{1-z}{2}\right) \quad (|1-z| < 2)^*$$

² The functions $Y_n^m(\theta, \varphi) = \frac{\cos m\varphi}{\sin m\varphi} P_n^m(\cos \theta)$ called surface harmonics of the first kind, tesseral for $m < n$ and sectoral for $m = n$. With $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$, they are everywhere one valued and continuous functions on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$ where $x = \sin \theta \cos \varphi$, $y = \sin \theta \sin \varphi$ and $z = \cos \theta$.

*See page II.

$$8.1.7 \quad e^{-i\mu\pi} Q_{\nu}^{\mu}(z) = \pi^{1/2} 2^{\mu} (z^2 - 1)^{-1/2} \left\{ \frac{\Gamma\left(\frac{1}{2} + \frac{\nu}{2} + \frac{\mu}{2}\right)}{2\Gamma\left(1 + \frac{\nu}{2} - \frac{\mu}{2}\right)} e^{\pm i\frac{1}{2}\pi(\mu - \nu - 1)} F\left(-\frac{\nu}{2} - \frac{\mu}{2}, \frac{1}{2} + \frac{\nu}{2} - \frac{\mu}{2}, \frac{1}{2}; z^2\right) \right. \\ \left. + \frac{z\Gamma\left(1 + \frac{\nu}{2} + \frac{\mu}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{\nu}{2} - \frac{\mu}{2}\right)} e^{\pm i\frac{1}{2}\pi(\mu - \nu)} F\left(\frac{1}{2} - \frac{\nu}{2} - \frac{\mu}{2}, 1 + \frac{\nu}{2} - \frac{\mu}{2}, \frac{3}{2}; z^2\right) \right\} \quad (|z^2| < 1)$$

Wronskian

8.1.8

$$W\{P_{\nu}^{\mu}(z), Q_{\nu}^{\mu}(z)\} = \frac{e^{i\mu\pi} 2^{\mu} \Gamma\left(\frac{\nu + \mu + 2}{2}\right) \Gamma\left(\frac{\nu + \mu + 1}{2}\right)}{(1 - z^2) \Gamma\left(\frac{\nu - \mu + 2}{2}\right) \Gamma\left(\frac{\nu - \mu + 1}{2}\right)}$$

$$8.1.9 \quad W\{P_{\nu}(z), Q_{\nu}(z)\} = -(z^2 - 1)^{-1}$$

8.2. Relations Between Legendre Functions

Negative Degree

$$8.2.1 \quad P_{-\nu-1}^{\mu}(z) = P_{\nu}^{\mu}(z)$$

8.2.2

$$Q_{-\nu-1}^{\mu}(z) = \{-\pi e^{i\mu\pi} \cos \nu\pi P_{\nu}^{\mu}(z) + Q_{\nu}^{\mu}(z) \sin [\pi(\nu + \mu)]\} / \sin [\pi(\nu - \mu)]$$

 Negative Argument ($\mathcal{J}z \geq 0$)

8.2.3

$$P_{\nu}^{\mu}(-z) = e^{\mp i\mu\pi} P_{\nu}^{\mu}(z) - \frac{2}{\pi} e^{-i\mu\pi} \sin [\pi(\nu + \mu)] Q_{\nu}^{\mu}(z)$$

8.2.4

$$Q_{\nu}^{\mu}(-z) = -e^{\pm i\mu\pi} Q_{\nu}^{\mu}(z)$$

Negative Order

8.2.5

$$P_{\nu}^{-\mu}(z) = \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} \left[P_{\nu}^{\mu}(z) - \frac{2}{\pi} e^{-i\mu\pi} \sin (\mu\pi) Q_{\nu}^{\mu}(z) \right]$$

8.2.6

$$Q_{\nu}^{-\mu}(z) = e^{-2i\mu\pi} \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} Q_{\nu}^{\mu}(z)$$

 Degree $\mu + \frac{1}{2}$ and Order $\nu + \frac{1}{2}$

$$\mathcal{R}z > 0$$

$$8.2.7 \quad P_{-\nu-\frac{1}{2}}^{-\frac{1}{2}}\left(\frac{z}{(z^2-1)^{1/2}}\right) = \frac{(z^2-1)^{1/4} e^{-i\mu\pi} Q_{\nu}^{\mu}(z)}{(\frac{1}{2}\pi)^{1/2} \Gamma(\nu + \mu + 1)}$$

8.2.8

$$Q_{-\nu-\frac{1}{2}}^{-\frac{1}{2}}\left(\frac{z}{(z^2-1)^{1/2}}\right) = -i(\frac{1}{2}\pi)^{1/2} \Gamma(-\nu - \mu) (z^2 - 1)^{1/4} e^{-i\mu\pi} P_{\nu}^{\mu}(z)$$

8.3. Values on the Cut

$$(-1 < x < 1)$$

8.3.1

$$P_{\nu}^{\mu}(x) = \frac{1}{2} [e^{i\mu\pi} P_{\nu}^{\mu}(x + i0) + e^{-i\mu\pi} P_{\nu}^{\mu}(x - i0)]$$

 (Upper and lower signs according as $\mathcal{J}z \geq 0$.)

8.3.2

$$P_{\nu}^{\mu}(x) = e^{\pm i\mu\pi} P_{\nu}^{\mu}(x \pm i0) \quad *$$

8.3.3

$$= i\pi^{-1} e^{-i\mu\pi} [e^{-i\mu\pi} Q_{\nu}^{\mu}(x + i0) - e^{i\mu\pi} Q_{\nu}^{\mu}(x - i0)] \quad *$$

8.3.4

$$Q_{\nu}^{\mu}(x) = \frac{1}{2} e^{-i\mu\pi} [e^{-i\mu\pi} Q_{\nu}^{\mu}(x + i0) + e^{i\mu\pi} Q_{\nu}^{\mu}(x - i0)]$$

(Formulas for $P_{\nu}^{\mu}(x)$ and $Q_{\nu}^{\mu}(x)$ are obtained with the replacement of $z-1$ by $(1-x)e^{\pm i\pi}$, (z^2-1) by $(1-x^2)e^{\pm i\pi}$, $z+1$ by $x+1$ for $z=x \pm i0$.)

8.4. Explicit Expressions

$$(x = \cos \theta)$$

8.4.1

$$P_0(z) = 1 \quad P_0(x) = 1$$

8.4.2

$$Q_0(z) = \frac{1}{2} \ln \left(\frac{z+1}{z-1} \right) \quad Q_0(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) = xF\left(\frac{1}{2}, 1; \frac{3}{2}; x^2\right)$$

8.4.3

$$P_1(z) = z \quad P_1(x) = x = \cos \theta$$

8.4.4

$$Q_1(z) = \frac{z}{2} \ln \left(\frac{z+1}{z-1} \right) - 1 \quad Q_1(x) = \frac{x}{2} \ln \left(\frac{1+x}{1-x} \right) - 1$$

8.4.5

$$P_2(z) = \frac{1}{2}(3z^2 - 1) \quad P_2(x) = \frac{1}{2}(3x^2 - 1) = \frac{1}{4}(3 \cos 2\theta + 1)$$

8.4.6

$$Q_2(z) = \frac{1}{2} P_2(z) \ln \left(\frac{z+1}{z-1} \right) - \frac{3z}{2} \quad Q_2(x) = \frac{(3x^2 - 1)}{4} \ln \left(\frac{1+x}{1-x} \right) - \frac{3x}{2}$$

8.5. Recurrence Relations

(Both P_{ν}^{μ} and Q_{ν}^{μ} satisfy the same recurrence relations.)

Varying Order

8.5.1

$$P_{\nu}^{\mu+1}(z) = (z^2 - 1)^{-1/2} \{ (\nu - \mu) z P_{\nu}^{\mu}(z) - (\nu + \mu) P_{\nu-1}^{\mu}(z) \}$$

*See page II.

8.5.2

$$(z^2-1) \frac{dP_\nu^\mu(z)}{dz} = (\nu+\mu)(\nu-\mu+1)(z^2-1)^{\frac{1}{2}} P_{\nu-1}^{\mu-1}(z) - \mu z P_\nu^\mu(z)$$

Varying Degree

8.5.3

$$(\nu-\mu+1)P_{\nu+1}^\mu(z) = (2\nu+1)zP_\nu^\mu(z) - (\nu+\mu)P_{\nu-1}^\mu(z)$$

$$8.5.4 \quad (z^2-1) \frac{dP_\nu^\mu(z)}{dz} = \nu z P_\nu^\mu(z) - (\nu+\mu)P_{\nu-1}^\mu(z)$$

Varying Order and Degree

$$8.5.5 \quad P_{\nu+1}^\mu(z) = P_{\nu-1}^\mu(z) + (2\nu+1)(z^2-1)^{\frac{1}{2}} P_\nu^{\mu-1}(z)$$

8.6. Special Values

$$x=0$$

8.6.1

$$P_\nu^\mu(0) = 2^\mu \pi^{-\frac{1}{2}} \cos \left[\frac{1}{2} \pi (\nu + \mu) \right] \Gamma\left(\frac{1}{2} \nu + \frac{1}{2} \mu + \frac{1}{2}\right) / \Gamma\left(\frac{1}{2} \nu - \frac{1}{2} \mu + 1\right)$$

8.6.2

$$Q_\nu^\mu(0) = -2^{\mu+1} \pi^{-\frac{1}{2}} \sin \left[\frac{1}{2} \pi (\nu + \mu) \right] \Gamma\left(\frac{1}{2} \nu + \frac{1}{2} \mu + \frac{1}{2}\right) / \Gamma\left(\frac{1}{2} \nu - \frac{1}{2} \mu + \frac{1}{2}\right)$$

8.6.3

$$\left[\frac{dP_\nu^\mu(x)}{dx} \right]_{x=0} = 2^{\mu+1} \pi^{-\frac{1}{2}} \sin \left[\frac{1}{2} \pi (\nu + \mu) \right] \Gamma\left(\frac{1}{2} \nu + \frac{1}{2} \mu + 1\right) / \Gamma\left(\frac{1}{2} \nu - \frac{1}{2} \mu + \frac{1}{2}\right)$$

8.6.4

$$\left[\frac{dQ_\nu^\mu(x)}{dx} \right]_{x=0} = 2^\mu \pi^{\frac{1}{2}} \cos \left[\frac{1}{2} \pi (\nu + \mu) \right] \Gamma\left(\frac{1}{2} \nu + \frac{1}{2} \mu + 1\right) / \Gamma\left(\frac{1}{2} \nu - \frac{1}{2} \mu + \frac{1}{2}\right)$$

8.6.5

$$W\{P_\nu^\mu(x), Q_\nu^\mu(x)\}_{x=0} = \frac{2^{2\mu} \Gamma\left(\frac{1}{2} \nu + \frac{1}{2} \mu + 1\right) \Gamma\left(\frac{1}{2} \nu + \frac{1}{2} \mu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2} \nu - \frac{1}{2} \mu + 1\right) \Gamma\left(\frac{1}{2} \nu - \frac{1}{2} \mu + \frac{1}{2}\right)}$$

$$\mu = m = 1, 2, 3, \dots$$

8.6.6

$$P_\nu^m(z) = (z^2-1)^{\frac{1}{2}m} \frac{d^m P_\nu(z)}{dz^m},$$

$$P_\nu^m(x) = (-1)^m (1-x^2)^{\frac{1}{2}m} \frac{d^m P_\nu(x)}{dx^m}$$

8.6.7

$$Q_\nu^m(z) = (z^2-1)^{\frac{1}{2}m} \frac{d^m Q_\nu(z)}{dz^m},$$

$$Q_\nu^m(x) = (-1)^m (1-x^2)^{\frac{1}{2}m} \frac{d^m Q_\nu(x)}{dx^m}$$

$$\mu = \pm \frac{1}{2}$$

8.6.8

$$P_\nu^{\frac{1}{2}}(z) = (z^2-1)^{-1/4} (2\pi)^{-1/2} \{ [z + (z^2-1)^{1/2}]^{\nu+\frac{1}{2}} + [z + (z^2-1)^{1/2}]^{-\nu-\frac{1}{2}} \}$$

8.6.9

$$P_\nu^{-\frac{1}{2}}(z) = \left(\frac{2}{\pi}\right)^{1/2} \frac{(z^2-1)^{-1/4}}{2\nu+1} \{ [z + (z^2-1)^{1/2}]^{\nu+\frac{1}{2}} - [z + (z^2-1)^{1/2}]^{-\nu-\frac{1}{2}} \}$$

8.6.10

$$Q_\nu^{\frac{1}{2}}(z) = i \left(\frac{1}{2}\pi\right)^{1/2} (z^2-1)^{-1/4} [z + (z^2-1)^{1/2}]^{-\nu-\frac{1}{2}}$$

8.6.11

$$Q_\nu^{-\frac{1}{2}}(z) = -i (2\pi)^{1/2} \frac{(z^2-1)^{-1/4}}{2\nu+1} [z + (z^2-1)^{1/2}]^{-\nu-\frac{1}{2}} \quad *$$

8.6.12

$$P_\nu^{\frac{1}{2}}(\cos \theta) = \left(\frac{1}{2}\pi\right)^{-\frac{1}{2}} (\sin \theta)^{-\frac{1}{2}} \cos \left[(\nu + \frac{1}{2})\theta\right]$$

8.6.13

$$Q_\nu^{\frac{1}{2}}(\cos \theta) = -\left(\frac{1}{2}\pi\right)^{\frac{1}{2}} (\sin \theta)^{-\frac{1}{2}} \sin \left[(\nu + \frac{1}{2})\theta\right]$$

8.6.14

$$P_\nu^{-\frac{1}{2}}(\cos \theta) = \left(\frac{1}{2}\pi\right)^{-\frac{1}{2}} (\nu + \frac{1}{2})^{-1} (\sin \theta)^{-\frac{1}{2}} \sin \left[(\nu + \frac{1}{2})\theta\right]$$

8.6.15

$$Q_\nu^{-\frac{1}{2}}(\cos \theta) = (2\pi)^{-\frac{1}{2}} (2\nu+1)^{-1} (\sin \theta)^{-\frac{1}{2}} \cos \left[(\nu + \frac{1}{2})\theta\right] \quad *$$

$$\mu = -\nu$$

8.6.16

$$P_\nu^{-\nu}(z) = \frac{2^{-\nu} (z^2-1)^{\frac{1}{2}\nu}}{\Gamma(\nu+1)}$$

8.6.17

$$P_\nu^{-\nu}(\cos \theta) = \frac{2^{-\nu} (\sin \theta)^\nu}{\Gamma(\nu+1)}$$

$$\mu = 0, \nu = n$$

(Rodrigues' Formula)

8.6.18

$$P_n(z) = \frac{1}{2^n n!} \frac{d^n (z^2-1)^n}{dz^n}$$

$$8.6.19 \quad Q_n(x) = \frac{1}{2} P_n(x) \ln \frac{1+x}{1-x} - W_{n-1}(x)$$

where

$$W_{n-1}(x) = \frac{2n-1}{1 \cdot n} P_{n-1}(x) + \frac{2n-5}{3(n-1)} P_{n-3}(x) + \frac{2n-9}{5(n-2)} P_{n-5}(x) + \dots$$

$$= \sum_{m=1}^n \frac{1}{m} P_{m-1}(x) P_{n-m}(x)$$

$$W_{-1}(x) = 0$$

*See page 11.

$$\nu=0, 1$$

$$8.6.20 \quad \left[\frac{\partial P_\nu(\cos \theta)}{\partial \nu} \right]_{\nu=0} = 2 \ln (\cos \tfrac{1}{2}\theta)$$

$$8.6.21 \quad \left[\frac{\partial P_\nu^{-1}(\cos \theta)}{\partial \nu} \right]_{\nu=0} = -\tan \tfrac{1}{2}\theta - 2 \cot \tfrac{1}{2}\theta \ln (\cos \tfrac{1}{2}\theta)$$

$$8.6.22 \quad \left[\frac{\partial P_\nu^{-1}(\cos \theta)}{\partial \nu} \right]_{\nu=-1} = -\tfrac{1}{2} \tan \tfrac{1}{2}\theta \sin^2 \tfrac{1}{2}\theta + \sin \theta \ln (\cos \tfrac{1}{2}\theta)$$

8.7. Trigonometric Expansions ($0 < \theta < \pi$)

$$8.7.1 \quad P_\nu^\mu(\cos \theta) = \pi^{-1/2} 2^{\mu+1} (\sin \theta)^\mu \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} \sum_{k=0}^{\infty} \frac{(\mu+\frac{1}{2})_k (\nu+\mu+1)_k}{k! (\nu+\frac{3}{2})_k} \sin [(\nu+\mu+2k+1)\theta]$$

$$8.7.2 \quad Q_\nu^\mu(\cos \theta) = \pi^{1/2} 2^\mu (\sin \theta)^\mu \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} \sum_{k=0}^{\infty} \frac{(\mu+\frac{1}{2})_k (\nu+\mu+1)_k}{k! (\nu+\frac{3}{2})_k} \cos [(\nu+\mu+2k+1)\theta]$$

$$8.7.3 \quad P_n(\cos \theta) = \frac{2^{2n+2} (n!)^2}{\pi (2n+1)!} \left[\sin (n+1)\theta + \frac{n+1}{2n+3} \sin (n+3)\theta + \frac{1 \cdot 3}{2!} \frac{(n+1)(n+2)}{(2n+3)(2n+5)} \sin (n+5)\theta + \dots \right]$$

$$8.7.4 \quad Q_n(\cos \theta) = \frac{2^{2n+1} (n!)^2}{(2n+1)!} \left[\cos (n+1)\theta + \frac{n+1}{2n+3} \cos (n+3)\theta + \frac{1 \cdot 3}{2!} \frac{(n+1)(n+2)}{(2n+3)(2n+5)} \cos (n+5)\theta + \dots \right]$$

8.8. Integral Representations

(z not on the real axis between -1 and $-\infty$)

$$8.8.1 \quad P_\nu^\mu(z) = \frac{2^{-\nu} (z^2-1)^{-\frac{1}{2}\mu}}{\Gamma(-\nu-\mu) \Gamma(\nu+1)} \int_0^\infty (z + \cosh t)^{\mu-\nu-1} (\sinh t)^{2\nu+1} dt \quad (\Re(-\mu) > \Re \nu > -1)$$

$$8.8.2 \quad Q_\nu^\mu(z) = \frac{e^{i\mu\pi} \sqrt{\pi} 2^{-\mu}}{\Gamma(\mu+\frac{1}{2})} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} (z^2-1)^{\frac{1}{2}\mu} \int_0^\infty [z + (z^2-1)^{\frac{1}{2}} \cosh t]^{-\nu-\mu-1} (\sinh t)^{2\mu} dt \quad (\Re(\nu \pm \mu + 1) > 0) \quad *$$

$$8.8.3 \quad Q_n(z) = \frac{1}{2} \int_{-1}^1 (z-t)^{-1} P_n(t) dt = (-1)^{n+1} Q_n(-z)$$

(For other integral representations see [8.2].)

8.9. Summation Formulas

$$8.9.1 \quad (\xi-z) \sum_{m=0}^n (2m+1) P_m(z) P_m(\xi) = (n+1) [P_{n+1}(\xi) P_n(z) - P_n(\xi) P_{n+1}(z)]$$

$$8.9.2 \quad (\xi-z) \sum_{m=0}^n (2m+1) P_m(z) Q_m(\xi) = 1 - (n+1) [P_{n+1}(z) Q_n(\xi) - P_n(z) Q_{n+1}(\xi)]$$

8.10. Asymptotic Expansions

For fixed z and ν and $\Re \mu \rightarrow \infty$, 8.10.1–8.10.3 are asymptotic expansions if z is not on the real axis between $-\infty$ and -1 and $+\infty$ and $+1$. (Upper or lower signs according as $\Im z \gtrless 0$.)

$$8.10.1 \quad P_\nu^\mu(z) = \frac{\Gamma(\nu+\mu+1) \Gamma(\mu-\nu)}{\pi \Gamma(\mu+1)} \left(\frac{z+1}{z-1} \right)^{\frac{1}{2}\mu} \sin \mu\pi \left[F(-\nu, \nu+1; 1+\mu; \tfrac{1}{2} + \tfrac{1}{2}z) - \frac{\sin \nu\pi}{\sin \mu\pi} e^{\mp i\mu\pi} \left(\frac{z-1}{z+1} \right)^\mu F(-\nu, \nu+1; 1+\mu; \tfrac{1}{2} - \tfrac{1}{2}z) \right]$$

$$8.10.2 \quad Q_\nu^\mu(z) = \tfrac{1}{2} e^{i\mu\pi} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)} \left(\frac{z+1}{z-1} \right)^{\frac{1}{2}\mu} \Gamma(\mu-\nu) \left[F(-\nu, \nu+1; 1+\mu; \tfrac{1}{2} + \tfrac{1}{2}z) - e^{\mp i\nu\pi} \left(\frac{z-1}{z+1} \right)^\mu F(-\nu, \nu+1; 1+\mu; \tfrac{1}{2} - \tfrac{1}{2}z) \right]$$

*See page II.

$$8.10.3 \quad Q_{\nu}^{-\mu}(z) = \frac{e^{-i\mu\pi} \csc[\pi(\nu-\mu)]}{2\pi\Gamma(1+\mu)} \left[e^{\mp i\nu\pi} \left(\frac{z+1}{z-1} \right)^{-\frac{1}{2}\mu} F(-\nu, \nu+1; 1+\mu; \frac{1}{2}-\frac{1}{2}z) \right. \\ \left. - \left(\frac{z-1}{z+1} \right)^{-\frac{1}{2}\mu} F(-\nu, \nu+1; 1+\mu; \frac{1}{2}+\frac{1}{2}z) \right]$$

With μ replaced by $-\mu$, 8.1.2 is an asymptotic expansion for $P_{\nu}^{-\mu}(z)$ for fixed z and ν and $\Re \mu \rightarrow \infty$ if z is not on the real axis between $-\infty$ and -1 .

For fixed z and μ and $\Re \nu \rightarrow \infty$, 8.10.4 and 8.10.6 are asymptotic expansions if z is not on the real axis between $-\infty$ and -1 and $+\infty$ and $+1$; 8.10.5 if z is not on the real axis between $-\infty$ and $+1$.

$$8.10.4 \quad P_{\nu}^{\mu}(z) = (2\pi)^{-\frac{1}{2}} (z^2-1)^{-1/4} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} \left\{ [z+(z^2-1)^{\frac{1}{2}}]^{\nu+\frac{1}{2}} F(\frac{1}{2}+\mu, \frac{1}{2}-\mu; \frac{3}{2}+\nu; \frac{z+(z^2-1)^{\frac{1}{2}}}{2(z^2-1)^{\frac{1}{2}}}) \right. \\ \left. + i e^{-i\mu\pi} [z-(z^2-1)^{\frac{1}{2}}]^{\nu+\frac{1}{2}} F(\frac{1}{2}+\mu, \frac{1}{2}-\mu; \frac{3}{2}+\nu; \frac{-z+(z^2-1)^{\frac{1}{2}}}{2(z^2-1)^{\frac{1}{2}}}) \right\}$$

$$8.10.5 \quad Q_{\nu}^{\mu}(z) = e^{i\mu\pi} (\frac{1}{2}\pi)^{\frac{1}{2}} (z^2-1)^{-1/4} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} [z-(z^2-1)^{\frac{1}{2}}]^{\nu+\frac{1}{2}} F(\frac{1}{2}+\mu, \frac{1}{2}-\mu; \frac{3}{2}+\nu; \frac{-z+(z^2-1)^{\frac{1}{2}}}{2(z^2-1)^{\frac{1}{2}}})$$

$$8.10.6 \quad Q_{\nu}^{-\mu}(z) = \frac{e^{i\mu\pi} (\frac{1}{2}\pi)^{\frac{1}{2}} (z^2-1)^{-1/4}}{\sin[\pi(\mu-\nu)]} \frac{\Gamma(\mu+\nu)}{\Gamma(\frac{1}{2}-\mu)} \left\{ \cos \nu\pi [z+(z^2-1)^{\frac{1}{2}}]^{\nu-\frac{1}{2}} F(\frac{1}{2}+\mu, \frac{1}{2}-\mu; \frac{1}{2}+\nu; \frac{z+(z^2-1)^{\frac{1}{2}}}{2(z^2-1)^{\frac{1}{2}}}) \right. \\ \left. + i e^{i\nu\pi} \cos \mu\pi [z-(z^2-1)^{\frac{1}{2}}]^{\nu-\frac{1}{2}} F(\frac{1}{2}+\mu, \frac{1}{2}-\mu; \frac{1}{2}+\nu; \frac{-z+(z^2-1)^{\frac{1}{2}}}{2(z^2-1)^{\frac{1}{2}}}) \right\}$$

The related asymptotic expansion for $P_{\nu}^{\mu}(z)$ may be derived from 8.10.4 together with 8.2.1.

$$8.10.7 \quad P_{\nu}^{\mu}(\cos \theta) = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} (\frac{1}{2}\pi \sin \theta)^{-\frac{1}{2}} \cos[(\nu+\frac{1}{2})\theta - \frac{\pi}{4} + \frac{\mu\pi}{2}] + O(\nu^{-1})$$

$$8.10.8 \quad Q_{\nu}^{\mu}(\cos \theta) = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} \left(\frac{\pi}{2 \sin \theta} \right)^{\frac{1}{2}} \cos[(\nu+\frac{1}{2})\theta + \frac{\pi}{4} + \frac{\mu\pi}{2}] + O(\nu^{-1}) \quad (\epsilon < \theta \leq \pi - \epsilon, \epsilon > 0)$$

For other asymptotic expansions, see [8.7] and [8.9].

8.11. Toroidal Functions (or Ring Functions)

(Only special properties are given; other properties and representations follow from the earlier sections.)

$$8.11.1 \quad P_{\nu-\frac{1}{2}}^{\mu}(\cosh \eta) = [\Gamma(1-\mu)]^{-1} 2^{2\mu} (1-e^{-2\eta})^{-\mu} e^{-(\nu+\frac{1}{2})\eta} F(\frac{1}{2}-\mu, \frac{1}{2}+\nu-\mu; 1-2\mu; 1-e^{-2\eta})$$

$$8.11.2 \quad P_{n-\frac{1}{2}}^m(\cosh \eta) = \frac{\Gamma(n+m+\frac{1}{2})(\sinh \eta)^m}{\Gamma(n-m+\frac{1}{2}) 2^m \sqrt{\pi} \Gamma(m+\frac{1}{2})} \int_0^{\pi} \frac{(\sin \varphi)^{2m} d\varphi}{(\cosh \eta + \cos \varphi \sinh \eta)^{n+m+\frac{1}{2}}}$$

$$8.11.3 \quad Q_{\nu-\frac{1}{2}}^{\mu}(\cosh \eta) = [\Gamma(1+\nu)]^{-1} \sqrt{\pi} e^{i\mu\pi} \Gamma(\frac{1}{2}+\nu+\mu) (1-e^{-2\eta})^{\mu} e^{-(\nu+\frac{1}{2})\eta} F(\frac{1}{2}+\mu, \frac{1}{2}+\nu+\mu; 1+\nu; e^{-2\eta}) \quad *$$

$$8.11.4 \quad Q_{n-\frac{1}{2}}^m(\cosh \eta) = \frac{(-1)^m \Gamma(n+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})} \int_0^{\infty} \frac{\cosh mt \, dt}{(\cosh \eta + \cosh t \sinh \eta)^{n+\frac{1}{2}}} \quad * \quad (n > m)$$

*See page 11.

8.12. Conical Functions

$$(P_{-\frac{1}{2}+i\lambda}(\cos \theta), Q_{-\frac{1}{2}+i\lambda}(\cos \theta))$$

(Only special properties are given as other properties and representations follow from earlier sections with $\nu = -\frac{1}{2} + i\lambda$ (λ , a real parameter) and $z = \cos \theta$.)

8.12.1

$$P_{-\frac{1}{2}+i\lambda}(\cos \theta) = 1 + \frac{4\lambda^2 + 1^2}{2^2} \sin^2 \frac{\theta}{2} + \frac{(4\lambda^2 + 1^2)(4\lambda^2 + 3^2)}{2^2 4^2} \sin^4 \frac{\theta}{2} + \dots \quad (0 \leq \theta < \pi)$$

$$8.12.2 \quad P_{-\frac{1}{2}+i\lambda}(\cos \theta) = P_{-\frac{1}{2}-i\lambda}(\cos \theta)$$

$$8.12.3 \quad P_{-\frac{1}{2}+i\lambda}(\cos \theta) = \frac{2}{\pi} \int_0^\theta \frac{\cosh \lambda t dt}{\sqrt{2}(\cos t - \cos \theta)}$$

8.12.4

$$Q_{-\frac{1}{2}+i\lambda}(\cos \theta) = \pm i \sinh \lambda \pi \int_0^\infty \frac{\cos \lambda t dt}{\sqrt{2}(\cosh t + \cos \theta)} + \int_0^\infty \frac{\cosh \lambda t dt}{\sqrt{2}(\cosh t - \cos \theta)}$$

8.12.5

$$P_{-\frac{1}{2}+i\lambda}(-\cos \theta) = \frac{\cosh \lambda \pi}{\pi} [Q_{-\frac{1}{2}+i\lambda}(\cos \theta) + Q_{-\frac{1}{2}-i\lambda}(\cos \theta)]$$

 * 8.13. Relation to Elliptic Integrals
(see chapter 17) ($\mathcal{R}\eta > 0$)

$$8.13.1 \quad P_{-\frac{1}{2}}(z) = \frac{2}{\pi} \sqrt{\frac{2}{z+1}} K\left(\sqrt{\frac{z-1}{z+1}}\right)$$

$$8.13.2 \quad P_{-\frac{1}{2}}(\cosh \eta) = \left[\frac{\pi}{2} \cosh \frac{\eta}{2}\right]^{-1} K\left(\tanh \frac{\eta}{2}\right)$$

$$8.13.3 \quad Q_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{z+1}} K\left(\sqrt{\frac{2}{z+1}}\right)$$

$$8.13.4 \quad Q_{-\frac{1}{2}}(\cosh \eta) = 2e^{-\eta/2} K(e^{-\eta})$$

8.13.5

$$P_{\frac{1}{2}}(z) = \frac{2}{\pi} (z + \sqrt{z^2 - 1})^{\frac{1}{2}} E\left(\sqrt{\frac{2(z^2 - 1)^{1/2}}{z + (z^2 - 1)^{1/2}}}\right)$$

$$8.13.6 \quad P_{\frac{1}{2}}(\cosh \eta) = \frac{2}{\pi} e^{\eta/2} E(\sqrt{1 - e^{-2\eta}})$$

8.13.7

$$Q_{\frac{1}{2}}(z) = z \sqrt{\frac{2}{z+1}} K\left(\sqrt{\frac{2}{z+1}}\right) - [2(z+1)]^{\frac{1}{2}} E\left(\sqrt{\frac{2}{z+1}}\right) \quad (-1 < x < 1) \quad *$$

$$8.13.8 \quad P_{-\frac{1}{2}}(x) = \frac{2}{\pi} K\left(\sqrt{\frac{1-x}{2}}\right)$$

$$8.13.9 \quad P_{-\frac{1}{2}}(\cos \theta) = \frac{2}{\pi} K\left(\sin \frac{\theta}{2}\right)$$

$$8.13.10 \quad Q_{-\frac{1}{2}}(x) = K\left(\sqrt{\frac{1+x}{2}}\right) \quad *$$

$$8.13.11 \quad P_{\frac{1}{2}}(x) = \frac{2}{\pi} \left[2E\left(\sqrt{\frac{1-x}{2}}\right) - K\left(\sqrt{\frac{1-x}{2}}\right) \right]$$

$$8.13.12 \quad Q_{\frac{1}{2}}(x) = K\left(\sqrt{\frac{1+x}{2}}\right) - 2E\left(\sqrt{\frac{1+x}{2}}\right) \quad *$$

8.14. Integrals

$$8.14.1 \quad \int_1^\infty P_\nu(x) Q_\rho(x) dx = [(\rho - \nu)(\rho + \nu + 1)]^{-1} \quad (\mathcal{R}\rho > \mathcal{R}\nu > 0)$$

$$8.14.2 \quad \int_1^\infty Q_\nu(x) Q_\rho(x) dx = [(\rho - \nu)(\rho + \nu + 1)]^{-1} [\psi(\rho + 1) - \psi(\nu + 1)] \quad (\mathcal{R}(\rho + \nu) > -1, \rho + \nu + 1 \neq 0; \nu, \rho \neq -1, -2, -3, \dots)$$

$$8.14.3 \quad \int_1^\infty [Q_\nu(x)]^2 dx = (2\nu + 1)^{-1} \psi'(\nu + 1) \quad (\mathcal{R}\nu > -\frac{1}{2})$$

$$8.14.4 \quad \int_{-1}^1 P_\nu(x) P_\rho(x) dx = \frac{2}{\pi^2} [(\rho - \nu)(\rho + \nu + 1)]^{-1} \{ 2 \sin \pi \nu \sin \pi \rho [\psi(\nu + 1) - \psi(\rho + 1)] + \pi \sin(\pi \rho - \pi \nu) \} \quad (\rho + \nu + 1 \neq 0)$$

$$8.14.5 \quad \int_{-1}^1 [P_\nu(x)]^2 dx = \frac{\pi^2 - 2(\sin \pi \nu)^2 \psi'(\nu + 1)}{\pi^2(\nu + \frac{1}{2})} \quad *$$

$$8.14.6 \quad \int_{-1}^1 Q_\nu(x) Q_\rho(x) dx = [(\rho - \nu)(\rho + \nu + 1)]^{-1} \{ [\psi(\nu + 1) - \psi(\rho + 1)][1 + \cos \rho \pi \cos \nu \pi] - \frac{1}{2} \pi \sin(\nu \pi - \rho \pi) \} \quad (\rho + \nu + 1 \neq 0; \nu, \rho \neq -1, -2, -3, \dots)$$

$$8.14.7 \quad \int_{-1}^1 [Q_\nu(x)]^2 dx = (2\nu + 1)^{-1} \{ \frac{1}{2} \pi^2 - \psi'(\nu + 1)[1 + (\cos \nu \pi)^2] \} \quad (\nu \neq -1, -2, -3, \dots)$$

*See page 11.

$$8.14.8 \quad \int_{-1}^1 P_\nu(x) Q_\rho(x) dx = [(\nu - \rho)(\rho + \nu + 1)]^{-1} \left\{ 1 - \cos(\rho\pi - \nu\pi) - \frac{2}{\pi} \sin \pi\nu \cos \pi\nu [\psi(\nu+1) - \psi(\rho+1)] \right\} \\ (\Re \nu > 0, \Re \rho > 0, \rho \neq \nu)$$

$$8.14.9 \quad \int_{-1}^1 P_\nu(x) Q_\nu(x) dx = -\frac{1}{\pi} (2\nu+1)^{-1} \sin 2\nu\pi \psi'(\nu+1) \quad (\Re \nu > 0) \\ (m, n, l \text{ positive integers})$$

8.14.10

$$\int_{-1}^1 Q_n^m(x) P_l^m(x) dx = (-1)^m \frac{1 - (-1)^{l+n} (n+m)!}{(l-n)(l+n+1)(n-m)!}$$

$$8.14.11 \quad \int_{-1}^1 P_n^m(x) P_l^m(x) dx = 0 \quad (l \neq n)$$

$$8.14.12 \quad \int_{-1}^1 P_n^m(x) P_n^l(x) (1-x^2)^{-1} dx = 0 \quad (l \neq m)$$

$$8.14.13 \quad \int_{-1}^1 [P_n^m(x)]^2 dx = (n + \frac{1}{2})^{-1} (n+m)! / (n-m)!$$

8.14.14

$$\int_{-1}^1 (1-x^2)^{-1} [P_n^m(x)]^2 dx = (n+m)! / m(n-m)!$$

8.14.15

$$\int_0^1 P_\nu(x) x^\rho dx = \frac{\pi^{1/2} 2^{-\rho-1} \Gamma(1+\rho)}{\Gamma(1+\frac{1}{2}\rho-\frac{1}{2}\nu) \Gamma(\frac{1}{2}\rho+\frac{1}{2}\nu+\frac{3}{2})} \quad (\Re \rho > -1)$$

8.14.16

$$\int_0^\pi (\sin t)^{\alpha-1} P_\nu^{-\mu}(\cos t) dt = \frac{2^{-\mu} \pi \Gamma(\frac{1}{2}\alpha + \frac{1}{2}\mu) \Gamma(\frac{1}{2}\alpha - \frac{1}{2}\mu)}{\Gamma(\frac{1}{2} + \frac{1}{2}\alpha + \frac{1}{2}\nu) \Gamma(\frac{1}{2}\alpha - \frac{1}{2}\nu) \Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu + 1) \Gamma(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2})} \quad (\Re(\alpha \pm \mu) > 0)$$

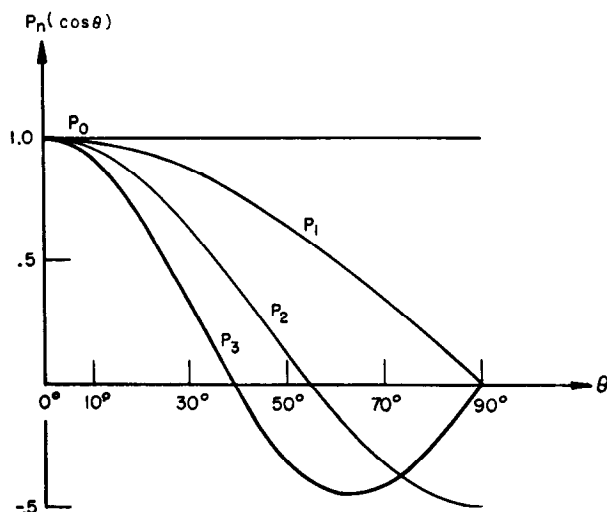
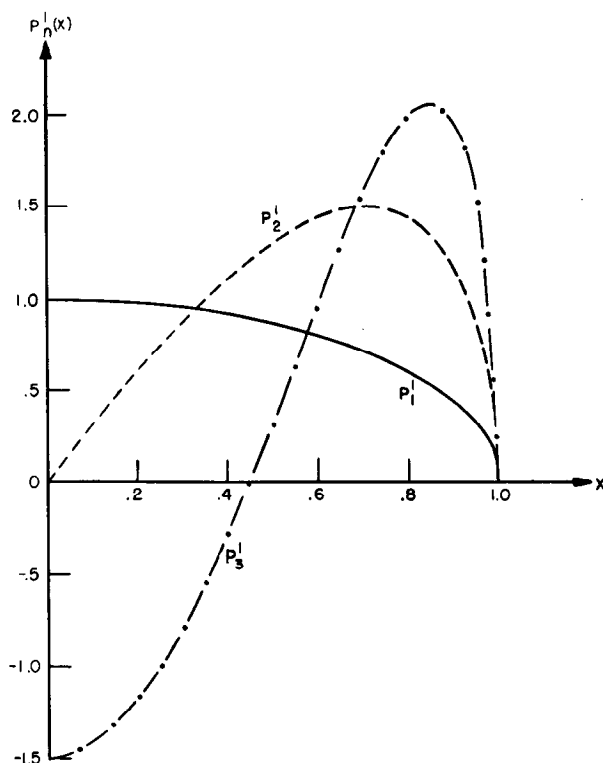
8.14.17

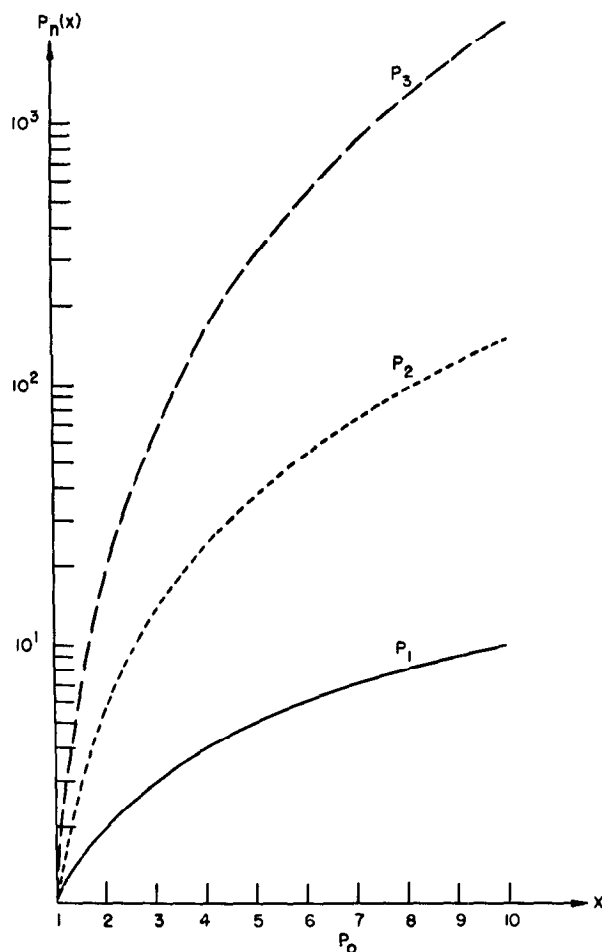
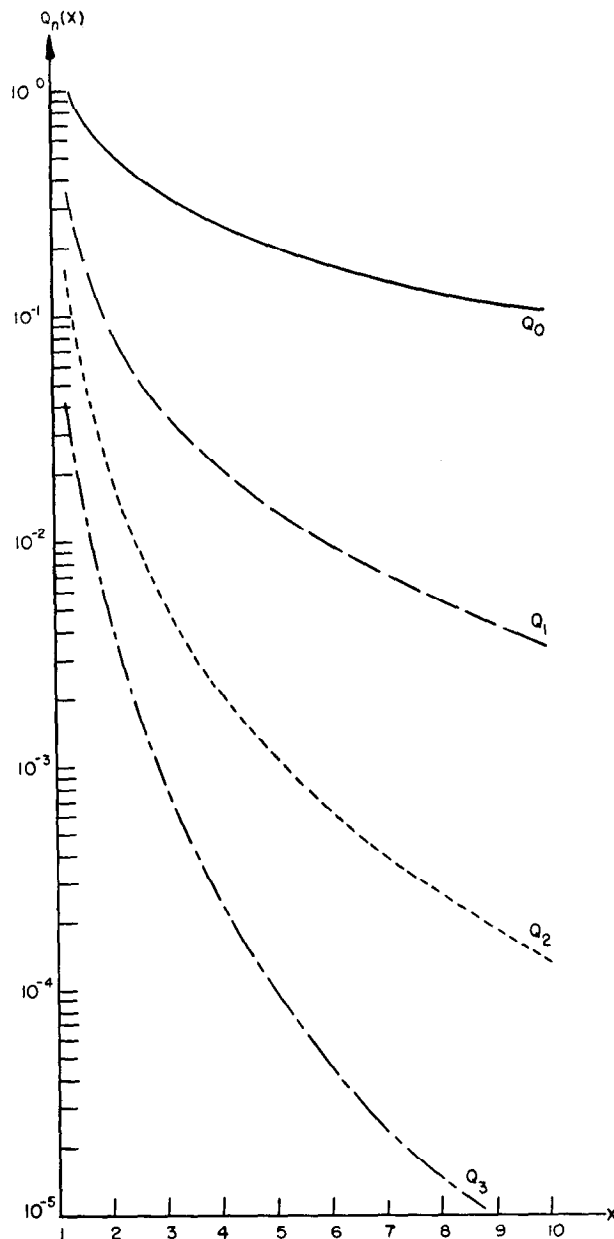
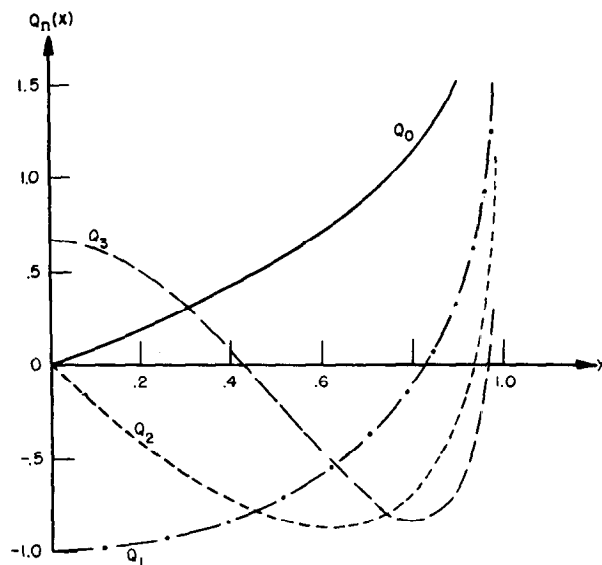
$$P_\nu^{-m}(z) = (z^2 - 1)^{-1/2m} \int_1^z \cdots \int_1^z P_\nu(z) (dz)^m$$

8.14.18

$$Q_\nu^{-m}(z) = (-1)^m (z^2 - 1)^{-1/2m} \int_z^\infty \cdots \int_z^\infty Q_\nu(z) (dz)^m$$

For other integrals, see [8.2], [8.4] and chapter 22.

FIGURE 8.1. $P_n(\cos \theta)$. $n=0(1)3$.FIGURE 8.2. $P_n^l(x)$. $n=1(1)3, x \leq 1$.


 FIGURE 8.3. $P_n(x)$. $n=0(1)3$, $x \geq 1$.

 FIGURE 8.5. $Q_n(x)$. $n=0(1)3$, $x > 1$.

 FIGURE 8.4. $Q_n(x)$. $n=0(1)3$, $x < 1$.

Numerical Methods

8.15. Use and Extension of the Tables

Computation of $P_n(x)$

For all values of x there is very little loss of significant figures (except at zeros) in using the recurrence relation 8.5.3 for increasing values of n .

Example 1. Compute $P_n(x)$ for $x=.31415\ 92654$ and $x=2.6$ for $n=2(1)8$.

n	$P_n(.31415\ 92654)$	$P_n(2.6)$
0	1	1
1	.31415 92654	2.6
2	-.35195 59340	9.64
3	-.39372 32064	40.04
4	.04750 63122	174.952
5	.34184 27517	786.74336
6	.15729 86975	3604.350016
7	-.20123 39354	16729.51005
8	-.25617 29328	78402.55522

Computing $P_8(x)$ using **Table 22.9** carrying ten significant figures, $P_8(.31415\ 92654) = -.25617\ 2933$ and $P_8(2.6) = 78402.55526$.

Computation of $Q_n(x)$

For $x < 1$, use of **8.5.3** for increasing values of n leads to very little loss of significant figures. However, for $x > 1$, the recurrence relation **8.5.3** should be used only for decreasing values of n , after having first obtained Q_n using the formulas in terms of hypergeometric functions.

Example 2. Compute $Q_n(x)$ for $x = .31415\ 92654$ and $n = 0(1)4$.

With the aid of **8.4.2** and **8.4.4** we obtain

n	$Q_n(.31415\ 92654)$
0	.32515 34813
1	-.89785 00212
2	-.58567 85953
3	.29190 60854
4	.59974 26989

Using the results of **Example 1** together with **8.6.19**, we find $Q_4(x) = \frac{1}{2}P_4(x)\ln\left(\frac{1+x}{1-x}\right) - W_3(x)$ where $W_3 = \frac{7}{4}P_3 + \frac{1}{3}P_1$, giving $Q_4(.31415\ 92654) = .59974\ 26989$.

Example 3. Compute $Q_5(x)$ for $x = 2.6$.

Ten terms in the series for $F\left(\frac{\nu+2}{2}, \frac{\nu+1}{2}; \nu+\frac{3}{2}, \frac{1}{z^2}\right)$ of **8.1.3** are necessary to obtain nine significant figures giving $Q_5(2.6) = 4.8182\ 4468 \times 10^{-5}$. Using **8.5.3** with increasing values of n carrying ten significant figures we obtain

n	$Q_n(2.6)$
0	.40546 51081
1	.05420 928
2	.00868 364
3	.00148 95
4	.00026 49
5	.00004 81

where Q_0 and Q_1 are obtained using **8.4.2** and **8.4.4**.

Computation of $P_{\pm\frac{1}{2}}(x)$, $Q_{\pm\frac{1}{2}}(x)$

For all values of x , $P_{\pm\frac{1}{2}}(x)$ and $Q_{\pm\frac{1}{2}}(x)$ are most easily computed by means of **8.13**.

Example 4. Compute $Q_{-\frac{1}{2}}(x)$ for $x = 2.6$.

Using **8.13.3** and interpolating in **Table 17.1** for $K(.5)$, we find

$$\begin{aligned} Q_{-\frac{1}{2}}(2.6) &= \sqrt{\frac{2}{x+1}} K\left(\sqrt{\frac{2}{x+1}}\right) \\ &= (.74535\ 59925)(1.90424\ 1417) \\ &= 1.41933\ 7751. \end{aligned}$$

On the other hand, at least nine terms in the expansion of $F\left(\frac{\nu+2}{2}, \frac{\nu+1}{2}; \nu+\frac{3}{2}, \frac{1}{z^2}\right)$ of **8.1.3** are necessary to obtain comparable accuracy.

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Texts

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Tables

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- $P_n^m(x)$, $\frac{d}{dx} P_n^m(x)$, $n = 1(1)10$, $(-1)^m Q_n^m(x)$, $(-1)^{m+1} \frac{d}{dx} Q_n^m(x)$, $n = 0(1)10$, $m(\leq n) = 0(1)4$, $x = 1(1)10$, 6S or exact; $i^{-n} P_n^m(ix)$, $i^{-n} \frac{d}{dx} P_n^m(ix)$, $n = 1(1)10$, $i^{n+2m+1} Q_n^m(ix)$, $i^{n+2m-1} \frac{d}{dx} Q_n^m(ix)$, $n = 0(1)10$, $m(\leq n) = 0(1)4$, $x = 0(1)10$, 6S; $P_{n+\frac{1}{2}}^m(x)$, $\frac{d}{dx} P_{n-\frac{1}{2}}^m(x)$, $(-1)^m Q_{n-\frac{1}{2}}^m(x)$, $(-1)^{m+1} \frac{d}{dx} Q_{n+\frac{1}{2}}^m(x)$, $n = -1(1)4$, $m = 0(1)4$, $x = 1(1)10$, 4-6S.
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9. Bessel Functions of Integer Order

F. W. J. OLVER¹

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$\lambda^{-1}=.8(-.2) .2, .1(-.02)0$, 5D (8D for $s=1$)	
sth zero of $J_1(x)Y_1(\lambda x) - Y_1(x)J_1(\lambda x)$	
$\lambda^{-1}=.8(-.2) .2, .1(-.02)0$, 5D (8D for $s=1$)	
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9. Bessel Functions of Integer Order

Mathematical Properties

Notation

The tables in this chapter are for Bessel functions of integer order; the text treats general orders. The conventions used are:

$z = x + iy$; x, y real.

n is a positive integer or zero.

ν, μ are unrestricted except where otherwise indicated; ν is supposed real in the sections devoted to Kelvin functions 9.9, 9.10, and 9.11.

The notation used for the Bessel functions is that of Watson [9.15] and the British Association and Royal Society Mathematical Tables. The function $Y_\nu(z)$ is often denoted $N_\nu(z)$ by physicists and European workers.

Other notations are those of:

Aldis, Airey:

$$G_n(z) \text{ for } -\frac{1}{2}\pi Y_n(z), K_n(z) \text{ for } (-)^n K_n(z).$$

Clifford:

$$C_n(x) \text{ for } x^{-1/2} J_n(2\sqrt{x}).$$

Gray, Mathews and MacRobert [9.9]:

$$Y_n(z) \text{ for } \frac{1}{2}\pi Y_n(z) + (\ln 2 - \gamma) J_n(z),$$

$$\bar{Y}_\nu(z) \text{ for } \pi e^{\nu\pi i} \sec(\nu\pi) Y_\nu(z),$$

$$G_\nu(z) \text{ for } \frac{1}{2}\pi i H_\nu^{(1)}(z).$$

Jahnke, Emde and Lösch [9.32]:

$$\Lambda_\nu(z) \text{ for } \Gamma(\nu+1) (\frac{1}{2}z)^{-\nu} J_\nu(z).$$

Jeffreys:

$$Hs_\nu(z) \text{ for } H_\nu^{(1)}(z), Hi_\nu(z) \text{ for } H_\nu^{(2)}(z),$$

$$Kh_\nu(z) \text{ for } (2/\pi) K_\nu(z).$$

Heine:

$$K_n(z) \text{ for } -\frac{1}{2}\pi Y_n(z).$$

Neumann:

$$Y^n(z) \text{ for } \frac{1}{2}\pi Y_n(z) + (\ln 2 - \gamma) J_n(z).$$

Whittaker and Watson [9.18]:

$$K_\nu(z) \text{ for } \cos(\nu\pi) K_\nu(z).$$

Bessel Functions J and Y

9.1. Definitions and Elementary Properties

Differential Equation

$$9.1.1 \quad z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2)w = 0$$

Solutions are the Bessel functions of the first kind $J_{\pm\nu}(z)$, of the second kind $Y_\nu(z)$ (also called Weber's function) and of the third kind $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ (also called the Hankel functions). Each is a regular (holomorphic) function of z throughout the z -plane cut along the negative real axis, and for fixed $z (\neq 0)$ each is an entire (integral) function of ν . When $\nu = \pm n$, $J_\nu(z)$ has no branch point and is an entire (integral) function of z .

Important features of the various solutions are as follows: $J_\nu(z) (\Re \nu \geq 0)$ is bounded as $z \rightarrow 0$ in any bounded range of $\arg z$. $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent except when ν is an integer. $J_\nu(z)$ and $Y_\nu(z)$ are linearly independent for all values of ν .

$H_\nu^{(1)}(z)$ tends to zero as $|z| \rightarrow \infty$ in the sector $0 < \arg z < \pi$; $H_\nu^{(2)}(z)$ tends to zero as $|z| \rightarrow \infty$ in the sector $-\pi < \arg z < 0$. For all values of ν , $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are linearly independent.

Relations Between Solutions

$$9.1.2 \quad Y_\nu(z) = \frac{J_\nu(z) \cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}$$

The right of this equation is replaced by its limiting value if ν is an integer or zero.

9.1.3

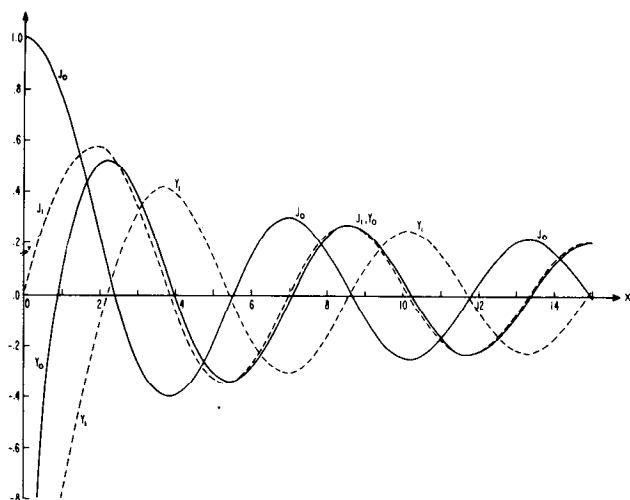
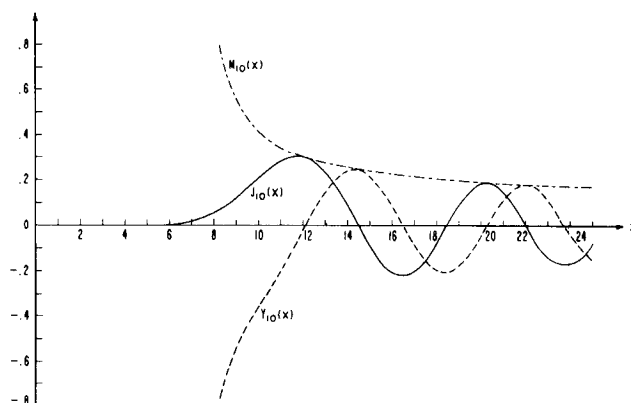
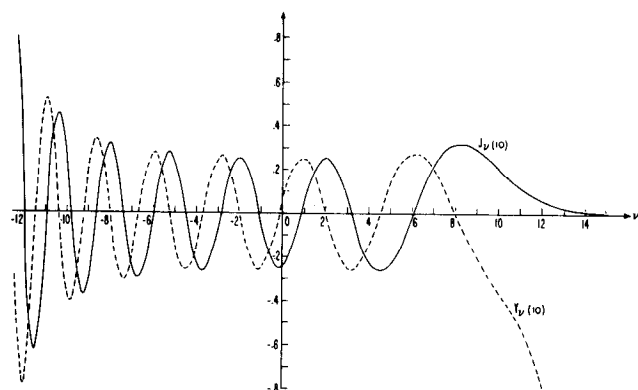
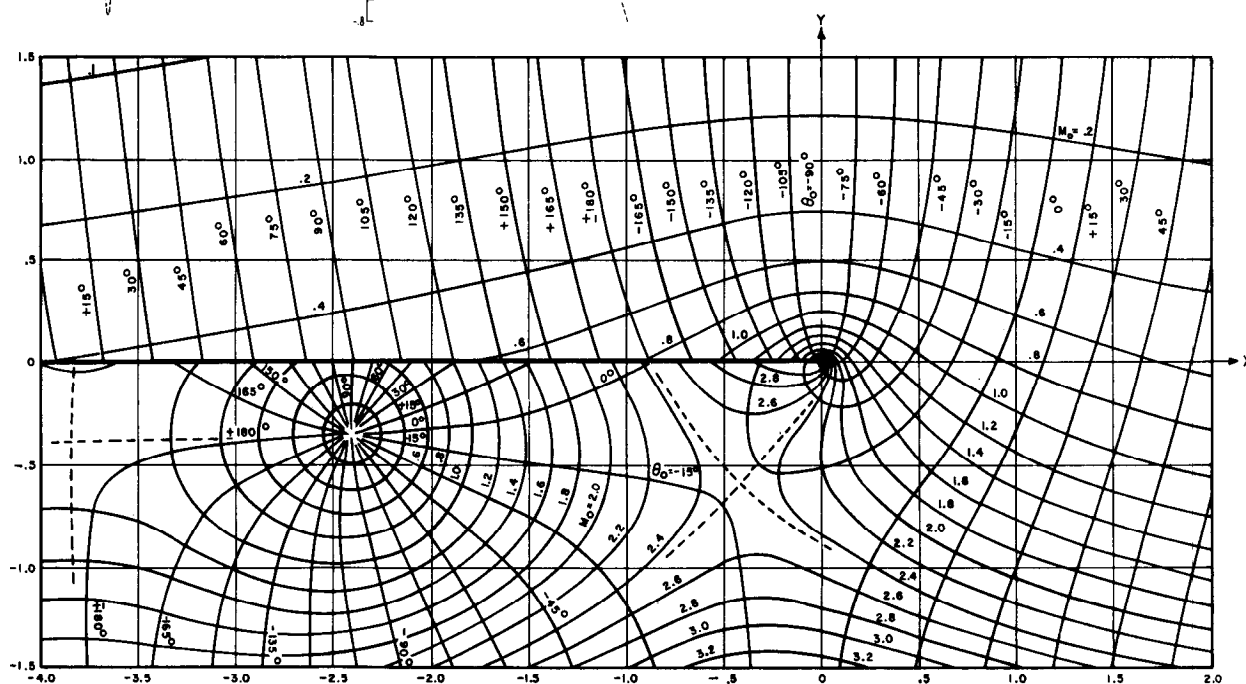
$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z) \\ = i \csc(\nu\pi) \{ e^{-\nu\pi i} J_\nu(z) - J_{-\nu}(z) \}$$

9.1.4

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z) \\ = i \csc(\nu\pi) \{ J_{-\nu}(z) - e^{\nu\pi i} J_\nu(z) \}$$

$$9.1.5 \quad J_{-n}(z) = (-)^n J_n(z) \quad Y_{-n}(z) = (-)^n Y_n(z)$$

$$9.1.6 \quad H_{-n}^{(1)}(z) = e^{\nu\pi i} H_n^{(1)}(z) \quad H_{-n}^{(2)}(z) = e^{-\nu\pi i} H_n^{(2)}(z)$$


 FIGURE 9.1. $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$.

 FIGURE 9.2. $J_{10}(x)$, $Y_{10}(x)$, and $M_{10}(x) = \sqrt{J_{10}^2(x) + Y_{10}^2(x)}$.

 FIGURE 9.3. $J_{\nu}(10)$ and $Y_{\nu}(10)$.

 FIGURE 9.4. Contour lines of the modulus and phase of the Hankel Function $H_0^{(1)}(x+iy) = M_0 e^{i\theta_0}$. From E. Jahnke, F. Emde, and F. Lösch, Tables of higher functions, McGraw-Hill Book Co., Inc., New York, N.Y., 1960 (with permission).

Limiting Forms for Small Arguments

When ν is fixed and $z \rightarrow 0$

9.1.7

$$J_\nu(z) \sim (\frac{1}{2}z)^\nu / \Gamma(\nu+1) \quad (\nu \neq -1, -2, -3, \dots)$$

$$9.1.8 \quad Y_0(z) \sim -iH_0^{(1)}(z) \sim iH_0^{(2)}(z) \sim (2/\pi) \ln z$$

9.1.9

$$Y_\nu(z) \sim -iH_\nu^{(1)}(z) \sim iH_\nu^{(2)}(z) \sim -(1/\pi) \Gamma(\nu) (\frac{1}{2}z)^{-\nu} \quad (\Re \nu > 0)$$

Ascending Series

$$9.1.10 \quad J_\nu(z) = (\frac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}z^2)^k}{k! \Gamma(\nu+k+1)}$$

9.1.11

$$Y_n(z) = -\frac{(\frac{1}{2}z)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} (\frac{1}{4}z^2)^k + \frac{2}{\pi} \ln(\frac{1}{2}z) J_n(z) - \frac{(\frac{1}{2}z)^n}{\pi} \sum_{k=0}^{\infty} \{\psi(k+1) + \psi(n+k+1)\} \frac{(-\frac{1}{4}z^2)^k}{k!(n+k)!}$$

where $\psi(n)$ is given by 6.3.2.

$$9.1.12 \quad J_0(z) = 1 - \frac{\frac{1}{4}z^2}{(1!)^2} + \frac{(\frac{1}{4}z^2)^2}{(2!)^2} - \frac{(\frac{1}{4}z^2)^3}{(3!)^2} + \dots$$

9.1.13

$$Y_0(z) = \frac{2}{\pi} \{ \ln(\frac{1}{2}z) + \gamma \} J_0(z) + \frac{2}{\pi} \{ \frac{\frac{1}{4}z^2}{(1!)^2} - (1+\frac{1}{2}) \frac{(\frac{1}{4}z^2)^2}{(2!)^2} + (1+\frac{1}{2}+\frac{1}{3}) \frac{(\frac{1}{4}z^2)^3}{(3!)^2} - \dots \}$$

9.1.14

$$J_\nu(z) J_\mu(z) =$$

$$(\frac{1}{2}z)^{\nu+\mu} \sum_{k=0}^{\infty} \frac{(-)^k \Gamma(\nu+\mu+2k+1) (\frac{1}{4}z^2)^k}{\Gamma(\nu+k+1) \Gamma(\mu+k+1) \Gamma(\nu+\mu+k+1) k!}$$

Wronskians

9.1.15

$$W\{J_\nu(z), J_{-\nu}(z)\} = J_{\nu+1}(z) J_{-\nu}(z) + J_\nu(z) J_{-(\nu+1)}(z) = -2 \sin(\nu\pi) / (\pi z)$$

9.1.16

$$W\{J_\nu(z), Y_\nu(z)\} = J_{\nu+1}(z) Y_\nu(z) - J_\nu(z) Y_{\nu+1}(z) = 2/(\pi z)$$

9.1.17

$$W\{H_\nu^{(1)}(z), H_\nu^{(2)}(z)\} = H_{\nu+1}^{(1)}(z) H_\nu^{(2)}(z) - H_\nu^{(1)}(z) H_{\nu+1}^{(2)}(z) = -4i/(\pi z)$$

Integral Representations

9.1.18

$$J_0(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta) d\theta = \frac{1}{\pi} \int_0^\pi \cos(z \cos \theta) d\theta$$

9.1.19

$$Y_0(z) = \frac{4}{\pi^2} \int_0^{\frac{1}{2}\pi} \cos(z \cos \theta) \{ \gamma + \ln(2z \sin^2 \theta) \} d\theta$$

9.1.20

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}} \Gamma(\nu + \frac{1}{2})} \int_0^\pi \cos(z \cos \theta) \sin^{2\nu} \theta d\theta = \frac{2(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}} \Gamma(\nu + \frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos(zt) dt \quad (\Re \nu > -\frac{1}{2})$$

9.1.21

$$J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta - n\theta) d\theta = \frac{i^{-n}}{\pi} \int_0^\pi e^{iz \cos \theta} \cos(n\theta) d\theta$$

9.1.22

$$J_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta - \nu\theta) d\theta - \frac{\sin(\nu\pi)}{\pi} \int_0^\infty e^{-z \sinh t - \nu t} dt \quad (|\arg z| < \frac{1}{2}\pi)$$

$$Y_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin(z \sin \theta - \nu\theta) d\theta$$

$$- \frac{1}{\pi} \int_0^\infty \{ e^{\nu t} + e^{-\nu t} \cos(\nu\pi) \} e^{-z \sinh t} dt \quad (|\arg z| < \frac{1}{2}\pi)$$

9.1.23

$$J_0(x) = \frac{2}{\pi} \int_0^\infty \sin(x \cosh t) dt \quad (x > 0)$$

$$Y_0(x) = -\frac{2}{\pi} \int_0^\infty \cos(x \cosh t) dt \quad (x > 0)$$

9.1.24

$$J_\nu(x) = \frac{2(\frac{1}{2}x)^{-\nu}}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2}-\nu)} \int_1^\infty \frac{\sin(xt) dt}{(t^2-1)^{\nu+\frac{1}{2}}} \quad (|\Re \nu| < \frac{1}{2}, x > 0)$$

$$Y_\nu(x) = -\frac{2(\frac{1}{2}x)^{-\nu}}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2}-\nu)} \int_1^\infty \frac{\cos(xt) dt}{(t^2-1)^{\nu+\frac{1}{2}}} \quad (|\Re \nu| < \frac{1}{2}, x > 0)$$

9.1.25

$$H_\nu^{(1)}(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty+\pi i} e^{z \sinh t - \nu t} dt \quad (|\arg z| < \frac{1}{2}\pi)$$

$$H_\nu^{(2)}(z) = -\frac{1}{\pi i} \int_{-\infty}^{\infty-\pi i} e^{z \sinh t - \nu t} dt \quad (|\arg z| < \frac{1}{2}\pi)$$

9.1.26

$$J_\nu(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(-t) (\frac{1}{2}x)^{\nu+2t}}{\Gamma(\nu+t+1)} dt \quad (\Re \nu > 0, x > 0)$$

In the last integral the path of integration must lie to the left of the points $t=0, 1, 2, \dots$

Recurrence Relations

9.1.27

$$\mathcal{C}_{\nu-1}(z) + \mathcal{C}_{\nu+1}(z) = \frac{2\nu}{z} \mathcal{C}_{\nu}(z)$$

$$\mathcal{C}_{\nu-1}(z) - \mathcal{C}_{\nu+1}(z) = 2\mathcal{C}'_{\nu}(z)$$

$$\mathcal{C}'_{\nu}(z) = \mathcal{C}_{\nu-1}(z) - \frac{\nu}{z} \mathcal{C}_{\nu}(z)$$

$$\mathcal{C}'_{\nu}(z) = -\mathcal{C}_{\nu+1}(z) + \frac{\nu}{z} \mathcal{C}_{\nu}(z)$$

$\mathcal{C}$ denotes $J, Y, H^{(1)}, H^{(2)}$ or any linear combination of these functions, the coefficients in which are independent of z and ν .

9.1.28 $J'_0(z) = -J_1(z) \quad Y'_0(z) = -Y_1(z)$

If $f_{\nu}(z) = z^p \mathcal{C}_{\nu}(\lambda z^q)$ where p, q, λ are independent of ν , then

9.1.29

$$f_{\nu-1}(z) + f_{\nu+1}(z) = (2\nu/\lambda) z^{-q} f_{\nu}(z)$$

$$(p + \nu q) f_{\nu-1}(z) + (p - \nu q) f_{\nu+1}(z) = (2\nu/\lambda) z^{1-q} f'_{\nu}(z)$$

$$z f'_{\nu}(z) = \lambda q z^q f_{\nu-1}(z) + (p - \nu q) f_{\nu}(z)$$

$$z f'_{\nu}(z) = -\lambda q z^q f_{\nu+1}(z) + (p + \nu q) f_{\nu}(z)$$

Formulas for Derivatives

9.1.30

$$\left(\frac{1}{z} \frac{d}{dz}\right)^k \{z^{\nu} \mathcal{C}_{\nu}(z)\} = z^{\nu-k} \mathcal{C}_{\nu-k}(z)$$

$$\left(\frac{1}{z} \frac{d}{dz}\right)^k \{z^{-\nu} \mathcal{C}_{\nu}(z)\} = (-)^k z^{-\nu-k} \mathcal{C}_{\nu+k}(z) \quad (k=0, 1, 2, \dots)$$

9.1.31

$$\begin{aligned} \mathcal{C}_{\nu}^{(k)}(z) &= \frac{1}{2^k} \left\{ \mathcal{C}_{\nu-k}(z) - \binom{k}{1} \mathcal{C}_{\nu-k+2}(z) \right. \\ &\quad \left. + \binom{k}{2} \mathcal{C}_{\nu-k+4}(z) - \dots + (-)^k \mathcal{C}_{\nu+k}(z) \right\} \\ &\quad (k=0, 1, 2, \dots) \end{aligned}$$

Recurrence Relations for Cross-Products

If

9.1.32

$$p_{\nu} = J_{\nu}(a) Y_{\nu}(b) - J_{\nu}(b) Y_{\nu}(a)$$

$$q_{\nu} = J_{\nu}(a) Y'_{\nu}(b) - J'_{\nu}(b) Y_{\nu}(a)$$

$$r_{\nu} = J'_{\nu}(a) Y_{\nu}(b) - J_{\nu}(b) Y'_{\nu}(a)$$

$$s_{\nu} = J'_{\nu}(a) Y'_{\nu}(b) - J'_{\nu}(b) Y'_{\nu}(a)$$

then

9.1.33

$$p_{\nu+1} - p_{\nu-1} = -\frac{2\nu}{a} q_{\nu} - \frac{2\nu}{b} r_{\nu}$$

$$q_{\nu+1} + r_{\nu} = \frac{\nu}{a} p_{\nu} - \frac{\nu+1}{b} p_{\nu+1}$$

$$r_{\nu+1} + q_{\nu} = \frac{\nu}{b} p_{\nu} - \frac{\nu+1}{a} p_{\nu+1}$$

$$s_{\nu} = \frac{1}{2} p_{\nu+1} + \frac{1}{2} p_{\nu-1} - \frac{\nu^2}{ab} p_{\nu}$$

and

9.1.34
$$p_{\nu} s_{\nu} - q_{\nu} r_{\nu} = \frac{4}{\pi^2 ab}$$

Analytic Continuation

 In 9.1.35 to 9.1.38, m is an integer.

9.1.35 $J_{\nu}(ze^{m\pi i}) = e^{m\nu\pi i} J_{\nu}(z)$

9.1.36

$$Y_{\nu}(ze^{m\pi i}) = e^{-m\nu\pi i} Y_{\nu}(z) + 2i \sin(m\nu\pi) \cot(\nu\pi) J_{\nu}(z)$$

9.1.37

$$\sin(\nu\pi) H_{\nu}^{(1)}(ze^{m\pi i}) = -\sin\{(m-1)\nu\pi\} H_{\nu}^{(1)}(z)$$

$$-e^{-\nu\pi i} \sin(m\nu\pi) H_{\nu}^{(2)}(z)$$

9.1.38

$$\sin(\nu\pi) H_{\nu}^{(2)}(ze^{m\pi i}) = \sin\{(m+1)\nu\pi\} H_{\nu}^{(2)}(z)$$

$$+ e^{\nu\pi i} \sin(m\nu\pi) H_{\nu}^{(1)}(z)$$

9.1.39

$$H_{\nu}^{(1)}(ze^{\pi i}) = -e^{-\nu\pi i} H_{\nu}^{(2)}(z)$$

$$H_{\nu}^{(2)}(ze^{-\pi i}) = -e^{\nu\pi i} H_{\nu}^{(1)}(z)$$

9.1.40

$$J_{\nu}(\bar{z}) = \overline{J_{\nu}(z)} \quad Y_{\nu}(\bar{z}) = \overline{Y_{\nu}(z)}$$

$$H_{\nu}^{(1)}(\bar{z}) = \overline{H_{\nu}^{(2)}(z)} \quad H_{\nu}^{(2)}(\bar{z}) = \overline{H_{\nu}^{(1)}(z)} \quad (\nu \text{ real})$$

Generating Function and Associated Series

9.1.41
$$e^{\frac{1}{2}z(t-1/t)} = \sum_{k=-\infty}^{\infty} t^k J_k(z) \quad (t \neq 0)$$

9.1.42
$$\cos(z \sin \theta) = J_0(z) + 2 \sum_{k=1}^{\infty} J_{2k}(z) \cos(2k\theta)$$

9.1.43
$$\sin(z \sin \theta) = 2 \sum_{k=0}^{\infty} J_{2k+1}(z) \sin\{(2k+1)\theta\}$$

9.1.44

$$\cos(z \cos \theta) = J_0(z) + 2 \sum_{k=1}^{\infty} (-)^k J_{2k}(z) \cos(2k\theta)$$

9.1.45

$$\sin(z \cos \theta) = 2 \sum_{k=0}^{\infty} (-)^k J_{2k+1}(z) \cos\{(2k+1)\theta\}$$

9.1.46
$$1 = J_0(z) + 2J_2(z) + 2J_4(z) + 2J_6(z) + \dots$$

9.1.47

$$\cos z = J_0(z) - 2J_2(z) + 2J_4(z) - 2J_6(z) + \dots$$

9.1.48
$$\sin z = 2J_1(z) - 2J_3(z) + 2J_5(z) - \dots$$

Other Differential Equations

$$9.1.49 \quad w'' + \left(\lambda^2 - \frac{\nu^2 - \frac{1}{4}}{z^2} \right) w = 0, \quad w = z^{\frac{1}{2}} \mathcal{C}_\nu(\lambda z)$$

$$9.1.50 \quad w'' + \left(\frac{\lambda^2}{4z} - \frac{\nu^2 - 1}{4z^2} \right) w = 0, \quad w = z^{\frac{1}{2}} \mathcal{C}_\nu(\lambda z^{\frac{1}{2}})$$

$$9.1.51 \quad w'' + \lambda^2 z^{\nu-2} w = 0, \quad w = z^{\frac{1}{2}} \mathcal{C}_{1/\nu}(2\lambda z^{\frac{1}{2}\nu}/p)$$

9.1.52

$$w'' - \frac{2\nu-1}{z} w' + \lambda^2 w = 0, \quad w = z^\nu \mathcal{C}_\nu(\lambda z)$$

9.1.53

$$z^2 w'' + (1-2p)zw' + (\lambda^2 q^2 z^{2q} + p^2 - \nu^2 q^2)w = 0, \\ w = z^p \mathcal{C}_\nu(\lambda z^q)$$

9.1.54

$$w'' + (\lambda^2 e^{2z} - \nu^2)w = 0, \quad w = \mathcal{C}_\nu(\lambda e^z)$$

9.1.55

$$z^2(z^2 - \nu^2)w'' + z(z^2 - 3\nu^2)w' \\ + \{(z^2 - \nu^2)^2 - (z^2 + \nu^2)\}w = 0, \quad w = \mathcal{C}'_\nu(z)$$

9.1.56

$$w^{(2n)} = (-)^n \lambda^{2n} z^{-n} w, \quad w = z^{\frac{1}{2n}} \mathcal{C}_n(2\lambda \alpha z^{\frac{1}{2}})$$

where α is any of the $2n$ roots of unity.

Differential Equations for Products

In the following $\vartheta \equiv z \frac{d}{dz}$ and $\mathcal{C}_\nu(z)$, $\mathcal{D}_\mu(z)$ are any cylinder functions of orders ν , μ respectively.

9.1.57

$$\{\vartheta^4 - 2(\nu^2 + \mu^2)\vartheta^2 + (\nu^2 - \mu^2)^2\}w \\ + 4z^2(\vartheta+1)(\vartheta+2)w = 0, \quad w = \mathcal{C}_\nu(z)\mathcal{D}_\mu(z)$$

9.1.58

$$\vartheta(\vartheta^2 - 4\nu^2)w + 4z^2(\vartheta+1)w = 0, \quad w = \mathcal{C}_\nu(z)\mathcal{D}_\nu(z)$$

9.1.59

$$z^3 w''' + z(4z^2 + 1 - 4\nu^2)w' + (4\nu^2 - 1)w = 0, \\ w = z \mathcal{C}_\nu(z)\mathcal{D}_\nu(z)$$

Upper Bounds

$$9.1.60 \quad |J_\nu(x)| \leq 1 \quad (\nu \geq 0), \quad |J_\nu(x)| \leq 1/\sqrt{2} \quad (\nu \geq 1)$$

$$9.1.61 \quad 0 < J_\nu(\nu) < \frac{2^{\frac{1}{2}}}{3^{\frac{1}{2}} \Gamma(\frac{3}{2}) \nu^{\frac{1}{2}}} \quad (\nu > 0)$$

$$9.1.62 \quad |J_\nu(z)| \leq \frac{|\frac{1}{2}z|^\nu e^{|\Re z|}}{\Gamma(\nu+1)} \quad (\nu \geq -\frac{1}{2}) \quad *$$

$$9.1.63 \quad |J_n(nz)| \leq \left| \frac{z^n \exp \{n\sqrt{1-z^2}\}}{\{1+\sqrt{1-z^2}\}^n} \right|$$

Derivatives With Respect to Order

9.1.64

$$\frac{\partial}{\partial \nu} J_\nu(z) = J_\nu(z) \ln(\tfrac{1}{2}z)$$

$$-(\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} (-)^k \frac{\psi(\nu+k+1)}{\Gamma(\nu+k+1)} \frac{(\tfrac{1}{4}z^2)^k}{k!}$$

9.1.65

$$\frac{\partial}{\partial \nu} Y_\nu(z) = \cot(\nu\pi) \left\{ \frac{\partial}{\partial \nu} J_\nu(z) - \pi Y_\nu(z) \right\} \\ - \csc(\nu\pi) \frac{\partial}{\partial \nu} J_{-\nu}(z) - \pi J_\nu(z) \\ (\nu \neq 0, \pm 1, \pm 2, \dots)$$

9.1.66

$$\left[\frac{\partial}{\partial \nu} J_\nu(z) \right]_{\nu=n} = \frac{\pi}{2} Y_n(z) + \frac{n! (\tfrac{1}{2}z)^{-n}}{2} \sum_{k=0}^{n-1} \frac{(\tfrac{1}{2}z)^k J_k(z)}{(n-k)k!}$$

9.1.67

$$\left[\frac{\partial}{\partial \nu} Y_\nu(z) \right]_{\nu=n} = -\frac{\pi}{2} J_n(z) + \frac{n! (\tfrac{1}{2}z)^{-n}}{2} \sum_{k=0}^{n-1} \frac{(\tfrac{1}{2}z)^k Y_k(z)}{(n-k)k!}$$

9.1.68

$$\left[\frac{\partial}{\partial \nu} J_\nu(z) \right]_{\nu=0} = \frac{\pi}{2} Y_0(z), \quad \left[\frac{\partial}{\partial \nu} Y_\nu(z) \right]_{\nu=0} = -\frac{\pi}{2} J_0(z)$$

Expressions in Terms of Hypergeometric Functions

9.1.69

$$J_\nu(z) = \frac{(\tfrac{1}{2}z)^\nu}{\Gamma(\nu+1)} {}_0F_1(\nu+1; -\tfrac{1}{4}z^2) \\ = \frac{(\tfrac{1}{2}z)^\nu e^{-iz}}{\Gamma(\nu+1)} M(\nu+\tfrac{1}{2}, 2\nu+1, 2iz)$$

9.1.70

$$J_\nu(z) = \frac{(\tfrac{1}{2}z)^\nu}{\Gamma(\nu+1)} \lim_{\lambda, \mu \rightarrow \infty} F\left(\lambda, \mu; \nu+1; -\frac{z^2}{4\lambda\mu}\right)$$

as $\lambda, \mu \rightarrow \infty$ through real or complex values; z, ν being fixed.

(${}_0F_1$ is the generalized hypergeometric function. For $M(a, b, z)$ and $F(a, b, c; z)$ see chapters 13 and 15.)

Connection With Legendre Functions

If μ and x are fixed and $\nu \rightarrow \infty$ through real positive values

9.1.71

$$\lim \{ \nu^\mu P_{-\nu}^\mu(\cos \frac{x}{\nu}) \} = J_\mu(x) \quad (x > 0)$$

9.1.72

$$\lim \{ \nu^\mu Q_\nu^{-\mu} \left(\cos \frac{x}{\nu} \right) \} = -\frac{1}{2} \pi Y_\mu(x) \quad (x > 0)$$

For $P_\nu^{-\mu}$ and $Q_\nu^{-\mu}$, see chapter 8.

Continued Fractions

9.1.73

$$\begin{aligned} \frac{J_\nu(z)}{J_{\nu-1}(z)} &= \frac{1}{2\nu z^{-1} - \frac{1}{2(\nu+1)z^{-1} - \frac{1}{2(\nu+2)z^{-1} - \dots}}} \\ &= \frac{\frac{1}{2}z/\nu}{1 - \frac{\frac{1}{2}z^2/\{\nu(\nu+1)\}}{1 - \frac{\frac{1}{2}z^2/\{(\nu+1)(\nu+2)\}}{1 - \dots}}} \end{aligned}$$

Multiplication Theorem

9.1.74

$$\mathcal{C}_\nu(\lambda z) = \lambda^{\pm \nu} \sum_{k=0}^{\infty} \frac{(\mp)^k (\lambda^2 - 1)^k (\frac{1}{2}z)^k}{k!} \mathcal{C}_{\nu \pm k}(z) \quad (|\lambda^2 - 1| < 1)$$

If $\mathcal{C} = J$ and the upper signs are taken, the restriction on λ is unnecessary.

This theorem will furnish expansions of $\mathcal{C}_\nu(re^{i\theta})$ in terms of $\mathcal{C}_{\nu \pm k}(r)$.

Addition Theorems

Neumann's

$$9.1.75 \quad \mathcal{C}_\nu(u \pm v) = \sum_{k=-\infty}^{\infty} \mathcal{C}_{\nu \mp k}(u) J_k(v) \quad (|v| < |u|)$$

The restriction $|v| < |u|$ is unnecessary when $\mathcal{C} = J$ and ν is an integer or zero. Special cases are

$$9.1.76 \quad 1 = J_0^2(z) + 2 \sum_{k=1}^{\infty} J_k^2(z)$$

9.1.77

$$0 = \sum_{k=0}^{2n} (-)^k J_k(z) J_{2n-k}(z) + 2 \sum_{k=1}^{\infty} J_k(z) J_{2n+k}(z) \quad (n \geq 1)$$

9.1.78

$$J_n(2z) = \sum_{k=0}^n J_k(z) J_{n-k}(z) + 2 \sum_{k=1}^{\infty} (-)^k J_k(z) J_{n+k}(z)$$

Graf's

9.1.79

$$\mathcal{C}_\nu(w) \frac{\cos \nu \chi}{\sin \nu \chi} = \sum_{k=-\infty}^{\infty} \mathcal{C}_{\nu+k}(u) J_k(v) \frac{\cos k \alpha}{\sin k \alpha} \quad (|ve^{\pm i \alpha}| < |u|)$$

Gegenbauer's

9.1.80

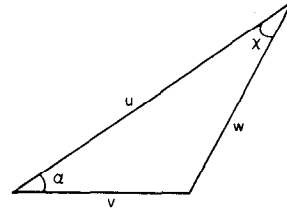
$$\frac{\mathcal{C}_\nu(w)}{w^\nu} = 2^\nu \Gamma(\nu) \sum_{k=0}^{\infty} (\nu+k) \frac{\mathcal{C}_{\nu+k}(u)}{u^\nu} \frac{J_{\nu+k}(v)}{v^\nu} C_k^{(\nu)}(\cos \alpha) \quad (\nu \neq 0, -1, \dots, |ve^{\pm i \alpha}| < |u|)$$

In 9.1.79 and 9.1.80,

$$w = \sqrt{(u^2 + v^2 - 2uv \cos \alpha)},$$

$$u - v \cos \alpha = w \cos \chi, \quad v \sin \alpha = w \sin \chi$$

the branches being chosen so that $w \rightarrow u$ and $\chi \rightarrow 0$ as $v \rightarrow 0$. $C_k^{(\nu)}(\cos \alpha)$ is Gegenbauer's polynomial (see chapter 22).



Gegenbauer's addition theorem.

If u, v are real and positive and $0 \leq \alpha \leq \pi$, then w, χ are real and non-negative, and the geometrical relationship of the variables is shown in the diagram.

The restrictions $|ve^{\pm i \alpha}| < |u|$ are unnecessary in 9.1.79 when $\mathcal{C} = J$ and ν is an integer or zero, and in 9.1.80 when $\mathcal{C} = J$.

Degenerate Form ($u = \infty$):

9.1.81

$$e^{i\nu \cos \alpha} = \Gamma(\nu) \left(\frac{1}{2}v \right)^{-\nu} \sum_{k=0}^{\infty} (\nu+k) i^k J_{\nu+k}(v) C_k^{(\nu)}(\cos \alpha) \quad (\nu \neq 0, -1, \dots)$$

Neumann's Expansion of an Arbitrary Function in a Series of Bessel Functions

$$9.1.82 \quad f(z) = a_0 J_0(z) + 2 \sum_{k=1}^{\infty} a_k J_k(z) \quad (|z| < c)$$

where c is the distance of the nearest singularity of $f(z)$ from $z=0$,

$$9.1.83 \quad a_k = \frac{1}{2\pi i} \int_{|z|=c'} f(t) O_k(t) dt \quad (0 < c' < c)$$

and $O_k(t)$ is Neumann's polynomial. The latter is defined by the generating function

9.1.84

$$\frac{1}{t-z} = J_0(z) O_0(t) + 2 \sum_{k=1}^{\infty} J_k(z) O_k(t) \quad (|z| < |t|)$$

$O_n(t)$ is a polynomial of degree $n+1$ in $1/t$; $O_0(t) = 1/t$,

9.1.85

$$O_n(t) = \frac{1}{4} \sum_{k=0}^{2n} \frac{n(n-k-1)!}{k!} \left(\frac{2}{t} \right)^{n-2k+1} \quad (n=1, 2, \dots)$$

The more general form of expansion

$$9.1.86 \quad f(z) = a_0 J_\nu(z) + 2 \sum_{k=1}^{\infty} a_k J_{\nu+k}(z)$$

also called a Neumann expansion, is investigated in [9.7] and [9.15] together with further generalizations. Examples of Neumann expansions are 9.1.41 to 9.1.48 and the Addition Theorems. Other examples are

9.1.87

$$\left(\frac{1}{2}z\right)^{\nu} = \sum_{k=0}^{\infty} \frac{(\nu+2k)\Gamma(\nu+k)}{k!} J_{\nu+2k}(z) \quad (\nu \neq 0, -1, -2, \dots)$$

9.1.88

$$Y_n(z) = -\frac{n!(\frac{1}{2}z)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(\frac{1}{2}z)^k J_k(z)}{(n-k)k!} + \frac{2}{\pi} \{\ln(\frac{1}{2}z) - \psi(n+1)\} J_n(z) - \frac{2}{\pi} \sum_{k=1}^{\infty} (-)^k \frac{(n+2k)J_{n+2k}(z)}{k(n+k)}$$

where $\psi(n)$ is given by 6.3.2.

9.1.89

$$Y_0(z) = \frac{2}{\pi} \{\ln(\frac{1}{2}z) + \gamma\} J_0(z) - \frac{4}{\pi} \sum_{k=1}^{\infty} (-)^k \frac{J_{2k}(z)}{k}$$

9.2. Asymptotic Expansions for Large Arguments

Principal Asymptotic Forms

When ν is fixed and $|z| \rightarrow \infty$

9.2.1

$$J_{\nu}(z) = \sqrt{2/(\pi z)} \{ \cos(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + e^{|\mathcal{I}z|} O(|z|^{-1}) \} \quad (|\arg z| < \pi)$$

9.2.2

$$Y_{\nu}(z) = \sqrt{2/(\pi z)} \{ \sin(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + e^{|\mathcal{I}z|} O(|z|^{-1}) \} \quad (|\arg z| < \pi)$$

9.2.3

$$H_{\nu}^{(1)}(z) \sim \sqrt{2/(\pi z)} e^{i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} \quad (-\pi < \arg z < 2\pi)$$

9.2.4

$$H_{\nu}^{(2)}(z) \sim \sqrt{2/(\pi z)} e^{-i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} \quad (-2\pi < \arg z < \pi)$$

Hankel's Asymptotic Expansions

When ν is fixed and $|z| \rightarrow \infty$

9.2.5

$$J_{\nu}(z) = \sqrt{2/(\pi z)} \{ P(\nu, z) \cos \chi - Q(\nu, z) \sin \chi \} \quad (|\arg z| < \pi)$$

9.2.6

$$Y_{\nu}(z) = \sqrt{2/(\pi z)} \{ P(\nu, z) \sin \chi + Q(\nu, z) \cos \chi \} \quad (|\arg z| < \pi)$$

9.2.7

$$H_{\nu}^{(1)}(z) = \sqrt{2/(\pi z)} \{ P(\nu, z) + iQ(\nu, z) \} e^{i\chi} \quad (-\pi < \arg z < 2\pi)$$

9.2.8

$$H_{\nu}^{(2)}(z) = \sqrt{2/(\pi z)} \{ P(\nu, z) - iQ(\nu, z) \} e^{-i\chi} \quad (-2\pi < \arg z < \pi)$$

where $\chi = z - (\frac{1}{2}\nu + \frac{1}{4})\pi$ and, with $4\nu^2$ denoted by μ ,

9.2.9

$$P(\nu, z) \sim \sum_{k=0}^{\infty} (-)^k \frac{(\nu, 2k)}{(2z)^{2k}} = 1 - \frac{(\mu-1)(\mu-9)}{2!(8z)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)(\mu-49)}{4!(8z)^4} - \dots$$

9.2.10

$$Q(\nu, z) \sim \sum_{k=0}^{\infty} (-)^k \frac{(\nu, 2k+1)}{(2z)^{2k+1}} = \frac{\mu-1}{8z} - \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + \dots$$

If ν is real and non-negative and z is positive, the remainder after k terms in the expansion of $P(\nu, z)$ does not exceed the $(k+1)$ th term in absolute value and is of the same sign, provided that $k > \frac{1}{2}\nu - \frac{1}{4}$. The same is true of $Q(\nu, z)$ provided that $k > \frac{1}{2}\nu - \frac{3}{4}$.

Asymptotic Expansions of Derivatives

With the conditions and notation of the preceding subsection

9.2.11

$$J'_{\nu}(z) = \sqrt{2/(\pi z)} \{ -R(\nu, z) \sin \chi - S(\nu, z) \cos \chi \} \quad (|\arg z| < \pi)$$

9.2.12

$$Y'_{\nu}(z) = \sqrt{2/(\pi z)} \{ R(\nu, z) \cos \chi - S(\nu, z) \sin \chi \} \quad (|\arg z| < \pi)$$

9.2.13

$$H_{\nu}^{(1)'}(z) = \sqrt{2/(\pi z)} \{ iR(\nu, z) - S(\nu, z) \} e^{i\chi} \quad (-\pi < \arg z < 2\pi)$$

9.2.14

$$H_{\nu}^{(2)'}(z) = \sqrt{2/(\pi z)} \{ -iR(\nu, z) - S(\nu, z) \} e^{-i\chi} \quad (-2\pi < \arg z < \pi)$$

9.2.15

$$R(\nu, z) \sim \sum_{k=0}^{\infty} (-)^k \frac{4\nu^2 + 16k^2 - 1}{4\nu^2 - (4k-1)^2} \frac{(\nu, 2k)}{(2z)^{2k}}$$

$$= 1 - \frac{(\mu-1)(\mu+15)}{2!(8z)^2} + \dots$$

9.2.16

$$S(\nu, z) \sim \sum_{k=0}^{\infty} (-)^k \frac{4\nu^2 + 4(2k+1)^2 - 1}{4\nu^2 - (4k+1)^2} \frac{(\nu, 2k+1)}{(2z)^{2k+1}}$$

$$= \frac{\mu+3}{8z} - \frac{(\mu-1)(\mu-9)(\mu+35)}{3!(8z)^3} + \dots$$

Modulus and Phase

For real ν and positive x

9.2.17

$$M_\nu = |H_\nu^{(1)}(x)| = \sqrt{\{J_\nu^2(x) + Y_\nu^2(x)\}}$$

$$\theta_\nu = \arg H_\nu^{(1)}(x) = \arctan \{Y_\nu(x)/J_\nu(x)\}$$

9.2.18

$$N_\nu = |H_\nu^{(1)'}(x)| = \sqrt{\{J_\nu'^2(x) + Y_\nu'^2(x)\}}$$

$$\varphi_\nu = \arg H_\nu^{(1)'}(x) = \arctan \{Y_\nu'(x)/J_\nu'(x)\}$$

$$9.2.19 \quad J_\nu(x) = M_\nu \cos \theta_\nu, \quad Y_\nu(x) = M_\nu \sin \theta_\nu,$$

$$9.2.20 \quad J_\nu'(x) = N_\nu \cos \varphi_\nu, \quad Y_\nu'(x) = N_\nu \sin \varphi_\nu.$$

In the following relations, primes denote differentiations with respect to x .

$$9.2.21 \quad M_\nu^2 \theta_\nu' = 2/(\pi x) \quad N_\nu^2 \varphi_\nu' = 2(x^2 - \nu^2)/(\pi x^3)$$

$$9.2.22 \quad N_\nu^2 = M_\nu'^2 + M_\nu^2 \theta_\nu'^2 = M_\nu'^2 + 4/(\pi x M_\nu)^2$$

$$9.2.23 \quad (x^2 - \nu^2) M_\nu M_\nu' + x^2 N_\nu N_\nu' + x N_\nu^2 = 0$$

9.2.24

$$\tan(\varphi_\nu - \theta_\nu) = M_\nu \theta_\nu' / M_\nu' = 2/(\pi x M_\nu M_\nu')$$

$$M_\nu N_\nu \sin(\varphi_\nu - \theta_\nu) = 2/(\pi x)$$

$$9.2.25 \quad x^2 M_\nu'' + x M_\nu' + (x^2 - \nu^2) M_\nu - 4/(\pi^2 M_\nu^3) = 0$$

9.2.26

$$x^3 w'''' + x(4x^2 + 1 - 4\nu^2)w' + (4\nu^2 - 1)w = 0, \quad w = x M_\nu^2$$

$$9.2.27 \quad \theta_\nu'^2 + \frac{1}{2} \frac{\theta_\nu'''}{\theta_\nu'} - \frac{3}{4} \left(\frac{\theta_\nu''}{\theta_\nu'} \right)^2 = 1 - \frac{\nu^2 - \frac{1}{4}}{x^2}$$

Asymptotic Expansions of Modulus and Phase

When ν is fixed, x is large and positive, and $\mu = 4\nu^2$

9.2.28

$$M_\nu^2 \sim \frac{2}{\pi x} \left\{ 1 + \frac{1}{2} \frac{\mu-1}{(2x)^2} + \frac{1 \cdot 3}{2 \cdot 4} \frac{(\mu-1)(\mu-9)}{(2x)^4} \right.$$

$$\left. + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{(\mu-1)(\mu-9)(\mu-25)}{(2x)^6} + \dots \right\}$$

9.2.29

$$\theta_\nu \sim x - \left(\frac{1}{2}\nu + \frac{1}{4} \right) \pi + \frac{\mu-1}{2(4x)}$$

$$+ \frac{(\mu-1)(\mu-25)}{6(4x)^3} + \frac{(\mu-1)(\mu^2-114\mu+1073)}{5(4x)^5}$$

$$+ \frac{(\mu-1)(5\mu^3-1535\mu^2+54703\mu-375733)}{14(4x)^7} + \dots$$

9.2.30

$$N_\nu^2 \sim \frac{2}{\pi x} \left\{ 1 - \frac{1}{2} \frac{\mu-3}{(2x)^2} - \frac{1 \cdot 1}{2 \cdot 4} \frac{(\mu-1)(\mu-45)}{(2x)^4} - \dots \right\}$$

The general term in the last expansion is given by

$$- \frac{1 \cdot 1 \cdot 3 \dots (2k-3)}{2 \cdot 4 \cdot 6 \dots (2k)}$$

$$\times \frac{(\mu-1)(\mu-9) \dots \{ \mu - (2k-3)^2 \} \{ \mu - (2k+1)(2k-1)^2 \}}{(2x)^{2k}} *$$

9.2.31

$$\phi_\nu \sim x - \left(\frac{1}{2}\nu - \frac{1}{4} \right) \pi + \frac{\mu+3}{2(4x)} + \frac{\mu^2+46\mu-63}{6(4x)^3}$$

$$+ \frac{\mu^3+185\mu^2-2053\mu+1899}{5(4x)^5} + \dots$$

If $\nu \geq 0$, the remainder after k terms in 9.2.28 does not exceed the $(k+1)$ th term in absolute value and is of the same sign, provided that $k > \nu - \frac{1}{2}$.

9.3. Asymptotic Expansions for Large Orders

Principal Asymptotic Forms

In the following equations it is supposed that $\nu \rightarrow \infty$ through real positive values, the other variables being fixed.

9.3.1

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ez}{2\nu} \right)^\nu$$

$$Y_\nu(z) \sim -\sqrt{\frac{2}{\pi\nu}} \left(\frac{ez}{2\nu} \right)^{-\nu}$$

9.3.2

$$J_\nu(\nu \operatorname{sech} \alpha) \sim \frac{e^{\nu(\tanh \alpha - \alpha)}}{\sqrt{2\pi\nu \tanh \alpha}} \quad (\alpha > 0)$$

$$Y_\nu(\nu \operatorname{sech} \alpha) \sim -\frac{e^{\nu(\alpha - \tanh \alpha)}}{\sqrt{\frac{1}{2}\pi\nu \tanh \alpha}} \quad (\alpha > 0)$$

*See page II.

9.3.3

$$J_\nu(\nu \sec \beta) = \sqrt{2/(\pi \nu \tan \beta)} \left\{ \cos(\nu \tan \beta - \nu \beta - \frac{1}{4}\pi) + O(\nu^{-1}) \right\} \\ (0 < \beta < \frac{1}{2}\pi)$$

$$Y_\nu(\nu \sec \beta) = \sqrt{2/(\pi \nu \tan \beta)} \left\{ \sin(\nu \tan \beta - \nu \beta - \frac{1}{4}\pi) + O(\nu^{-1}) \right\} \\ (0 < \beta < \frac{1}{2}\pi)$$

9.3.4

$$J_\nu(\nu + z\nu^{1/2}) = 2^{1/2}\nu^{-1/2} \text{Ai}(-2^{1/2}z) + O(\nu^{-1})$$

$$Y_\nu(\nu + z\nu^{1/2}) = -2^{1/2}\nu^{-1/2} \text{Bi}(-2^{1/2}z) + O(\nu^{-1})$$

9.3.5

$$J_\nu(\nu) \sim \frac{2^{1/2}}{3^{3/2}\Gamma(\frac{2}{3})} \frac{1}{\nu^{1/2}}$$

$$Y_\nu(\nu) \sim -\frac{2^{1/2}}{3^{1/2}\Gamma(\frac{2}{3})} \frac{1}{\nu^{1/2}}$$

9.3.6

$$J_\nu(\nu z) = \left(\frac{4\zeta}{1-z^2}\right)^{1/2} \left\{ \frac{\text{Ai}(\nu^{2/3}\zeta)}{\nu^{1/6}} + \frac{\exp(-\frac{2}{3}\nu\zeta^{3/2})}{1+\nu^{1/6}|\zeta|^{1/2}} O\left(\frac{1}{\nu^{1/6}}\right) \right\} \quad (|\arg z| < \pi)$$

$$Y_\nu(\nu z) = -\left(\frac{4\zeta}{1-z^2}\right)^{1/2} \left\{ \frac{\text{Bi}(\nu^{2/3}\zeta)}{\nu^{1/6}} + \frac{\exp|\mathcal{R}(\frac{2}{3}\nu\zeta^{3/2})|}{1+\nu^{1/6}|\zeta|^{1/2}} O\left(\frac{1}{\nu^{1/6}}\right) \right\} \quad (|\arg z| < \pi)$$

In the last two equations ζ is given by 9.3.38 and 9.3.39 below.

Debye's Asymptotic Expansions

(i) If α is fixed and positive and ν is large and positive

9.3.7

$$J_\nu(\nu \text{sech } \alpha) \sim \frac{e^{\nu(\tanh \alpha - \alpha)}}{\sqrt{2\pi\nu \tanh \alpha}} \left\{ 1 + \sum_{k=1}^{\infty} \frac{u_k(\coth \alpha)}{\nu^k} \right\}$$

9.3.8

$$Y_\nu(\nu \text{sech } \alpha) \sim -\frac{e^{\nu(\alpha - \tanh \alpha)}}{\sqrt{\frac{1}{2}\pi\nu \tanh \alpha}} \left\{ 1 + \sum_{k=1}^{\infty} (-)^k \frac{u_k(\coth \alpha)}{\nu^k} \right\}$$

where

9.3.9

$$u_0(t) = 1$$

$$u_1(t) = (3t - 5t^3)/24$$

$$u_2(t) = (81t^2 - 462t^4 + 385t^6)/1152$$

$$u_3(t) = (30375t^3 - 3\ 69603t^5 + 7\ 65765t^7 - 4\ 25425t^9)/4\ 14720$$

$$u_4(t) = (44\ 65125t^4 - 941\ 21676t^6 + 3499\ 22430t^8 - 4461\ 85740t^{10} + 1859\ 10725t^{12})/398\ 13120$$

For $u_5(t)$ and $u_6(t)$ see [9.4] or [9.21].

9.3.10

$$u_{k+1}(t) = \frac{1}{2}t^2(1-t^2)u'_k(t) + \frac{1}{8}\int_0^t (1-5t^2)u_k(t)dt \quad (k=0, 1, \dots)$$

Also

9.3.11

$$J'_\nu(\nu \text{sech } \alpha) \sim \sqrt{\frac{\sinh 2\alpha}{4\pi\nu}} e^{\nu(\tanh \alpha - \alpha)} \left\{ 1 + \sum_{k=1}^{\infty} \frac{v_k(\coth \alpha)}{\nu^k} \right\}$$

9.3.12

$$Y'_\nu(\nu \text{sech } \alpha) \sim \sqrt{\frac{\sinh 2\alpha}{\pi\nu}} e^{\nu(\alpha - \tanh \alpha)} \left\{ 1 + \sum_{k=1}^{\infty} (-)^k \frac{v_k(\coth \alpha)}{\nu^k} \right\}$$

where

9.3.13

$$v_0(t) = 1$$

$$v_1(t) = (-9t + 7t^3)/24$$

$$v_2(t) = (-135t^2 + 594t^4 - 455t^6)/1152$$

$$v_3(t) = (-42525t^3 + 4\ 51737t^5 - 8\ 83575t^7 + 4\ 75475t^9)/4\ 14720$$

9.3.14

$$v_k(t) = u_k(t) + t(t^2 - 1) \left\{ \frac{1}{2}u_{k-1}(t) + tu'_{k-1}(t) \right\} \quad (k=1, 2, \dots)$$

(ii) If β is fixed, $0 < \beta < \frac{1}{2}\pi$ and ν is large and positive

9.3.15

$$J_\nu(\nu \sec \beta) = \sqrt{2/(\pi \nu \tan \beta)} \{ L(\nu, \beta) \cos \Psi + M(\nu, \beta) \sin \Psi \}$$

9.3.16

$$Y_\nu(\nu \sec \beta) = \sqrt{2/(\pi \nu \tan \beta)} \{ L(\nu, \beta) \sin \Psi - M(\nu, \beta) \cos \Psi \}$$

where $\Psi = \nu(\tan \beta - \beta) - \frac{1}{4}\pi$

9.3.17

$$L(\nu, \beta) \sim \sum_{k=0}^{\infty} \frac{u_{2k}(i \cot \beta)}{\nu^{2k}} \\ = 1 - \frac{81 \cot^2 \beta + 462 \cot^4 \beta + 385 \cot^6 \beta}{1152\nu^2} + \dots$$

9.3.18

$$M(\nu, \beta) \sim -i \sum_{k=0}^{\infty} \frac{u_{2k+1}(i \cot \beta)}{\nu^{2k+1}} \\ = \frac{3 \cot \beta + 5 \cot^3 \beta}{24\nu} - \dots$$

Also

9.3.19

$$J'_{\nu}(\nu \sec \beta) = \sqrt{(\sin 2\beta)/(\pi\nu)} \{ -N(\nu, \beta) \sin \Psi \\ - O(\nu, \beta) \cos \Psi \}$$

9.3.20

$$Y'_{\nu}(\nu \sec \beta) = \sqrt{(\sin 2\beta)/(\pi\nu)} \{ N(\nu, \beta) \cos \Psi \\ - O(\nu, \beta) \sin \Psi \}$$

where

9.3.21

$$N(\nu, \beta) \sim \sum_{k=0}^{\infty} \frac{v_{2k}(i \cot \beta)}{\nu^{2k}} \\ = 1 + \frac{135 \cot^2 \beta + 594 \cot^4 \beta + 455 \cot^6 \beta}{1152\nu^2} - \dots$$

9.3.22

$$O(\nu, \beta) \sim i \sum_{k=0}^{\infty} \frac{v_{2k+1}(i \cot \beta)}{\nu^{2k+1}} = \frac{9 \cot \beta + 7 \cot^3 \beta}{24\nu} - \dots$$

Asymptotic Expansions in the Transition Regions

When z is fixed, $|\nu|$ is large and $|\arg \nu| < \frac{1}{2}\pi$

9.3.23

$$J_{\nu}(\nu + z\nu^{1/3}) \sim \frac{2^{1/3}}{\nu^{1/3}} \text{Ai}(-2^{1/3}z) \left\{ 1 + \sum_{k=1}^{\infty} \frac{f_k(z)}{\nu^{2k/3}} \right\} \\ + \frac{2^{2/3}}{\nu} \text{Ai}'(-2^{1/3}z) \sum_{k=0}^{\infty} \frac{g_k(z)}{\nu^{2k/3}}$$

9.3.24

$$Y_{\nu}(\nu + z\nu^{1/3}) \sim -\frac{2^{1/3}}{\nu^{1/3}} \text{Bi}(-2^{1/3}z) \left\{ 1 + \sum_{k=1}^{\infty} \frac{f_k(z)}{\nu^{2k/3}} \right\} \\ - \frac{2^{2/3}}{\nu} \text{Bi}'(-2^{1/3}z) \sum_{k=0}^{\infty} \frac{g_k(z)}{\nu^{2k/3}}$$

where

9.3.25

$$f_1(z) = -\frac{1}{5}z \\ f_2(z) = -\frac{9}{100}z^5 + \frac{3}{35}z^2 \\ f_3(z) = \frac{957}{7000}z^6 - \frac{173}{3150}z^3 - \frac{1}{225} \\ f_4(z) = \frac{27}{20000}z^{10} - \frac{23573}{147000}z^7 + \frac{5903}{138600}z^4 + \frac{947}{346500}z$$

9.3.26

$$g_0(z) = \frac{3}{10}z^2 \\ g_1(z) = -\frac{17}{70}z^3 + \frac{1}{70} \\ g_2(z) = -\frac{9}{1000}z^7 + \frac{611}{3150}z^4 - \frac{37}{3150}z \\ g_3(z) = \frac{549}{28000}z^8 - \frac{110767}{693000}z^5 + \frac{79}{12375}z^2$$

The corresponding expansions for $H_{\nu}^{(1)}(\nu + z\nu^{1/3})$ and $H_{\nu}^{(2)}(\nu + z\nu^{1/3})$ are obtained by use of 9.1.3 and 9.1.4; they are valid for $-\frac{1}{2}\pi < \arg \nu < \frac{3}{2}\pi$ and $-\frac{3}{2}\pi < \arg \nu < \frac{1}{2}\pi$, respectively.

9.3.27

$$J'_{\nu}(\nu + z\nu^{1/3}) \sim -\frac{2^{2/3}}{\nu^{2/3}} \text{Ai}'(-2^{1/3}z) \left\{ 1 + \sum_{k=1}^{\infty} \frac{h_k(z)}{\nu^{2k/3}} \right\} \\ + \frac{2^{1/3}}{\nu^{4/3}} \text{Ai}(-2^{1/3}z) \sum_{k=0}^{\infty} \frac{l_k(z)}{\nu^{2k/3}}$$

9.3.28

$$Y'_{\nu}(\nu + z\nu^{1/3}) \sim \frac{2^{2/3}}{\nu^{2/3}} \text{Bi}'(-2^{1/3}z) \left\{ 1 + \sum_{k=1}^{\infty} \frac{h_k(z)}{\nu^{2k/3}} \right\} \\ - \frac{2^{1/3}}{\nu^{4/3}} \text{Bi}(-2^{1/3}z) \sum_{k=0}^{\infty} \frac{l_k(z)}{\nu^{2k/3}}$$

where

9.3.29

$$h_1(z) = -\frac{4}{5}z \\ h_2(z) = -\frac{9}{100}z^5 + \frac{57}{70}z^2 \\ h_3(z) = \frac{699}{3500}z^6 - \frac{2617}{3150}z^3 + \frac{23}{3150} \\ h_4(z) = \frac{27}{20000}z^{10} - \frac{46631}{147000}z^7 + \frac{3889}{4620}z^4 - \frac{1159}{115500}z$$

9.3.30

$$l_0(z) = \frac{3}{5}z^3 - \frac{1}{5} \\ l_1(z) = -\frac{131}{140}z^4 + \frac{1}{5}z \\ l_2(z) = -\frac{9}{500}z^8 + \frac{5437}{4500}z^5 - \frac{593}{3150}z^2 \\ l_3(z) = \frac{369}{7000}z^9 - \frac{999443}{693000}z^6 + \frac{31727}{173250}z^3 + \frac{947}{346500}$$

$$9.3.31 \quad J_\nu(\nu) \sim \frac{a}{\nu^{1/3}} \left\{ 1 + \sum_{k=1}^{\infty} \frac{\alpha_k}{\nu^{2k}} \right\} - \frac{b}{\nu^{5/3}} \sum_{k=0}^{\infty} \frac{\beta_k}{\nu^{2k}}$$

$$9.3.32 \quad Y_\nu(\nu) \sim -\frac{3^{1/2}a}{\nu^{1/3}} \left\{ 1 + \sum_{k=1}^{\infty} \frac{\alpha_k}{\nu^{2k}} \right\} - \frac{3^{1/2}b}{\nu^{5/3}} \sum_{k=0}^{\infty} \frac{\beta_k}{\nu^{2k}}$$

$$9.3.33 \quad J'_\nu(\nu) \sim \frac{b}{\nu^{2/3}} \left\{ 1 + \sum_{k=1}^{\infty} \frac{\gamma_k}{\nu^{2k}} \right\} - \frac{a}{\nu^{4/3}} \sum_{k=0}^{\infty} \frac{\delta_k}{\nu^{2k}}$$

$$9.3.34 \quad Y'_\nu(\nu) \sim \frac{3^{1/2}b}{\nu^{2/3}} \left\{ 1 + \sum_{k=1}^{\infty} \frac{\gamma_k}{\nu^{2k}} \right\} + \frac{3^{1/2}a}{\nu^{4/3}} \sum_{k=0}^{\infty} \frac{\delta_k}{\nu^{2k}}$$

where

$$a = \frac{2^{1/3}}{3^{2/3}\Gamma(\frac{2}{3})} = .44730 \ 73184, \quad 3^{\frac{1}{2}}a = .77475 \ 90021$$

$$b = \frac{2^{2/3}}{3^{1/3}\Gamma(\frac{1}{3})} = .41085 \ 01939, \quad 3^{\frac{1}{2}}b = .71161 \ 34101$$

$$\alpha_0 = 1, \quad \alpha_1 = -\frac{1}{225} = -.004\dot{4},$$

$$\alpha_2 = .00069 \ 3735 \dots, \quad \alpha_3 = -.00035 \ 38 \dots$$

$$\beta_0 = \frac{1}{70} = .01428 \ 57143 \dots,$$

$$\beta_1 = -\frac{1213}{10 \ 23750} = -.00118 \ 48596 \dots,$$

$$\beta_2 = .00043 \ 78 \dots, \quad \beta_3 = -.00038 \dots$$

$$\gamma_0 = 1, \quad \gamma_1 = \frac{23}{3150} = .00730 \ 15873 \dots,$$

$$\gamma_2 = -.00093 \ 7300 \dots, \quad \gamma_3 = .00044 \ 40 \dots$$

$$\delta_0 = \frac{1}{5}, \quad \delta_1 = -\frac{947}{3 \ 46500} = -.00273 \ 30447 \dots,$$

$$\delta_2 = .00060 \ 47 \dots, \quad \delta_3 = -.00038 \dots$$

Uniform Asymptotic Expansions

These are more powerful than the previous expansions of this section, save for 9.3.31 and 9.3.32, but their coefficients are more complicated. They reduce to 9.3.31 and 9.3.32 when the argument equals the order.

9.3.35

$$J_\nu(\nu z) \sim \left(\frac{4\zeta}{1-z^2} \right)^{1/4} \left\{ \frac{\text{Ai}(\nu^{2/3}\zeta)}{\nu^{1/3}} \sum_{k=0}^{\infty} \frac{a_k(\zeta)}{\nu^{2k}} + \frac{\text{Ai}'(\nu^{2/3}\zeta)}{\nu^{5/3}} \sum_{k=0}^{\infty} \frac{b_k(\zeta)}{\nu^{2k}} \right\}$$

9.3.36

$$Y_\nu(\nu z) \sim -\left(\frac{4\zeta}{1-z^2} \right)^{1/4} \left\{ \frac{\text{Bi}(\nu^{2/3}\zeta)}{\nu^{1/3}} \sum_{k=0}^{\infty} \frac{a_k(\zeta)}{\nu^{2k}} + \frac{\text{Bi}'(\nu^{2/3}\zeta)}{\nu^{5/3}} \sum_{k=0}^{\infty} \frac{b_k(\zeta)}{\nu^{2k}} \right\}$$

9.3.37

$$H_\nu^{(1)}(\nu z) \sim 2e^{-\pi i/3} \left(\frac{4\zeta}{1-z^2} \right)^{1/4} \left\{ \frac{\text{Ai}(e^{2\pi i/3}\nu^{2/3}\zeta)}{\nu^{1/3}} \sum_{k=0}^{\infty} \frac{a_k(\zeta)}{\nu^{2k}} + \frac{e^{2\pi i/3}\text{Ai}'(e^{2\pi i/3}\nu^{2/3}\zeta)}{\nu^{5/3}} \sum_{k=0}^{\infty} \frac{b_k(\zeta)}{\nu^{2k}} \right\}$$

When $\nu \rightarrow +\infty$, these expansions hold uniformly with respect to z in the sector $|\arg z| \leq \pi - \epsilon$, where ϵ is an arbitrary positive number. The corresponding expansion for $H_\nu^{(2)}(\nu z)$ is obtained by changing the sign of i in 9.3.37.

Here

9.3.38

$$\frac{2}{3} \zeta^{3/2} = \int_z^1 \frac{\sqrt{1-t^2}}{t} dt = \ln \frac{1+\sqrt{1-z^2}}{z} - \sqrt{1-z^2}$$

equivalently,

9.3.39

$$\frac{2}{3} (-\zeta)^{3/2} = \int_1^z \frac{\sqrt{t^2-1}}{t} dt = \sqrt{z^2-1} - \arccos\left(\frac{1}{z}\right)$$

the branches being chosen so that ζ is real when z is positive. The coefficients are given by

9.3.40

$$a_k(\zeta) = \sum_{s=0}^{2k} \mu_s \zeta^{-3s/2} u_{2k-s} \{ (1-z^2)^{-\frac{1}{2}} \}$$

$$b_k(\zeta) = -\zeta^{-\frac{1}{2}} \sum_{s=0}^{2k+1} \lambda_s \zeta^{-3s/2} u_{2k-s+1} \{ (1-z^2)^{-\frac{1}{2}} \}$$

where u_k is given by 9.3.9 and 9.3.10, $\lambda_0 = \mu_0 = 1$ and

9.3.41

$$\lambda_s = \frac{(2s+1)(2s+3)\dots(6s-1)}{s!(144)^s}, \quad \mu_s = -\frac{6s+1}{6s-1} \lambda_s$$

Thus $a_0(\zeta) = 1$,

9.3.42

$$b_0(\zeta) = -\frac{5}{48\zeta^2} + \frac{1}{\zeta^{\frac{1}{2}}} \left\{ \frac{5}{24(1-z^2)^{3/2}} - \frac{1}{8(1-z^2)^{\frac{1}{2}}} \right\} \\ = -\frac{5}{48\zeta^2} + \frac{1}{(-\zeta)^{\frac{1}{2}}} \left\{ \frac{5}{24(z^2-1)^{3/2}} + \frac{1}{8(z^2-1)^{\frac{1}{2}}} \right\}$$

Tables of the early coefficients are given below. For more extensive tables of the coefficients and for bounds on the remainder terms in 9.3.35 and 9.3.36 see [9.38].

Uniform Expansions of the Derivatives

With the conditions of the preceding subsection

9.3.43

$$J'_\nu(\nu z) \sim -\frac{2}{z} \left(\frac{1-z^2}{4\zeta} \right)^{\frac{1}{2}} \left\{ \frac{\text{Ai}(\nu^{2/3}\zeta)}{\nu^{4/3}} \sum_{k=0}^{\infty} \frac{c_k(\zeta)}{\nu^{2k}} + \frac{\text{Ai}'(\nu^{2/3}\zeta)}{\nu^{2/3}} \sum_{k=0}^{\infty} \frac{d_k(\zeta)}{\nu^{2k}} \right\}$$

9.3.44

$$Y'_\nu(\nu z) \sim \frac{2}{z} \left(\frac{1-z^2}{4\zeta} \right)^{\frac{1}{2}} \left\{ \frac{\text{Bi}(\nu^{2/3}\zeta)}{\nu^{4/3}} \sum_{k=0}^{\infty} \frac{c_k(\zeta)}{\nu^{2k}} + \frac{\text{Bi}'(\nu^{2/3}\zeta)}{\nu^{2/3}} \sum_{k=0}^{\infty} \frac{d_k(\zeta)}{\nu^{2k}} \right\}$$

9.3.45

$$H^{(1)\prime}_\nu(\nu z) \sim \frac{4e^{2\pi i/3}}{z} \left(\frac{1-z^2}{4\zeta} \right)^{\frac{1}{2}} \left\{ \frac{\text{Ai}(e^{2\pi i/3}\nu^{2/3}\zeta)}{\nu^{4/3}} \sum_{k=0}^{\infty} \frac{c_k(\zeta)}{\nu^{2k}} + \frac{e^{2\pi i/3}\text{Ai}'(e^{2\pi i/3}\nu^{2/3}\zeta)}{\nu^{2/3}} \sum_{k=0}^{\infty} \frac{d_k(\zeta)}{\nu^{2k}} \right\}$$

where

9.3.46

$$c_k(\zeta) = -\zeta^{\frac{1}{2}} \sum_{s=0}^{2k+1} \mu_s \zeta^{-3s/2} \nu_{2k-s+1} \left\{ (1-z^2)^{-\frac{1}{2}} \right\}$$

$$d_k(\zeta) = \sum_{s=0}^{2k} \lambda_s \zeta^{-3s/2} \nu_{2k-s} \left\{ (1-z^2)^{-\frac{1}{2}} \right\}$$

 and ν_k is given by 9.3.13 and 9.3.14. For bounds on the remainder terms in 9.3.43 and 9.3.44 see [9.38].

ζ	$b_0(\zeta)$	$a_1(\zeta)$	$c_0(\zeta)$	$d_1(\zeta)$
0	0.0180	-0.004	0.1587	0.007
1	.0278	-.004	.1785	.009
2	.0351	-.001	.1862	.007
3	.0366	+.002	.1927	.005
4	.0352	.003	.2031	.004
5	.0331	.004	.2155	.003
6	.0311	.004	.2284	.003
7	.0294	.004	.2413	.003
8	.0278	.004	.2539	.003
9	.0265	.004	.2662	.003
10	.0253	.004	.2781	.003

$-\zeta$	$b_0(\zeta)$	$a_1(\zeta)$	$c_0(\zeta)$	$d_1(\zeta)$
0	0.0180	-0.004	0.1587	0.007
1	.0109	-.003	.1323	.004
2	.0067	-.002	.1087	.002
3	.0044	-.001	.0903	.001
4	.0031	-.001	.0764	.001
5	.0022	-.000	.0658	.000
6	.0017	-.000	.0576	.000
7	.0013	-.000	.0511	.000
8	.0011	-.000	.0459	.000
9	.0009	-.000	.0415	.000
10	.0007	-.000	.0379	.000

 For $\zeta > 10$ use

$$b_0(\zeta) \sim \frac{1}{12} \zeta^{-\frac{1}{2}} - .104 \zeta^{-2}, \quad a_1(\zeta) = .003,$$

$$c_0(\zeta) \sim \frac{1}{12} \zeta^{\frac{1}{2}} + .146 \zeta^{-1}, \quad d_1(\zeta) = .003.$$

 For $\zeta < -10$ use

$$b_0(\zeta) \sim \frac{1}{12} \zeta^{-2}, \quad a_1(\zeta) = .000,$$

$$c_0(\zeta) \sim -\frac{5}{12} \zeta^{-1} - 1.33(-\zeta)^{-5/2}, \quad d_1(\zeta) = .000.$$

Maximum values of higher coefficients:

$$|b_1(\zeta)| = .003, \quad |a_2(\zeta)| = .0008, \quad |d_2(\zeta)| = .001$$

$$|c_1(\zeta)| = .008 \quad (\zeta < 10), \quad c_1(\zeta) \sim -.003 \zeta^{\frac{1}{2}} \text{ as } \zeta \rightarrow +\infty.$$

 9.4. Polynomial Approximations²

9.4.1

$$-3 \leq x \leq 3$$

$$J_0(x) = 1 - 2.24999 \ 97(x/3)^2 + 1.26562 \ 08(x/3)^4 - .31638 \ 66(x/3)^6 + .04444 \ 79(x/3)^8 - .00394 \ 44(x/3)^{10} + .00021 \ 00(x/3)^{12} + \epsilon$$

$$|\epsilon| < 5 \times 10^{-8}$$

9.4.2

$$0 < x \leq 3$$

$$Y_0(x) = (2/\pi) \ln(\frac{1}{2}x) J_0(x) + .36746 \ 691 + .60559 \ 366(x/3)^2 - .74350 \ 384(x/3)^4 + .25300 \ 117(x/3)^6 - .04261 \ 214(x/3)^8 + .00427 \ 916(x/3)^{10} - .00024 \ 846(x/3)^{12} + \epsilon$$

$$|\epsilon| < 1.4 \times 10^{-8}$$

9.4.3

$$3 \leq x < \infty$$

$$J_0(x) = x^{-\frac{1}{2}} f_0 \cos \theta_0 \quad Y_0(x) = x^{-\frac{1}{2}} f_0 \sin \theta_0$$

$$f_0 = .79788 \ 456 - .00000 \ 077(3/x) - .00552 \ 740(3/x)^2 - .00009 \ 512(3/x)^3 + .00137 \ 237(3/x)^4 - .00072 \ 805(3/x)^5 + .00014 \ 476(3/x)^6 + \epsilon$$

$$|\epsilon| < 1.6 \times 10^{-8}$$

² Equations 9.4.1 to 9.4.6 and 9.8.1 to 9.8.8 are taken from E. E. Allen, Analytical approximations, Math. Tables Aids Comp. 8, 240-241 (1954), and Polynomial approximations to some modified Bessel functions, Math. Tables Aids Comp. 10, 162-164 (1956) (with permission). They were checked at the National Physical Laboratory by systematic tabulation; new bounds for the errors, ϵ , given here were obtained as a result.

$$\begin{aligned}\theta_0 = & x - .78539\ 816 - .04166\ 397(3/x) \\ & - .00003\ 954(3/x)^2 + .00262\ 573(3/x)^3 \\ & - .00054\ 125(3/x)^4 - .00029\ 333(3/x)^5 \\ & + .00013\ 558(3/x)^6 + \epsilon\end{aligned}$$

$$|\epsilon| < 7 \times 10^{-8}$$

$$9.4.4 \quad -3 \leq x \leq 3$$

$$\begin{aligned}x^{-1} J_1(x) = & \frac{1}{2} - .56249\ 985(x/3)^2 + .21093\ 573(x/3)^4 \\ & - .03954\ 289(x/3)^6 + .00443\ 319(x/3)^8 \\ & - .00031\ 761(x/3)^{10} + .00001\ 109(x/3)^{12} + \epsilon\end{aligned}$$

$$|\epsilon| < 1.3 \times 10^{-8}$$

$$9.4.5 \quad 0 < x \leq 3$$

$$\begin{aligned}x Y_1(x) = & (2/\pi)x \ln(\frac{1}{2}x) J_1(x) - .63661\ 98 \\ & + .22120\ 91(x/3)^2 + .216827\ 09(x/3)^4 \\ & - .131648\ 27(x/3)^6 + .31239\ 51(x/3)^8 \\ & - .04009\ 76(x/3)^{10} + .00278\ 73(x/3)^{12} + \epsilon\end{aligned}$$

$$|\epsilon| < 1.1 \times 10^{-7}$$

$$9.4.6 \quad 3 \leq x < \infty$$

$$\begin{aligned}J_1(x) = & x^{-\frac{1}{2}} f_1 \cos \theta_1, \quad Y_1(x) = x^{-\frac{1}{2}} f_1 \sin \theta_1 \\ f_1 = & .79788\ 456 + .00000\ 156(3/x) + .01659\ 667(3/x)^2 \\ & + .00017\ 105(3/x)^3 - .00249\ 511(3/x)^4 \\ & + .00113\ 653(3/x)^5 - .00020\ 033(3/x)^6 + \epsilon\end{aligned}$$

$$|\epsilon| < 4 \times 10^{-8}$$

$$\begin{aligned}\theta_1 = & x - 2.35619\ 449 + .12499\ 612(3/x) \\ & + .00005\ 650(3/x)^2 - .00637\ 879(3/x)^3 \\ & + .00074\ 348(3/x)^4 + .00079\ 824(3/x)^5 \\ & - .00029\ 166(3/x)^6 + \epsilon\end{aligned}$$

$$|\epsilon| < 9 \times 10^{-8}$$

For expansions of $J_0(x)$, $Y_0(x)$, $J_1(x)$, and $Y_1(x)$ in series of Chebyshev polynomials for the ranges $0 \leq x \leq 8$ and $0 \leq 8/x \leq 1$, see [9.37].

9.5. Zeros

Real Zeros

When ν is real, the functions $J_\nu(z)$, $J'_\nu(z)$, $Y_\nu(z)$ and $Y'_\nu(z)$ each have an infinite number of real zeros, all of which are simple with the possible exception of $z=0$. For non-negative ν the s th positive zeros of these functions are denoted by

$j_{\nu,s}$, $j'_{\nu,s}$, $y_{\nu,s}$ and $y'_{\nu,s}$ respectively, except that $z=0$ is counted as the first zero of $J'_0(z)$. Since $J'_0(z) = -J_1(z)$, it follows that

$$9.5.1 \quad j'_{0,1} = 0, \quad j'_{0,s} = j_{1,s-1} \quad (s=2, 3, \dots)$$

The zeros interlace according to the inequalities

9.5.2

$$j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < j_{\nu,3} < \dots$$

$$y_{\nu,1} < y_{\nu+1,1} < y_{\nu,2} < y_{\nu+1,2} < y_{\nu,3} < \dots$$

$$\nu \leq j'_{\nu,1} < y_{\nu,1} < y'_{\nu,1} < j_{\nu,1} < j'_{\nu,2}$$

$$< y_{\nu,2} < y'_{\nu,2} < j_{\nu,2} < j'_{\nu,3} < \dots$$

The positive zeros of any two real distinct cylinder functions of the same order are interlaced, as are the positive zeros of any real cylinder function $\mathcal{C}_\nu(z)$, defined as in 9.1.27, and the contiguous function $\mathcal{C}_{\nu+1}(z)$.

If ρ_ν is a zero of the cylinder function

$$9.5.3 \quad \mathcal{C}_\nu(z) = J_\nu(z) \cos(\pi t) + Y_\nu(z) \sin(\pi t)$$

where t is a parameter, then

$$9.5.4 \quad \mathcal{C}'_\nu(\rho_\nu) = \mathcal{C}_{\nu-1}(\rho_\nu) = -\mathcal{C}_{\nu+1}(\rho_\nu)$$

If σ_ν is a zero of $\mathcal{C}'_\nu(z)$ then

$$9.5.5 \quad \mathcal{C}_\nu(\sigma_\nu) = \frac{\sigma_\nu}{\nu} \mathcal{C}_{\nu-1}(\sigma_\nu) = \frac{\sigma_\nu}{\nu} \mathcal{C}_{\nu+1}(\sigma_\nu)$$

The parameter t may be regarded as a continuous variable and ρ_ν , σ_ν as functions $\rho_\nu(t)$, $\sigma_\nu(t)$ of t . If these functions are fixed by

$$9.5.6 \quad \rho_\nu(0) = 0, \quad \sigma_\nu(0) = j'_{\nu,1}$$

then

9.5.7

$$j_{\nu,s} = \rho_\nu(s), \quad y_{\nu,s} = \rho_\nu(s - \frac{1}{2}) \quad (s=1, 2, \dots)$$

9.5.8

$$j'_{\nu,s} = \sigma_\nu(s-1), \quad y'_{\nu,s} = \sigma_\nu(s - \frac{1}{2}) \quad (s=1, 2, \dots)$$

$$9.5.9 \quad \mathcal{C}'_\nu(\rho_\nu) = \left(\frac{\rho_\nu}{2} \frac{d\rho_\nu}{dt} \right)^{-\frac{1}{2}}, \quad \mathcal{C}_\nu(\sigma_\nu) = \left(\frac{\sigma_\nu^2 - \nu^2}{2\sigma_\nu} \frac{d\sigma_\nu}{dt} \right)^{-\frac{1}{2}}$$

Infinite Products

$$9.5.10 \quad J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)} \prod_{s=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,s}^2} \right)$$

$$9.5.11 \quad J'_\nu(z) = \frac{(\frac{1}{2}z)^{\nu-1}}{2\Gamma(\nu)} \prod_{s=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,s}^2} \right) \quad (\nu > 0)$$

McMahon's Expansions for Large Zeros

When ν is fixed, $s \gg \nu$ and $\mu = 4\nu^2$

9.5.12

$$j_{\nu,s}, y_{\nu,s} \sim \beta - \frac{\mu-1}{8\beta} - \frac{4(\mu-1)(7\mu-31)}{3(8\beta)^3} - \frac{32(\mu-1)(83\mu^2-982\mu+3779)}{15(8\beta)^5} - \frac{64(\mu-1)(6949\mu^3-153855\mu^2+1585743\mu-6277237)}{105(8\beta)^7} - \dots$$

where $\beta = (s + \frac{1}{2}\nu - \frac{1}{4})\pi$ for $j_{\nu,s}$, $\beta = (s + \frac{1}{2}\nu - \frac{3}{4})\pi$ for $y_{\nu,s}$. With $\beta = (t + \frac{1}{2}\nu - \frac{1}{4})\pi$, the right of 9.5.12 is the asymptotic expansion of $\rho_\nu(t)$ for large t .

9.5.13

$$j'_{\nu,s}, y'_{\nu,s} \sim \beta' - \frac{\mu+3}{8\beta'} - \frac{4(7\mu^2+82\mu-9)}{3(8\beta')^3} - \frac{32(83\mu^3+2075\mu^2-3039\mu+3537)}{15(8\beta')^5} - \frac{64(6949\mu^4+296492\mu^3-1248002\mu^2+7414380\mu-5853627)}{105(8\beta')^7} - \dots$$

where $\beta' = (s + \frac{1}{2}\nu - \frac{3}{4})\pi$ for $j'_{\nu,s}$, $\beta' = (s + \frac{1}{2}\nu - \frac{1}{4})\pi$ for $y'_{\nu,s}$, $\beta' = (t + \frac{1}{2}\nu + \frac{1}{4})\pi$ for $\sigma_\nu(t)$. For higher terms in 9.5.12 and 9.5.13 see [9.4] or [9.40].

Asymptotic Expansions of Zeros and Associated Values for Large Orders

9.5.14

$$j_{\nu,1} \sim \nu + 1.85575 \, 71\nu^{1/3} + 1.03315 \, 0\nu^{-1/3} - .00397\nu^{-1} - .0908\nu^{-5/3} + .043\nu^{-7/3} + \dots$$

9.5.15

$$y_{\nu,1} \sim \nu + .93157 \, 68\nu^{1/3} + .26035 \, 1\nu^{-1/3} + .01198\nu^{-1} - .0060\nu^{-5/3} - .001\nu^{-7/3} + \dots$$

9.5.16

$$j'_{\nu,1} \sim \nu + .80861 \, 65\nu^{1/3} + .07249 \, 0\nu^{-1/3} - .05097\nu^{-1} + .0094\nu^{-5/3} + \dots$$

9.5.17

$$y'_{\nu,1} \sim \nu + 1.82109 \, 80\nu^{1/3} + .94000 \, 7\nu^{-1/3} - .05808\nu^{-1} - .0540\nu^{-5/3} + \dots$$

9.5.18

$$J'_\nu(j_{\nu,1}) \sim -1.11310 \, 28\nu^{-2/3}/(1 + 1.48460 \, 6\nu^{-2/3} + .43294\nu^{-4/3} - .1943\nu^{-2} + .019\nu^{-8/3} + \dots)$$

9.5.19

$$Y'_\nu(y_{\nu,1}) \sim .95554 \, 86\nu^{-2/3}/(1 + .74526 \, 1\nu^{-2/3} + .10910\nu^{-4/3} - .0185\nu^{-2} - .003\nu^{-8/3} + \dots)$$

9.5.20

$$J_\nu(j'_{\nu,1}) \sim .67488 \, 51\nu^{-1/3}(1 - .16172 \, 3\nu^{-2/3} + .02918\nu^{-4/3} - .0068\nu^{-2} + \dots)$$

9.5.21

$$Y_\nu(y'_{\nu,1}) \sim .57319 \, 40\nu^{-1/3}(1 - .36422 \, 0\nu^{-2/3} + .09077\nu^{-4/3} + .0237\nu^{-2} + \dots)$$

Corresponding expansions for $s=2, 3$ are given in [9.40]. These expansions become progressively weaker as s increases; those which follow do not suffer from this defect.

Uniform Asymptotic Expansions of Zeros and Associated Values for Large Orders

$$9.5.22 \quad j_{\nu,s} \sim \nu z(\zeta) + \sum_{k=1}^{\infty} \frac{f_k(\zeta)}{\nu^{2k-1}} \text{ with } \zeta = \nu^{-2/3} a_s$$

9.5.23

$$J'_\nu(j_{\nu,s}) \sim -\frac{2}{\nu^{2/3}} \frac{\text{Ai}'(a_s)}{z(\zeta)h(\zeta)} \left\{ 1 + \sum_{k=1}^{\infty} \frac{F_k(\zeta)}{\nu^{2k}} \right\} \text{ with } \zeta = \nu^{-2/3} a_s$$

$$9.5.24 \quad j'_{\nu,s} \sim \nu z(\zeta) + \sum_{k=1}^{\infty} \frac{g_k(\zeta)}{\nu^{2k-1}} \text{ with } \zeta = \nu^{-2/3} a'_s$$

9.5.25

$$J_\nu(j'_{\nu,s}) \sim \text{Ai}(a'_s) \frac{h(\zeta)}{\nu^{1/3}} \left\{ 1 + \sum_{k=1}^{\infty} \frac{G_k(\zeta)}{\nu^{2k}} \right\} \text{ with } \zeta = \nu^{-2/3} a'_s$$

where a_s, a'_s are the s th negative zeros of $\text{Ai}(z)$, $\text{Ai}'(z)$ (see 10.4), $z = z(\zeta)$ is the inverse function defined implicitly by 9.3.39, and

9.5.26

$$\begin{aligned} h(\zeta) &= \{4\zeta/(1-z^2)\}^{\frac{1}{2}} \\ f_1(\zeta) &= \frac{1}{2}z(\zeta)\{h(\zeta)\}^2 b_0(\zeta) \\ g_1(\zeta) &= \frac{1}{2}\zeta^{-1}z(\zeta)\{h(\zeta)\}^2 c_0(\zeta) \end{aligned}$$

where $b_0(\zeta), c_0(\zeta)$ appear in 9.3.42 and 9.3.46. Tables of the leading coefficients follow. More extensive tables are given in [9.40].

The expansions of $y_{\nu,s}, Y'_\nu(y_{\nu,s}), y'_{\nu,s}$ and $Y_\nu(y'_{\nu,s})$ corresponding to 9.5.22 to 9.5.25 are obtained by changing the symbols $j, J, \text{Ai}, \text{Ai}', a_s$ and a'_s to $y, Y, -\text{Bi}, -\text{Bi}', b_s$ and b'_s respectively.

$-\zeta$	$z(\zeta)$	$h(\zeta)$	$f_1(\zeta)$	$F_1(\zeta)$	$(-\zeta)g_1(\zeta)$	$(-\zeta)^3g_2(\zeta)$	$(-\zeta)^5G_1(\zeta)$
0.0	1.000000	1.25992	0.0143	-0.007	-0.1260	-0.010	0.000
0.2	1.166284	1.22076	.0142	-.005	-.1335	-.010	.002
0.4	1.347557	1.18337	.0139	-.004	-.1399	-.009	.004
0.6	1.543615	1.14780	.0135	-.003	-.1453	-.009	.005
0.8	1.754187	1.11409	.0131	-.003	-.1498	-.008	.006
1.0	1.978963	1.08220	0.0126	-0.002	-0.1533	-0.008	0.006

$-\zeta$	$z(\zeta)$	$h(\zeta)$	$f_1(\zeta)$	$F_1(\zeta)$	$g_1(\zeta)$	$g_2(\zeta)$	$G_1(\zeta)$
1.0	1.978963	1.08220	0.0126	-0.002	-0.1533	-0.008	0.006
1.2	2.217607	1.05208	.0121	-.002	-.1301	-.004	.004
1.4	2.469770	1.02367	.0115	-.001	-.1130	-.002	.003
1.6	2.735103	0.99687	.0110	-.001	-.0998	-.001	.002
1.8	3.013256	.97159	.0105	-.001	-.0893	-.001	.002
2.0	3.303889	0.94775	0.0100	-0.001	-0.0807	-0.001	0.001
2.2	3.606673	.92524	.0095	-0.001	-.0734		.001
2.4	3.921292	.90397	.0091		-.0673		.001
2.6	4.247441	.88387	.0086		-.0619		.001
2.8	4.584833	.86484	.0082		-.0573		0.001
3.0	4.933192	0.84681	0.0078		-0.0533		
3.2	5.292257	.82972	.0075		-.0497		
3.4	5.661780	.81348	.0071		-.0464		
3.6	6.041525	.79806	.0068		-.0436		
3.8	6.431269	.78338	.0065		-.0410		
4.0	6.830800	0.76939	0.0062		-0.0386		
4.2	7.239917	.75605	.0060		-.0365		
4.4	7.658427	.74332	.0057		-.0345		
4.6	8.086150	.73115	.0055		-.0328		
4.8	8.522912	.71951	.0052		-.0311		
5.0	8.968548	0.70836	0.0050		-0.0296		
5.2	9.422900	.69768	.0048		-.0282		
5.4	9.885820	.68742	.0047		-.0270		
5.6	10.357162	.67758	.0045		-.0258		
5.8	10.836791	.66811	.0043		-.0246		
6.0	11.324575	0.65901	0.0042		-0.0236		
6.2	11.820388	.65024	.0040		-.0227		
6.4	12.324111	.64180	.0039		-.0218		
6.6	12.835627	.63366	.0037		-.0209		
6.8	13.354826	.62580	.0036		-.0201		
7.0	13.881601	0.61821	0.0035		-0.0194		

$(-\zeta)^{-\frac{1}{2}}$	$z(\zeta) - \frac{1}{2}(-\zeta)^{\frac{3}{2}}$	$(-\zeta)^{\frac{1}{2}}h(\zeta)$	$f_1(\zeta)$	$g_1(\zeta)$
0.40	1.528915	1.62026	0.0040	-0.0224
.35	1.541532	1.65351	.0029	-.0158
.30	1.551741	1.68067	.0020	-.0104
.25	1.559490	1.70146	.0012	-.0062
.20	1.564907	1.71607	.0006	-.0033
0.15	1.568285	1.72523	0.0003	-0.0014
.10	1.570048	1.73002	.0001	-.0004
.05	1.570703	1.73180	.0000	-.0001
.00	1.570796	1.73205	.0000	-.0000

Maximum Values of Higher Coefficients

$$|f_2(\zeta)| = .001, |F_2(\zeta)| = .0004 \quad (0 \leq -\zeta < \infty)$$

$$|g_3(\zeta)| = .001, |G_2(\zeta)| = .0007 \quad (1 \leq -\zeta < \infty)$$

$$|(-\zeta)^5g_3(\zeta)| = .002, |(-\zeta)^4G_2(\zeta)| = .0007 \quad (0 \leq -\zeta \leq 1)$$

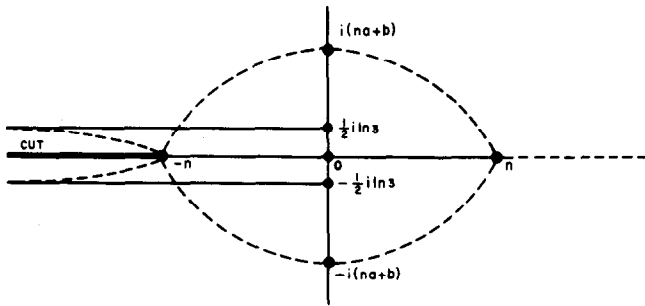
Complex Zeros of $J_\nu(z)$

When $\nu \geq -1$ the zeros of $J_\nu(z)$ are all real. If $\nu < -1$ and ν is not an integer the number of complex zeros of $J_\nu(z)$ is twice the integer part of $(-\nu)$; if the integer part of $(-\nu)$ is odd two of these zeros lie on the imaginary axis.

If $\nu \geq 0$, all zeros of $J'_\nu(z)$ are real.

Complex Zeros of $Y_\nu(z)$

When ν is real the pattern of the complex zeros of $Y_\nu(z)$ and $Y'_\nu(z)$ depends on the non-integer part of ν . Attention is confined here to the case $\nu = n$, a positive integer or zero.


 FIGURE 9.5. Zeros of $Y_n(z)$ and $Y'_n(z)$. . .

$$|\arg z| \leq \pi.$$

Figure 9.5 shows the approximate distribution of the complex zeros of $Y_n(z)$ in the region $|\arg z| \leq \pi$. The figure is symmetrical about the real axis. The two curves on the left extend to infinity, having the asymptotes

$$\mathcal{J}z = \pm \frac{1}{2} \ln 3 = \pm .54931 \dots$$

There are an infinite number of zeros near each of these curves.

The two curves extending from $z = -n$ to $z = n$ and bounding an eye-shaped domain intersect the imaginary axis at the points $\pm i(na+b)$, where

$$a = \sqrt{t_0^2 - 1} = .66274 \dots$$

$$b = \frac{1}{2} \sqrt{1 - t_0^2} \ln 2 = .19146 \dots$$

and $t_0 = 1.19968 \dots$ is the positive root of $\coth t = t$. There are n zeros near each of these curves. Asymptotic expansions of these zeros for large n

are given by the right of 9.5.22 with $\nu = n$ and $\zeta = n^{-2/3}\beta$, or $n^{-2/3}\bar{\beta}$, where β , $\bar{\beta}$ are the complex zeros of $\text{Bi}(z)$ (see 10.4).

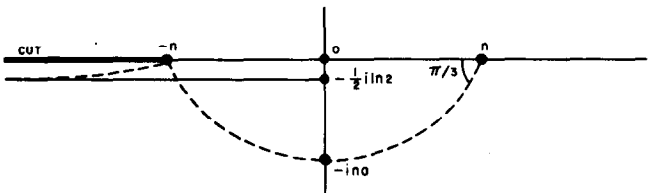
Figure 9.5 is also applicable to the zeros of $Y'_n(z)$. There are again an infinite number near the infinite curves, and n near each of the finite curves. Asymptotic expansions of the latter for large n are given by the right of 9.5.24 with $\nu = n$ and $\zeta = n^{-2/3}\beta'$, or $n^{-2/3}\bar{\beta}'$; where β' and $\bar{\beta}'$ are the complex zeros of $\text{Bi}'(z)$.

Numerical values of the three smallest complex zeros of $Y_0(z)$, $Y_1(z)$ and $Y'_1(z)$ in the region $0 < \arg z < \pi$ are given below.

For further details see [9.36] and [9.13]. The latter reference includes tables to facilitate computation.

Complex Zeros of the Hankel Functions

The approximate distribution of the zeros of $H_n^{(1)}(z)$ and its derivative in the region $|\arg z| \leq \pi$ is indicated in a similar manner on Figure 9.6.


 FIGURE 9.6. Zeros of $H_n^{(1)}(z)$ and $H_n^{(1)'}(z)$. . .

$$|\arg z| \leq \pi.$$

The asymptote of the solitary infinite curve is given by

$$\mathcal{J}z = -\frac{1}{2} \ln 2 = -.34657 \dots$$

Zeros of $Y_0(z)$ and Values of $Y_1(z)$ at the Zeros³
Zero Y_1

Real	Imag.	Real	Imag.
-2.40301 6632	+.53988 2313	+.10074 7689	-.88196 7710
-5.51987 6702	+.54718 0011	-.02924 6418	+.58716 9503
-8.65367 2403	+.54841 2067	+.01490 8063	-.46945 8752

Zeros of $Y_1(z)$ and Values of $Y_0(z)$ at the Zeros
Zero Y_0

Real	Imag.	Real	Imag.
-0.50274 3273	+.78624 3714	-.45952 7684	+1.31710 1937
-3.83353 5193	+.56235 6538	+.04830 1909	-0.69251 2884
-7.01590 3683	+.55339 3046	-.02012 6949	+0.51864 2833

Zeros of $Y'_1(z)$ and Values of $Y_1(z)$ at the Zeros
Zero Y_1

Real	Imag.	Real	Imag.
+0.57678 5129	+.90398 4792	-.76349 7088	+.58924 4865
-1.94047 7342	+.72118 5919	+.16206 4006	-.95202 7886
-5.33347 8617	+.56721 9637	-.03179 4008	+.59685 3673

³ From National Bureau of Standards, Tables of the Bessel functions $Y_0(z)$ and $Y_1(z)$ for complex arguments, Columbia Univ. Press, New York, N.Y., 1950 (with permission).

There are n zeros of each function near the finite curve extending from $z=-n$ to $z=n$; the asymptotic expansions of these zeros for large n are given by the right side of 9.5.22 or 9.5.24 with $\nu=n$ and $\zeta=e^{-2\pi i/3}n^{-2/3}a_s$ or $\zeta=e^{-2\pi i/3}n^{-2/3}a'_s$.

Zeros of Cross-Products

If ν is real and λ is positive, the zeros of the function

$$9.5.27 \quad J_\nu(z)Y_\nu(\lambda z) - J_\nu(\lambda z)Y_\nu(z)$$

are real and simple. If $\lambda > 1$, the asymptotic expansion of the s th zero is

$$9.5.28 \quad \beta + \frac{p}{\beta} + \frac{q-p^2}{\beta^3} + \frac{r-4pq+2p^3}{\beta^5} + \dots$$

where with $4\nu^2$ denoted by μ ,

$$9.5.29$$

$$\begin{aligned} \beta &= s\pi/(\lambda-1) \\ p &= \frac{\mu-1}{8\lambda}, \quad q = \frac{(\mu-1)(\mu-25)(\lambda^3-1)}{6(4\lambda)^3(\lambda-1)} \\ r &= \frac{(\mu-1)(\mu^2-114\mu+1073)(\lambda^5-1)}{5(4\lambda)^5(\lambda-1)} \end{aligned}$$

The asymptotic expansion of the large positive zeros (not necessarily the s th) of the function

$$9.5.30 \quad J'_\nu(z)Y'_\nu(\lambda z) - J'_\nu(\lambda z)Y'_\nu(z) \quad (\lambda > 1)$$

is given by 9.5.28 with the same value of β , but instead of 9.5.29 we have

$$9.5.31$$

$$\begin{aligned} p &= \frac{\mu+3}{8\lambda}, \quad q = \frac{(\mu^2+46\mu-63)(\lambda^3-1)}{6(4\lambda)^3(\lambda-1)} \\ r &= \frac{(\mu^3+185\mu^2-2053\mu+1899)(\lambda^5-1)}{5(4\lambda)^5(\lambda-1)} \end{aligned}$$

The asymptotic expansion of the large positive zeros of the function

$$9.5.32 \quad J'_\nu(z)Y_\nu(\lambda z) - Y'_\nu(z)J_\nu(\lambda z)$$

is given by 9.5.28 with

$$9.5.33$$

$$\begin{aligned} \beta &= (s-\frac{1}{2})\pi/(\lambda-1) \\ p &= \frac{(\mu+3)\lambda-(\mu-1)}{8\lambda(\lambda-1)} \\ q &= \frac{(\mu^2+46\mu-63)\lambda^3-(\mu-1)(\mu-25)}{6(4\lambda)^3(\lambda-1)} \end{aligned}$$

$$5(4\lambda)^5(\lambda-1)r = (\mu^3+185\mu^2-2053\mu+1899)\lambda^5 - (\mu-1)(\mu^2-114\mu+1073)$$

Modified Bessel Functions I and K

9.6. Definitions and Properties

Differential Equation

$$9.6.1 \quad z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} - (z^2 + \nu^2)w = 0$$

Solutions are $I_{\pm\nu}(z)$ and $K_\nu(z)$. Each is a regular function of z throughout the z -plane cut along the negative real axis, and for fixed z ($\neq 0$) each is an entire function of ν . When $\nu = \pm n$, $I_\nu(z)$ is an entire function of z .

$I_\nu(z)$ ($\Re \nu \geq 0$) is bounded as $z \rightarrow 0$ in any bounded range of $\arg z$. $I_\nu(z)$ and $I_{-\nu}(z)$ are linearly independent except when ν is an integer. $K_\nu(z)$ tends to zero as $|z| \rightarrow \infty$ in the sector $|\arg z| < \frac{1}{2}\pi$, and for all values of ν , $I_\nu(z)$ and $K_\nu(z)$ are linearly independent. $I_\nu(z)$, $K_\nu(z)$ are real and positive when $\nu > -1$ and $z > 0$.

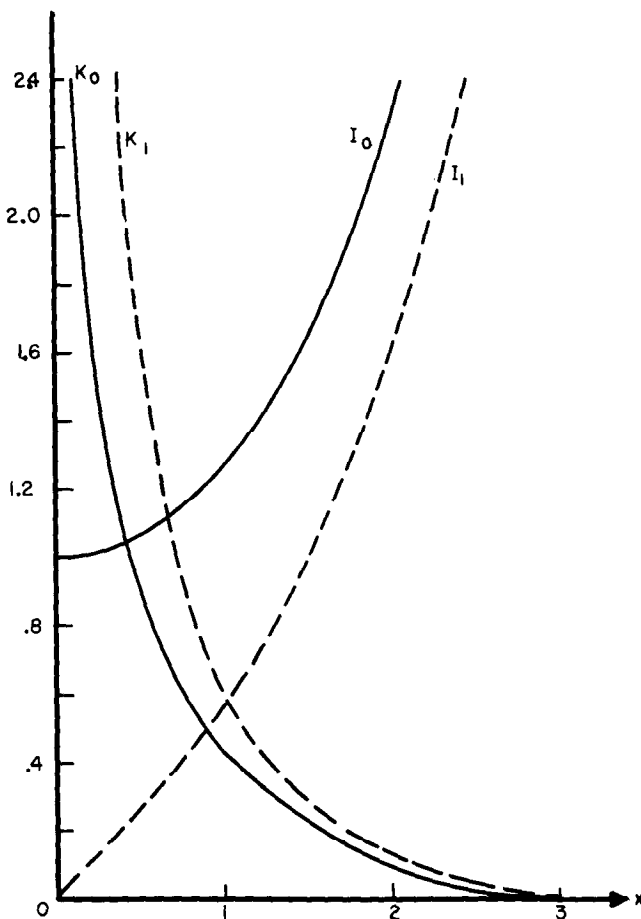
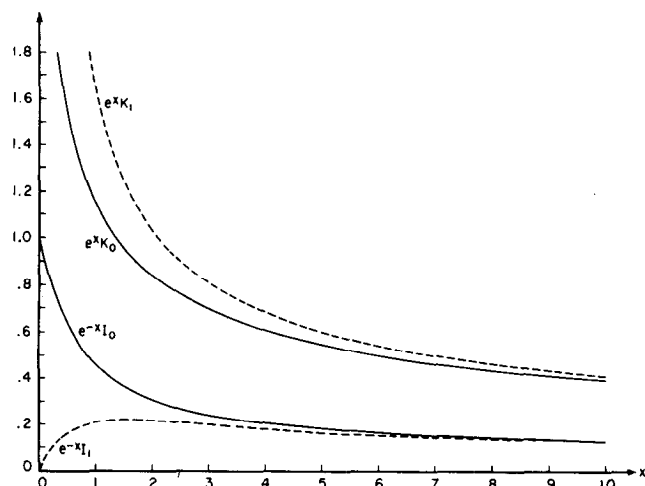
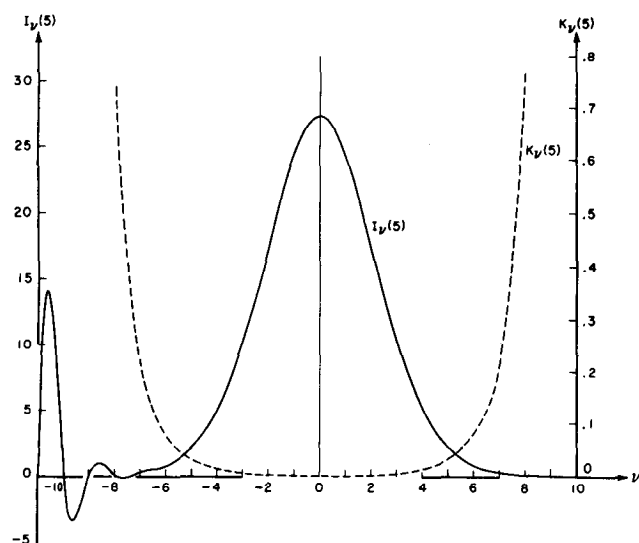


FIGURE 9.7. $I_0(x)$, $K_0(x)$, $I_1(x)$ and $K_1(x)$.


 FIGURE 9.8. $e^{-x}I_0(x)$, $e^{-x}I_1(x)$, $e^xK_0(x)$ and $e^xK_1(x)$.

 FIGURE 9.9. $I_5(5)$ and $K_5(5)$.

Relations Between Solutions

$$9.6.2 \quad K_\nu(z) = \frac{1}{2}\pi \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}$$

The right of this equation is replaced by its limiting value if ν is an integer or zero.

9.6.3

$$I_\nu(z) = e^{-\frac{1}{2}\nu\pi i} J_\nu(ze^{\frac{1}{2}\pi i}) \quad (-\pi < \arg z \leq \frac{1}{2}\pi)$$

$$I_\nu(z) = e^{\frac{3}{2}\nu\pi i} J_\nu(ze^{-\frac{3}{2}\pi i/2}) \quad (\frac{1}{2}\pi < \arg z \leq \pi)$$

9.6.4

$$K_\nu(z) = \frac{1}{2}\pi i e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(ze^{\frac{1}{2}\pi i}) \quad (-\pi < \arg z \leq \frac{1}{2}\pi)$$

$$K_\nu(z) = -\frac{1}{2}\pi i e^{-\frac{1}{2}\nu\pi i} H_\nu^{(2)}(ze^{-\frac{1}{2}\pi i}) \quad (-\frac{1}{2}\pi < \arg z \leq \pi)$$

9.6.5

$$Y_\nu(ze^{\frac{1}{2}\pi i}) = e^{\frac{1}{2}(\nu+1)\pi i} I_\nu(z) - (2/\pi) e^{-\frac{1}{2}\nu\pi i} K_\nu(z) \quad (-\pi < \arg z \leq \frac{1}{2}\pi)$$

9.6.6

$$I_{-\nu}(z) = I_\nu(z), \quad K_{-\nu}(z) = K_\nu(z)$$

Most of the properties of modified Bessel functions can be deduced immediately from those of ordinary Bessel functions by application of these relations.

Limiting Forms for Small Arguments

When ν is fixed and $z \rightarrow 0$

9.6.7

$$I_\nu(z) \sim (\frac{1}{2}z)^\nu / \Gamma(\nu+1) \quad (\nu \neq -1, -2, \dots)$$

9.6.8

$$K_0(z) \sim -\ln z$$

9.6.9

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu) (\frac{1}{2}z)^{-\nu} \quad (\Re \nu > 0)$$

Ascending Series

$$9.6.10 \quad I_\nu(z) = (\frac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k! \Gamma(\nu+k+1)}$$

9.6.11

$$K_\nu(z) = \frac{1}{2} (\frac{1}{2}z)^{-\nu} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} (-\frac{1}{4}z^2)^k$$

$$+ (-)^{n+1} \ln(\frac{1}{2}z) I_n(z)$$

$$+ (-)^n \frac{1}{2} (\frac{1}{2}z)^n \sum_{k=0}^{\infty} \{ \psi(k+1) + \psi(n+k+1) \} \frac{(\frac{1}{4}z^2)^k}{k!(n+k)!}$$

where $\psi(n)$ is given by 6.3.2.

$$9.6.12 \quad I_0(z) = 1 + \frac{\frac{1}{4}z^2}{(1!)^2} + \frac{(\frac{1}{4}z^2)^2}{(2!)^2} + \frac{(\frac{1}{4}z^2)^3}{(3!)^2} + \dots$$

9.6.13

$$K_0(z) = -\{ \ln(\frac{1}{2}z) + \gamma \} I_0(z) + \frac{\frac{1}{4}z^2}{(1!)^2}$$

$$+ (1 + \frac{1}{2}) \frac{(\frac{1}{4}z^2)^2}{(2!)^2} + (1 + \frac{1}{2} + \frac{1}{3}) \frac{(\frac{1}{4}z^2)^3}{(3!)^2} + \dots$$

Wronskians

9.6.14

$$W\{I_\nu(z), I_{-\nu}(z)\} = I_\nu(z) I_{-(\nu+1)}(z) - I_{\nu+1}(z) I_{-\nu}(z)$$

$$= -2 \sin(\nu\pi) / (\pi z)$$

9.6.15

$$W\{K_\nu(z), I_\nu(z)\} = I_\nu(z) K_{\nu+1}(z) + I_{\nu+1}(z) K_\nu(z) = 1/z$$

Integral Representations

9.6.16

$$I_0(z) = \frac{1}{\pi} \int_0^\pi e^{\pm z \cos \theta} d\theta = \frac{1}{\pi} \int_0^\pi \cosh(z \cos \theta) d\theta$$

$$9.6.17 \quad K_0(z) = -\frac{1}{\pi} \int_0^\pi e^{\pm z \cos \theta} \{\gamma + \ln(2z \sin^2 \theta)\} d\theta$$

9.6.18

$$I_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}}\Gamma(\nu + \frac{1}{2})} \int_0^\pi e^{\pm z \cos \theta} \sin^{2\nu} \theta d\theta$$

$$= \frac{(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}}\Gamma(\nu + \frac{1}{2})} \int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} e^{\pm zt} dt \quad (\Re \nu > -\frac{1}{2})$$

$$9.6.19 \quad I_n(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos(n\theta) d\theta$$

9.6.20

$$I_\nu(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos(\nu\theta) d\theta$$

$$- \frac{\sin(\nu\pi)}{\pi} \int_0^\infty e^{-z \cosh t - \nu t} dt \quad (|\arg z| < \frac{1}{2}\pi)$$

9.6.21

$$K_0(x) = \int_0^\infty \cos(x \sinh t) dt = \int_0^\infty \frac{\cos(xt)}{\sqrt{t^2+1}} dt$$

$$(x > 0)$$

9.6.22

$$K_\nu(x) = \sec(\frac{1}{2}\nu\pi) \int_0^\infty \cos(x \sinh t) \cosh(\nu t) dt$$

$$= \csc(\frac{1}{2}\nu\pi) \int_0^\infty \sin(x \sinh t) \sinh(\nu t) dt$$

$$(|\Re \nu| < 1, x > 0)$$

9.6.23

$$K_\nu(z) = \frac{\pi^{\frac{1}{2}}(\frac{1}{2}z)^\nu}{\Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{-z \cosh t} \sinh^{2\nu} t dt$$

$$= \frac{\pi^{\frac{1}{2}}(\frac{1}{2}z)^\nu}{\Gamma(\nu + \frac{1}{2})} \int_1^\infty e^{-zt} (t^2-1)^{\nu-\frac{1}{2}} dt$$

$$(\Re \nu > -\frac{1}{2}, |\arg z| < \frac{1}{2}\pi)$$

$$9.6.24 \quad K_\nu(z) = \int_0^\infty e^{-z \cosh t} \cosh(\nu t) dt \quad (|\arg z| < \frac{1}{2}\pi)$$

9.6.25

$$K_\nu(xz) = \frac{\Gamma(\nu + \frac{1}{2})(2z)^\nu}{\pi^{\frac{1}{2}}x^\nu} \int_0^\infty \frac{\cos(xt) dt}{(t^2+z^2)^{\nu+\frac{1}{2}}}$$

$$(\Re \nu > -\frac{1}{2}, x > 0, |\arg z| < \frac{1}{2}\pi)^*$$

Recurrence Relations

9.6.26

$$\mathcal{Z}_{\nu-1}(z) - \mathcal{Z}_{\nu+1}(z) = \frac{2\nu}{z} \mathcal{Z}_\nu(z)$$

$$\mathcal{Z}'_\nu(z) = \mathcal{Z}_{\nu-1}(z) - \frac{\nu}{z} \mathcal{Z}_\nu(z)$$

$$\mathcal{Z}_{\nu-1}(z) + \mathcal{Z}_{\nu+1}(z) = 2\mathcal{Z}'_\nu(z)$$

$$\mathcal{Z}'_\nu(z) = \mathcal{Z}_{\nu+1}(z) + \frac{\nu}{z} \mathcal{Z}_\nu(z)$$

$\mathcal{Z}_\nu$ denotes I_ν , $e^{\nu\pi i}K_\nu$, or any linear combination of these functions, the coefficients in which are independent of z and ν .

$$9.6.27 \quad I'_0(z) = I_1(z), \quad K'_0(z) = -K_1(z)$$

Formulas for Derivatives

9.6.28

$$\left(\frac{1}{z} \frac{d}{dz}\right)^k \{z^\nu \mathcal{Z}_\nu(z)\} = z^{\nu-k} \mathcal{Z}_{\nu-k}(z)$$

$$\left(\frac{1}{z} \frac{d}{dz}\right)^k \{z^{-\nu} \mathcal{Z}_\nu(z)\} = z^{-\nu-k} \mathcal{Z}_{\nu+k}(z) \quad (k=0,1,2,\dots)$$

9.6.29

$$\mathcal{Z}_{\nu}^{(k)}(z) = \frac{1}{2^k} \{ \mathcal{Z}_{\nu-k}(z) + \binom{k}{1} \mathcal{Z}_{\nu-k+2}(z) \\ + \binom{k}{2} \mathcal{Z}_{\nu-k+4}(z) + \dots + \mathcal{Z}_{\nu+k}(z) \}$$

$$(k=0,1,2,\dots)$$

Analytic Continuation

$$9.6.30 \quad I_\nu(ze^{m\pi i}) = e^{m\nu\pi i} I_\nu(z) \quad (m \text{ an integer})$$

9.6.31

$$K_\nu(ze^{m\pi i}) = e^{-m\nu\pi i} K_\nu(z) - \pi i \sin(m\nu\pi) \csc(\nu\pi) I_\nu(z)$$

$$(m \text{ an integer})$$

$$9.6.32 \quad I_\nu(\bar{z}) = \overline{I_\nu(z)}, \quad K_\nu(\bar{z}) = \overline{K_\nu(z)} \quad (\nu \text{ real})$$

Generating Function and Associated Series

$$9.6.33 \quad e^{\frac{1}{2}z(t+1/t)} = \sum_{k=-\infty}^{\infty} t^k I_k(z) \quad (t \neq 0)$$

$$9.6.34 \quad e^{z \cos \theta} = I_0(z) + 2 \sum_{k=1}^{\infty} I_k(z) \cos(k\theta)$$

9.6.35

$$e^{z \sin \theta} = I_0(z) + 2 \sum_{k=1}^{\infty} (-1)^k I_{2k+1}(z) \sin\{(2k+1)\theta\}$$

$$+ 2 \sum_{k=1}^{\infty} (-1)^k I_{2k}(z) \cos(2k\theta)$$

$$9.6.36 \quad 1 = I_0(z) - 2I_2(z) + 2I_4(z) - 2I_6(z) + \dots$$

$$9.6.37 \quad e^z = I_0(z) + 2I_1(z) + 2I_2(z) + 2I_3(z) + \dots$$

$$9.6.38 \quad e^{-z} = I_0(z) - 2I_1(z) + 2I_2(z) - 2I_3(z) + \dots$$

9.6.39

$$\cosh z = I_0(z) + 2I_2(z) + 2I_4(z) + 2I_6(z) + \dots$$

$$9.6.40 \quad \sinh z = 2I_1(z) + 2I_3(z) + 2I_5(z) + \dots$$

*See page 11.

Other Differential Equations

The quantity λ^2 in equations 9.1.49 to 9.1.54 and 9.1.56 can be replaced by $-\lambda^2$ if at the same time the symbol $\mathcal{C}$ in the given solutions is replaced by $\mathcal{L}$.

9.6.41

$$z^2 w'' + z(1 \pm 2z)w' + (\pm z - \nu^2)w = 0, \quad w = e^{\mp i\mathcal{L}} \mathcal{L}_\nu(z)$$

Differential equations for products may be obtained from 9.1.57 to 9.1.59 by replacing z by iz .

Derivatives With Respect to Order

9.6.42

$$\frac{\partial}{\partial \nu} I_\nu(z) = I_\nu(z) \ln\left(\frac{1}{2}z\right) - \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\psi(\nu+k+1)}{\Gamma(\nu+k+1)} \frac{(\frac{1}{2}z^2)^k}{k!}$$

9.6.43

$$\begin{aligned} \frac{\partial}{\partial \nu} K_\nu(z) &= \frac{1}{2}\pi \csc(\nu\pi) \left\{ \frac{\partial}{\partial \nu} I_{-\nu}(z) - \frac{\partial}{\partial \nu} I_\nu(z) \right\} \\ &\quad - \pi \cot(\nu\pi) K_\nu(z) \quad (\nu \neq 0, \pm 1, \pm 2, \dots) \end{aligned}$$

9.6.44

$$\begin{aligned} (-)^n \left[\frac{\partial}{\partial \nu} I_\nu(z) \right]_{\nu=n} &= \\ &= -K_n(z) + \frac{n! (\frac{1}{2}z)^{-n}}{2} \sum_{k=0}^{n-1} (-)^k \frac{(\frac{1}{2}z)^k I_k(z)}{(n-k)k!} \end{aligned}$$

9.6.45

$$\left[\frac{\partial}{\partial \nu} K_\nu(z) \right]_{\nu=n} = \frac{n! (\frac{1}{2}z)^{-n}}{2} \sum_{k=0}^{n-1} \frac{(\frac{1}{2}z)^k K_k(z)}{(n-k)k!}$$

9.6.46

$$\left[\frac{\partial}{\partial \nu} I_\nu(z) \right]_{\nu=0} = -K_0(z), \quad \left[\frac{\partial}{\partial \nu} K_\nu(z) \right]_{\nu=0} = 0$$

Expressions in Terms of Hypergeometric Functions

9.6.47

$$\begin{aligned} I_\nu(z) &= \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)} {}_0F_1(\nu+1; \frac{1}{2}z^2) \\ &= \frac{(\frac{1}{2}z)^\nu e^{-z}}{\Gamma(\nu+1)} M\left(\nu+\frac{1}{2}, 2\nu+1, 2z\right) = \frac{z^{-\frac{1}{2}} M_{0,\nu}(2z)}{2^{2\nu+1} \Gamma(\nu+1)} \end{aligned}$$

9.6.48

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} W_{0,\nu}(2z)$$

(${}_0F_1$ is the generalized hypergeometric function. For $M(a, b, z)$, $M_{0,\nu}(z)$ and $W_{0,\nu}(z)$ see chapter 13.)

Connection With Legendre Functions

If μ and z are fixed, $\Re z > 0$, and $\nu \rightarrow \infty$ through real positive values

$$9.6.49 \quad \lim \{ \nu^\mu P_\nu^{-\mu} \left(\cosh \frac{z}{\nu} \right) \} = I_\mu(z)$$

$$9.6.50 \quad \lim \{ \nu^{-\mu} e^{-\mu\pi i} Q_\nu^\mu \left(\cosh \frac{z}{\nu} \right) \} = K_\mu(z)$$

For the definition of $P_\nu^{-\mu}$ and Q_ν^μ , see chapter 8.

Multiplication Theorems

9.6.51

$$\mathcal{L}_\nu(\lambda z) = \lambda^{\pm \nu} \sum_{k=0}^{\infty} \frac{(\lambda^2 - 1)^k (\frac{1}{2}z)^k}{k!} \mathcal{L}_{\nu \pm k}(z) \quad (|\lambda^2 - 1| < 1)$$

If $\mathcal{L} = I$ and the upper signs are taken, the restriction on λ is unnecessary.

9.6.52

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} J_{\nu+k}(z), \quad J_\nu(z) = \sum_{k=0}^{\infty} (-)^k \frac{z^k}{k!} I_{\nu+k}(z)$$

Neumann Series for $K_n(z)$

9.6.53

$$\begin{aligned} K_n(z) &= (-)^{n-1} \{ \ln(\frac{1}{2}z) - \psi(n+1) \} I_n(z) \\ &\quad + \frac{n! (\frac{1}{2}z)^{-n}}{2} \sum_{k=0}^{n-1} (-)^k \frac{(\frac{1}{2}z)^k I_k(z)}{(n-k)k!} \\ &\quad + (-)^n \sum_{k=1}^{\infty} \frac{(n+2k) I_{n+2k}(z)}{k(n+k)} \end{aligned}$$

$$9.6.54 \quad K_0(z) = - \{ \ln(\frac{1}{2}z) + \gamma \} I_0(z) + 2 \sum_{k=1}^{\infty} \frac{I_{2k}(z)}{k}$$

Zeros

Properties of the zeros of $I_\nu(z)$ and $K_\nu(z)$ may be deduced from those of $J_\nu(z)$ and $H_\nu^{(1)}(z)$ respectively, by application of the transformations 9.6.3 and 9.6.4.

For example, if ν is real the zeros of $I_\nu(z)$ are all complex unless $-2k < \nu < -(2k-1)$ for some positive integer k , in which event $I_\nu(z)$ has two real zeros.

The approximate distribution of the zeros of $K_n(z)$ in the region $-\frac{3}{2}\pi \leq \arg z \leq \frac{1}{2}\pi$ is obtained on rotating Figure 9.6 through an angle $-\frac{1}{2}\pi$ so that the cut lies along the positive imaginary axis. The zeros in the region $-\frac{1}{2}\pi \leq \arg z \leq \frac{3}{2}\pi$ are their conjugates. $K_n(z)$ has no zeros in the region $|\arg z| \leq \frac{1}{2}\pi$; this result remains true when n is replaced by any real number ν .

9.7. Asymptotic Expansions

Asymptotic Expansions for Large Arguments

When ν is fixed, $|z|$ is large and $\mu = 4\nu^2$

9.7.1

$$\begin{aligned} I_\nu(z) &\sim \frac{e^z}{\sqrt{2\pi z}} \left\{ 1 - \frac{\mu-1}{8z} + \frac{(\mu-1)(\mu-9)}{2!(8z)^2} \right. \\ &\quad \left. - \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + \dots \right\} \quad (|\arg z| < \frac{1}{2}\pi) \end{aligned}$$

9.7.2

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left\{ 1 + \frac{\mu-1}{8z} + \frac{(\mu-1)(\mu-9)}{2!(8z)^2} + \frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + \dots \right\} \quad (|\arg z| < \frac{3}{2}\pi)$$

9.7.3

$$I'_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left\{ 1 - \frac{\mu+3}{8z} + \frac{(\mu-1)(\mu+15)}{2!(8z)^2} - \frac{(\mu-1)(\mu-9)(\mu+35)}{3!(8z)^3} + \dots \right\} \quad (|\arg z| < \frac{1}{2}\pi)$$

9.7.4

$$K'_\nu(z) \sim -\sqrt{\frac{\pi}{2z}} e^{-z} \left\{ 1 + \frac{\mu+3}{8z} + \frac{(\mu-1)(\mu+15)}{2!(8z)^2} + \frac{(\mu-1)(\mu-9)(\mu+35)}{3!(8z)^3} + \dots \right\} \quad (|\arg z| < \frac{3}{2}\pi)$$

The general terms in the last two expansions can be written down by inspection of 9.2.15 and 9.2.16.

If ν is real and non-negative and z is positive the remainder after k terms in the expansion 9.7.2 does not exceed the $(k+1)$ th term in absolute value and is of the same sign, provided that $k \geq \nu - \frac{1}{2}$.

9.7.5

$$I_\nu(z)K_\nu(z) \sim \frac{1}{2z} \left\{ 1 - \frac{1}{2} \frac{\mu-1}{(2z)^2} + \frac{1 \cdot 3}{2 \cdot 4} \frac{(\mu-1)(\mu-9)}{(2z)^4} - \dots \right\} \quad (|\arg z| < \frac{1}{2}\pi)$$

9.7.6

$$I'_\nu(z)K'_\nu(z) \sim -\frac{1}{2z} \left\{ 1 + \frac{1}{2} \frac{\mu-3}{(2z)^2} - \frac{1 \cdot 1}{2 \cdot 4} \frac{(\mu-1)(\mu-45)}{(2z)^4} + \dots \right\} \quad (|\arg z| < \frac{1}{2}\pi)$$

The general terms can be written down by inspection of 9.2.28 and 9.2.30.

Uniform Asymptotic Expansions for Large Orders

$$9.7.7 \quad I_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{e^{\nu\eta}}{(1+z^2)^{1/4}} \left\{ 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right\}$$

9.7.8

$$K_\nu(\nu z) \sim \sqrt{\frac{\pi}{2\nu}} \frac{e^{-\nu\eta}}{(1+z^2)^{1/4}} \left\{ 1 + \sum_{k=1}^{\infty} (-)^k \frac{u_k(t)}{\nu^k} \right\}$$

$$9.7.9 \quad I'_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{(1+z^2)^{1/4}}{z} e^{\nu\eta} \left\{ 1 + \sum_{k=1}^{\infty} \frac{v_k(t)}{\nu^k} \right\}$$

9.7.10

$$K'_\nu(\nu z) \sim -\sqrt{\frac{\pi}{2\nu}} \frac{(1+z^2)^{1/4}}{z} e^{-\nu\eta} \left\{ 1 + \sum_{k=1}^{\infty} (-)^k \frac{v_k(t)}{\nu^k} \right\}$$

When $\nu \rightarrow +\infty$, these expansions hold uniformly with respect to z in the sector $|\arg z| \leq \frac{1}{2}\pi - \epsilon$, where ϵ is an arbitrary positive number. Here

$$9.7.11 \quad t = 1/\sqrt{1+z^2}, \quad \eta = \sqrt{1+z^2} + \ln \frac{z}{1+\sqrt{1+z^2}}$$

and $u_k(t)$, $v_k(t)$ are given by 9.3.9, 9.3.10, 9.3.13 and 9.3.14. See [9.38] for tables of η , $u_k(t)$, $v_k(t)$, and also for bounds on the remainder terms in 9.7.7 to 9.7.10.

9.8. Polynomial Approximations⁴

In equations 9.8.1 to 9.8.4, $t = x/3.75$.

$$9.8.1 \quad -3.75 \leq x \leq 3.75$$

$$I_0(x) = 1 + 3.51562 \, 29t^2 + 3.08994 \, 24t^4 + 1.20674 \, 92t^6 + .26597 \, 32t^8 + .03607 \, 68t^{10} + .00458 \, 13t^{12} + \epsilon$$

$$|\epsilon| < 1.6 \times 10^{-7}$$

$$9.8.2 \quad 3.75 \leq x < \infty$$

$$x^2 e^{-x} I_0(x) = .39894 \, 228 + .01328 \, 592t^{-1} + .00225 \, 319t^{-2} - .00157 \, 565t^{-3} + .00916 \, 281t^{-4} - .02057 \, 706t^{-5} + .02635 \, 537t^{-6} - .01647 \, 633t^{-7} + .00392 \, 377t^{-8} + \epsilon$$

$$|\epsilon| < 1.9 \times 10^{-7}$$

$$9.8.3 \quad -3.75 \leq x \leq 3.75$$

$$x^{-1} I_1(x) = \frac{1}{2} + .87890 \, 594t^2 + .51498 \, 869t^4 + .15084 \, 934t^6 + .02658 \, 733t^8 + .00301 \, 532t^{10} + .00032 \, 411t^{12} + \epsilon$$

$$|\epsilon| < 8 \times 10^{-9}$$

$$9.8.4 \quad 3.75 \leq x < \infty$$

$$x^2 e^{-x} I_1(x) = .39894 \, 228 - .03988 \, 024t^{-1} - .00362 \, 018t^{-2} + .00163 \, 801t^{-3} - .01031 \, 555t^{-4} + .02282 \, 967t^{-5} - .02895 \, 312t^{-6} + .01787 \, 654t^{-7} - .00420 \, 059t^{-8} + \epsilon$$

$$|\epsilon| < 2.2 \times 10^{-7}$$

⁴ See footnote 2, section 9.4.

9.8.5 $0 < x \leq 2$

$$\begin{aligned} K_0(x) = & -\ln(x/2)I_0(x) - .57721\ 566 \\ & + .42278\ 420(x/2)^2 + .23069\ 756(x/2)^4 \\ & + .03488\ 590(x/2)^6 + .00262\ 698(x/2)^8 \\ & + .00010\ 750(x/2)^{10} + .00000\ 740(x/2)^{12} + \epsilon \\ & |\epsilon| < 1 \times 10^{-8} \end{aligned}$$

9.8.6 $2 \leq x < \infty$

$$\begin{aligned} x^{\frac{1}{2}}e^x K_0(x) = & 1.25331\ 414 - .07832\ 358(2/x) \\ & + .02189\ 568(2/x)^2 - .01062\ 446(2/x)^3 \\ & + .00587\ 872(2/x)^4 - .00251\ 540(2/x)^5 \\ & + .00053\ 208(2/x)^6 + \epsilon \\ & |\epsilon| < 1.9 \times 10^{-7} \end{aligned}$$

9.8.7 $0 < x \leq 2$

$$\begin{aligned} xK_1(x) = & x \ln(x/2)I_1(x) + 1 + .15443\ 144(x/2)^2 \\ & - .67278\ 579(x/2)^4 - .18156\ 897(x/2)^6 \\ & - .01919\ 402(x/2)^8 - .00110\ 404(x/2)^{10} \\ & - .00004\ 686(x/2)^{12} + \epsilon \\ & |\epsilon| < 8 \times 10^{-9} \end{aligned}$$

9.8.8 $2 \leq x < \infty$

$$\begin{aligned} x^{\frac{1}{2}}e^x K_1(x) = & 1.25331\ 414 + .23498\ 619(2/x) \\ & - .03655\ 620(2/x)^2 + .01504\ 268(2/x)^3 \\ & - .00780\ 353(2/x)^4 + .00325\ 614(2/x)^5 \\ & - .00068\ 245(2/x)^6 + \epsilon \\ & |\epsilon| < 2.2 \times 10^{-7} \end{aligned}$$

For expansions of $I_0(x)$, $K_0(x)$, $I_1(x)$, and $K_1(x)$ in series of Chebyshev polynomials for the ranges $0 \leq x \leq 8$ and $0 \leq 8/x \leq 1$, see [9.37].

Kelvin Functions
9.9. Definitions and Properties

In this and the following section ν is real, x is real and non-negative, and n is again a positive integer or zero.

Definitions
9.9.1

$$\begin{aligned} \text{ber}, x + i \text{bei}, x = & J_\nu(xe^{3\pi i/4}) = e^{\nu\pi i} J_\nu(xe^{-\pi i/4}) \\ = & e^{\frac{1}{2}\nu\pi i} I_\nu(xe^{\pi i/4}) = e^{\frac{3}{2}\nu\pi i/2} I_\nu(xe^{-3\pi i/4}) \end{aligned}$$

9.9.2

$$\begin{aligned} \text{ker}, x + i \text{kei}, x = & e^{-\frac{1}{2}\nu\pi i} K_\nu(xe^{\pi i/4}) \\ = & \frac{1}{2}\pi i H_\nu^{(1)}(xe^{3\pi i/4}) = -\frac{1}{2}\pi i e^{-\nu\pi i} H_\nu^{(2)}(xe^{-\pi i/4}) \end{aligned}$$

When $\nu=0$, suffices are usually suppressed.

Differential Equations
9.9.3

$$\begin{aligned} x^2 w'' + x w' - (ix^2 + \nu^2) w = & 0, \\ w = & \text{ber}, x + i \text{bei}, x, \quad \text{ber}_-, x + i \text{bei}_-, x, \\ & \text{ker}, x + i \text{kei}, x, \quad \text{ker}_-, x + i \text{kei}_-, x \end{aligned}$$

9.9.4

$$\begin{aligned} x^4 w^{(4)} + 2x^3 w''' - (1 + 2\nu^2)(x^2 w'' - x w') \\ + (\nu^4 - 4\nu^2 + x^4) w = & 0, \\ w = & \text{ber}_{\pm}, x, \text{bei}_{\pm}, x, \text{ker}_{\pm}, x, \text{kei}_{\pm}, x \end{aligned}$$

Relations Between Solutions
9.9.5

$$\begin{aligned} \text{ber}_-, x = & \cos(\nu\pi) \text{ber}, x + \sin(\nu\pi) \text{bei}, x \\ & + (2/\pi) \sin(\nu\pi) \text{ker}, x \\ \text{bei}_-, x = & -\sin(\nu\pi) \text{ber}, x + \cos(\nu\pi) \text{bei}, x \\ & + (2/\pi) \sin(\nu\pi) \text{kei}, x \end{aligned}$$

9.9.6

$$\begin{aligned} \text{ker}_-, x = & \cos(\nu\pi) \text{ker}, x - \sin(\nu\pi) \text{kei}, x \\ \text{kei}_-, x = & \sin(\nu\pi) \text{ker}, x + \cos(\nu\pi) \text{kei}, x \end{aligned}$$

$$9.9.7 \quad \text{ber}_{-n} x = (-)^n \text{ber}_n x, \quad \text{bei}_{-n} x = (-)^n \text{bei}_n x$$

$$9.9.8 \quad \text{ker}_{-n} x = (-)^n \text{ker}_n x, \quad \text{kei}_{-n} x = (-)^n \text{kei}_n x$$

Ascending Series
9.9.9

$$\begin{aligned} \text{ber}, x = & \left(\frac{1}{2}x\right)^\nu \sum_{k=0}^{\infty} \frac{\cos\left\{\left(\frac{3}{4}\nu + \frac{1}{2}k\right)\pi\right\}}{k! \Gamma(\nu + k + 1)} \left(\frac{1}{4}x^2\right)^k \\ \text{bei}, x = & \left(\frac{1}{2}x\right)^\nu \sum_{k=0}^{\infty} \frac{\sin\left\{\left(\frac{3}{4}\nu + \frac{1}{2}k\right)\pi\right\}}{k! \Gamma(\nu + k + 1)} \left(\frac{1}{4}x^2\right)^k \end{aligned}$$

9.9.10

$$\text{ber } x = 1 - \frac{(\frac{1}{4}x^2)^2}{(2!)^2} + \frac{(\frac{1}{4}x^2)^4}{(4!)^2} - \dots$$

$$\text{bei } x = \frac{1}{4}x^2 - \frac{(\frac{1}{4}x^2)^3}{(3!)^2} + \frac{(\frac{1}{4}x^2)^5}{(5!)^2} - \dots$$

9.9.11

$$\begin{aligned} \text{ker}_n x = & \frac{1}{2} \left(\frac{1}{2}x\right)^{-n} \sum_{k=0}^{n-1} \cos\left\{\left(\frac{3}{4}n + \frac{1}{2}k\right)\pi\right\} \\ & \times \frac{(n-k-1)!}{k!} \left(\frac{1}{4}x^2\right)^k - \ln\left(\frac{1}{2}x\right) \text{ber}_n x + \frac{1}{4}\pi \text{bei}_n x \\ & + \frac{1}{2} \left(\frac{1}{2}x\right)^n \sum_{k=0}^{\infty} \cos\left\{\left(\frac{3}{4}n + \frac{1}{2}k\right)\pi\right\} \\ & \times \frac{\{\psi(k+1) + \psi(n+k+1)\}}{k!(n+k)!} \left(\frac{1}{4}x^2\right)^k \end{aligned}$$

$$\begin{aligned} \text{kei}_n x &= -\frac{1}{2}(\frac{1}{2}x)^{-n} \sum_{k=0}^{n-1} \sin\{(\frac{3}{2}n + \frac{1}{2}k)\pi\} \\ &\times \frac{(n-k-1)!}{k!} (\frac{1}{4}x^2)^k - \ln(\frac{1}{2}x) \text{bei}_n x - \frac{1}{4}\pi \text{ber}_n x \\ &+ \frac{1}{2}(\frac{1}{2}x)^n \sum_{k=0}^{\infty} \sin\{(\frac{3}{2}n + \frac{1}{2}k)\pi\} \\ &\times \frac{\{\psi(k+1) + \psi(n+k+1)\}}{k!(n+k)!} (\frac{1}{4}x^2)^k \end{aligned}$$

where $\psi(n)$ is given by 6.3.2.

9.9.12

$$\begin{aligned} \ker x &= -\ln(\frac{1}{2}x) \text{ber} x + \frac{1}{4}\pi \text{bei} x \\ &+ \sum_{k=0}^{\infty} (-)^k \frac{\psi(2k+1)}{\{(2k)!\}^2} (\frac{1}{4}x^2)^{2k} \\ \text{kei} x &= -\ln(\frac{1}{2}x) \text{bei} x - \frac{1}{4}\pi \text{ber} x \\ &+ \sum_{k=0}^{\infty} (-)^k \frac{\psi(2k+2)}{\{(2k+1)!\}^2} (\frac{1}{4}x^2)^{2k+1} \end{aligned}$$

Functions of Negative Argument

In general Kelvin functions have a branch point at $x=0$ and individual functions with arguments $xe^{\pm\pi i}$ are complex. The branch point is absent however in the case of ber_ν and bei_ν , when ν is an integer, and

9.9.13

$$\text{ber}_n(-x) = (-)^n \text{ber}_n x, \quad \text{bei}_n(-x) = (-)^n \text{bei}_n x$$

Recurrence Relations

9.9.14

$$\begin{aligned} f_{\nu+1} + f_{\nu-1} &= -\frac{\nu\sqrt{2}}{x} (f_\nu - g_\nu) \\ f'_\nu &= \frac{1}{2\sqrt{2}} (f_{\nu+1} + g_{\nu+1} - f_{\nu-1} - g_{\nu-1}) \\ f'_\nu - \frac{\nu}{x} f_\nu &= \frac{1}{\sqrt{2}} (f_{\nu+1} + g_{\nu+1}) \\ f'_\nu + \frac{\nu}{x} f_\nu &= -\frac{1}{\sqrt{2}} (f_{\nu-1} + g_{\nu-1}) \end{aligned}$$

where

9.9.15

$$\begin{aligned} \left. \begin{aligned} f_\nu &= \text{ber}_\nu x \\ g_\nu &= \text{bei}_\nu x \end{aligned} \right\} & \left. \begin{aligned} f_\nu &= \text{bei}_\nu x \\ g_\nu &= -\text{ber}_\nu x \end{aligned} \right\} \\ \left. \begin{aligned} f_\nu &= \ker_\nu x \\ g_\nu &= \text{kei}_\nu x \end{aligned} \right\} & \left. \begin{aligned} f_\nu &= \text{kei}_\nu x \\ g_\nu &= -\ker_\nu x \end{aligned} \right\} \end{aligned}$$

9.9.16

$$\sqrt{2} \text{ber}'_\nu x = \text{ber}_\nu x + \text{bei}_\nu x$$

9.9.17

$$\sqrt{2} \text{bei}'_\nu x = -\text{ber}_\nu x + \text{bei}_\nu x$$

$$\sqrt{2} \ker'_\nu x = \ker_\nu x + \text{kei}_\nu x$$

$$\sqrt{2} \text{kei}'_\nu x = -\ker_\nu x + \text{kei}_\nu x$$

Recurrence Relations for Cross-Products

If

9.9.18

$$\begin{aligned} p_\nu &= \text{ber}_\nu^2 x + \text{bei}_\nu^2 x \\ q_\nu &= \text{ber}_\nu x \text{bei}'_\nu x - \text{ber}'_\nu x \text{bei}_\nu x \\ r_\nu &= \text{ber}_\nu x \text{ber}'_\nu x + \text{bei}_\nu x \text{bei}'_\nu x \\ s_\nu &= \text{ber}_\nu'^2 x + \text{bei}_\nu'^2 x \end{aligned}$$

then

9.9.19

$$\begin{aligned} p_{\nu+1} &= p_{\nu-1} - \frac{4\nu}{x} r_\nu \\ q_{\nu+1} &= -\frac{\nu}{x} p_\nu + r_\nu = -q_{\nu-1} + 2r_\nu \\ r_{\nu+1} &= -\frac{(\nu+1)}{x} p_{\nu+1} + q_\nu \\ s_\nu &= \frac{1}{2} p_{\nu+1} + \frac{1}{2} p_{\nu-1} - \frac{\nu^2}{x^2} p_\nu \end{aligned}$$

and

9.9.20

$$p_\nu s_\nu = r_\nu^2 + q_\nu^2$$

The same relations hold with ber , bei replaced throughout by ker , kei , respectively.

Indefinite Integrals

In the following f_ν , g_ν are any one of the pairs given by equations 9.9.15 and f'_ν , g'_ν are either the same pair or any other pair.

9.9.21

$$\int x^{1+\nu} f_\nu dx = -\frac{x^{1+\nu}}{\sqrt{2}} (f_{\nu+1} - g_{\nu+1}) = -x^{1+\nu} \left(\frac{\nu}{x} g_\nu - g'_\nu \right)$$

9.9.22

$$\int x^{1-\nu} f_\nu dx = \frac{x^{1-\nu}}{\sqrt{2}} (f_{\nu-1} - g_{\nu-1}) = x^{1-\nu} \left(\frac{\nu}{x} g_\nu + g'_\nu \right)$$

9.9.23

$$\begin{aligned} \int x(f_\nu g'_\nu - g_\nu f'_\nu) dx &= \frac{x}{2\sqrt{2}} \{ f'_\nu (f_{\nu+1} + g_{\nu+1}) \\ &- g'_\nu (f_{\nu+1} - g_{\nu+1}) - f_\nu (f'_{\nu+1} + g'_{\nu+1}) + g_\nu (f'_{\nu+1} - g'_{\nu+1}) \} \\ &= \frac{1}{2} x (f'_\nu f'_\nu - f_\nu f'_\nu + g'_\nu g'_\nu - g_\nu g'_\nu) \end{aligned}$$

9.9.24

$$\int x(f, g_v^* + g, f_v^*) dx = \frac{1}{4} x^2 (2f, g_v^* - f_{v-1} g_{v+1}^* - f_{v+1} g_{v-1}^* + 2g, f_v^* - g_{v-1} f_{v+1}^* - g_{v+1} f_{v-1}^*)$$

9.9.25

$$\int x(f_v^2 + g_v^2) dx = x(f, g_v' - f_v' g_v) = -(x/\sqrt{2})(f, f_{v+1} + g, g_{v+1} - f, g_{v+1} + f_{v+1} g_v)$$

9.9.26

$$\int x f, g dx = \frac{1}{4} x^2 (2f, g_v - f_{v-1} g_{v+1} - f_{v+1} g_{v-1})$$

9.9.27

$$\int x(f_v^2 - g_v^2) dx = \frac{1}{2} x^2 (f_v^2 - f_{v-1} f_{v+1} - g_v^2 + g_{v-1} g_{v+1})$$

Ascending Series for Cross-Products

9.9.28

$$\text{ber}_v^2 x + \text{bei}_v^2 x =$$

$$\left(\frac{1}{2}x\right)^{2\nu} \sum_{k=0}^{\infty} \frac{1}{\Gamma(\nu+k+1)\Gamma(\nu+2k+1)} \frac{\left(\frac{1}{2}x^2\right)^{2k}}{k!}$$

9.9.29

$$\text{ber}_v x \text{ bei}_v' x - \text{ber}_v' x \text{ bei}_v x$$

$$= \left(\frac{1}{2}x\right)^{2\nu+1} \sum_{k=0}^{\infty} \frac{1}{\Gamma(\nu+k+1)\Gamma(\nu+2k+2)} \frac{\left(\frac{1}{2}x^2\right)^{2k}}{k!}$$

9.9.30

$$\text{ber}_v x \text{ ber}_v' x + \text{bei}_v x \text{ bei}_v' x$$

$$= \frac{1}{2} \left(\frac{1}{2}x\right)^{2\nu-1} \sum_{k=0}^{\infty} \frac{1}{\Gamma(\nu+k+1)\Gamma(\nu+2k)} \frac{\left(\frac{1}{2}x^2\right)^{2k}}{k!}$$

9.9.31

$$\text{ber}_v'^2 x + \text{bei}_v'^2 x$$

$$= \left(\frac{1}{2}x\right)^{2\nu-2} \sum_{k=0}^{\infty} \frac{(2k^2 + 2\nu k + \frac{1}{2}\nu^2)}{\Gamma(\nu+k+1)\Gamma(\nu+2k+1)} \frac{\left(\frac{1}{2}x^2\right)^{2k}}{k!}$$

Expansions in Series of Bessel Functions

9.9.32

$$\begin{aligned} \text{ber}_v x + i \text{bei}_v x &= \sum_{k=0}^{\infty} \frac{e^{(3\nu+k)\pi i/4} x^k J_{\nu+k}(x)}{2^{k/2} k!} \\ &= \sum_{k=0}^{\infty} \frac{e^{(3\nu+3k)\pi i/4} x^k I_{\nu+k}(x)}{2^{k/2} k!} \end{aligned}$$

9.9.33

$$\text{ber}_n(x\sqrt{2}) = \sum_{k=-\infty}^{\infty} (-)^{n+k} J_{n+2k}(x) I_{2k}(x)$$

$$\text{bei}_n(x\sqrt{2}) = \sum_{k=-\infty}^{\infty} (-)^{n+k} J_{n+2k+1}(x) I_{2k+1}(x)$$

 Zeros of Functions of Order Zero ⁵

	ber x	bei x	ker x	kei x
1st zero	2. 84892	5. 02622	1. 71854	3. 91467
2nd zero	7. 23883	9. 45541	6. 12728	8. 34422
3rd zero	11. 67396	13. 89349	10. 56294	12. 78256
4th zero	16. 11356	18. 33398	15. 00269	17. 22314
5th zero	20. 55463	22. 77544	19. 44381	21. 66464

	ber' x	bei' x	ker' x	kei' x
1st zero	6. 03871	3. 77320	2. 66584	4. 93181
2nd zero	10. 51364	8. 28099	7. 17212	9. 40405
3rd zero	14. 96844	12. 74215	11. 63218	13. 85827
4th zero	19. 41758	17. 19343	16. 08312	18. 30717
5th zero	23. 86430	21. 64114	20. 53068	22. 75379

9.10. Asymptotic Expansions

Asymptotic Expansions for Large Arguments

When ν is fixed and x is large

9.10.1

$$\text{ber}_v x = \frac{e^{x/\sqrt{2}}}{\sqrt{2\pi x}} \{f_v(x) \cos \alpha + g_v(x) \sin \alpha\}$$

$$-\frac{1}{\pi} \{\sin(2\nu\pi) \text{ker}_v x + \cos(2\nu\pi) \text{kei}_v x\}$$

9.10.2

$$\text{bei}_v x = \frac{e^{x/\sqrt{2}}}{\sqrt{2\pi x}} \{f_v(x) \sin \alpha - g_v(x) \cos \alpha\}$$

$$+\frac{1}{\pi} \{\cos(2\nu\pi) \text{ker}_v x - \sin(2\nu\pi) \text{kei}_v x\}$$

9.10.3

$$\text{ker}_v x = \sqrt{\pi/(2x)} e^{-x/\sqrt{2}} \{f_v(-x) \cos \beta - g_v(-x) \sin \beta\}$$

9.10.4

$$\text{kei}_v x = \sqrt{\pi/(2x)} e^{-x/\sqrt{2}} \{-f_v(-x) \sin \beta - g_v(-x) \cos \beta\}$$

where

9.10.5

$$\alpha = (x/\sqrt{2}) + (\frac{1}{2}\nu - \frac{1}{8})\pi, \quad \beta = (x/\sqrt{2}) + (\frac{1}{2}\nu + \frac{1}{8})\pi = \alpha + \frac{1}{4}\pi$$

and, with $4\nu^2$ denoted by μ ,

9.10.6

$$f_v(\pm x)$$

$$\sim 1 + \sum_{k=1}^{\infty} (\mp)^k \frac{(\mu-1)(\mu-9) \dots \{\mu-(2k-1)^2\}}{k!(8x)^k} \cos\left(\frac{k\pi}{4}\right)$$

⁵ From British Association for the Advancement of Science, Annual Report (J. R. Airey), 254 (1927) with permission. This reference also gives 5-decimal values of the next five zeros of each function.

9.10.7

 $g_r(\pm x)$

$$\sim \sum_{k=1}^{\infty} (\mp)^k \frac{(\mu-1)(\mu-9) \dots \{\mu-(2k-1)^2\}}{k!(8x)^k} \sin\left(\frac{k\pi}{4}\right)$$

The terms⁶ in $\ker, x$ and $\ker i, x$ in equations 9.10.1 and 9.10.2 are asymptotically negligible compared with the other terms, but their inclusion in numerical calculations yields improved accuracy.

The corresponding series for $\text{ber}'_r x$, $\text{bei}'_r x$, $\ker'_r x$ and $\ker i'_r x$ can be derived from 9.2.11 and 9.2.13 with $z = xe^{3\pi i/4}$; the extra terms in the expansions of $\text{ber}'_r x$ and $\text{bei}'_r x$ are respectively

$$-(1/\pi) \{ \sin(2\nu\pi) \ker'_r x + \cos(2\nu\pi) \ker i'_r x \}$$

and

$$(1/\pi) \{ \cos(2\nu\pi) \ker'_r x - \sin(2\nu\pi) \ker i'_r x \}.$$

Modulus and Phase

9.10.8

$$M_r = \sqrt{(\text{ber}_r^2 x + \text{bei}_r^2 x)}, \quad \theta_r = \arctan(\text{bei}_r x / \text{ber}_r x)$$

$$9.10.9 \quad \text{ber}_r x = M_r \cos \theta_r, \quad \text{bei}_r x = M_r \sin \theta_r$$

$$9.10.10 \quad M_{-n} = M_n, \quad \theta_{-n} = \theta_n - n\pi$$

9.10.11

$$\begin{aligned} \text{ber}'_r x &= \frac{1}{2} M_{r+1} \cos(\theta_{r+1} - \frac{1}{2}\pi) - \frac{1}{2} M_{r-1} \cos(\theta_{r-1} - \frac{1}{2}\pi) \\ &= (\nu/x) M_r \cos \theta_r + M_{r+1} \cos(\theta_{r+1} - \frac{1}{2}\pi) \\ &= -(\nu/x) M_r \cos \theta_r - M_{r-1} \cos(\theta_{r-1} - \frac{1}{2}\pi) \end{aligned}$$

9.10.12

$$\begin{aligned} \text{bei}'_r x &= \frac{1}{2} M_{r+1} \sin(\theta_{r+1} - \frac{1}{2}\pi) - \frac{1}{2} M_{r-1} \sin(\theta_{r-1} - \frac{1}{2}\pi) \\ &= (\nu/x) M_r \sin \theta_r + M_{r+1} \sin(\theta_{r+1} - \frac{1}{2}\pi) \\ &= -(\nu/x) M_r \sin \theta_r - M_{r-1} \sin(\theta_{r-1} - \frac{1}{2}\pi) \end{aligned}$$

9.10.13

$$\text{ber}'_r x = M_1 \cos(\theta_1 - \frac{1}{4}\pi), \quad \text{bei}'_r x = M_1 \sin(\theta_1 - \frac{1}{4}\pi)$$

9.10.14

$$\begin{aligned} M'_r &= (\nu/x) M_r + M_{r+1} \cos(\theta_{r+1} - \theta_r - \frac{1}{4}\pi) \\ &= -(\nu/x) M_r - M_{r-1} \cos(\theta_{r-1} - \theta_r - \frac{1}{4}\pi) \end{aligned}$$

9.10.15

$$\begin{aligned} \theta'_r &= (M_{r+1}/M_r) \sin(\theta_{r+1} - \theta_r - \frac{1}{4}\pi) \\ &= -(M_{r-1}/M_r) \sin(\theta_{r-1} - \theta_r - \frac{1}{4}\pi) \end{aligned}$$

⁶ The coefficients of these terms given in [9.17] are incorrect. The present results are due to Mr. G. F. Miller.

9.10.16

$$\begin{aligned} M'_0 &= M_1 \cos(\theta_1 - \theta_0 - \frac{1}{4}\pi) \\ \theta'_0 &= (M_1/M_0) \sin(\theta_1 - \theta_0 - \frac{1}{4}\pi) \end{aligned}$$

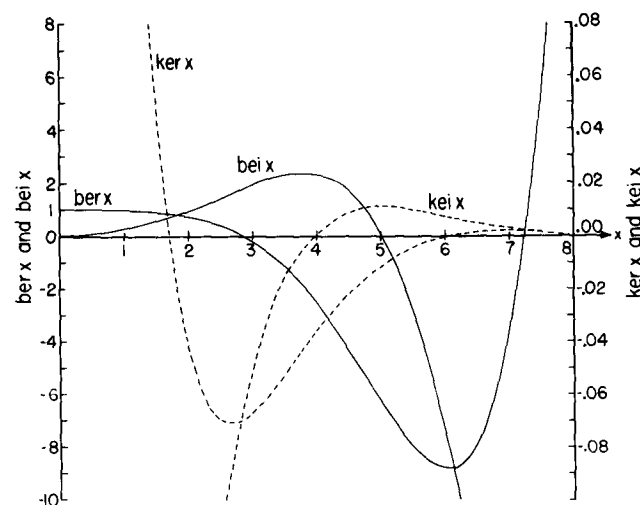
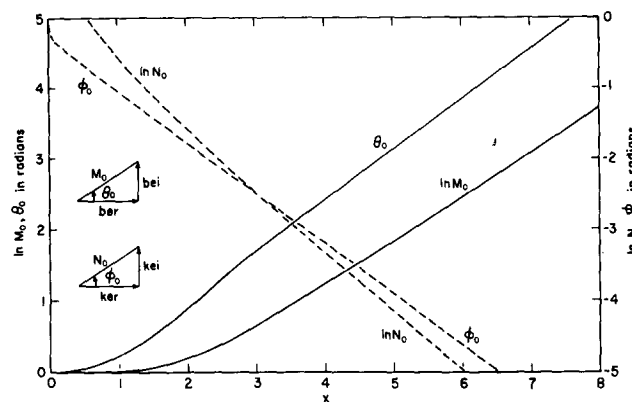
9.10.17

$$d(xM'_r/\theta'_r)/dx = xM_r^2, \quad x^2 M'_r + xM'_r - \nu^2 M_r = x^2 M_r \theta'_r{}^2$$

9.10.18

$$N_r = \sqrt{(\ker_r^2 x + \ker i_r^2 x)}, \quad \phi_r = \arctan(\ker i_r x / \ker_r x)$$

$$9.10.19 \quad \ker_r x = N_r \cos \phi_r, \quad \ker i_r x = N_r \sin \phi_r$$

FIGURE 9.10. $\text{ber } x$, $\text{bei } x$, $\ker x$ and $\ker i x$.FIGURE 9.11. $\ln M_0(x)$, $\theta_0(x)$, $\ln N_0(x)$ and $\phi_0(x)$.

Equations 9.10.11 to 9.10.17 hold with the symbols b, M, θ replaced throughout by k, N, ϕ , respectively. In place of 9.10.10

$$9.10.20 \quad N_{-r} = N_r, \quad \phi_{-r} = \phi_r + \nu\pi$$

Asymptotic Expansions of Modulus and Phase

When ν is fixed, x is large and $\mu=4\nu^2$

9.10.21

$$M_\nu = \frac{e^{x/\sqrt{2}}}{\sqrt{2\pi x}} \left\{ 1 - \frac{\mu-1}{8\sqrt{2}} \frac{1}{x} + \frac{(\mu-1)^2}{256} \frac{1}{x^2} - \frac{(\mu-1)(\mu^2+14\mu-399)}{6144\sqrt{2}} \frac{1}{x^3} + O\left(\frac{1}{x^4}\right) \right\}$$

9.10.22

$$\ln M_\nu = \frac{x}{\sqrt{2}} - \frac{1}{2} \ln(2\pi x) - \frac{\mu-1}{8\sqrt{2}} \frac{1}{x} - \frac{(\mu-1)(\mu-25)}{384\sqrt{2}} \frac{1}{x^2} - \frac{(\mu-1)(\mu-13)}{128} \frac{1}{x^3} + O\left(\frac{1}{x^4}\right)$$

9.10.23

$$\theta_\nu = \frac{x}{\sqrt{2}} + \left(\frac{1}{2}\nu - \frac{1}{8}\right)\pi + \frac{\mu-1}{8\sqrt{2}} \frac{1}{x} + \frac{\mu-1}{16} \frac{1}{x^2} - \frac{(\mu-1)(\mu-25)}{384\sqrt{2}} \frac{1}{x^3} + O\left(\frac{1}{x^4}\right)$$

9.10.24

$$N_\nu = \sqrt{\frac{\pi}{2x}} e^{-x/\sqrt{2}} \left\{ 1 + \frac{\mu-1}{8\sqrt{2}} \frac{1}{x} + \frac{(\mu-1)^2}{256} \frac{1}{x^2} + \frac{(\mu-1)(\mu^2+14\mu-399)}{6144\sqrt{2}} \frac{1}{x^3} + O\left(\frac{1}{x^4}\right) \right\}$$

9.10.25

$$\ln N_\nu = -\frac{x}{\sqrt{2}} + \frac{1}{2} \ln\left(\frac{\pi}{2x}\right) + \frac{\mu-1}{8\sqrt{2}} \frac{1}{x} + \frac{(\mu-1)(\mu-25)}{384\sqrt{2}} \frac{1}{x^2} - \frac{(\mu-1)(\mu-13)}{128} \frac{1}{x^3} + O\left(\frac{1}{x^4}\right)$$

9.10.26

$$\phi_\nu = -\frac{x}{\sqrt{2}} - \left(\frac{1}{2}\nu + \frac{1}{8}\right)\pi - \frac{\mu-1}{8\sqrt{2}} \frac{1}{x} + \frac{\mu-1}{16} \frac{1}{x^2} + \frac{(\mu-1)(\mu-25)}{384\sqrt{2}} \frac{1}{x^3} + O\left(\frac{1}{x^4}\right)$$

Asymptotic Expansions of Cross-Products

If x is large

9.10.27

$$\text{ber}^2 x + \text{bei}^2 x \sim \frac{e^{x\sqrt{2}}}{2\pi x} \left(1 + \frac{1}{4\sqrt{2}} \frac{1}{x} + \frac{1}{64} \frac{1}{x^2} - \frac{33}{256\sqrt{2}} \frac{1}{x^3} - \frac{1797}{8192} \frac{1}{x^4} + \dots \right)$$

9.10.28

$$\text{ber } x \text{ bei}' x - \text{ber}' x \text{ bei } x \sim \frac{e^{x\sqrt{2}}}{2\pi x} \left(\frac{1}{\sqrt{2}} + \frac{1}{8} \frac{1}{x} + \frac{9}{64\sqrt{2}} \frac{1}{x^2} + \frac{39}{512} \frac{1}{x^3} + \frac{75}{8192\sqrt{2}} \frac{1}{x^4} + \dots \right)$$

9.10.29

$$\text{ber } x \text{ ber}' x + \text{bei } x \text{ bei}' x \sim \frac{e^{x\sqrt{2}}}{2\pi x} \left(\frac{1}{\sqrt{2}} - \frac{3}{8} \frac{1}{x} - \frac{15}{64\sqrt{2}} \frac{1}{x^2} - \frac{45}{512} \frac{1}{x^3} + \frac{315}{8192\sqrt{2}} \frac{1}{x^4} + \dots \right)$$

9.10.30

$$\text{ber}'^2 x + \text{bei}'^2 x \sim \frac{e^{x\sqrt{2}}}{2\pi x} \left(1 - \frac{3}{4\sqrt{2}} \frac{1}{x} + \frac{9}{64} \frac{1}{x^2} + \frac{75}{256\sqrt{2}} \frac{1}{x^3} + \frac{2475}{8192} \frac{1}{x^4} + \dots \right)$$

9.10.31

$$\text{ker}^2 x + \text{kei}^2 x \sim \frac{\pi}{2x} e^{-x\sqrt{2}} \left(1 - \frac{1}{4\sqrt{2}} \frac{1}{x} + \frac{1}{64} \frac{1}{x^2} + \frac{33}{256\sqrt{2}} \frac{1}{x^3} - \frac{1797}{8192} \frac{1}{x^4} + \dots \right)$$

9.10.32

$$\text{ker } x \text{ kei}' x - \text{ker}' x \text{ kei } x \sim -\frac{\pi}{2x} e^{-x\sqrt{2}} \left(\frac{1}{\sqrt{2}} - \frac{1}{8} \frac{1}{x} + \frac{9}{64\sqrt{2}} \frac{1}{x^2} - \frac{39}{512} \frac{1}{x^3} + \frac{75}{8192\sqrt{2}} \frac{1}{x^4} + \dots \right)$$

9.10.33

$$\text{ker } x \text{ ker}' x + \text{kei } x \text{ kei}' x \sim -\frac{\pi}{2x} e^{-x\sqrt{2}} \left(\frac{1}{\sqrt{2}} + \frac{3}{8} \frac{1}{x} - \frac{15}{64\sqrt{2}} \frac{1}{x^2} + \frac{45}{512} \frac{1}{x^3} + \frac{315}{8192\sqrt{2}} \frac{1}{x^4} + \dots \right)$$

9.10.34

$$\text{ker}'^2 x + \text{kei}'^2 x \sim \frac{\pi}{2x} e^{-x\sqrt{2}} \left(1 + \frac{3}{4\sqrt{2}} \frac{1}{x} + \frac{9}{64} \frac{1}{x^2} - \frac{75}{256\sqrt{2}} \frac{1}{x^3} + \frac{2475}{8192} \frac{1}{x^4} + \dots \right)$$

Asymptotic Expansions of Large Zeros

Let

9.10.35

$$f(\delta) = \frac{\mu-1}{16\delta} + \frac{\mu-1}{32\delta^2} + \frac{(\mu-1)(5\mu+19)}{1536\delta^3} + \frac{3(\mu-1)^2}{512\delta^4} + \dots$$

where $\mu=4\nu^2$. Then if s is a large positive integer

9.10.36

$$\text{Zeros of ber}, x \sim \sqrt{2}\{\delta - f(\delta)\}, \quad \delta = (s - \frac{1}{2}\nu - \frac{3}{8})\pi$$

$$\text{Zeros of bei}, x \sim \sqrt{2}\{\delta - f(\delta)\}, \quad \delta = (s - \frac{1}{2}\nu + \frac{1}{8})\pi$$

$$\text{Zeros of ker}, x \sim \sqrt{2}\{\delta + f(-\delta)\}, \quad \delta = (s - \frac{1}{2}\nu - \frac{5}{8})\pi$$

$$\text{Zeros of kei}, x \sim \sqrt{2}\{\delta + f(-\delta)\}, \quad \delta = (s - \frac{1}{2}\nu - \frac{1}{8})\pi$$

For $\nu=0$ these expressions give the s th zero of each function; for other values of ν the zeros represented may not be the s th.

Uniform Asymptotic Expansions for Large Orders

When ν is large and positive

9.10.37

$\text{ber},(\nu x) + i \text{bei},(\nu x) \sim$

$$\frac{e^{\nu\xi}}{\sqrt{2\pi\nu\xi}} \left(\frac{x e^{3\pi i/4}}{1+\xi} \right)^{\nu} \left\{ 1 + \sum_{k=1}^{\infty} \frac{u_k(\xi^{-1})}{\nu^k} \right\}$$

9.10.38

$\text{ker},(\nu x) + i \text{kei},(\nu x)$

$$\sim \sqrt{\frac{\pi}{2\nu\xi}} e^{-\nu\xi} \left(\frac{x e^{3\pi i/4}}{1+\xi} \right)^{-\nu} \left\{ 1 + \sum_{k=1}^{\infty} (-)^k \frac{u_k(\xi^{-1})}{\nu^k} \right\}$$

9.10.39

$\text{ber}',(\nu x) + i \text{bei}',(\nu x)$

$$\sim \sqrt{\frac{\xi}{2\pi\nu}} \frac{e^{\nu\xi}}{x} \left(\frac{x e^{3\pi i/4}}{1+\xi} \right)^{\nu} \left\{ 1 + \sum_{k=1}^{\infty} \frac{v_k(\xi^{-1})}{\nu^k} \right\}$$

9.10.40

$\text{ker}',(\nu x) + i \text{kei}',(\nu x)$

$$\sim -\sqrt{\frac{\pi\xi}{2\nu}} \frac{e^{-\nu\xi}}{x} \left(\frac{x e^{3\pi i/4}}{1+\xi} \right)^{-\nu} \left\{ 1 + \sum_{k=1}^{\infty} (-)^k \frac{v_k(\xi^{-1})}{\nu^k} \right\}$$

where

$$\xi = \sqrt{1+ix^2}$$

and $u_k(t), v_k(t)$ are given by 9.3.9 and 9.3.13. All fractional powers take their principal values.

9.11. Polynomial Approximations

9.11.1 $-8 \leq x \leq 8$

$$\begin{aligned} \text{ber } x = & 1 - 64(x/8)^4 + 113.77777 \ 774(x/8)^8 \\ & - 32.36345 \ 652(x/8)^{12} + 2.64191 \ 397(x/8)^{16} \\ & - .08349 \ 609(x/8)^{20} + .00122 \ 552(x/8)^{24} \\ & - .00000 \ 901(x/8)^{28} + \epsilon \\ & |\epsilon| < 1 \times 10^{-9} \end{aligned}$$

9.11.2 $-8 \leq x \leq 8$

$$\begin{aligned} \text{bei } x = & 16(x/8)^2 - 113.77777 \ 774(x/8)^6 \\ & + 72.81777 \ 742(x/8)^{10} - 10.56765 \ 779(x/8)^{14} \\ & + .52185 \ 615(x/8)^{18} - .01103 \ 667(x/8)^{22} \\ & + .00011 \ 346(x/8)^{26} + \epsilon \\ & |\epsilon| < 6 \times 10^{-9} \end{aligned}$$

9.11.3 $0 < x \leq 8$

$$\begin{aligned} \text{ker } x = & -\ln(\tfrac{1}{2}x) \text{ber } x + \tfrac{1}{4}\pi \text{bei } x - .57721 \ 566 \\ & - 59.05819 \ 744(x/8)^4 + 171.36272 \ 133(x/8)^8 \\ & - 60.60977 \ 451(x/8)^{12} + 5.65539 \ 121(x/8)^{16} \\ & - .19636 \ 347(x/8)^{20} + .00309 \ 699(x/8)^{24} \\ & - .00002 \ 458(x/8)^{28} + \epsilon \\ & |\epsilon| < 1 \times 10^{-8} \end{aligned}$$

9.11.4 $0 < x \leq 8$

$$\begin{aligned} \text{kei } x = & -\ln(\tfrac{1}{2}x) \text{bei } x - \tfrac{1}{4}\pi \text{ber } x + 6.76454 \ 936(x/8)^2 \\ & - 142.91827 \ 687(x/8)^6 + 124.23569 \ 650(x/8)^{10} \\ & - 21.30060 \ 904(x/8)^{14} + 1.17509 \ 064(x/8)^{18} \\ & - .02695 \ 875(x/8)^{22} + .00029 \ 532(x/8)^{26} + \epsilon \\ & |\epsilon| < 3 \times 10^{-9} \end{aligned}$$

9.11.5 $-8 \leq x \leq 8$

$$\begin{aligned} \text{ber}' x = & x[-4(x/8)^2 + 14.22222 \ 222(x/8)^6 \\ & - 6.06814 \ 810(x/8)^{10} + .66047 \ 849(x/8)^{14} \\ & - .02609 \ 253(x/8)^{18} + .00045 \ 957(x/8)^{22} \\ & - .00000 \ 394(x/8)^{26}] + \epsilon \\ & |\epsilon| < 2.1 \times 10^{-8} \end{aligned}$$

9.11.6 $-8 \leq x \leq 8$

$$\begin{aligned} \text{bei}' x = & x[\tfrac{1}{4} - 10.66666 \ 666(x/8)^4 \\ & + 11.37777 \ 772(x/8)^8 - 2.31167 \ 514(x/8)^{12} \\ & + .14677 \ 204(x/8)^{16} - .00379 \ 386(x/8)^{20} \\ & + .00004 \ 609(x/8)^{24}] + \epsilon \\ & |\epsilon| < 7 \times 10^{-8} \end{aligned}$$

9.11.7 $0 < x \leq 8$

$$\begin{aligned} \text{ker}' x = & -\ln(\tfrac{1}{2}x) \text{ber}' x - x^{-1} \text{ber } x + \tfrac{1}{4}\pi \text{bei}' x \\ & + x[-3.69113 \ 734(x/8)^2 + 21.42034 \ 017(x/8)^6 \\ & - 11.36433 \ 272(x/8)^{10} + 1.41384 \ 780(x/8)^{14} \\ & - .06136 \ 358(x/8)^{18} + .00116 \ 137(x/8)^{22} \\ & - .00001 \ 075(x/8)^{26}] + \epsilon \\ & |\epsilon| < 8 \times 10^{-8} \end{aligned}$$

$$\begin{aligned}
 &9.11.8 \quad 0 < x \leq 8 \\
 &\text{kei}' x = -\ln\left(\frac{1}{2}x\right) \text{bei}' x - x^{-1} \text{bei } x - \frac{1}{4}\pi \text{ber}' x \\
 &\quad + x[.21139 \ 217 - 13.39858 \ 846(x/8)^4 \\
 &\quad + 19.41182 \ 758(x/8)^8 - 4.65950 \ 823(x/8)^{12} \\
 &\quad + .33049 \ 424(x/8)^{16} - .00926 \ 707(x/8)^{20} \\
 &\quad + .00011 \ 997(x/8)^{24}] + \epsilon \\
 &\quad |\epsilon| < 7 \times 10^{-8}
 \end{aligned}$$

$$\begin{aligned}
 &9.11.9 \quad 8 \leq x < \infty \\
 &\text{ker } x + i \text{kei } x = f(x)(1 + \epsilon_1) \\
 &f(x) = \sqrt{\frac{\pi}{2x}} \exp\left[-\frac{1+i}{\sqrt{2}} x + \theta(-x)\right] \\
 &\quad |\epsilon_1| < 1 \times 10^{-7}
 \end{aligned}$$

$$\begin{aligned}
 &9.11.10 \quad 8 \leq x < \infty \\
 &\text{ber } x + i \text{bei } x - \frac{i}{\pi} (\text{ker } x + i \text{kei } x) = g(x)(1 + \epsilon_2) \\
 &g(x) = \frac{1}{\sqrt{2\pi x}} \exp\left[\frac{1+i}{\sqrt{2}} x + \theta(x)\right] \\
 &\quad |\epsilon_2| < 3 \times 10^{-7}
 \end{aligned}$$

where

$$\begin{aligned}
 &9.11.11 \\
 &\theta(x) = (.00000 \ 00 - .39269 \ 91i) \\
 &\quad + (.01104 \ 86 - .01104 \ 85i)(8/x) \\
 &\quad + (.00000 \ 00 - .00097 \ 65i)(8/x)^2 \\
 &\quad + (-.00009 \ 06 - .00009 \ 01i)(8/x)^3 \\
 &\quad + (-.00002 \ 52 + .00000 \ 00i)(8/x)^4 \\
 &\quad + (-.00000 \ 34 + .00000 \ 51i)(8/x)^5 \\
 &\quad + (.00000 \ 06 + .00000 \ 19i)(8/x)^6
 \end{aligned}$$

$$\begin{aligned}
 &9.11.12 \quad 8 \leq x < \infty \\
 &\text{ker}' x + i \text{kei}' x = -f(x)\phi(-x)(1 + \epsilon_3) \\
 &\quad |\epsilon_3| < 2 \times 10^{-7}
 \end{aligned}$$

$$\begin{aligned}
 &9.11.13 \quad 8 \leq x < \infty \\
 &\text{ber}' x + i \text{bei}' x - \frac{i}{\pi} (\text{ker}' x + i \text{kei}' x) = g(x)\phi(x)(1 + \epsilon_4) \\
 &\quad |\epsilon_4| < 3 \times 10^{-7}
 \end{aligned}$$

where

$$\begin{aligned}
 &9.11.14 \\
 &\phi(x) = (.70710 \ 68 + .70710 \ 68i) \\
 &\quad + (-.06250 \ 01 - .00000 \ 01i)(8/x) \\
 &\quad + (-.00138 \ 13 + .00138 \ 11i)(8/x)^2 \\
 &\quad + (.00000 \ 05 + .00024 \ 52i)(8/x)^3 \\
 &\quad + (.00003 \ 46 + .00003 \ 38i)(8/x)^4 \\
 &\quad + (.00001 \ 17 - .00000 \ 24i)(8/x)^5 \\
 &\quad + (.00000 \ 16 - .00000 \ 32i)(8/x)^6
 \end{aligned}$$

Numerical Methods

9.12. Use and Extension of the Tables

Example 1. To evaluate $J_n(1.55)$, $n=0, 1, 2, \dots$, each to 5 decimals.
The recurrence relation

$$J_{n-1}(x) + J_{n+1}(x) = (2n/x)J_n(x)$$

can be used to compute $J_0(x)$, $J_1(x)$, $J_2(x)$, $\dots$, successively provided that $n < x$, otherwise severe accumulation of rounding errors will occur. Since, however, $J_n(x)$ is a decreasing function of n when $n > x$, recurrence can always be carried out in the direction of decreasing n .

Inspection of **Table 9.2** shows that $J_n(1.55)$ vanishes to 5 decimals when $n > 7$. Taking arbitrary values zero for J_0 and unity for J_8 , we compute by recurrence the entries in the second column of the following table, rounding off to the nearest integer at each step.

n	<i>Trial values</i>	$J_n(1.55)$
9	0	.00000
8	1	.00000
7	10	.00003
6	89	.00028
5	679	.00211
4	4292	.01331
3	21473	.06661
2	78829	.24453
1	181957	.56442
0	155954	.48376

We normalize the results by use of the equation **9.1.46**, namely

$$J_0(x) + 2J_2(x) + 2J_4(x) + \dots = 1$$

This yields the normalization factor

$$1/322376 = .00000 \ 31019 \ 7$$

and multiplying the trial values by this factor we obtain the required results, given in the third column. As a check we may verify the value of $J_0(1.55)$ by interpolation in **Table 9.1**.

Remarks. (i) In this example it was possible to estimate immediately the value of $n=N$, say, at which to begin the recurrence. This may not always be the case and an arbitrary value of N may have to be taken. The number of correct significant figures in the final values is the same as the number of digits in the respective trial values. If the chosen N is too small the trial values will have too few digits and insufficient accuracy is obtained in the results. The calculation must then be repeated taking a higher value. On the other hand if N were too large unnecessary effort would be expended. This could be offset to some extent by discarding significant figures in the trial values which are in excess of the number of decimals required in J_n .

(ii) If we had required, say, $J_0(1.55)$, $J_1(1.55)$, . . . , $J_{10}(1.55)$, each to 5 significant figures, we would have found the values of $J_{10}(1.55)$ and $J_{11}(1.55)$ to 5 significant figures by interpolation in **Table 9.3** and then computed by recurrence $J_9, J_8, \dots, J_0$, no normalization being required.

Alternatively, we could begin the recurrence at a higher value of N and retain only 5 significant figures in the trial values for $n \leq 10$.

(iii) Exactly similar methods can be used to compute the modified Bessel function $I_n(x)$ by means of the relations 9.6.26 and 9.6.36. If x is large, however, considerable cancellation will take place in using the latter equation, and it is preferable to normalize by means of 9.6.37.

Example 2. To evaluate $Y_n(1.55)$, $n=0, 1, 2, \dots, 10$, each to 5 significant figures.

The recurrence relation

$$Y_{n-1}(x) + Y_{n+1}(x) = (2n/x) Y_n(x)$$

can be used to compute $Y_n(x)$ in the direction of increasing n both for $n < x$ and $n > x$, because in the latter event $Y_n(x)$ is a numerically increasing function of n .

We therefore compute $Y_0(1.55)$ and $Y_1(1.55)$ by interpolation in **Table 9.1**, generate $Y_2(1.55)$, $Y_3(1.55)$, . . . , $Y_{10}(1.55)$ by recurrence and check $Y_{10}(1.55)$ by interpolation in **Table 9.3**.

n	$Y_n(1.55)$	n	$Y_n(1.55)$
0	+0.40225	6	-1.9917 $\times 10^2$
1	-0.37970	7	-1.5100 $\times 10^3$
2	-0.89218	8	-1.3440 $\times 10^4$
3	-1.9227	9	-1.3722 $\times 10^5$
4	-6.5505	10	-1.5801 $\times 10^6$
5	-31.886		

Remarks. (i) An alternative way of computing $Y_0(x)$, should $J_0(x)$, $J_2(x)$, $J_4(x)$, . . . , be available (see **Example 1**), is to use formula 9.1.89. The other starting value for the recurrence, $Y_1(x)$, can then be found from the Wronskian relation $J_1(x)Y_0(x) - J_0(x)Y_1(x) = 2/(\pi x)$. This is a convenient procedure for use with an automatic computer.

(ii) Similar methods can be used to compute the modified Bessel function $K_n(x)$ by means of the recurrence relation 9.6.26 and the relation 9.6.54, except that if x is large severe cancellation will occur in the use of 9.6.54 and other methods for evaluating $K_0(x)$ may be preferable, for example, use of the asymptotic expansion 9.7.2 or the polynomial approximation 9.8.6.

Example 3. To evaluate $J_0(.36)$ and $Y_0(.36)$ each to 5 decimals, using the multiplication theorem.

From 9.1.74 we have

$$\mathcal{C}_0(\lambda z) = \sum_{k=0}^{\infty} a_k \mathcal{C}_k(z), \text{ where } a_k = \frac{(-)^k (\lambda^2 - 1)^k (\frac{1}{2}z)^k}{k!}.$$

We take $z=.4$. Then $\lambda=.9$, $(\lambda^2 - 1)(\frac{1}{2}z) = -.038$, and extracting the necessary values of $J_k(.4)$ and $Y_k(.4)$ from **Tables 9.1** and **9.2**, we compute the required results as follows:

k	a_k	$a_k J_k(.4)$	$a_k Y_k(.4)$
0	+1.0	+ .96040	- .60602
1	+0.038	+ .00745	- .06767
2	+0.7220 $\times 10^{-3}$	+ .00001	- .00599
3	+0.914 $\times 10^{-5}$		- .00074
4	+0.87 $\times 10^{-7}$		- .00011
5	+0.7 $\times 10^{-9}$		- .00002
		$J_0(.36) = +.96786$	$Y_0(.36) = -.68055$

Remark. This procedure is equivalent to interpolating by means of the Taylor series

$$\mathcal{C}_0(z+h) = \sum_{k=0}^{\infty} \frac{h^k}{k!} \mathcal{C}_0^{(k)}(z)$$

at $z=.4$, and expressing the derivatives $\mathcal{C}_0^{(k)}(z)$ in terms of $\mathcal{C}_k(z)$ by means of the recurrence relations and differential equation for the Bessel functions.

Example 4. To evaluate $J_\nu(x)$, $J'_\nu(x)$, $Y_\nu(x)$ and $Y'_\nu(x)$ for $\nu=50$, $x=75$, each to 6 decimals.

We use the asymptotic expansions 9.3.35, 9.3.36, 9.3.43, and 9.3.44. Here $z=x/\nu=3/2$. From 9.3.39 we find

$$\frac{2}{3} (-\frac{1}{2})^{3/2} = \frac{1}{2} \sqrt{5} - \arccos \frac{2}{3} = +.2769653.$$

Hence

$$\zeta = -.5567724 \text{ and } \left(\frac{4\zeta}{1-\zeta^2}\right)^{1/4} = +1.155332.$$

Next,

$$\nu^{1/3} = 3.684031, \quad \nu^{2/3}\zeta = -7.556562.$$

Interpolating in **Table 10.11**, we find that

$$\text{Ai}(\nu^{2/3}\zeta) = +.299953, \quad \text{Ai}'(\nu^{2/3}\zeta) = +.451441,$$

$$\text{Bi}(\nu^{2/3}\zeta) = -.160565, \quad \text{Bi}'(\nu^{2/3}\zeta) = +.819542.$$

As a check on the interpolation, we may verify that $\text{Ai Bi}' - \text{Ai}' \text{Bi} = 1/\pi$.

Interpolating in the table following **9.3.46** we obtain

$$b_0(\zeta) = +.0136, \quad c_0(\zeta) = +.1442.$$

The contributions of the terms involving $a_1(\zeta)$ and $d_1(\zeta)$ are negligible, and substituting in the asymptotic expansions we find that

$$J_{50}(75) = +1.155332(50^{-1/3} \times .299953 \\ + 50^{-5/3} \times .451441 \times .0136) = +.094077,$$

$$J'_{50}(75) = -(4/3)(1.155332)^{-1}(50^{-4/3} \times .299953 \\ \times .1442 + 50^{-2/3} \times .451441) = -.038658,$$

$$Y_{50}(75) = -1.155332(-50^{-1/3} \times .160565 \\ + 50^{-5/3} \times .819542 \times .0136) = +.050335,$$

$$Y'_{50}(75) = +(4/3)(1.155332)^{-1}(-50^{-4/3} \times .160565 \\ \times .1442 + 50^{-2/3} \times .819542) = +.069543.$$

As a check we may verify that

$$JY' - J'Y = 2/(75\pi).$$

Remarks. This example may also be computed using the Debye expansions **9.3.15**, **9.3.16**, **9.3.19**, and **9.3.20**. Four terms of each of these series are required, compared with two in the computations above. The closer the argument-order ratio is to unity, the less effective the Debye expansions become. In the neighborhood of unity the expansions **9.3.23**, **9.3.24**, **9.3.27**, and **9.3.28** will furnish results of moderate accuracy; for high-accuracy work the uniform expansions should again be used.

Example 5. To evaluate the 5th positive zero of $J_{10}(x)$ and the corresponding value of $J'_{10}(x)$, each to 5 decimals.

We use the asymptotic expansions **9.5.22** and **9.5.23** setting $\nu=10$, $s=5$. From **Table 10.11**

we find

$$a_5 = -7.944134, \quad \text{Ai}'(a_5) = +.947336.$$

Hence

$$\zeta = 10^{-2/3}a_5 = .21544347a_5 = -1.7115118.$$

Interpolating in the table following **9.5.26** we obtain

$$z(\zeta) = +2.888631, \quad h(\zeta) = +.98259, \\ f_1(\zeta) = +.0107, \quad F_1(\zeta) = -.001.$$

The bounds given at the foot of the table show that the contributions of higher terms to the asymptotic series are negligible. Hence

$$j_{10,5} = 28.88631 + .00107 + \dots = 28.88738,$$

$$J'_{10}(j_{10,5}) = -\frac{2}{10^{2/3}} \frac{.947336}{2.888631 \times .98259} \\ \times (1 - .00001 + \dots) = -.14381.$$

Example 6. To evaluate the first root of $J_0(x)Y_0(\lambda x) - Y_0(x)J_0(\lambda x) = 0$ for $\lambda = \frac{3}{2}$ to 4 significant figures.

Let $\alpha_\lambda^{(1)}$ denote the root. Direct interpolation in **Table 9.7** is impracticable owing to the divergence of the differences. Inspection of **9.5.28** suggests that a smoother function is $(\lambda-1)\alpha_\lambda^{(1)}$. Using **Table 9.7** we compute the following values

$1/\lambda$	$(\lambda-1)\alpha_\lambda^{(1)}$	δ	δ^2
0.4	3.110		
		+21	
0.6	3.131		-12
		+9	
0.8	3.140		-7
		+2	
1.0	3.142(π)		

Interpolating for $1/\lambda = .667$, we obtain $(\lambda-1)\alpha_\lambda^{(1)} = 3.134$ and thence the required root $\alpha_{1.5}^{(1)} = 6.268$.

Example 7. To evaluate $\text{ber}_n 1.55$, $\text{bei}_n 1.55$, $n=0, 1, 2, \dots$, each to 5 decimals.

We use the recurrence relation

$$J_{n-1}(xe^{3\pi i/4}) + J_{n+1}(xe^{3\pi i/4}) \\ = -\frac{n\sqrt{2}}{x}(1+i)J_n(xe^{3\pi i/4}),$$

taking arbitrary values zero for $J_0(xe^{3\pi i/4})$ and $1+0i$ for $J_1(xe^{3\pi i/4})$ (see **Example 1**).

n	Real trial values	Imag. trial values	$\text{ber}_n x$	$\text{bei}_n x$
9	0	0	.00000	.00000
8	+1	0	.00000	.00000
7	-7	-7	-.00002	-.00003
6	-1	+89	-.00003	+.00030
5	+500	-475	+.00181	-.00148
4	-4447	-203	-.01494	-.00180
3	+14989	+17446	+.04614	+.06258
2	+11172	-88578	+.05994	-.29580
1	-197012	+123804	-.69531	+.36781
0	+281539	+155373	+.91004	+.59461
Σ	+106734	+207449	+.30763	+.72619

The values of $\text{ber}_n x$ and $\text{bei}_n x$ are computed by multiplication of the trial values by the normalizing factor

$$1/(294989 - 22011i) = (.337119 + .025155i) \times 10^{-5},$$

obtained from the relation

$$J_0(xe^{3\pi i/4}) + 2J_2(xe^{3\pi i/4}) + 2J_4(xe^{3\pi i/4}) + \dots = 1.$$

Adequate checks are furnished by interpolating in Table 9.12 for ber 1.55 and bei 1.55, and the use of a simple sum check on the normalization.

Should $\ker_n x$ and $\text{kei}_n x$ be required they can be computed by forward recurrence using formulas 9.9.14, taking the required starting values for $n=0$ and 1 from Table 9.12 (see Example 2). If an independent check on the recurrence is required the asymptotic expansion 9.10.38 can be used.

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10. Bessel Functions of Fractional Order

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$n=9, 10; x=5(.1)10, 6S$	
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$\text{Ai}(x) = \frac{1}{2}x^{-1/4}e^{-\xi}f(-\xi); \text{Bi}(x) = x^{-1/4}e^{\xi}f(\xi)$	
$\text{Ai}'(x) = -\frac{1}{2}x^{1/4}e^{-\xi}g(-\xi); \text{Bi}'(x) = x^{1/4}e^{\xi}g(\xi)$	
$f(\pm\xi), g(\pm\xi); \xi = \frac{2}{3}x^{3/2}, \xi^{-1} = 1.5(-.1).5(-.05)0, 6D$	
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$\text{Bi}(-x) = x^{-1/4}[f_2(\xi) \cos \xi - f_1(\xi) \sin \xi]$	
$\text{Ai}'(-x) = x^{1/4}[g_1(\xi) \sin \xi - g_2(\xi) \cos \xi]$	
$\text{Bi}'(-x) = x^{1/4}[g_2(\xi) \sin \xi + g_1(\xi) \cos \xi]$	
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10. Bessel Functions of Fractional Order

Mathematical Properties

10.1. Spherical Bessel Functions

Definitions

Differential Equation

10.1.1

$$z^2 w'' + 2zw' + [z^2 - n(n+1)]w = 0$$

$$(n=0, \pm 1, \pm 2, \dots)$$

Particular solutions are the *Spherical Bessel functions of the first kind*

$$j_n(z) = \sqrt{\frac{1}{2}\pi/z} J_{n+\frac{1}{2}}(z),$$

the *Spherical Bessel functions of the second kind*

$$y_n(z) = \sqrt{\frac{1}{2}\pi/z} Y_{n+\frac{1}{2}}(z),$$

and the *Spherical Bessel functions of the third kind*

$$h_n^{(1)}(z) = j_n(z) + iy_n(z) = \sqrt{\frac{1}{2}\pi/z} H_{n+\frac{1}{2}}^{(1)}(z),$$

$$h_n^{(2)}(z) = j_n(z) - iy_n(z) = \sqrt{\frac{1}{2}\pi/z} H_{n+\frac{1}{2}}^{(2)}(z).$$

The pairs $j_n(z)$, $y_n(z)$ and $h_n^{(1)}(z)$, $h_n^{(2)}(z)$ are linearly independent solutions for every n . For general properties see the remarks after 9.1.1.

Ascending Series (See 9.1.2, 9.1.10)

10.1.2

$$j_n(z) = \frac{z^n}{1 \cdot 3 \cdot 5 \dots (2n+1)} \left\{ 1 - \frac{\frac{1}{2}z^2}{1!(2n+3)} + \frac{(\frac{1}{2}z^2)^2}{2!(2n+3)(2n+5)} - \dots \right\}$$

10.1.3

$$y_n(z) = -\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{z^{n+1}} \left\{ 1 - \frac{\frac{1}{2}z^2}{1!(1-2n)} + \frac{(\frac{1}{2}z^2)^2}{2!(1-2n)(3-2n)} - \dots \right\}$$

$$(n=0, 1, 2, \dots)$$

Limiting Values as $z \rightarrow 0$

10.1.4

$$z^{-n} j_n(z) \rightarrow \frac{1}{1 \cdot 3 \cdot 5 \dots (2n+1)}$$

10.1.5

$$z^{n+1} y_n(z) \rightarrow -1 \cdot 3 \cdot 5 \dots (2n-1) \quad (n=0, 1, 2, \dots)$$

Wronskians

10.1.6

$$W\{j_n(z), y_n(z)\} = z^{-2}$$

10.1.7

$$W\{h_n^{(1)}(z), h_n^{(2)}(z)\} = -2iz^{-2} \quad (n=0, 1, 2, \dots)$$

Representations by Elementary Functions

10.1.8

$$j_n(z) = z^{-1} [P(n+\frac{1}{2}, z) \sin(z - \frac{1}{2}n\pi) + Q(n+\frac{1}{2}, z) \cos(z - \frac{1}{2}n\pi)]$$

10.1.9

$$y_n(z) = (-1)^{n+1} z^{-1} [P(n+\frac{1}{2}, z) \cos(z + \frac{1}{2}n\pi) - Q(n+\frac{1}{2}, z) \sin(z + \frac{1}{2}n\pi)]$$

$$P(n+\frac{1}{2}, z) = 1 - \frac{(n+2)!}{2! \Gamma(n-1)} (2z)^{-2} + \frac{(n+4)!}{4! \Gamma(n-3)} (2z)^{-4} - \dots$$

$$= \sum_0^{\lfloor \frac{1}{2}n \rfloor} (-1)^k (n+\frac{1}{2}, 2k) (2z)^{-2k}$$

$$Q(n+\frac{1}{2}, z) = \frac{(n+1)!}{1! \Gamma(n)} (2z)^{-1} - \frac{(n+3)!}{3! \Gamma(n-2)} (2z)^{-3} + \frac{(n+5)!}{5! \Gamma(n-4)} (2z)^{-5} - \dots$$

$$= \sum_0^{\lfloor \frac{1}{2}(n-1) \rfloor} (-1)^k (n+\frac{1}{2}, 2k+1) (2z)^{-2k-1}$$

$$(n=0, 1, 2, \dots)$$

$$(n+\frac{1}{2}, k) = \frac{(n+k)!}{k! \Gamma(n-k+1)}$$

$n \backslash k$	1	2	3	4	5
1	2				
2	6	12			
3	12	60	120		
4	20	180	840	1680	
5	30	420	3360	15120	30240

10.1.10

$$j_n(z) = f_n(z) \sin z + (-1)^{n+1} f_{-n-1}(z) \cos z$$

$$f_0(z) = z^{-1}, \quad f_1(z) = z^{-2}$$

$$f_{n-1}(z) + f_{n+1}(z) = (2n+1)z^{-1}f_n(z) \\ (n=0, \pm 1, \pm 2, \dots)$$

The Functions $j_n(z)$, $y_n(z)$ for $n=0, 1, 2$

$$10.1.11 \quad j_0(z) = \frac{\sin z}{z}$$

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}$$

$$j_2(z) = \left(\frac{3}{z^3} - \frac{1}{z}\right) \sin z - \frac{3}{z^2} \cos z$$

10.1.12

$$y_0(z) = -j_{-1}(z) = -\frac{\cos z}{z}$$

$$y_1(z) = j_{-2}(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z}$$

$$y_2(z) = -j_{-3}(z) = \left(-\frac{3}{z^3} + \frac{1}{z}\right) \cos z - \frac{3}{z^2} \sin z$$

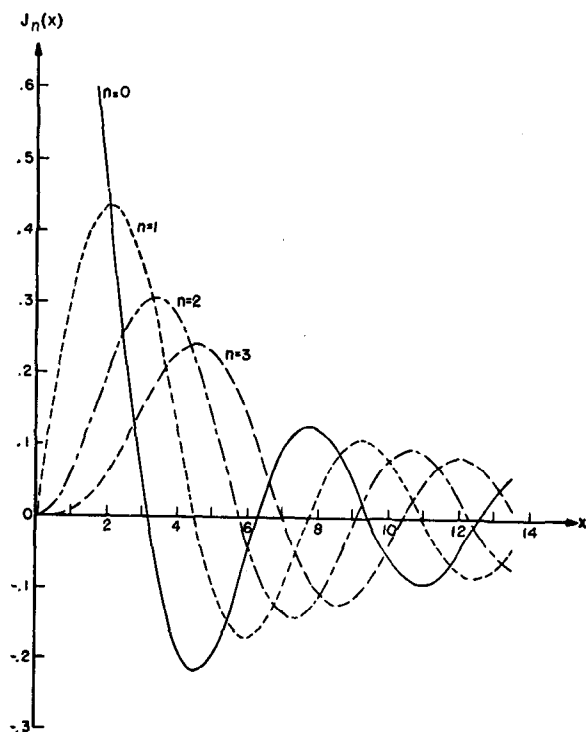


FIGURE 10.1. $j_n(x)$. $n=0(1)3$.

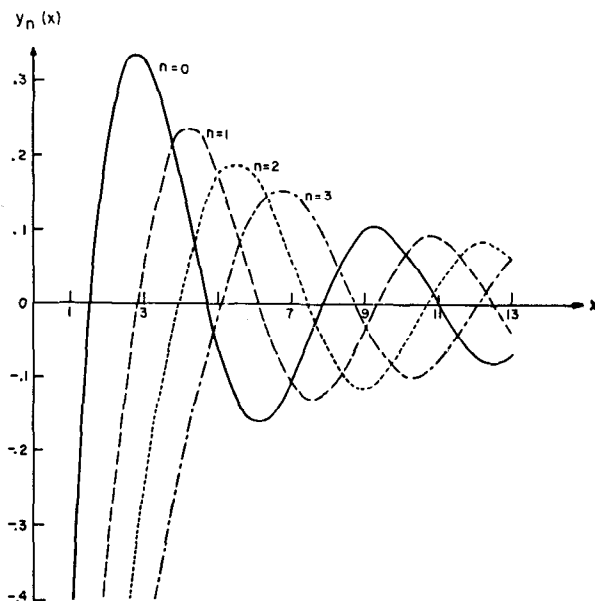


FIGURE 10.2. $y_n(x)$. $n=0(1)3$.

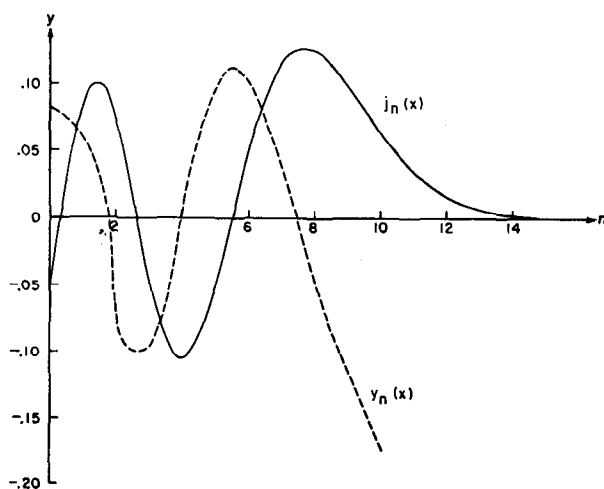


FIGURE 10.3. $j_n(x)$, $y_n(x)$. $x=10$.

Poisson's Integral and Gegenbauer's Generalization

$$10.1.13 \quad j_n(z) = \frac{z^n}{2^{n+1}n!} \int_0^\pi \cos(z \cos \theta) \sin^{2n+1} \theta d\theta \\ (\text{See 9.1.20.})$$

10.1.14

$$= \frac{1}{2} (-i)^n \int_0^\pi e^{iz \cos \theta} P_n(\cos \theta) \sin \theta d\theta \\ (n=0, 1, 2, \dots)$$

Spherical Bessel Functions of the Second and Third Kind

10.1.15

$$y_n(z) = (-1)^{n+1} j_{-n-1}(z) \quad (n=0, \pm 1, \pm 2, \dots)$$

10.1.16

$$h_n^{(1)}(z) = i^{-n-1} z^{-1} e^{iz} \sum_0^n (n + \frac{1}{2}, k) (-2iz)^{-k}$$

10.1.17

$$h_n^{(2)}(z) = i^{n+1} z^{-1} e^{-iz} \sum_0^n (n + \frac{1}{2}, k) (2iz)^{-k} \quad *$$

10.1.18

$$h_{-n-1}^{(1)}(z) = i(-1)^n h_n^{(1)}(z)$$

$$h_{-n-1}^{(2)}(z) = -i(-1)^n h_n^{(2)}(z) \quad (n=0, 1, 2, \dots)$$

 Elementary Properties
Recurrence Relations

$$f_n(z) : j_n(z), y_n(z), h_n^{(1)}(z), h_n^{(2)}(z) \quad (n=0, \pm 1, \pm 2, \dots)$$

$$10.1.19 \quad f_{n-1}(z) + f_{n+1}(z) = (2n+1)z^{-1}f_n(z)$$

$$10.1.20 \quad nf_{n-1}(z) - (n+1)f_{n+1}(z) = (2n+1) \frac{d}{dz} f_n(z)$$

$$10.1.21 \quad \frac{n+1}{z} f_n(z) + \frac{d}{dz} f_n(z) = f_{n-1}(z)$$

(See 10.1.23.)

$$10.1.22 \quad \frac{n}{z} f_n(z) - \frac{d}{dz} f_n(z) = f_{n+1}(z)$$

(See 10.1.24.)

Differentiation Formulas

$$f_n(z) : j_n(z), y_n(z), h_n^{(1)}(z), h_n^{(2)}(z) \quad (n=0, \pm 1, \pm 2, \dots)$$

$$10.1.23 \quad \left(\frac{1}{z} \frac{d}{dz}\right)^m [z^{n+1} f_n(z)] = z^{n-m+1} f_{n-m}(z)$$

$$10.1.24 \quad \left(\frac{1}{z} \frac{d}{dz}\right)^m [z^{-n} f_n(z)] = (-1)^m z^{-n-m} f_{n+m}(z) \quad (m=1, 2, 3, \dots)$$

Rayleigh's Formulas

10.1.25

$$j_n(z) = z^n \left(-\frac{1}{z} \frac{d}{dz}\right)^n \frac{\sin z}{z}$$

10.1.26

$$y_n(z) = -z^n \left(-\frac{1}{z} \frac{d}{dz}\right)^n \frac{\cos z}{z} \quad (n=0, 1, 2, \dots)$$

Modulus and Phase

$$j_n(z) = \sqrt{\frac{1}{2}\pi/z} M_{n+\frac{1}{2}}(z) \cos \theta_{n+\frac{1}{2}}(z),$$

$$y_n(z) = \sqrt{\frac{1}{2}\pi/z} M_{n+\frac{1}{2}}(z) \sin \theta_{n+\frac{1}{2}}(z)$$

(See 9.2.17.)

10.1.27

$$(\frac{1}{2}\pi/z) M_{n+\frac{1}{2}}^2(z) = \frac{1}{z^2} \sum_0^n \frac{(2n-k)!(2n-2k)!}{k![(n-k)!]^2} (2z)^{2k-2n}$$

(See 9.2.28.)

$$10.1.28 \quad (\frac{1}{2}\pi/z) M_{1/2}^2(z) = j_0^2(z) + y_0^2(z) = z^{-2}$$

10.1.29

$$(\frac{1}{2}\pi/z) M_{3/2}^2(z) = j_1^2(z) + y_1^2(z) = z^{-2} + z^{-4}$$

10.1.30

$$(\frac{1}{2}\pi/z) M_{5/2}^2(z) = j_2^2(z) + y_2^2(z) = z^{-2} + 3z^{-4} + 9z^{-6}$$

Cross Products

$$10.1.31 \quad j_n(z)y_{n-1}(z) - j_{n-1}(z)y_n(z) = z^{-2}$$

10.1.32

$$j_{n+1}(z)y_{n-1}(z) - j_{n-1}(z)y_{n+1}(z) = (2n+1)z^{-3}$$

10.1.33

$$\begin{aligned} j_0(z)j_n(z) + y_0(z)y_n(z) \\ = z^{-2} \sum_0^{[n]} (-1)^k 2^{n-2k} \binom{k+\frac{1}{2}}{n-2k} \binom{n-k}{k} z^{2k-n} \end{aligned} \quad (n=0, 1, 2, \dots)$$

Analytic Continuation

$$10.1.34 \quad j_n(ze^{m\pi i}) = e^{m\pi i} j_n(z)$$

$$10.1.35 \quad y_n(ze^{m\pi i}) = (-1)^m e^{m\pi i} y_n(z)$$

$$10.1.36 \quad h_n^{(1)}(ze^{(2m+1)\pi i}) = (-1)^n h_n^{(2)}(z)$$

$$10.1.37 \quad h_n^{(2)}(ze^{(2m+1)\pi i}) = (-1)^n h_n^{(1)}(z)$$

$$10.1.38 \quad h_n^{(1)}(ze^{2m\pi i}) = h_n^{(1)}(z) \quad (l=1, 2; m, n=0, 1, 2, \dots)$$

Generating Functions

10.1.39

$$\frac{1}{z} \sin \sqrt{z^2 + 2zt} = \sum_0^\infty \frac{(-t)^n}{n!} y_{n-1}(z) \quad (2|t| < |z|)$$

$$10.1.40 \quad \frac{1}{z} \cos \sqrt{z^2 - 2zt} = \sum_0^\infty \frac{t^n}{n!} j_{n-1}(z)$$

*See page II.

Derivatives With Respect to Order

10.1.41

$$\left[\frac{\partial}{\partial \nu} j_\nu(x) \right]_{\nu=0} = (\tfrac{1}{2}\pi/x) \{ \text{Ci}(2x) \sin x - \text{Si}(2x) \cos x \}$$

10.1.42

$$\left[\frac{\partial}{\partial \nu} j_\nu(x) \right]_{\nu=-1} = (\tfrac{1}{2}\pi/x) \{ \text{Ci}(2x) \cos x + \text{Si}(2x) \sin x \}$$

10.1.43

$$\left[\frac{\partial}{\partial \nu} y_\nu(x) \right]_{\nu=0} = (\tfrac{1}{2}\pi/x) \{ \text{Ci}(2x) \cos x + [\text{Si}(2x) - \pi] \sin x \}$$

10.1.44

$$\left[\frac{\partial}{\partial \nu} y_\nu(x) \right]_{\nu=-1} = (\tfrac{1}{2}\pi/x) \{ \text{Ci}(2x) \sin x - [\text{Si}(2x) - \pi] \cos x \}$$

Addition Theorems and Degenerate Forms

r, ρ, θ, λ arbitrary complex; $R = \sqrt{(r^2 + \rho^2 - 2r\rho \cos \theta)}$

$$10.1.45 \quad \frac{\sin \lambda R}{\lambda R} = \sum_0^\infty (2n+1) j_n(\lambda r) j_n(\lambda \rho) P_n(\cos \theta)$$

$$*10.1.46 \quad -\frac{\cos \lambda R}{\lambda R} = \sum_0^\infty (2n+1) j_n(\lambda r) y_n(\lambda \rho) P_n(\cos \theta)$$

$|re^{\pm i\theta}| < |\rho|$

$$10.1.47 \quad e^{iz \cos \theta} = \sum_0^\infty (2n+1) e^{in\pi t} j_n(z) P_n(\cos \theta)$$

10.1.48

$$J_0(z \sin \theta) = \sum_0^\infty (4n+1) \frac{(2n)!}{2^{2n}(n!)^2} j_{2n}(z) P_{2n}(\cos \theta)$$

Duplication Formula

10.1.49

$$j_n(2z) =$$

$$* \quad -n! z^{n+1} \sum_0^n \frac{2n-2k+1}{k!(2n-k+1)!} j_{n-k}(z) y_{n-k}(z)$$

Some Infinite Series Involving $j_n^2(z)$

$$10.1.50 \quad \sum_0^\infty (2n+1) j_n^2(z) = 1$$

$$10.1.51 \quad \sum_0^\infty (-1)^n (2n+1) j_n^2(z) = \frac{\sin 2z}{2z}$$

$$10.1.52 \quad \sum_0^\infty j_n^2(z) = \frac{\text{Si}(2z)}{2z}$$

Fresnel Integrals

10.1.53

$$C(\sqrt{2x/\pi}) = \frac{1}{2} \int_0^x J_{-1/2}(t) dt$$

$$= \sqrt{2} [\cos \tfrac{1}{2}x \sum_0^\infty (-1)^n J_{2n+1/2}(\tfrac{1}{2}x) + \sin \tfrac{1}{2}x \sum_0^\infty (-1)^n J_{2n+3/2}(\tfrac{1}{2}x)]$$

10.1.54

$$S(\sqrt{2x/\pi}) = \frac{1}{2} \int_0^x J_{1/2}(t) dt$$

$$= \sqrt{2} [\sin \tfrac{1}{2}x \sum_0^\infty (-1)^n J_{2n+1/2}(\tfrac{1}{2}x) - \cos \tfrac{1}{2}x \sum_0^\infty (-1)^n J_{2n+3/2}(\tfrac{1}{2}x)].$$

(See also 11.1.1, 11.1.2.)

Zeros and Their Asymptotic Expansions

The zeros of $j_n(x)$ and $y_n(x)$ are the same as the zeros of $J_{n+1/2}(x)$ and $Y_{n+1/2}(x)$ and the formulas for $j_{\nu,s}$ and $y_{\nu,s}$ given in 9.5 are applicable with $\nu = n + \frac{1}{2}$. There are, however, no simple relations connecting the zeros of the derivatives. Accordingly, we now give formulas for $a'_{n,s}$, $b'_{n,s}$, the s -th positive zero of $j'_n(z)$, $y'_n(z)$, respectively; $z=0$ is counted as the first zero of $j'_0(z)$.

(Tables of $a'_{n,s}$, $b'_{n,s}$, $j_n(a'_{n,s})$, $y_n(b'_{n,s})$ are given in [10.31].)

Elementary Relations

$$f_n(z) = j_n(z) \cos \pi t + y_n(z) \sin \pi t$$

(t a real parameter, $0 \leq t \leq 1$)

If τ_n is a zero of $f'_n(z)$ then

$$10.1.55 \quad f_n(\tau_n) = [\tau_n/(n+1)] f_{n-1}(\tau_n)$$

(See 10.1.21.)

$$10.1.56 \quad = (\tau_n/n) f_{n+1}(\tau_n)$$

(See 10.1.22.)

$$10.1.57 \quad = \left\{ \frac{1}{\pi} [\tau_n^2 - n(n+1)] \frac{d\tau_n}{d\tau} \right\}^{-1}$$

*See page II.

McMahon's Expansions for n Fixed and s Large

10.1.58

$$\begin{aligned} a'_{n,s}, b'_{n,s} &\sim \beta - (\mu + 7)(8\beta)^{-1} \\ &\quad - \frac{4}{3}(7\mu^2 + 154\mu + 95)(8\beta)^{-3} \\ &\quad - \frac{32}{15}(85\mu^3 + 3535\mu^2 + 3561\mu + 6133)(8\beta)^{-5} \\ &\quad - \frac{64}{105}(6949\mu^4 + 474908\mu^3 + 330638\mu^2 \\ &\quad + 9046780\mu - 5075147)(8\beta)^{-7} - \dots \end{aligned}$$

$$\begin{aligned} \beta &= \pi(s + \frac{1}{2}n - \frac{1}{2}) \text{ for } a'_{n,s}, \beta = \pi(s + \frac{1}{2}n) \text{ for } b'_{n,s}; \\ \mu &= (2n + 1)^2 \end{aligned}$$

 Asymptotic Expansions of Zeros and Associated Values for n Large

10.1.59

$$\begin{aligned} a'_{n,1} &\sim (n + \frac{1}{2}) + .8086165(n + \frac{1}{2})^{1/3} - .236680(n + \frac{1}{2})^{-1/3} \\ &\quad - .20736(n + \frac{1}{2})^{-1} + .0233(n + \frac{1}{2})^{-5/3} + \dots \end{aligned}$$

10.1.60

$$\begin{aligned} b'_{n,1} &\sim (n + \frac{1}{2}) + 1.8210980(n + \frac{1}{2})^{1/3} \\ &\quad + .802728(n + \frac{1}{2})^{-1/3} - .11740(n + \frac{1}{2})^{-1} \\ &\quad + .0249(n + \frac{1}{2})^{-5/3} + \dots \end{aligned}$$

10.1.61

$$\begin{aligned} j_n(a'_{n,1}) &\sim .8458430(n + \frac{1}{2})^{-5/6} \{ 1 - .566032(n + \frac{1}{2})^{-2/3} \\ &\quad + .38081(n + \frac{1}{2})^{-4/3} - .2203(n + \frac{1}{2})^{-2} + \dots \} \end{aligned}$$

10.1.62

$$\begin{aligned} y_n(b'_{n,1}) &\sim .7183921(n + \frac{1}{2})^{-5/6} \{ 1 - 1.274769(n + \frac{1}{2})^{-2/3} \\ &\quad + 1.23038(n + \frac{1}{2})^{-4/3} - 1.0070(n + \frac{1}{2})^{-2} + \dots \} \end{aligned}$$

See [10.31] for corresponding expansions for $s=2, 3$.

 Uniform Asymptotic Expansions of Zeros and Associated Values for n Large

10.1.63

$$\begin{aligned} a'_{n,s} &\sim (n + \frac{1}{2}) \{ z[(n + \frac{1}{2})^{-2/3} a'_s] \\ &\quad + \sum_{k=1}^{\infty} h_k[(n + \frac{1}{2})^{-2/3} a'_s](n + \frac{1}{2})^{-2k} \} \end{aligned}$$

10.1.64

$$\begin{aligned} b'_{n,s} &\sim (n + \frac{1}{2}) \{ z[(n + \frac{1}{2})^{-2/3} b'_s] \\ &\quad + \sum_{k=1}^{\infty} h_k[(n + \frac{1}{2})^{-2/3} b'_s](n + \frac{1}{2})^{-2k} \} \end{aligned}$$

10.1.65

$$\begin{aligned} j_n(a'_{n,s}) &\sim \sqrt{\frac{1}{2}\pi} \text{Ai}(a'_s)(n + \frac{1}{2})^{-5/6} \\ &\quad h[(n + \frac{1}{2})^{-2/3} a'_s] \{ z[(n + \frac{1}{2})^{-2/3} a'_s] \}^{-1/2} \\ &\quad \{ 1 + \sum_{k=1}^{\infty} H_k[(n + \frac{1}{2})^{-2/3} a'_s](n + \frac{1}{2})^{-2k} \} \end{aligned}$$

10.1.66

$$\begin{aligned} y_n(b'_{n,s}) &\sim -\sqrt{\frac{1}{2}\pi} \text{Bi}(b'_s)(n + \frac{1}{2})^{-5/6} \\ &\quad h[(n + \frac{1}{2})^{-2/3} b'_s] \{ z[(n + \frac{1}{2})^{-2/3} b'_s] \}^{-1/2} \\ &\quad \{ 1 + \sum_{k=1}^{\infty} H_k[(n + \frac{1}{2})^{-2/3} b'_s](n + \frac{1}{2})^{-2k} \} \end{aligned}$$

$h(\xi)$, $z(\xi)$ are defined as in 9.5.26, 9.3.38, 9.3.39.
 a'_s , b'_s s -th (negative) real zero of $\text{Ai}'(z)$, $\text{Bi}'(z)$
 (see 10.4.95, 10.4.99.)

 Complex Zeros of $h_n^{(1)}(z)$, $h_n^{(1)'}(z)$

$h_n^{(1)}(z)$ and $h_n^{(1)}(ze^{2\pi m \tau i})$, m any integer, have the same zeros.

$h_n^{(1)}(z)$ has n zeros, symmetrically distributed with respect to the imaginary axis and lying approximately on the finite arc joining $z=-n$ and $z=n$ shown in Figure 9.6. If n is odd, one zero lies on the imaginary axis.

$h_n^{(1)'}(z)$ has $n+1$ zeros lying approximately on the same curve. If n is even, one zero lies on the imaginary axis.

$-\zeta$	$(-\zeta)h_1(\zeta)$	$(-\zeta)h_2(\zeta)$	$(-\zeta)h_3(\zeta)$	$(-\zeta)^2H_1(\zeta)$	$(-\zeta)^4H_2(\zeta)$	$(-\zeta)^6H_3(\zeta)$
0.0	-.4409724	-.122500	-.06806	.000000	.00000	.0000
0.2	-.4572444	-.114201	-.05986	.027518	.00575	.0023
0.4	-.4702250	-.107243	-.05279	.049069	.01118	.0043
0.6	-.4802184	-.101318	-.04674	.065677	.01592	.0061
0.8	-.4875705	-.096159	-.04160	.078255	.01983	.0075
1.0	-.4926355	-.091561	-.03725	.087587	.02290	.0085
$-\zeta$	$h_1(\zeta)$	$h_2(\zeta)$	$h_3(\zeta)$	$H_1(\zeta)$	$H_2(\zeta)$	
1.0	-.4926355	-.09156	-.037	.087587	.0229	
1.2	-.4131280	-.05056	-.014	.065507	.0121	
1.4	-.3551700	-.03043	-.006	.050524	.0070	
1.6	-.3108548	-.01950	-.003	.039890	.0042	
1.8	-.2757704	-.01310	-.001	.032085	.0027	
2.0	-.2472521	-.00914		.026206	.0018	
2.2	-.2235898	-.00658		.021682	.0012	
2.4	-.2036314	-.00485		.018141	.0008	
2.6	-.1865701	-.00366		.015326	.0006	
2.8	-.1718217	-.00280		.013061	.0004	
3.0	-.1589519	-.00219		.011217	.0003	
3.2	-.1476304	-.00173		.009701	.0002	
3.4	-.1376005	-.00138		.008443	.0002	
3.6	-.1286601	-.00112		.007391	.0001	
3.8	-.1206469	-.00091		.006505	.0001	
4.0	-.1134296	-.00075		.005753		
4.2	-.1069004	-.00062		.005111		
4.4	-.1009699	-.00052		.004560		
4.6	-.0955634	-.00044		.004085		
4.8	-.0906180	-.00037		.003672		
5.0	-.0860804	-.00032		.003313		
5.2	-.0819049	-.00027		.002998		
5.4	-.0780523	-.00023		.002722		
5.6	-.0744888	-.00020		.002478		
5.8	-.0711850	-.00018		.002262		
6.0	-.0681152	-.00015		.002070		
6.2	-.0652570	-.00013		.001899		
6.4	-.0625905	-.00012		.001746		
6.6	-.0600985	-.00010		.001609		
6.8	-.0577653	-.00009		.001486		
7.0	-.0555773	-.00008		.001375		

$(-\zeta) \rightarrow$	$h_1(\zeta)$	$h_2(\zeta)$	$H_1(\zeta)$
0.40	-.0645731	-.00013	.001859
.36	-.0487592	-.00005	.001056
.32	-.0352949	-.00002	.000551
.28	-.0242415	-.00001	.000259
.24	-.0155683		.000106
.20	-.0091416		.000037
.16	-.0047276		.000010
.12	-.0020068		.000002
.08	-.0005965		
.04	-.0000747		
.00	-.0000000		

10.2. Modified Spherical Bessel Functions

Definitions

Differential Equation

10.2.1

$$z^2 w'' + 2zw' - [z^2 + n(n+1)]w = 0$$

$$(n=0, \pm 1, \pm 2, \dots)$$

Particular solutions are the *Modified Spherical Bessel functions of the first kind*,

10.2.2

$$\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z) = e^{-n\pi i/2} j_n(ze^{\pi i/2}) \quad (-\pi < \arg z \leq \frac{1}{2}\pi)$$

$$= e^{3n\pi i/2} j_n(ze^{-3\pi i/2}) \quad (\frac{1}{2}\pi < \arg z \leq \pi)$$

of the second kind,

10.2.3

$$\sqrt{\frac{1}{2}\pi/z} I_{-n-\frac{1}{2}}(z) = e^{3(n+1)\pi i/2} y_n(ze^{\pi i/2})$$

$$= e^{-(n+1)\pi i/2} y_n(ze^{-3\pi i/2})$$

$$(-\pi < \arg z \leq \frac{1}{2}\pi)$$

$$(\frac{1}{2}\pi < \arg z \leq \pi)$$

of the third kind,

10.2.4

$$\sqrt{\frac{1}{2}\pi/z} K_{n+\frac{1}{2}}(z) = \frac{1}{2}\pi (-1)^{n+1} \sqrt{\frac{1}{2}\pi/z} [I_{n+\frac{1}{2}}(z) - I_{-n-\frac{1}{2}}(z)]$$

The pairs

$$\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z), \sqrt{\frac{1}{2}\pi/z} I_{-n-\frac{1}{2}}(z)$$

and

$$\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z), \sqrt{\frac{1}{2}\pi/z} K_{n+\frac{1}{2}}(z)$$

are linearly independent solutions for every n .

Most properties of the Modified Spherical Bessel functions can be derived from those of the Spherical Bessel functions by use of the above relations.

Ascending Series

10.2.5

$$\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z) = \frac{z^n}{1 \cdot 3 \cdot 5 \dots (2n+1)}$$

$$\left\{ 1 + \frac{\frac{1}{2}z^2}{1!(2n+3)} + \frac{(\frac{1}{2}z^2)^2}{2!(2n+3)(2n+5)} + \dots \right\}$$

10.2.6

$$\sqrt{\frac{1}{2}\pi/z} I_{-n-\frac{1}{2}}(z) = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{(-1)^n z^{n+1}}$$

$$\left\{ 1 + \frac{\frac{1}{2}z^2}{1!(1-2n)} + \frac{(\frac{1}{2}z^2)^2}{2!(1-2n)(3-2n)} + \dots \right\}$$

$$(n=0, 1, 2, \dots)$$

Wronskians

10.2.7

$$W\{\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z), \sqrt{\frac{1}{2}\pi/z} I_{-n-\frac{1}{2}}(z)\} = (-1)^{n+1} z^{-2}$$

10.2.8

$$W\{\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z), \sqrt{\frac{1}{2}\pi/z} K_{n+\frac{1}{2}}(z)\} = -\frac{1}{2}\pi z^{-2}$$

Representations by Elementary Functions

10.2.9

$$\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z) = (2z)^{-1} [R(n+\frac{1}{2}, -z) e^z$$

$$- (-1)^n R(n+\frac{1}{2}, z) e^{-z}]$$

10.2.10

$$\sqrt{\frac{1}{2}\pi/z} I_{-n-\frac{1}{2}}(z) = (2z)^{-1} [R(n+\frac{1}{2}, -z) e^z$$

$$+ (-1)^n R(n+\frac{1}{2}, z) e^{-z}]$$

10.2.11

$$R(n+\frac{1}{2}, z) = 1 + \frac{(n+1)!}{1!\Gamma(n)} (2z)^{-1}$$

$$+ \frac{(n+2)!}{2!\Gamma(n-1)} (2z)^{-2} + \dots$$

$$= \sum_0^n (n+\frac{1}{2}, k) (2z)^{-k}$$

$$(n=0, 1, 2, \dots)$$

(See 10.1.9.)

10.2.12

$$\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z) = g_n(z) \sinh z + g_{-n-1}(z) \cosh z$$

$$g_0(z) = z^{-1}, g_1(z) = -z^{-2}$$

$$g_{n-1}(z) - g_{n+1}(z) = (2n+1) z^{-1} g_n(z)$$

$$(n=0, \pm 1, \pm 2, \dots)$$

The Functions $\sqrt{\frac{1}{2}\pi/z} I_{\pm(n+\frac{1}{2})}(z)$, $n=0, 1, 2$

10.2.13

$$\sqrt{\frac{1}{2}\pi/z} I_{1/2}(z) = \frac{\sinh z}{z}$$

$$\sqrt{\frac{1}{2}\pi/z} I_{3/2}(z) = -\frac{\sinh z}{z^2} + \frac{\cosh z}{z}$$

$$\sqrt{\frac{1}{2}\pi/z} I_{5/2}(z) = \left(\frac{3}{z^3} + \frac{1}{z}\right) \sinh z - \frac{3}{z^2} \cosh z$$

*

10.2.14

$$\sqrt{\frac{1}{2}\pi/z} I_{-1/2}(z) = \frac{\cosh z}{z}$$

$$\sqrt{\frac{1}{2}\pi/z} I_{-3/2}(z) = \frac{\sinh z}{z} - \frac{\cosh z}{z^2}$$

$$\sqrt{\frac{1}{2}\pi/z} I_{-5/2}(z) = -\frac{3}{z^2} \sinh z + \left(\frac{3}{z^3} + \frac{1}{z}\right) \cosh z$$

*

Modified Spherical Bessel Functions of the Third Kind

10.2.15

$$\begin{aligned}
 \sqrt{\frac{1}{2}\pi/z} K_{n+\frac{1}{2}}(z) &= \frac{1}{2}\pi i e^{(n+1)\pi i/2} h_n^{(1)}(ze^{\frac{1}{2}\pi i}) \\
 &\quad (-\pi < \arg z \leq \frac{1}{2}\pi) \\
 &= -\frac{1}{2}\pi i e^{-(n+1)\pi i/2} h_n^{(2)}(ze^{-\frac{1}{2}\pi i}) \\
 &\quad (\frac{1}{2}\pi < \arg z \leq \pi) \\
 &= (\frac{1}{2}\pi/z) e^{-z} \sum_0^n (n+\frac{1}{2}, k) (2z)^{-k}
 \end{aligned}$$

10.2.16

$$K_{n+\frac{1}{2}}(z) = K_{-n-\frac{1}{2}}(z) \quad (n=0, 1, 2, \dots)$$

The Functions $\sqrt{\frac{1}{2}\pi/z} K_{n+\frac{1}{2}}(z), n=0, 1, 2$

$$10.2.17 \quad \sqrt{\frac{1}{2}\pi/z} K_{1/2}(z) = (\frac{1}{2}\pi/z) e^{-z}$$

$$\sqrt{\frac{1}{2}\pi/z} K_{3/2}(z) = (\frac{1}{2}\pi/z) e^{-z} (1+z^{-1})$$

$$\sqrt{\frac{1}{2}\pi/z} K_{5/2}(z) = (\frac{1}{2}\pi/z) e^{-z} (1+3z^{-1}+3z^{-2})$$

Elementary Properties

Recurrence Relations

$$f_n(z): \sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z), (-1)^{n+1} \sqrt{\frac{1}{2}\pi/z} K_{n+\frac{1}{2}}(z)$$

$$(n=0, \pm 1, \pm 2, \dots)$$

$$10.2.18 \quad f_{n-1}(z) - f_{n+1}(z) = (2n+1) z^{-1} f_n(z)$$

$$10.2.19 \quad n f_{n-1}(z) + (n+1) f_{n+1}(z) = (2n+1) \frac{d}{dz} f_n(z)$$

$$10.2.20 \quad \frac{n+1}{z} f_n(z) + \frac{d}{dz} f_n(z) = f_{n-1}(z)$$

(See 10.2.22.)

$$10.2.21 \quad -\frac{n}{z} f_n(z) + \frac{d}{dz} f_n(z) = f_{n+1}(z)$$

(See 10.2.23.)

Differentiation Formulas

$$f_n(z): \sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z), (-1)^{n+1} \sqrt{\frac{1}{2}\pi/z} K_{n+\frac{1}{2}}(z)$$

$$(n=0, \pm 1, \pm 2, \dots)$$

$$10.2.22 \quad \left(\frac{1}{z} \frac{d}{dz}\right)^m [z^{n+1} f_n(z)] = z^{n-m+1} f_{n-m}(z)$$

$$10.2.23 \quad \left(\frac{1}{z} \frac{d}{dz}\right)^m [z^{-n} f_n(z)] = z^{-n-m} f_{n+m}(z)$$

$$(m=1, 2, 3, \dots)$$

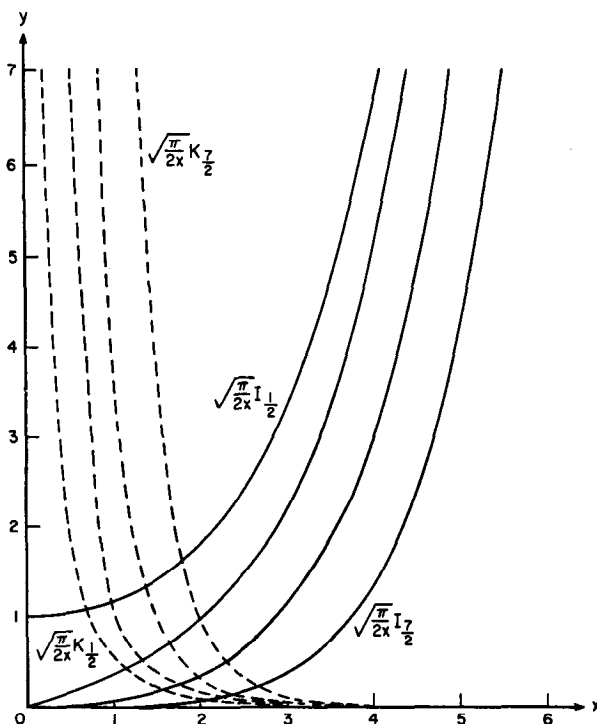


FIGURE 10.4. $\sqrt{\frac{\pi}{2x}} I_{n+\frac{1}{2}}(x), \sqrt{\frac{\pi}{2x}} K_{n+\frac{1}{2}}(x), n=0(1)3.$

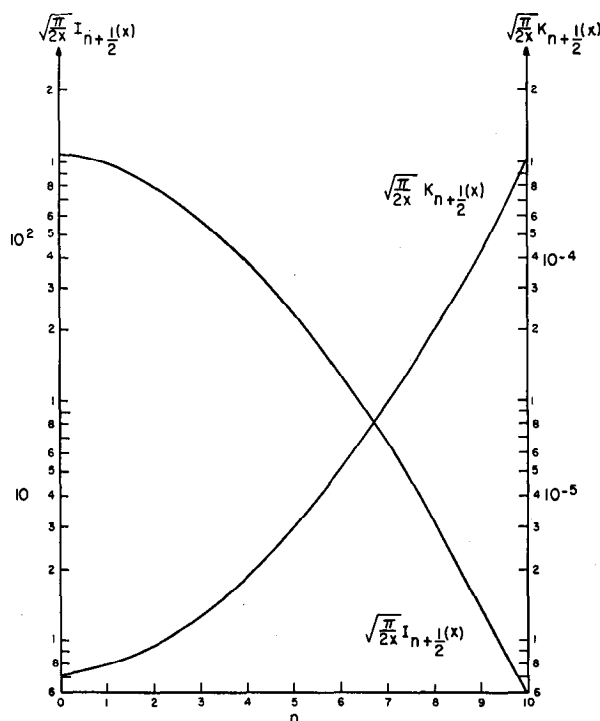


FIGURE 10.5. $\sqrt{\frac{\pi}{2x}} I_{n+\frac{1}{2}}(x), \sqrt{\frac{\pi}{2x}} K_{n+\frac{1}{2}}(x), x=10.$

Formulas of Rayleigh's Type

$$10.2.24 \quad \sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z) = z^n \left(\frac{1}{z} \frac{d}{dz} \right)^n \frac{\sinh z}{z}$$

10.2.25

$$\sqrt{\frac{1}{2}\pi/z} I_{-n-\frac{1}{2}}(z) = z^n \left(\frac{1}{z} \frac{d}{dz} \right)^n \frac{\cosh z}{z} \quad (n=0, 1, 2, \dots)$$

Formulas for $I_{n+\frac{1}{2}}^2(z) - I_{-n-\frac{1}{2}}^2(z)$

10.2.26

$$\begin{aligned} & \left(\frac{1}{2}\pi/z \right) [I_{n+\frac{1}{2}}^2(z) - I_{-n-\frac{1}{2}}^2(z)] \\ &= \frac{1}{z^2} \sum_0^n (-1)^{k+1} \frac{(2n-k)! (2n-2k)!}{k! [(n-k)!]^2} (2z)^{2k-2n} \\ & \quad (n=0, 1, 2, \dots) \end{aligned}$$

$$10.2.27 \quad \left(\frac{1}{2}\pi/z \right) [I_{1/2}^2(z) - I_{-1/2}^2(z)] = -z^{-2}$$

$$10.2.28 \quad \left(\frac{1}{2}\pi/z \right) [I_{3/2}^2(z) - I_{-3/2}^2(z)] = z^{-2} - z^{-4}$$

10.2.29

$$\left(\frac{1}{2}\pi/z \right) [I_{5/2}^2(z) - I_{-5/2}^2(z)] = -z^{-2} + 3z^{-4} - 9z^{-6}$$

Generating Functions

10.2.30

$$\frac{1}{z} \sinh \sqrt{z^2 - 2izt} = \sum_0^\infty \frac{(-it)^n}{n!} \left[\sqrt{\frac{1}{2}\pi/z} I_{-n-\frac{1}{2}}(z) \right] \quad (2|t| < |z|)$$

10.2.31

$$\frac{1}{z} \cosh \sqrt{z^2 + 2izt} = \sum_0^\infty \frac{(it)^n}{n!} \left[\sqrt{\frac{1}{2}\pi/z} I_{n-\frac{1}{2}}(z) \right]$$

Derivatives With Respect to Order

10.2.32

$$\left[\frac{\partial}{\partial \nu} I_\nu(x) \right]_{\nu=-\frac{1}{2}} = -\frac{1}{2\pi x} [\text{Ei}(2x)e^{-x} - E_1(-2x)e^x]$$

10.2.33

$$\left[\frac{\partial}{\partial \nu} I_\nu(x) \right]_{\nu=-\frac{1}{2}} = \frac{1}{2\pi x} [\text{Ei}(2x)e^{-x} + E_1(-2x)e^x]$$

$$10.2.34 \quad \left[\frac{\partial}{\partial \nu} K_\nu(x) \right]_{\nu=\pm\frac{1}{2}} = \mp \sqrt{\pi/2x} \text{Ei}(-2x)e^x$$

For $E_1(x)$ and $\text{Ei}(x)$, see 5.1.1, 5.1.2.

Addition Theorems and Degenerate Forms

r, ρ, θ, λ arbitrary complex; $R = \sqrt{r^2 + \rho^2 - 2r\rho \cos \theta}$

10.2.35

$$\begin{aligned} \frac{e^{-\lambda R}}{\lambda R} &= \frac{2}{\pi} \sum_0^\infty (2n+1) \left[\sqrt{\frac{1}{2}\pi/\lambda r} I_{n+\frac{1}{2}}(\lambda r) \right] \\ & \quad \left[\sqrt{\frac{1}{2}\pi/\lambda \rho} K_{n+\frac{1}{2}}(\lambda \rho) \right] P_n(\cos \theta) \end{aligned}$$

10.2.36

$$e^{z \cos \theta} = \sum_0^\infty (2n+1) \left[\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z) \right] P_n(\cos \theta)$$

10.2.37

$$e^{-z \cos \theta} = \sum_0^\infty (-1)^n (2n+1) \left[\sqrt{\frac{1}{2}\pi/z} I_{n+\frac{1}{2}}(z) \right] P_n(\cos \theta)$$

Duplication Formula

10.2.38

$$\begin{aligned} K_{n+\frac{1}{2}}(2z) &= \\ n! \pi^{-\frac{1}{2}} z^{n+\frac{1}{2}} & \sum_0^n \frac{(-1)^k (2n-2k+1)}{k! (2n-k+1)!} K_{n-k+\frac{1}{2}}^2(z) \end{aligned}$$

10.3. Riccati-Bessel Functions

Differential Equation

10.3.1

$$z^2 w'' + [z^2 - n(n+1)]w = 0$$

$$(n=0, \pm 1, \pm 2, \dots)$$

Pairs of linearly independent solutions are

$$zj_n(z), zy_n(z)$$

$$zh_n^{(1)}(z), zh_n^{(2)}(z)$$

All properties of these functions follow directly from those of the Spherical Bessel functions.

The Functions $zj_n(z), zy_n(z), n=0, 1, 2$

10.3.2

$$zj_0(z) = \sin z, \quad zj_1(z) = z^{-1} \sin z - \cos z$$

$$zj_2(z) = (3z^{-2} - 1) \sin z - 3z^{-1} \cos z \quad *$$

10.3.3

$$zy_0(z) = -\cos z, \quad zy_1(z) = -\sin z - z^{-1} \cos z$$

$$zy_2(z) = -3z^{-1} \sin z - (3z^{-2} - 1) \cos z \quad *$$

Wronskians

$$10.3.4 \quad W\{zj_n(z), zy_n(z)\} = 1$$

$$10.3.5 \quad W\{zh_n^{(1)}(z), zh_n^{(2)}(z)\} = -2i \quad (n=0, 1, 2, \dots)$$

*See page II.

10.4. Airy Functions

Definitions and Elementary Properties

Differential Equation

$$10.4.1 \quad w'' - zw = 0$$

Pairs of linearly independent solutions are

$$\text{Ai}(z), \text{Bi}(z),$$

$$\text{Ai}(z), \text{Ai}(ze^{2\pi i/3}),$$

$$\text{Ai}(z), \text{Ai}(ze^{-2\pi i/3}).$$

Ascending Series

$$10.4.2 \quad \text{Ai}(z) = c_1 f(z) - c_2 g(z)$$

$$10.4.3 \quad \text{Bi}(z) = \sqrt{3} [c_1 f(z) + c_2 g(z)]$$

$$f(z) = 1 + \frac{1}{3!} z^3 + \frac{1 \cdot 4}{6!} z^6 + \frac{1 \cdot 4 \cdot 7}{9!} z^9 + \dots$$

$$= \sum_0^{\infty} 3^k \left(\frac{1}{3}\right)_k \frac{z^{3k}}{(3k)!}$$

$$g(z) = z + \frac{2}{4!} z^4 + \frac{2 \cdot 5}{7!} z^7 + \frac{2 \cdot 5 \cdot 8}{10!} z^{10} + \dots$$

$$= \sum_0^{\infty} 3^k \left(\frac{2}{3}\right)_k \frac{z^{3k+1}}{(3k+1)!}$$

$$\left(\alpha + \frac{1}{3}\right)_0 = 1$$

$$3^k \left(\alpha + \frac{1}{3}\right)_k = (3\alpha + 1)(3\alpha + 4) \dots (3\alpha + 3k - 2)$$

(α arbitrary; $k = 1, 2, 3, \dots$)

(See 6.1.22.)

10.4.4

$$c_1 = \text{Ai}(0) = \text{Bi}(0)/\sqrt{3} = 3^{-2/3}/\Gamma(2/3)$$

$$= .35502 \ 80538 \ 87817$$

10.4.5

$$c_2 = -\text{Ai}'(0) = \text{Bi}'(0)/\sqrt{3} = 3^{-1/3}/\Gamma(1/3)$$

$$= .25881 \ 94037 \ 92807$$

Relations Between Solutions

$$10.4.6 \quad \text{Bi}(z) = e^{\pi i/6} \text{Ai}(ze^{2\pi i/3}) + e^{-\pi i/6} \text{Ai}(ze^{-2\pi i/3})$$

10.4.7

$$\text{Ai}(z) + e^{2\pi i/3} \text{Ai}(ze^{2\pi i/3}) + e^{-2\pi i/3} \text{Ai}(ze^{-2\pi i/3}) = 0$$

10.4.8

$$\text{Bi}(z) + e^{2\pi i/3} \text{Bi}(ze^{2\pi i/3}) + e^{-2\pi i/3} \text{Bi}(ze^{-2\pi i/3}) = 0$$

$$10.4.9 \quad \text{Ai}(ze^{\pm 2\pi i/3}) = \frac{1}{2} e^{\pm \pi i/3} [\text{Ai}(z) \mp i \text{Bi}(z)]$$

Wronskians

$$10.4.10 \quad W\{\text{Ai}(z), \text{Bi}(z)\} = \pi^{-1}$$

$$10.4.11 \quad W\{\text{Ai}(z), \text{Ai}(ze^{2\pi i/3})\} = \frac{1}{2} \pi^{-1} e^{-\pi i/6}$$

$$10.4.12 \quad W\{\text{Ai}(z), \text{Ai}(ze^{-2\pi i/3})\} = \frac{1}{2} \pi^{-1} e^{\pi i/6}$$

$$10.4.13 \quad W\{\text{Ai}(ze^{2\pi i/3}), \text{Ai}(ze^{-2\pi i/3})\} = \frac{1}{2} i \pi^{-1}$$

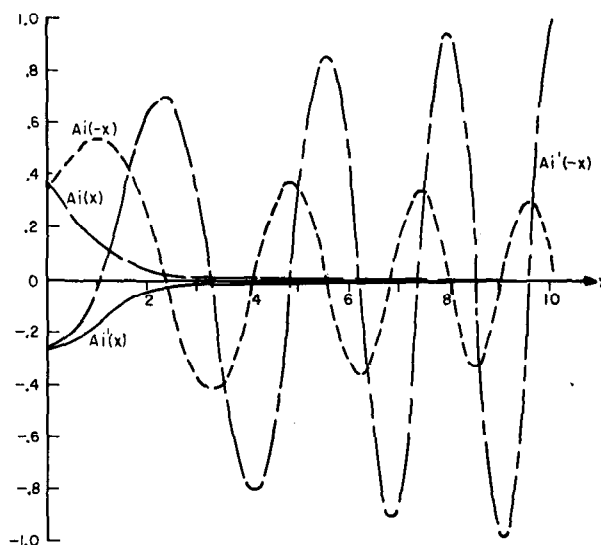


FIGURE 10.6. $\text{Ai}(\pm x), \text{Ai}'(\pm x)$.

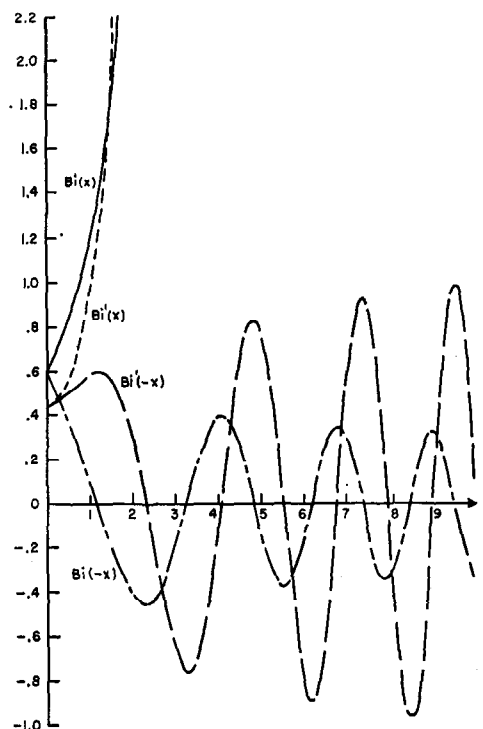


FIGURE 10.7. $\text{Bi}(\pm x), \text{Bi}'(\pm x)$.

Representations in Terms of Bessel Functions

$$\zeta = \frac{2}{3} z^{3/2}$$

10.4.14

$$\text{Ai}(z) = \frac{1}{3}\sqrt{z}[I_{-1/3}(\zeta) - I_{1/3}(\zeta)] = \pi^{-1}\sqrt{z/3}K_{1/3}(\zeta)$$

10.4.15

$$\begin{aligned}\text{Ai}(-z) &= \frac{1}{3}\sqrt{z}[J_{1/3}(\zeta) + J_{-1/3}(\zeta)] \\ &= \frac{1}{3}\sqrt{z/3}[e^{\pi i/6}H_{1/3}^{(1)}(\zeta) + e^{-\pi i/6}H_{1/3}^{(2)}(\zeta)]\end{aligned}$$

10.4.16

$$* -\text{Ai}'(z) = \frac{1}{3}z[I_{-2/3}(\zeta) - I_{2/3}(\zeta)] = \pi^{-1}(z/\sqrt{3})K_{2/3}(\zeta)$$

10.4.17

$$\begin{aligned}\text{Ai}'(-z) &= -\frac{1}{3}z[J_{-2/3}(\zeta) - J_{2/3}(\zeta)] \\ &= \frac{1}{3}i(z/\sqrt{3})[e^{-\pi i/6}H_{2/3}^{(1)}(\zeta) + e^{\pi i/6}H_{2/3}^{(2)}(\zeta)]\end{aligned}$$

$$10.4.18 \quad \text{Bi}(z) = \sqrt{z/3}[I_{-1/3}(\zeta) + I_{1/3}(\zeta)]$$

10.4.19

$$\begin{aligned}\text{Bi}(-z) &= \sqrt{z/3}[J_{-1/3}(\zeta) - J_{1/3}(\zeta)] \\ &= \frac{1}{3}i\sqrt{z/3}[e^{\pi i/6}H_{1/3}^{(1)}(\zeta) - e^{-\pi i/6}H_{1/3}^{(2)}(\zeta)]\end{aligned}$$

$$10.4.20 \quad \text{Bi}'(z) = (z/\sqrt{3})[I_{-2/3}(\zeta) + I_{2/3}(\zeta)]$$

10.4.21

$$\begin{aligned}\text{Bi}'(-z) &= (z/\sqrt{3})[J_{-2/3}(\zeta) + J_{2/3}(\zeta)] \\ &= \frac{1}{3}i(z/\sqrt{3})[e^{-\pi i/6}H_{2/3}^{(1)}(\zeta) - e^{\pi i/6}H_{2/3}^{(2)}(\zeta)]\end{aligned}$$

Representations of Bessel Functions in Terms of Airy Functions

$$z = \left(\frac{3}{2}\zeta\right)^{2/3}$$

$$10.4.22 \quad J_{\pm 1/3}(\zeta) = \frac{1}{3}\sqrt{3/z}[\sqrt{3}\text{Ai}(-z) \mp \text{Bi}(-z)]$$

$$*10.4.23 \quad H_{\pm 1/3}^{(1)}(\zeta) = e^{\mp \pi i/6}\sqrt{3/z}[\text{Ai}(-z) - i\text{Bi}(-z)]$$

$$10.4.24 \quad H_{\pm 1/3}^{(2)}(\zeta) = e^{\pm \pi i/6}\sqrt{3/z}[\text{Ai}(-z) + i\text{Bi}(-z)]$$

$$10.4.25 \quad I_{\pm 1/3}(\zeta) = \frac{1}{3}\sqrt{3/z}[\mp \sqrt{3}\text{Ai}(z) + \text{Bi}(z)]$$

$$10.4.26 \quad K_{\pm 1/3}(\zeta) = \pi\sqrt{3/z}\text{Ai}(z)$$

$$10.4.27 \quad J_{\pm 2/3}(\zeta) = (\sqrt{3}/2z)[\pm \sqrt{3}\text{Ai}'(-z) + \text{Bi}'(-z)]$$

10.4.28

$$\begin{aligned}H_{2/3}^{(1)}(\zeta) &= e^{-2\pi i/3}H_{-2/3}^{(1)}(\zeta) \\ &= e^{\pi i/6}(\sqrt{3}/z)[\text{Ai}'(-z) - i\text{Bi}'(-z)]\end{aligned}$$

10.4.29

$$\begin{aligned}H_{2/3}^{(2)}(\zeta) &= e^{2\pi i/3}H_{-2/3}^{(2)}(\zeta) \\ &= e^{-\pi i/6}(\sqrt{3}/z)[\text{Ai}'(-z) + i\text{Bi}'(-z)]\end{aligned}$$

$$10.4.30 \quad I_{\pm 2/3}(\zeta) = (\sqrt{3}/2z)[\pm \sqrt{3}\text{Ai}'(z) + \text{Bi}'(z)]$$

$$10.4.31 \quad K_{\pm 2/3}(\zeta) = -\pi(\sqrt{3}/z)\text{Ai}'(z)$$

Integral Representations

10.4.32

$$(3a)^{-1/3}\pi \text{Ai}[\pm (3a)^{-1/3}x] = \int_0^\infty \cos(at^3 \pm xt)dt$$

10.4.33

$$\begin{aligned}(3a)^{-1/3}\pi \text{Bi}[\pm (3a)^{-1/3}x] \\ = \int_0^\infty [\exp(-at^3 \pm xt) + \sin(at^3 \pm xt)]dt\end{aligned}$$

$$\text{The Integrals } \int_0^z \text{Ai}(\pm t)dt, \int_0^z \text{Bi}(\pm t)dt$$

$$\zeta = \frac{2}{3}z^{3/2}$$

$$10.4.34 \quad \int_0^z \text{Ai}(t)dt = \frac{1}{3} \int_0^\zeta [I_{-1/3}(t) - I_{1/3}(t)]dt$$

$$10.4.35 \quad \int_0^z \text{Ai}(-t)dt = \frac{1}{3} \int_0^\zeta [J_{-1/3}(t) + J_{1/3}(t)]dt$$

$$10.4.36 \quad \int_0^z \text{Bi}(t)dt = \frac{1}{\sqrt{3}} \int_0^\zeta [I_{-1/3}(t) + I_{1/3}(t)]dt$$

$$10.4.37 \quad \int_0^z \text{Bi}(-t)dt = \frac{1}{\sqrt{3}} \int_0^\zeta [J_{-1/3}(t) - J_{1/3}(t)]dt$$

$$\text{Ascending Series for } \int_0^z \text{Ai}(\pm t)dt, \int_0^z \text{Bi}(\pm t)dt$$

$$10.4.38 \quad \int_0^z \text{Ai}(t)dt = c_1 F(z) - c_2 G(z)$$

(See 10.4.2.)

$$10.4.39 \quad \int_0^z \text{Ai}(-t)dt = -c_1 F(-z) + c_2 G(-z)$$

$$10.4.40 \quad \int_0^z \text{Bi}(t)dt = \sqrt{3}[c_1 F(z) + c_2 G(z)]$$

(See 10.4.3.)

10.4.41

$$\int_0^z \text{Bi}(-t)dt = -\sqrt{3}[c_1 F(-z) + c_2 G(-z)]$$

$$F(z) = z + \frac{1}{4!}z^4 + \frac{1 \cdot 4}{7!}z^7 + \frac{1 \cdot 4 \cdot 7}{10!}z^{10} + \dots$$

$$= \sum_0^\infty 3^k \left(\frac{1}{3}\right)_k \frac{z^{3k+1}}{(3k+1)!}$$

$$G(z) = \frac{1}{2!}z^2 + \frac{2}{5!}z^5 + \frac{2 \cdot 5}{8!}z^8 + \frac{2 \cdot 5 \cdot 8}{11!}z^{11} + \dots$$

$$= \sum_0^\infty 3^k \left(\frac{2}{3}\right)_k \frac{z^{3k+2}}{(3k+2)!}$$

The constants c_1, c_2 are given in 10.4.4, 10.4.5.

*See page II.

The Functions $\text{Gi}(z)$, $\text{Hi}(z)$

10.4.42

$$\begin{aligned}\text{Gi}(z) &= \pi^{-1} \int_0^\infty \sin\left(\frac{1}{3}t^3 + zt\right) dt \\ &= \frac{1}{3} \text{Bi}(z) + \int_0^z [\text{Ai}(z) \text{Bi}(t) - \text{Ai}(t) \text{Bi}(z)] dt\end{aligned}$$

10.4.43

$$\text{Gi}'(z) = \frac{1}{3} \text{Bi}'(z) + \int_0^z [\text{Ai}'(z) \text{Bi}(t) - \text{Ai}(t) \text{Bi}'(z)] dt$$

10.4.44

$$\begin{aligned}\text{Hi}(z) &= \pi^{-1} \int_0^\infty \exp\left(-\frac{1}{3}t^3 + zt\right) dt \\ &= \frac{2}{3} \text{Bi}(z) + \int_0^z [\text{Ai}(t) \text{Bi}(z) - \text{Ai}(z) \text{Bi}(t)] dt\end{aligned}$$

10.4.45

$$\text{Hi}'(z) = \frac{2}{3} \text{Bi}'(z) + \int_0^z [\text{Ai}(t) \text{Bi}'(z) - \text{Ai}'(z) \text{Bi}(t)] dt$$

$$10.4.46 \quad \text{Gi}(z) + \text{Hi}(z) = \text{Bi}(z)$$

Representations of $\int_0^z \text{Ai}(\pm t) dt$, $\int_0^z \text{Bi}(\pm t) dt$
by $\text{Gi}(\pm z)$, $\text{Hi}(\pm z)$

10.4.47

$$\int_0^z \text{Ai}(t) dt = \frac{1}{3} + \pi [\text{Ai}'(z) \text{Gi}(z) - \text{Ai}(z) \text{Gi}'(z)]$$

10.4.48

$$= -\frac{2}{3} - \pi [\text{Ai}'(z) \text{Hi}(z) - \text{Ai}(z) \text{Hi}'(z)]$$

10.4.49

$$\begin{aligned}\int_0^z \text{Ai}(-t) dt &= -\frac{1}{3} - \pi [\text{Ai}'(-z) \text{Gi}(-z) \\ &\quad - \text{Ai}(-z) \text{Gi}'(-z)]\end{aligned}$$

10.4.50

$$\begin{aligned}&= \frac{2}{3} + \pi [\text{Ai}'(-z) \text{Hi}(-z) \\ &\quad - \text{Ai}(-z) \text{Hi}'(-z)]\end{aligned}$$

10.4.51

$$\int_0^z \text{Bi}(t) dt = \pi [\text{Bi}'(z) \text{Gi}(z) - \text{Bi}(z) \text{Gi}'(z)]$$

$$10.4.52 \quad = -\pi [\text{Bi}'(z) \text{Hi}(z) - \text{Bi}(z) \text{Hi}'(z)]$$

10.4.53

$$\begin{aligned}\int_0^z \text{Bi}(-t) dt &= -\pi [\text{Bi}'(-z) \text{Gi}(-z) \\ &\quad - \text{Bi}(-z) \text{Gi}'(-z)]\end{aligned}$$

$$\begin{aligned}10.4.54 \quad &= \pi [\text{Bi}'(-z) \text{Hi}(-z) \\ &\quad - \text{Bi}(-z) \text{Hi}'(-z)]\end{aligned}$$

Differential Equations for $\text{Gi}(z)$, $\text{Hi}(z)$

10.4.55

$$w'' - zw = -\pi^{-1}$$

$$w(0) = \frac{1}{3} \text{Bi}(0) = \frac{1}{\sqrt{3}} \text{Ai}(0) = .20497 \ 55424 \ 78$$

$$w'(0) = \frac{1}{3} \text{Bi}'(0) = -\frac{1}{\sqrt{3}} \text{Ai}'(0) = .14942 \ 94524 \ 49$$

$$w(z) = \text{Gi}(z)$$

10.4.56

$$w'' - zw = \pi^{-1}$$

$$w(0) = \frac{2}{3} \text{Bi}(0) = \frac{2}{\sqrt{3}} \text{Ai}(0) = .40995 \ 10849 \ 56$$

$$w'(0) = \frac{2}{3} \text{Bi}'(0) = -\frac{2}{\sqrt{3}} \text{Ai}'(0) = .29885 \ 89048 \ 98$$

$$w(z) = \text{Hi}(z)$$

Differential Equation for Products of Airy Functions

10.4.57

$$w''' - 4zw' - 2w = 0$$

Linearly independent solutions are $\text{Ai}^2(z)$, $\text{Ai}(z) \text{Bi}(z)$, $\text{Bi}^2(z)$.

Wronskian for Products of Airy Functions

$$10.4.58 \quad W\{\text{Ai}^2(z), \text{Ai}(z) \text{Bi}(z), \text{Bi}^2(z)\} = 2\pi^{-3}$$

Asymptotic Expansions for $|z|$ Large

$$c_0 = 1, c_k = \frac{\Gamma(3k + \frac{1}{2})}{54^k k! \Gamma(k + \frac{1}{2})} = \frac{(2k+1)(2k+3) \dots (6k-1)}{216^k k!},$$

$$d_0 = 1, d_k = -\frac{6k+1}{6k-1} c_k \quad (k=1, 2, 3, \dots)$$

$$\zeta = \frac{2}{3} z^{3/2}$$

10.4.59

$$\text{Ai}(z) \sim \frac{1}{2} \pi^{-1/2} z^{-1/4} e^{-\zeta} \sum_0^\infty (-1)^k c_k \zeta^{-k} \quad (|\arg z| < \pi)$$

10.4.60

$$\begin{aligned}\text{Ai}(-z) &\sim \pi^{-1/2} z^{-1/4} \left[\sin\left(\zeta + \frac{\pi}{4}\right) \sum_0^\infty (-1)^k c_{2k} \zeta^{-2k} \right. \\ &\quad \left. - \cos\left(\zeta + \frac{\pi}{4}\right) \sum_0^\infty (-1)^k c_{2k+1} \zeta^{-2k-1} \right]\end{aligned}$$

$$(|\arg z| < \frac{3}{2}\pi)$$

10.4.61

$$\begin{aligned}\text{Ai}'(z) &\sim -\frac{1}{2} \pi^{-1/2} z^{1/4} e^{-\zeta} \sum_0^\infty (-1)^k d_k \zeta^{-k} \\ &\quad (|\arg z| < \pi)\end{aligned}$$

10.4.62

$$\begin{aligned} \text{Ai}'(-z) \sim -\pi^{-1/2} z^{\frac{1}{2}} \left[\cos\left(z + \frac{\pi}{4}\right) \sum_{k=0}^{\infty} (-1)^k d_{2k} z^{-2k} \right. \\ \left. + \sin\left(z + \frac{\pi}{4}\right) \sum_{k=0}^{\infty} (-1)^k d_{2k+1} z^{-2k-1} \right] \\ (|\arg z| < \frac{3}{2}\pi) \end{aligned}$$

10.4.63

$$\text{Bi}(z) \sim \pi^{-1/2} z^{-1/2} e^z \sum_{k=0}^{\infty} c_k z^{-k} \quad (|\arg z| < \frac{1}{2}\pi)$$

10.4.64

$$\begin{aligned} \text{Bi}(-z) \sim \pi^{-1/2} z^{-1/2} \left[\cos\left(z + \frac{\pi}{4}\right) \sum_{k=0}^{\infty} (-1)^k c_{2k} z^{-2k} \right. \\ \left. + \sin\left(z + \frac{\pi}{4}\right) \sum_{k=0}^{\infty} (-1)^k c_{2k+1} z^{-2k-1} \right] \\ (|\arg z| < \frac{3}{2}\pi) \end{aligned}$$

10.4.65

$$\text{Bi}(ze^{\pm\pi i/3})$$

$$\begin{aligned} \sim \sqrt{2/\pi} e^{\pm\pi i/6} z^{-1/2} \left[\sin\left(z + \frac{\pi}{4} \mp \frac{i}{2} \ln 2\right) \sum_{k=0}^{\infty} (-1)^k c_{2k} z^{-2k} \right. \\ \left. - \cos\left(z + \frac{\pi}{4} \mp \frac{i}{2} \ln 2\right) \sum_{k=0}^{\infty} (-1)^k c_{2k+1} z^{-2k-1} \right] \\ (|\arg z| < \frac{3}{2}\pi) \end{aligned}$$

10.4.66

$$* \text{Bi}'(z) \sim \pi^{-1/2} z^{\frac{1}{2}} e^z \sum_{k=0}^{\infty} d_k z^{-k} \quad (|\arg z| < \frac{1}{2}\pi)$$

10.4.67

$$\begin{aligned} \text{Bi}'(-z) \sim \pi^{-1/2} z^{\frac{1}{2}} \left[\sin\left(z + \frac{\pi}{4}\right) \sum_{k=0}^{\infty} (-1)^k d_{2k} z^{-2k} \right. \\ \left. - \cos\left(z + \frac{\pi}{4}\right) \sum_{k=0}^{\infty} (-1)^k d_{2k+1} z^{-2k-1} \right] \\ (|\arg z| < \frac{3}{2}\pi) \end{aligned}$$

10.4.68

$$\text{Bi}'(ze^{\pm\pi i/3})$$

$$\begin{aligned} \sim \sqrt{2/\pi} e^{\mp\pi i/6} z^{\frac{1}{2}} \left[\cos\left(z + \frac{\pi}{4} \mp \frac{i}{2} \ln 2\right) \sum_{k=0}^{\infty} (-1)^k d_{2k} z^{-2k} \right. \\ \left. + \sin\left(z + \frac{\pi}{4} \mp \frac{i}{2} \ln 2\right) \sum_{k=0}^{\infty} (-1)^k d_{2k+1} z^{-2k-1} \right] \\ (|\arg z| < \frac{3}{2}\pi) \end{aligned}$$

Modulus and Phase

10.4.69

$$\text{Ai}(-x) = M(x) \cos \theta(x), \text{Bi}(-x) = M(x) \sin \theta(x)$$

$$\begin{aligned} M(x) = \sqrt{[\text{Ai}^2(-x) + \text{Bi}^2(-x)]}, \\ \theta(x) = \arctan [\text{Bi}(-x)/\text{Ai}(-x)] \end{aligned}$$

10.4.70

$$\text{Ai}'(-x) = N(x) \cos \phi(x), \text{Bi}'(-x) = N(x) \sin \phi(x)$$

$$\begin{aligned} N(x) = \sqrt{[\text{Ai}'^2(-x) + \text{Bi}'^2(-x)]}, \\ \phi(x) = \arctan [\text{Bi}'(-x)/\text{Ai}'(-x)] \end{aligned}$$

Differential Equations for Modulus and Phase

Primes denote differentiation with respect to x

$$10.4.71 \quad M^2 \theta' = -\pi^{-1}, N^2 \phi' = -\pi^{-1} x$$

$$10.4.72 \quad N^2 = M'^2 + M^2 \theta'^2 = M'^2 + \pi^{-2} M^{-2} \quad *$$

$$10.4.73 \quad NN' = -xMM'$$

10.4.74

$$\begin{aligned} \tan(\phi - \theta) = M\theta'/M' = -(\pi MM')^{-1}, \\ MN \sin(\phi - \theta) = \pi^{-1} \end{aligned}$$

$$10.4.75 \quad M'' + xM - \pi^{-2} M^{-3} = 0$$

$$10.4.76 \quad (M^2)''' + 4x(M^2)' - 2M^2 = 0$$

$$10.4.77 \quad \theta'^2 + \frac{1}{2}(\theta''/\theta') - \frac{3}{4}(\theta'/\theta')^2 = x$$

 Asymptotic Expansions of Modulus and Phase for Large x

$$10.4.78 \quad M^2(x) \sim \frac{1}{\pi} x^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{12^k k!} 2^{3k} \left(\frac{1}{2}\right)_{3k} (2x)^{-3k}$$

10.4.79

$$\begin{aligned} \theta(x) \sim \frac{1}{4}\pi - \frac{2}{3}x^{3/2} \left[1 - \frac{5}{4}(2x)^{-3} + \frac{1105}{96}(2x)^{-6} \right. \\ \left. - \frac{82825}{128}(2x)^{-9} + \frac{1282031525}{14336}(2x)^{-12} - \dots \right] \end{aligned}$$

10.4.80

$$N^2(x) \sim \frac{1}{\pi} x^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{12^k k!} \frac{6k+1}{6k-1} 2^{3k} \left(\frac{1}{2}\right)_{3k} (2x)^{-3k}$$

10.4.81

$$\begin{aligned} \phi(x) \sim \frac{3}{4}\pi - \frac{2}{3}x^{3/2} \left[1 + \frac{7}{4}(2x)^{-3} - \frac{1463}{96}(2x)^{-6} \right. \\ \left. + \frac{495271}{640}(2x)^{-9} - \frac{206530429}{2048}(2x)^{-12} + \dots \right] \end{aligned}$$

 Asymptotic Forms of $\int_0^x \text{Ai}(\pm t) dt, \int_0^x \text{Bi}(\pm t) dt$ for Large x

$$10.4.82 \quad \int_0^x \text{Ai}(t) dt \sim \frac{1}{3} - \frac{1}{2}\pi^{-1/2} x^{-3/4} \exp\left(-\frac{2}{3}x^{3/2}\right)$$

10.4.83

$$\int_0^x \text{Ai}(-t) dt \sim \frac{2}{3} - \pi^{-1/2} x^{-3/4} \cos\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right)$$

*See page II.

$$10.4.84 \quad \int_0^z \text{Bi}(t) dt \sim \pi^{-1/2} x^{-3/4} \exp\left(\frac{2}{3} x^{3/2}\right)$$

$$10.4.85 \quad \int_0^z \text{Bi}(-t) dt \sim \pi^{-1/2} x^{-3/4} \sin\left(\frac{2}{3} x^{3/2} + \frac{\pi}{4}\right)$$

Asymptotic Forms of $\text{Gi}(\pm x)$, $\text{Gi}'(\pm x)$, $\text{Hi}(\pm x)$, $\text{Hi}'(\pm x)$
for Large x

$$10.4.86 \quad \text{Gi}(x) \sim \pi^{-1} x^{-1}$$

$$10.4.87 \quad \text{Gi}(-x) \sim \pi^{-1/2} x^{-1/4} \cos\left(\frac{2}{3} x^{3/2} + \frac{\pi}{4}\right)$$

$$10.4.88 \quad \text{Gi}'(x) \sim \frac{7}{96} \pi^{-1} x^{-2}$$

$$10.4.89 \quad \text{Gi}'(-x) \sim \pi^{-1/2} x^{1/4} \sin\left(\frac{2}{3} x^{3/2} + \frac{\pi}{4}\right)$$

$$10.4.90 \quad \text{Hi}(x) \sim \pi^{-1/2} x^{-1/4} \exp\left(\frac{2}{3} x^{3/2}\right)$$

$$10.4.91 \quad \text{Hi}(-x) \sim \pi^{-1} x^{-1}$$

$$10.4.92 \quad \text{Hi}'(x) \sim \pi^{-1/2} x^{1/4} \exp\left(\frac{2}{3} x^{3/2}\right)$$

$$10.4.93 \quad \text{Hi}'(-x) \sim -\frac{3}{2} \pi^{-1} x^{-2}$$

Zeros and Their Asymptotic Expansions

$\text{Ai}(z)$, $\text{Ai}'(z)$ have zeros on the negative real axis only. $\text{Bi}(z)$, $\text{Bi}'(z)$ have zeros on the negative real axis and in the sector $\frac{1}{2}\pi < |\arg z| < \frac{3}{2}\pi$. a_s , a'_s ; b_s , b'_s s -th (real) negative zero of $\text{Ai}(z)$, $\text{Ai}'(z)$; $\text{Bi}(z)$, $\text{Bi}'(z)$, respectively. β_s , β'_s ; $\bar{\beta}_s$, $\bar{\beta}'_s$ s -th complex zero of $\text{Bi}(z)$, $\text{Bi}'(z)$ in the sectors $\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$, $-\frac{1}{2}\pi < \arg z < -\frac{3}{2}\pi$, respectively.

$$10.4.94 \quad a_s = -f[3\pi(4s-1)/8]$$

$$10.4.95 \quad a'_s = -g[3\pi(4s-3)/8]$$

$$10.4.96 \quad \text{Ai}'(a_s) = (-1)^{s-1} f_1[3\pi(4s-1)/8]$$

$$10.4.97 \quad \text{Ai}(a'_s) = (-1)^{s-1} g_1[3\pi(4s-3)/8]$$

$$10.4.98 \quad b_s = -f[3\pi(4s-3)/8]$$

$$10.4.99 \quad b'_s = -g[3\pi(4s-1)/8]$$

$$10.4.100 \quad \text{Bi}'(b_s) = (-1)^{s-1} f_1[3\pi(4s-3)/8]$$

$$10.4.101 \quad \text{Bi}(b'_s) = (-1)^s g_1[3\pi(4s-1)/8]$$

$$10.4.102 \quad \beta_s = e^{\pi i/3} f \left[\frac{3\pi}{8} (4s-1) + \frac{3i}{4} \ln 2 \right]$$

$$10.4.103 \quad \beta'_s = e^{\pi i/3} g \left[\frac{3\pi}{8} (4s-3) + \frac{3i}{4} \ln 2 \right]$$

10.4.104

$$\text{Bi}'(\beta_s) = (-1)^s \sqrt{2} e^{-\pi i/6} f_1 \left[\frac{3\pi}{8} (4s-1) + \frac{3i}{4} \ln 2 \right]$$

10.4.105

$$\text{Bi}(\beta'_s) = (-1)^{s-1} \sqrt{2} e^{\pi i/6} g_1 \left[\frac{3\pi}{8} (4s-3) + \frac{3i}{4} \ln 2 \right]$$

$|z|$ sufficiently large

$$f(z) \sim z^{2/3} \left(1 + \frac{5}{48} z^{-2} - \frac{5}{36} z^{-4} + \frac{77125}{82944} z^{-6} - \frac{108056875}{6967296} z^{-8} + \frac{162375596875}{334430208} z^{-10} - \dots \right)$$

$$g(z) \sim z^{2/3} \left(1 - \frac{7}{48} z^{-2} + \frac{35}{288} z^{-4} - \frac{181223}{207360} z^{-6} + \frac{18683371}{1244160} z^{-8} - \frac{91145884361}{191102976} z^{-10} + \dots \right)$$

$$f_1(z) \sim \pi^{-1/2} z^{1/6} \left(1 + \frac{5}{48} z^{-2} - \frac{1525}{4608} z^{-4} + \frac{2397875}{663552} z^{-6} - \dots \right)$$

$$g_1(z) \sim \pi^{-1/2} z^{-1/6} \left(1 - \frac{7}{96} z^{-2} + \frac{1673}{6144} z^{-4} - \frac{84394709}{26542080} z^{-6} + \dots \right) *$$

Formal and Asymptotic Solutions of Ordinary Differential Equations of Second Order With Turning Points

An equation

$$10.4.106 \quad W'' + a(z, \lambda)W' + b(z, \lambda)W = 0$$

in which λ is a real or complex parameter and, for fixed λ , $a(z, \lambda)$ is analytic in z and $b(z, \lambda)$ is continuous in z in some region of the z -plane, may be reduced by the transformation

$$10.4.107 \quad W(z) = w(z) \exp \left(-\frac{1}{2} \int^z a(t, \lambda) dt \right)$$

to the equation

10.4.108

$$w'' + \varphi(z, \lambda)w = 0$$

$$\varphi(z, \lambda) = b(z, \lambda) - \frac{1}{4} a^2(z, \lambda) - \frac{1}{2} \frac{d}{dz} a(z, \lambda).$$

*See page 11.

If $\varphi(z, \lambda)$ can be written in the form

$$10.4.109 \quad \varphi(z, \lambda) = \lambda^2 p(z) + q(z, \lambda)$$

where $q(z, \lambda)$ is bounded in a region R of the z -plane, then the zeros of $p(z)$ in R are said to be turning points of the equation 10.4.108.

The Special Case $w'' + [\lambda^2 z + q(z, \lambda)]w = 0$

Let $\lambda = |\lambda|e^{i\omega}$ vary over a sectorial domain S : $|\lambda| \geq \lambda_0 (> 0)$, $\omega_1 \leq \omega \leq \omega_2$, and suppose that $q(z, \lambda)$ is continuous in z for $|z| < r$ and λ in S , and $q(z, \lambda) \sim \sum_0^\infty q_n(z)\lambda^{-n}$ as $\lambda \rightarrow \infty$ in S .

Formal Series Solution

10.4.110

$$w(z) = u(z) \sum_0^\infty \varphi_n(z)\lambda^{-n} + \lambda^{-1}u'(z) \sum_0^\infty \psi_n(z)\lambda^{-n}$$

$$u'' + \lambda^2 zu = 0$$

$$\varphi_0(z) = c_0, \quad \psi_0(z) = z^{-1}c_1, \quad c_0, c_1 \text{ constants}$$

$$\varphi_{n+1}(z) = -\frac{1}{2}\psi_n'(z) - \frac{1}{2}\int_0^z \sum_0^n q_{n-k}(t)\psi_k(t)dt$$

$$\psi_n(z) = \frac{1}{2}z^{-1}\int_0^z t^{-1}\left[\varphi_n''(t) + \sum_0^n q_{n-k}(t)\varphi_k(t)\right]dt$$

$$(n=0, 1, 2, \dots)$$

Uniform Asymptotic Expansions of Solutions

For z real, i.e. for the equation

$$10.4.111 \quad y'' + [\lambda^2 x + q(x, \lambda)]y = 0$$

where x varies in a bounded interval $a \leq x \leq b$ that includes the origin and where, for each fixed λ in S , $q(x, \lambda)$ is continuous in x for $a \leq x \leq b$, the following asymptotic representations hold.

(i) If λ is real and positive, there are solutions $y_0(x)$, $y_1(x)$ such that, uniformly in x on $a \leq x \leq 0$,

10.4.112

$$y_0(x) = \text{Ai}(-\lambda^{2/3}x)[1 + O(\lambda^{-1})] \quad (\lambda \rightarrow \infty)$$

$$y_1(x) = \text{Bi}(-\lambda^{2/3}x)[1 + O(\lambda^{-1})]$$

and, uniformly in x on $0 \leq x \leq b$

10.4.113

$$y_0(x) = \text{Ai}(-\lambda^{2/3}x)[1 + O(\lambda^{-1})] + \text{Bi}(-\lambda^{2/3}x)O(\lambda^{-1}),$$

$$y_1(x) = \text{Bi}(-\lambda^{2/3}x)[1 + O(\lambda^{-1})] + \text{Ai}(-\lambda^{2/3}x)O(\lambda^{-1})$$

($\lambda \rightarrow \infty$)

(ii) If $\mathcal{R}\lambda \geq 0$, $\mathcal{I}\lambda \neq 0$, there are solutions $y_0(x)$, $y_1(x)$ such that, uniformly in x on $a \leq x \leq b$,

10.4.114

$$y_0(x) = \text{Ai}(-\lambda^{2/3}x)[1 + O(\lambda^{-1})]$$

$$y_1(x) = \text{Bi}(-\lambda^{2/3}x)[1 + O(\lambda^{-1})] \quad (|\lambda| \rightarrow \infty)$$

For further representations and details, we refer to [10.4].

When z is complex (bounded or unbounded), conditions under which the formal series 10.4.110 yields a uniform asymptotic expansion of a solution are given in [10.12] if $q(z, \lambda)$ is independent of λ and $|\lambda| \rightarrow \infty$ with fixed ω , and in [10.14] if λ lies in any region of the complex plane. Further references are [10.2; 10.9; 10.10].

The General Case $w'' + [\lambda^2 p(z) + q(z, \lambda)]w = 0$

Let $\lambda = |\lambda|e^{i\omega}$ where $|\lambda| \geq \lambda_0 (> 0)$ and $-\pi \leq \omega \leq \pi$; suppose that $p(z)$ is analytic in a region R and has a zero $z = z_0$ in R , and that, for fixed λ , $q(z, \lambda)$ is analytic in z for z in R . The transformation $\xi = \xi(z)$, $v = [p(z)/\xi]^{1/4}w(z)$, where ξ is defined as the (unique) solution of the equation

$$10.4.115 \quad \xi \left(\frac{d\xi}{dz} \right)^2 = p(z),$$

yields the special case

$$10.4.116 \quad \frac{d^2 v}{d\xi^2} + [\lambda^2 \xi + f(\xi, \lambda)]v = 0, \quad *$$

$$f(\xi, \lambda) = \left(\frac{d\xi}{dz} \right)^{-2} q(z, \lambda) - \left(\frac{d\xi}{dz} \right)^{-1} \frac{d^2}{d\xi^2} \left[\left(\frac{d\xi}{dz} \right)^{\frac{1}{2}} \right].$$

Example:

Consider the equation

$$10.4.117 \quad y'' + [\lambda^2 - (\lambda^2 - \frac{1}{4})x^{-2}]y = 0$$

for which the points $x=0, \infty$ are singular points and $x=1$ is a turning point. It has the functions $x^{\frac{1}{2}}J_{\lambda}(\lambda x)$, $x^{\frac{1}{2}}Y_{\lambda}(\lambda x)$ as particular solutions (see 9.1.49).

The equation 10.4.115 becomes

$$\xi \left(\frac{d\xi}{dx} \right)^2 = \frac{x^2 - 1}{x^2}$$

whence

$$\frac{2}{3}(-\xi)^{3/2} = -\sqrt{1-x^2} + \ln x^{-1}(1 + \sqrt{1-x^2}) \quad (0 < x \leq 1)$$

$$\frac{2}{3}\xi^{3/2} = \sqrt{x^2 - 1} - \arccos x^{-1} \quad (1 \leq x < \infty).$$

Thus

$$10.4.118 \quad v(\xi) = \left(\frac{x^2 - 1}{x^2 \xi} \right)^{1/4} y(x)$$

satisfies the equation

$$10.4.119 \quad \frac{d^2 v}{d\xi^2} + \left[\lambda^2 \xi - \frac{5}{16\xi^2} + \frac{\xi^2}{4} \frac{x^2(x^2+4)}{(x^2-1)^3} \right] v = 0$$

which is of the form 10.4.111 with x replaced by ξ and $q(\xi, \lambda)$ independent of λ .

Suppose $\Re \lambda \geq 0$, $\Im \lambda \neq 0$. By the first equation of 10.4.114 there is a solution $v_0(\xi)$ of 10.4.119, i.e., a solution $y_0(x)$ of 10.4.117 for which the representation

10.4.120

$$v_0(\xi) = \left(\frac{x^2-1}{x^2\xi} \right)^{1/4} y_0(x) = \text{Ai}(-\lambda^{2/3}\xi)[1 + O(\lambda^{-1})]$$

holds uniformly in x on $0 < x < \infty$ as $|\lambda| \rightarrow \infty$.

To identify $y_0(x)$ in terms of $x^{\frac{1}{2}}J_{\lambda}(\lambda x)$, $x^{\frac{1}{2}}Y_{\lambda}(\lambda x)$, restrict x to $0 < x \leq b < 1$ so that by 10.4.118 ξ is negative, and replace the Airy function by its asymptotic representation 10.4.59. This yields

10.4.121

$$\begin{aligned} y_0(x) &= \left(\frac{x^2-1}{x^2\xi} \right)^{-1/4} \frac{1}{2} \pi^{-1/2} \lambda^{-1/6} (-\xi)^{1/4} \exp \left(\frac{2}{3} \lambda (-\xi)^{3/2} \right) \\ &\quad [1 + O(\lambda^{-1})] \\ &= \frac{1}{2} \pi^{-1/2} \lambda^{-1/6} \left(\frac{1-x^2}{x^2} \right)^{-1/4} \exp \left(\frac{2}{3} \lambda (-\xi)^{3/2} \right) \\ &\quad [1 + O(\lambda^{-1})] \end{aligned}$$

Let now λ be fixed and $x \rightarrow 0$ in 10.4.121. There results

$$10.4.122 \quad y_0(x) \sim \frac{1}{2} \pi^{-1/2} \lambda^{-1/6} x^{1/2} \left(\frac{1}{2} x \right)^{\lambda} e^{\lambda}.$$

On the other hand, $y_0(x)$ is a solution of 10.4.117 and therefore it can be written in the form

$$10.4.123 \quad y_0(x) = x^{1/2} [c_1 J_{\lambda}(\lambda x) + c_2 Y_{\lambda}(\lambda x)]$$

where, from 9.1.7 for λ fixed and $x \rightarrow 0$

$$\begin{aligned} J_{\lambda}(\lambda x) &\sim \frac{(\frac{1}{2}\lambda x)^{\lambda}}{\Gamma(\lambda+1)}, \\ Y_{\lambda}(\lambda x) &\sim \frac{(\frac{1}{2}\lambda x)^{\lambda}}{\Gamma(\lambda+1)} \cot \lambda\pi - \frac{(\frac{1}{2}\lambda x)^{-\lambda}}{\Gamma(1-\lambda)} \csc \lambda\pi. \end{aligned}$$

Thus, letting $x \rightarrow 0$ in 10.4.123 and comparing the resulting relation with 10.4.122 one finds that $c_2 = 0$ and

$$10.4.124 \quad y_0(x) = \frac{1}{2} \pi^{-1/2} \lambda^{-\lambda-1/6} e^{\lambda} \Gamma(\lambda+1) x^{1/2} J_{\lambda}(\lambda x).$$

It follows from 10.4.120 that uniformly in x on $0 < x < \infty$

10.4.125

$$\begin{aligned} J_{\lambda}(\lambda x) &= \frac{2\pi^{1/2}}{\Gamma(\lambda+1)} \lambda^{\lambda+1/6} e^{-\lambda} \left(\frac{x^2-1}{\xi} \right)^{-1/4} \text{Ai}(-\lambda^{2/3}\xi) [1 + O(\lambda^{-1})] \\ &\quad (|\lambda| \rightarrow \infty). \end{aligned}$$

Numerical Methods

10.5. Use and Extension of the Tables

Spherical Bessel Functions

To compute $j_n(x)$, $y_n(x)$, $n=0, 1, 2$, for values of x outside the range of Table 10.1, use formulas 10.1.11, 10.1.12 and obtain values for the circular functions from Tables 4.6–4.8.

Example 1. Compute $j_1(x)$ for $x=11.425$.

From 10.1.11, $j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$. Hence, using Tables 4.6 and 4.8,

$$\begin{aligned} j_1(11.425) &= -\frac{.90920 \ 500}{(11.425)^2} - \frac{.41634 \ 873}{11.425} \\ &= -.00696 \ 54535 - .03644 \ 1902 \\ &= -.04340 \ 7356. \end{aligned}$$

To compute $j_n(x)$, $11 \leq n \leq 20$, for a value of x within the range of Table 10.3, obtain from Table 10.3, directly or possibly by linear interpolation, $j_{21}(x)$, $j_{20}(x)$ and use these as starting values in the recurrence relation 10.1.19 for decreasing n .

An alternative procedure which often yields better accuracy and which also applies to computations of $j_n(x)$ when both n and x are outside the range of Table 10.1 is the following device essentially due to J. C. P. Miller [9.20].

At some value N larger than the desired value n , assume tentatively $F_{N+1}=0$, $F_N=1$ and use recurrence relation 10.1.19 for decreasing N to obtain the sequence $F_{N-1}, \dots, F_0$. If N was chosen large enough, each term of this sequence up to F_n is proportional, to a certain number of significant figures, to the corresponding term in the sequence $j_{N-1}(x), \dots, j_0(x)$ of true values. The factor of proportionality, p , may be obtained by comparing, say, F_0 with the true value $j_0(x)$ computed separately. The terms in the sequence $pF_0, \dots, pF_n$ are then accurate to the number of significant figures present in the tentative values. If the accuracy obtained is not sufficient, the process may be repeated by starting from a larger value N .

Example 2. Compute $j_{15}(x)$ for $x=24.6$.
Interpolation in Table 10.3 yields for $x=24.6$

$$x^{-21}e^{x^2/86}j_{21}(x)=(-28)3.934616$$

$$x^{-20}e^{x^2/82}j_{20}(x)=(-27)9.48683$$

whence

$$j_{21}(24.6)=.05604\ 29, j_{20}(24.6)=.03896\ 98.$$

From the recurrence relation 10.1.19 there results

$$j_{19}(24.6)=.00890\ 67660\quad[.00890\ 70]$$

$$j_{18}(24.6)=-.02484\ 93173\quad[-.02485\ 90]$$

$$j_{17}(24.6)=-.04628\ 17554\quad[-.04628\ 16]$$

$$j_{16}(24.6)=-.04099\ 87086\quad[-.04099\ 88]$$

$$j_{15}(24.6)=-.00871\ 65122\quad[-.00871\ 67]$$

For comparison, the correct values are shown in brackets.

To compute $j_{15}(x)$ for $x=24.6$ by Miller's device, take, for example, $N=39$ and assume $F_{40}=0$, $F_{39}=1$. Using 10.1.19 with decreasing N , i.e., $F_{N-1}=[(2N+1)/x]F_N-F_{N+1}$, $N=39, 38, \dots, 1, 0$, generate the sequence $F_{38}, F_{37}, \dots, F_1, F_0$, compute from Table 4.6, $j_0(24.6)=(\sin 24.6)/24.6=-.02064\ 620296$, and obtain the factor of proportionality

$$p=j_0(24.6)/F_0=.00000\ 03839\ 17642.$$

The value pF_{15} equals $j_{15}(24.6)$ to 8 decimals. The final part of the computations is shown in the following table, in which the correct values are given for comparison.

N	F_N	pF_N	$j_N(24.6)$
15	-22704.71107	-.00871 67391	-.00871 674
14	+78178.88236	+.03001 42522	+.03001 425
13	+114866.80811	+.04409 93941	+.04409 939
12	+47894.44353	+.01838 75218	+.01838 752
11	-66193.59317	-.02541 28882	-.02541 289
10	-109782.76234	-.04214 75392	-.04214 754
9	-27523.39903	-.01056 67185	-.01056 672
8	+88524.85252	+.03398 62526	+.03398 625
7	+88699.11017	+.03405 31532	+.03405 315
6	-34440.02929	-.01322 21348	-.01322 213
5	-106899.12565	-.04104 04602	-.04104 046
4	-13360.39272	-.00512 92905	-.00512 929
3	+102011.17704	+.03916 38905	+.03916 389
2	+42387.96341	+.01627 34870	+.01627 349
1	-93395.73728	-.03585 62712	-.03585 627
0	-53777.68747	-.02064 62030	-.02064 620

It may be observed that the normalization of the sequence $F_N, F_{N-1}, \dots, F_0$ can also be obtained from formula 10.1.50 by computing the sum $\sigma=\sum_{k=0}^N (2k+1)F_k^2$ and finding $p=1/\sqrt{\sigma}$. This yields, in the case of the example, $p=1/\sqrt{\sigma}=.00000\ 03839\ 177$.

Modified Spherical Bessel Functions

To compute $\sqrt{\frac{1}{2}\pi/x}I_{n+\frac{1}{2}}(x)$, $\sqrt{\frac{1}{2}\pi/x}K_{n+\frac{1}{2}}(x)$, $n=0, 1, 2, \dots$ for values of x outside the range of Table 10.8, use formulas 10.2.13, 10.2.14 together with 10.2.4 and obtain values for the hyperbolic and exponential functions from Tables 4.4 and 4.15. In those cases when $\sqrt{\frac{1}{2}\pi/x}I_{n+\frac{1}{2}}(x)$ and $\sqrt{\frac{1}{2}\pi/x}I_{-n-\frac{1}{2}}(x)$ are nearly equal, i.e., when x is sufficiently large, compute $\sqrt{\frac{1}{2}\pi/x}K_{n+\frac{1}{2}}(x)$ from formula 10.2.15, for which the coefficients $(n+\frac{1}{2}, k)$ are given in 10.1.9.

Example 3. Compute $\sqrt{\frac{1}{2}\pi/x}I_{5/2}(x)$, $\sqrt{\frac{1}{2}\pi/x}K_{5/2}(x)$ for $x=16.2$.

From 10.2.13, $\sqrt{\frac{1}{2}\pi/x}I_{5/2}(x)=(3+x^2)\sinh x/x^3-3\cosh x/x^2$; from Table 4.4, $\cosh 16.2=(6)5.4267\ 59950$ and this equals the value of $\sinh 16.2$ to the same number of significant figures. Hence

$$\begin{aligned}\sqrt{\frac{1}{2}\pi/16.2}I_{5/2}(16.2) &= (.06243\ 402371 \\ &\quad - .01143\ 118427)[(6)5.4267\ 59950] \\ &= 338814.4594 - 62034.29298 \\ &= 276780.1664.\end{aligned}$$

To compute $\sqrt{\frac{1}{2}\pi/16.2}K_{5/2}(16.2)$ use 10.2.17 and obtain

$$\begin{aligned}\sqrt{\frac{1}{2}\pi/16.2}K_{5/2}(16.2) &= \pi e^{-16.2} \left[\frac{1}{32.4} + \frac{6}{(32.4)^2} + \frac{12}{(32.4)^3} \right] \\ &= (-7)2.8945\ 38069[.036932\ 60400] \\ &= (-8)1.0690\ 28283.\end{aligned}$$

To compute $\sqrt{\frac{1}{2}\pi/x}I_{n+\frac{1}{2}}(x)$, $3 \leq n \leq 8$, for a value of x within the range of Table 10.9, obtain from Table 10.9, $\sqrt{\frac{1}{2}\pi/x}I_{19/2}(x)$, $\sqrt{\frac{1}{2}\pi/x}I_{21/2}(x)$ for the desired value of x and use these as starting values in the recurrence relation 10.2.18 for decreasing n .

To compute $\sqrt{\frac{1}{2}\pi/x}K_{n+\frac{1}{2}}(x)$ for some integer n outside the range of Table 10.9, obtain from 10.2.15 or from Table 10.8, $\sqrt{\frac{1}{2}\pi/x}K_{\frac{1}{2}}(x)$, $\sqrt{\frac{1}{2}\pi/x}K_{3/2}(x)$ for the desired value of x and use these as starting values in the recurrence relation 10.2.18 for increasing n . If x lies within the range of Table 10.9 and $n > 10$, the recurrence may be started with $\sqrt{\frac{1}{2}\pi/x}K_{19/2}(x)$, $\sqrt{\frac{1}{2}\pi/x}K_{21/2}(x)$ obtained from Table 10.9.

Example 4. Compute $\sqrt{\frac{1}{2}\pi/x}K_{11/2}(x)$ for $x=3.6$. Obtain from Table 10.8 for $x=3.6$

$$\sqrt{\frac{1}{2}\pi/x}K_{1/2}(x)=.01192\ 222$$

$$\sqrt{\frac{1}{2}\pi/x}K_{3/2}(x)=.01523\ 3952$$

The recurrence relation 10.2.18 yields successively

$$\begin{aligned}
 -\sqrt{\frac{1}{2}\pi/3.6}K_{5/2}(3.6) &= -.01192\ 222 \\
 &\quad -\frac{3}{3.6} (.01523\ 3952) \\
 &= -.02461\ 718 \\
 \sqrt{\frac{1}{2}\pi/3.6}K_{7/2}(3.6) &= .01523\ 3952 \\
 &\quad +\frac{5}{3.6} (.02461\ 718) \\
 &= .04942\ 4480 \\
 -\sqrt{\frac{1}{2}\pi/3.6}K_{9/2}(3.6) &= -.02461\ 718 \\
 &\quad -\frac{7}{3.6} (.04942\ 4480) \\
 &= -.12072\ 034 \\
 \sqrt{\frac{1}{2}\pi/3.6}K_{11/2}(3.6) &= .04942\ 4480 \\
 &\quad +\frac{9}{3.6} (.12072\ 034) \\
 &= .35122\ 533.
 \end{aligned}$$

As a check, the recurrence can be carried out until $n=9$ and the value of $\sqrt{\frac{1}{2}\pi/3.6}K_{19/2}(3.6)$ so obtained can be compared with the corresponding value from Table 10.9.

To compute $\sqrt{\frac{1}{2}\pi/x}I_{n+1/2}(x)$ when both n and x are outside the range of Table 10.9, use the device described in [9.20].

Airy Functions

To compute $\text{Ai}(x)$, $\text{Bi}(x)$ for values of x beyond 1, use auxiliary functions from Table 10.11.

Example 5. Compute $\text{Ai}(x)$ for $x=4.5$.

First, for $x=4.5$,

$$\xi = \frac{2}{3}x^{3/2} = 6.36396\ 1029, \quad \xi^{-1} = .15713\ 48403.$$

Hence, from Table 10.11, $f(-\xi) = .55848\ 24$ and thus

$$\begin{aligned}
 \text{Ai}(4.5) &= \frac{1}{2}(4.5)^{-1/4}(.55848\ 24) \exp(-6.36396\ 1029) \\
 &= \frac{1}{2}(.68658\ 905)(.55848\ 24)(.00172\ 25302) \\
 &= .00033\ 02503.
 \end{aligned}$$

To compute the zeros c , c' of a solution $y(x)$ of the equation $y'' - xy = 0$ and of its derivative

$y'(x)$, respectively, the following formulas may be used, in which d , d' denote approximations to c , c' and $u = y(d)/y'(d)$, $v = y'(d')/d'^2 y(d')$.

$$\begin{aligned}
 c &= d - u - 2d \frac{u^3}{3!} + 2 \frac{u^4}{4!} - 24d^2 \frac{u^5}{5!} \\
 &\quad + 88d \frac{u^6}{6!} - (88 + 720d^3) \frac{u^7}{7!} \\
 &\quad + 5856d^2 \frac{u^8}{8!} - (16640d + 40320d^4) \frac{u^9}{9!} + \dots \\
 c' &= d' \left\{ 1 - v - \frac{v^2}{2!} - (3 + 2d'^3) \frac{v^3}{3!} - (15 + 10d'^3) \frac{v^4}{4!} \right. \\
 &\quad \left. - (105 + 76d'^3 + 24d'^6) \frac{v^5}{5!} \right. \\
 &\quad \left. - (945 + 756d'^3 + 272d'^6) \frac{v^6}{6!} - \dots \right\} \\
 y'(c) &= y'(d) \left\{ 1 - d \frac{u^2}{2!} + \frac{u^3}{3!} - 3d^2 \frac{u^4}{4!} + 14d \frac{u^5}{5!} \right. \\
 &\quad \left. - (14 + 45d^3) \frac{u^6}{6!} + 471d^2 \frac{u^7}{7!} \right. \\
 &\quad \left. - (1432d + 1575d^4) \frac{u^8}{8!} + \dots \right\} \\
 y(c') &= y(d') \left\{ 1 - d'^2 \frac{v^2}{2!} - d'^3 \frac{v^3}{3!} - (3d'^3 + 3d'^6) \frac{v^4}{4!} \right. \\
 &\quad \left. - (15d'^3 + 14d'^6) \frac{v^5}{5!} \right. \\
 &\quad \left. - (105d'^3 + 101d'^6 + 45d'^4) \frac{v^6}{6!} - \dots \right\}
 \end{aligned}$$

Example 6. Compute the zero of $y(x) = \text{Ai}(x) - \text{Bi}(x)$ near $d = -.4$.

From Table 10.11,

$$y(-.4) = .02420\ 467, \quad y'(-.4) = -.71276\ 627$$

whence $u = y(-.4)/y'(-.4) = -.03395\ 8776$. From the above formulas

$$\begin{aligned}
 c &= -.4 + .03395\ 8776 - .00000\ 5221 \\
 &\quad + .00000\ 0111 + .00000\ 0001 \\
 &= -.36604\ 6333. \\
 y'(c) &= (-.71276\ 627) \{ 1 + .00023\ 0640 \\
 &\quad - .00000\ 6527 - .00000\ 0027 + .00000\ 0002 \} \\
 &= (-.71276\ 627)(1.00022\ 4088) \\
 &= -.71292\ 599.
 \end{aligned}$$

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and the tabulation of the function

$$\text{Gi}(z) = (1/\pi) \int_0^\infty \sin(uz + 1/3u^3) du,$$

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11. Integrals of Bessel Functions

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$$\left. \begin{array}{l} \int_0^x J_0(t)dt, \int_0^x Y_0(t)dt, 10D \\ e^{-x} \int_0^x I_0(t)dt, e^x \int_x^\infty K_0(t)dt, 7D \end{array} \right\} x=0(.1)10$$

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$$\left. \begin{array}{l} \int_0^x \frac{[1-J_0(t)]dt}{t}, \int_x^\infty \frac{Y_0(t)dt}{t}, 8D \\ e^{-x} \int_0^x \frac{[I_0(t)-1]dt}{t}, 8D; xe^x \int_x^\infty \frac{K_0(t)dt}{t}, 6D \end{array} \right\} x=0(.1)5$$

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11. Integrals of Bessel Functions

Mathematical Properties

11.1. Simple Integrals of Bessel Functions

$$\int_0^z t^\mu J_\nu(t) dt$$

11.1.1

$$\int_0^z t^\mu J_\nu(t) dt = \frac{z^\mu \Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)} \times \sum_{k=0}^{\infty} \frac{(\nu+2k+1) \Gamma\left(\frac{\nu-\mu+1}{2}+k\right)}{\Gamma\left(\frac{\nu+\mu+3}{2}+k\right)} J_{\nu+2k+1}(z) \quad (\Re(\mu+\nu+1) > 0)$$

11.1.2

$$\int_0^z J_\nu(t) dt = 2 \sum_{k=0}^{\infty} J_{\nu+2k+1}(z) \quad (\Re \nu > -1)$$

$$11.1.3 \quad \int_0^z J_{2n}(t) dt = \int_0^z J_0(t) dt - 2 \sum_{k=0}^{n-1} J_{2k+1}(z)$$

$$11.1.4 \quad \int_0^z J_{2n+1}(t) dt = 1 - J_0(z) - 2 \sum_{k=1}^n J_{2k}(z)$$

Recurrence Relations

11.1.5

$$\int_0^z J_{n+1}(t) dt = \int_0^z J_{n-1}(t) dt - 2J_n(z) \quad (n > 0)$$

$$11.1.6 \quad \int_0^z J_1(t) dt = 1 - J_0(z)$$

$$\int J_0(t) dt, \int Y_0(t) dt, \int I_0(t) dt, \int K_0(t) dt$$

11.1.7

$$\int_0^z \mathcal{C}_0(t) dt = x \mathcal{C}_0(x) + \frac{1}{2} \pi x \{ \mathbf{H}_0(x) \mathcal{C}_1(x) - \mathbf{H}_1(x) \mathcal{C}_0(x) \}$$

$$\mathcal{C}_\nu(x) = A J_\nu(x) + B Y_\nu(x), \nu = 0, 1$$

A and B are constants.

11.1.8

$$\int_0^z Z_0(t) dt = x Z_0(x) + \frac{1}{2} \pi x \{ -\mathbf{L}_0(x) Z_1(x) + \mathbf{L}_1(x) Z_0(x) \}$$

$$Z_\nu(x) = A I_\nu(x) + B e^{i\nu\pi} K_\nu(x), \nu = 0, 1$$

A and B are constants.

$\mathbf{H}_\nu(x)$ and $\mathbf{L}_\nu(x)$ are Struve functions (see chapter 12).

11.1.9

$$\int_0^z K_0(t) dt = -\left(\gamma + \ln \frac{x}{2}\right) x \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{(k!)^2 (2k+1)^2} + x \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{(k!)^2 (2k+1)^2} + x \sum_{k=1}^{\infty} \frac{(x/2)^{2k}}{(k!)^2 (2k+1)^2} \left(1 + \frac{1}{2} + \dots + \frac{1}{k}\right)$$

$$\gamma \text{ (Euler's constant)} = .57721 \ 56649 \dots$$

In this and all other integrals of 11.1, x is real and positive although all the results remain valid for extended portions of the complex plane unless stated to the contrary.

11.1.10

$$\int_0^{-iz} K_0(t) dt = \frac{\pi}{2} \int_0^z J_0(t) dt + i \frac{\pi}{2} \int_0^z Y_0(t) dt$$

Asymptotic Expansions

11.1.11

$$\int_x^\infty [J_0(t) + i Y_0(t)] dt \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{i(x-\pi/4)} \times \left[\sum_{k=0}^{\infty} (-)^k a_{2k+1} x^{-2k-1} + i \sum_{k=0}^{\infty} (-)^k a_{2k} x^{-2k} \right]$$

$$11.1.12 \quad a_k = \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})} \sum_{s=0}^k \frac{\Gamma(s+\frac{1}{2})}{2^s s! \Gamma(\frac{1}{2})}$$

11.1.13

$$2(k+1)a_{k+1} = 3 \left(k + \frac{1}{2}\right) \left(k + \frac{5}{6}\right) a_k - \left(k + \frac{1}{2}\right)^2 \left(k - \frac{1}{2}\right) a_{k-1}$$

$$11.1.14 \quad x^{\frac{1}{2}} e^{-x} \int_0^x I_0(t) dt \sim (2\pi)^{-\frac{1}{2}} \sum_{k=0}^{\infty} a_k x^{-k}$$

where the a_k are defined as in 11.1.12.

$$11.1.15 \quad x^{\frac{1}{2}} e^x \int_x^{\infty} K_0(t) dt \sim \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} (-)^k a_k x^{-k}$$

where the a_k are defined as in 11.1.12.

Polynomial Approximations²

$$11.1.16 \quad 8 \leq x \leq \infty$$

$$\begin{aligned} \int_x^{\infty} [J_0(t) + iY_0(t)] dt \\ = x^{-\frac{1}{2}} e^{i(x-\pi/4)} \left[\sum_{k=0}^7 (-)^k a_k (x/8)^{-2k-1} \right. \\ \left. + i \sum_{k=0}^7 (-)^k b_k (x/8)^{-2k} + \epsilon(x) \right] \end{aligned}$$

$$|\epsilon(x)| \leq 2 \times 10^{-9}$$

k	a_k	b_k
0	.06233 47304	.79788 45600
1	.00404 03539	.01256 42405
2	.00100 89872	.00178 70944
3	.00053 66169	.00067 40148
4	.00039 92825	.00041 00676
5	.00027 55037	.00025 43955
6	.00012 70039	.00011 07299
7	.00002 68482	.00002 26238

$$11.1.17 \quad 8 \leq x \leq \infty$$

$$x^{\frac{1}{2}} e^{-x} \int_0^x I_0(t) dt = \sum_{k=0}^6 d_k (x/8)^{-k} + \epsilon(x)$$

$$|\epsilon(x)| \leq 2 \times 10^{-5}$$

k	d_k
0	.39894 23
1	.03117 34
2	.00591 91
3	.00559 56
4	-.01148 58
5	.01774 40
6	-.00739 95

² Approximation 11.1.16 is from A. J. M. Hitchcock. Polynomial approximations to Bessel functions of order zero and one and to related functions, Math. Tables Aids Comp. 11, 86-88 (1957) (with permission).

$$11.1.18 \quad 7 \leq x \leq \infty$$

$$x^{\frac{1}{2}} e^x \int_x^{\infty} K_0(t) dt = \sum_{k=0}^6 (-)^k e_k (x/7)^{-k} + \epsilon(x)$$

$$|\epsilon(x)| \leq 2 \times 10^{-7}$$

k	e_k
0	1.25331 414
1	0.11190 289
2	.02576 646
3	.00933 994
4	.00417 454
5	.00163 271
6	.00033 934

$$\int \frac{J_0(t) dt}{t}, \int \frac{Y_0(t) dt}{t}, \int \frac{K_0(t) dt}{t}$$

$$11.1.19$$

$$\begin{aligned} \int_0^x \frac{1-J_0(t)}{t} dt \\ = 2x^{-1} \sum_{k=0}^{\infty} (2k+3) [\psi(k+2) - \psi(1)] J_{2k+3}(x) \\ = 1 - 2x^{-1} J_1(x) \\ + 2x^{-1} \sum_{k=0}^{\infty} (2k+5) [\psi(k+3) - \psi(1) - 1] J_{2k+5}(x) \end{aligned}$$

For $\psi(z)$, see 6.3.

$$11.1.20$$

$$\begin{aligned} \int_x^{\infty} \frac{J_0(t) dt}{t} + \gamma + \ln \frac{x}{2} = \int_0^x \frac{[1-J_0(t)] dt}{t} \\ = - \sum_{k=1}^{\infty} \frac{(-)^k \left(\frac{x}{2}\right)^{2k}}{2k(k!)^2} \end{aligned}$$

$$11.1.21$$

$$\begin{aligned} \int_x^{\infty} \frac{Y_0(t) dt}{t} = -\frac{1}{\pi} \left(\ln \frac{x}{2}\right)^2 - \frac{2\gamma}{\pi} \left(\ln \frac{x}{2}\right) + \frac{1}{\pi} \left(\frac{\pi^2}{6} - \gamma^2\right) \\ + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-)^k \left(\frac{x}{2}\right)^{2k}}{2k(k!)^2} \left\{ \psi(k+1) + \frac{1}{2k} - \ln \frac{x}{2} \right\} \end{aligned}$$

$$11.1.22$$

$$\begin{aligned} \int_x^{\infty} \frac{K_0(t) dt}{t} = \frac{1}{2} \left(\ln \frac{x}{2}\right)^2 + \gamma \ln \frac{x}{2} + \frac{\pi^2}{24} + \frac{\gamma^2}{2} \\ - \sum_{k=1}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k}}{2k(k!)^2} \left\{ \psi(k+1) + \frac{1}{2k} - \ln \frac{x}{2} \right\} \end{aligned}$$

$$11.1.23$$

$$\int_{-ix}^{-i\infty} \frac{K_0(t) dt}{t} = \frac{i\pi}{2} \int_x^{\infty} \frac{J_0(t) dt}{t} - \frac{\pi}{2} \int_x^{\infty} \frac{Y_0(t) dt}{t}$$

Asymptotic Expansions

$$11.1.24 \quad \int_x^\infty \frac{\mathcal{C}_0(t)dt}{t} = \frac{2g_1(x)\mathcal{C}_0(x)}{x^2} - \frac{g_0(x)\mathcal{C}_1(x)}{x}$$

where

$$g_0(x) \sim \sum_{k=0}^{\infty} (-)^k \left(\frac{x}{2}\right)^{-2k} (k!)^2,$$

$$g_1(x) \sim \sum_{k=0}^{\infty} (-)^k \left(\frac{x}{2}\right)^{-2k} k!(k+1)!$$

$$11.1.25 \quad g_0(x) = 2x^2 \int_x^\infty \frac{g_1(t)dt}{t^3}$$

$$11.1.26 \quad x^{3/2}e^x \int_x^\infty \frac{K_0(t)dt}{t} \sim \left(\frac{\pi}{2}\right)^{1/2} \sum_{k=0}^{\infty} (-)^k c_k x^{-k}$$

where

$$11.1.27 \quad c_0 = 1, c_1 = \frac{13}{8}$$

$$2(k+1)c_{k+1} = \left[3(k+1)^2 + \frac{1}{4}\right]c_k - \left(k + \frac{1}{2}\right)^3 c_{k-1}$$

$$11.1.28 \quad x^{3/2}e^{-x} \int_0^x \frac{[I_0(t)-1]dt}{t} \sim (2\pi)^{-1/2} \sum_{k=0}^{\infty} c_k x^{-k}$$

where c_k is defined as in 11.1.27.

Polynomial Approximations

$$11.1.29 \quad 5 \leq x \leq \infty$$

$$\int_x^\infty \frac{\mathcal{C}_0(t)dt}{t} = \frac{2g_1(x)\mathcal{C}_0(x)}{x^2} - \frac{g_0(x)\mathcal{C}_1(x)}{x}$$

where

$$g_0(x) = \sum_{k=0}^9 (-)^k a_k (x/5)^{-2k} + \epsilon(x),$$

$$g_1(x) = \sum_{k=0}^9 (-)^k b_k (x/5)^{-2k} + \epsilon(x)$$

$$|\epsilon(x)| \leq 2 \times 10^{-7}$$

k	a_k	b_k
0	1.0	1.0
1	0.15999 2815	0.31998 5629
2	.10161 9385	.30485 8155
3	.13081 1585	.52324 6341
4	.20740 4022	1.03702 0112
5	.28330 0508	1.69980 3050
6	.27902 9488	1.95320 6413
7	.17891 5710	1.43132 5684
8	.06622 8328	0.59605 4956
9	.01070 2234	.10702 2336

$$11.1.30$$

$$4 \leq x \leq \infty$$

$$x^{3/2}e^x \int_x^\infty \frac{K_0(t)dt}{t} = \sum_{k=0}^6 (-)^k d_k \left(\frac{x}{4}\right)^{-k} + \epsilon(x)$$

$$|\epsilon(x)| \leq 6 \times 10^{-6}$$

k	d_k
0	1.25331 41
1	0.50913 39
2	.32191 84
3	.26214 46
4	.20601 26
5	.11103 96
6	.02724 00

$$11.1.31$$

$$5 \leq x \leq \infty$$

$$x^{3/2}e^{-x} \int_0^x \frac{[I_0(t)-1]dt}{t} = \sum_{k=0}^{10} f_k \left(\frac{x}{5}\right)^{-k} + \epsilon(x)$$

$$|\epsilon(x)| \leq 1.1 \times 10^{-5}$$

k	f_k
0	0.39893 14
1	.13320 55
2	-.04938 43
3	1.47800 44
4	-8.65560 13
5	28.12214 78
6	-48.05241 15
7	40.39473 40
8	-11.90943 95
9	-3.51950 09
10	2.19454 64

11.2. Repeated Integrals of $J_n(z)$ and $K_0(z)$ Repeated Integrals of $J_n(z)$

Let

$$11.2.1$$

$$f_{0,n}(z) = J_n(z),$$

$$f_{1,n}(z) = \int_0^z J_n(t)dt, \dots, f_{r,n}(z) = \int_0^z f_{r-1,n}(t)dt$$

$$11.2.2$$

$$f_{-r,n}(z) = \frac{d^r}{dz^r} J_n(z)$$

Then

$$11.2.3$$

$$f_{r,n}(z) = \frac{1}{\Gamma(r)} \int_0^z (z-t)^{r-1} J_n(t)dt \quad (\Re r > 0)$$

$$11.2.4 \quad f_{r,n}(z) = \frac{2^r}{\Gamma(r)} \sum_{k=0}^{\infty} \frac{\Gamma(k+r)}{k!} J_{n+r+2k}(z)$$

Recurrence Relations

11.2.5

$$r(r-1)f_{r+1,n}(z) = 2(r-1)zf_{r,n}(z) - [(1-r)^2 - n^2 + z^2]f_{r-1,n}(z) + (2r-3)zf_{r-2,n}(z) - z^2f_{r-3,n}(z)$$

11.2.6

$$rf_{r+1,0}(z) = zf_{r,0}(z) - (r-1)f_{r-1,0}(z) + zf_{r-2,0}(z)$$

$$11.2.7 \quad f_{r+1,n+1}(z) = f_{r+1,n-1}(z) - 2f_{r,n}(z)$$

 Repeated Integrals of $K_0(z)$

Let

11.2.8

$$Ki_0(z) = K_0(z),$$

$$Ki_1(z) = \int_z^\infty K_0(t)dt, \dots, Ki_r(z) = \int_z^\infty Ki_{r-1}(t)dt$$

$$11.2.9 \quad Ki_{-r}(z) = (-1)^r \frac{d^r}{dz^r} K_0(z)$$

Then

11.2.10

$$Ki_r(z) = \int_0^\infty \frac{e^{-z \cosh t} dt}{\cosh^r t} \quad (\Re z \geq 0, \Re r > 0, \Re z > 0, r=0)$$

11.2.11

$$Ki_r(z) = \frac{1}{\Gamma(r)} \int_z^\infty (t-z)^{r-1} K_0(t)dt \quad (\Re z \geq 0, \Re r > 0)$$

$$11.2.12 \quad Ki_{2r}(0) = \frac{\Gamma(r)\Gamma(\frac{3}{2})}{\Gamma(r+\frac{1}{2})} \quad (\Re r > 0)$$

$$11.2.13 \quad Ki_{2r+1}(0) = \frac{\pi}{2} \frac{\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(r+1)} \quad \left(\Re r > -\frac{1}{2}\right)$$

11.2.14

$$rKi_{r+1}(z) = -zKi_r(z) + (r-1)Ki_{r-1}(z) + zKi_{r-2}(z)$$

11.3. Reduction Formulas for Indefinite Integrals

Let

$$11.3.1 \quad g_{\mu,\nu}(z) = \int_z^\infty e^{-zt} t^\mu Z_\nu(t) dt$$

where $Z_\nu(z)$ represents any of the Bessel functions of the first three kinds or the modified Bessel functions. The parameters a and b appearing in the reduction formulae are associated with the particular type of Bessel function as delineated in the following table.

$Z_\nu(z)$	a	b
$J_\nu(z), Y_\nu(z), H_\nu^{(1)}(z), H_\nu^{(2)}(z)$	1	1
$I_\nu(z)$	-1	1
$K_\nu(z)$	1	-1

11.3.3

$$pg_{\mu,\nu}(z) = -e^{-pz} z^\mu Z_\nu(z) + (\mu+\nu)g_{\mu-1,\nu}(z) - ag_{\mu,\nu+1}(z)$$

11.3.4

$$pg_{\mu,\nu+1}(z) = -e^{-pz} z^\mu Z_{\nu+1}(z) + (\mu-\nu-1)g_{\mu-1,\nu+1}(z) + bg_{\mu,\nu}(z)$$

11.3.5

$$(p^2+ab)g_{\mu,\nu}(z) = ae^{-pz} z^\mu Z_{\nu+1}(z) + (\mu-\nu-1)e^{-pz} z^{\mu-1} Z_\nu(z) - pe^{-pz} z^\mu Z_\nu(z) + p(2\mu-1)g_{\mu-1,\nu}(z) + [\nu^2 - (\mu-1)^2]g_{\mu-2,\nu}(z)$$

11.3.6

$$a(\nu-\mu)g_{\mu,\nu+1}(z) = -2ve^{-pz} z^\mu Z_\nu(z) - 2vp g_{\mu,\nu}(z) + b(\mu+\nu)g_{\mu,\nu-1}(z)$$

$$\text{Case 1:} \quad p^2+ab=0, \nu=\pm(\mu-1)$$

$$11.3.7 \quad g_{\nu,\nu}(z) = \frac{e^{-pz} z^{\nu+1}}{2\nu+1} \left\{ Z_\nu(z) - \frac{a}{p} Z_{\nu+1}(z) \right\}$$

$$11.3.8 \quad g_{-\nu,\nu}(z) = -\frac{e^{-pz} z^{-\nu+1}}{2\nu-1} \left\{ Z_\nu(z) + \frac{b}{p} Z_{\nu-1}(z) \right\}$$

11.3.9

$$\int_0^z e^{it} t^\nu J_\nu(t) dt = \frac{e^{iz} z^{\nu+1}}{2\nu+1} [J_\nu(z) - iJ_{\nu+1}(z)] \quad (\Re \nu > -\frac{1}{2})$$

11.3.10

$$\int_0^z e^{it} t^{-\nu} J_\nu(t) dt = -\frac{e^{iz} z^{-\nu+1}}{2\nu-1} [J_\nu(z) + iJ_{\nu-1}(z)] + \frac{i}{2^{\nu-1}(2\nu-1)\Gamma(\nu)} \quad (\nu \neq \frac{1}{2})$$

11.3.11

$$\int_0^z e^{it} t^\nu Y_\nu(t) dt = \frac{e^{iz} z^{\nu+1}}{2\nu+1} [Y_\nu(z) - iY_{\nu+1}(z)] - \frac{i2^{\nu+1}\Gamma(\nu+1)}{\pi(2\nu+1)} \quad (\Re \nu > -\frac{1}{2})$$

11.3.12

$$\int_0^z e^{\pm it} t^\nu I_\nu(t) dt = \frac{e^{\pm iz} z^{\nu+1}}{2\nu+1} [I_\nu(z) \mp I_{\nu+1}(z)] \quad (\Re \nu > -\frac{1}{2})$$

11.3.13

$$\int_0^z e^{-t} I_n(t) dt = z e^{-z} [I_0(z) + I_1(z)] \\ + n[e^{-z} I_0(z) - 1] + 2e^{-z} \sum_{k=1}^{n-1} (n-k) I_k(z)$$

11.3.14

$$\int_0^z e^{\pm t} t^{-\nu} I_\nu(t) dt = -\frac{e^{\pm z} z^{-\nu+1}}{2\nu-1} [I_\nu(z) \mp I_{\nu-1}(z)] \\ \mp \frac{1}{2^{\nu-1}(2\nu-1)\Gamma(\nu)} \quad (\nu \neq \frac{1}{2})$$

11.3.15

$$\int_0^z e^{\pm t} t^\nu K_\nu(t) dt = \frac{e^{\pm z} z^{\nu+1}}{2\nu+1} [K_\nu(z) \pm K_{\nu+1}(z)] \\ \mp \frac{2^\nu \Gamma(\nu+1)}{2\nu+1} \quad (\Re \nu > -\frac{1}{2})$$

King's integral (see [11.5])

$$11.3.16 \quad \int_0^z e^t K_0(t) dt = z e^z [K_0(z) + K_1(z)] - 1$$

11.3.17

$$\int_z^\infty e^t t^{-\nu} K_\nu(t) dt \\ = \frac{e^z z^{-\nu+1}}{2\nu-1} [K_\nu(z) + K_{\nu-1}(z)] \quad (\Re \nu > \frac{1}{2})$$

Case 2: $p=0, \mu=\pm\nu$

$$11.3.18 \quad b g_{\nu, \nu-1}(z) = z^\nu Z_\nu(z)$$

$$11.3.19 \quad a g_{-\nu, \nu+1}(z) = -z^{-\nu} Z_\nu(z)$$

$$11.3.20 \quad \int_0^z t^\nu J_{\nu-1}(t) dt = z^\nu J_\nu(z) \quad (\Re \nu > 0)$$

$$11.3.21 \quad \int_0^z t^{-\nu} J_{\nu+1}(t) dt = \frac{1}{2^\nu \Gamma(\nu+1)} - z^{-\nu} J_\nu(z)$$

11.3.22

$$2n \int_0^z \frac{J_{2n}(t) dt}{t} = 1 - \frac{2}{z} \sum_{k=1}^n (2k-1) J_{2k-1}(z) \\ = \frac{2}{z} \sum_{k=n+1}^\infty (2k-1) J_{2k-1}(z) \quad (n > 0)$$

11.3.23

$$(2n+1) \int_0^z \frac{J_{2n+1}(t) dt}{t} = \int_0^z J_0(t) dt \\ - J_1(z) - \frac{4}{z} \sum_{k=1}^n k J_{2k}(z)$$

11.3.24

$$\int_0^z t^\nu Y_{\nu-1}(t) dt = z^\nu Y_\nu(z) + \frac{2^\nu \Gamma(\nu)}{\pi} \quad (\Re \nu > 0)$$

$$11.3.25 \quad \int_0^z t^\nu I_{\nu-1}(t) dt = z^\nu I_\nu(z) \quad (\Re \nu > 0)$$

$$11.3.26 \quad \int_0^z t^{-\nu} I_{\nu+1}(t) dt = z^{-\nu} I_\nu(z) - \frac{1}{2^\nu \Gamma(\nu+1)}$$

11.3.27

$$\int_0^z t^\nu K_{\nu-1}(t) dt = -z^\nu K_\nu(z) + 2^{\nu-1} \Gamma(\nu) \quad (\Re \nu > 0)$$

$$11.3.28 \quad \int_z^\infty t^{-\nu} K_{\nu+1}(t) dt = z^{-\nu} K_\nu(z)$$

Indefinite Integrals of Products of Bessel Functions

Let $\mathcal{C}_\mu(z)$ and $\mathcal{D}_\nu(z)$ denote any two cylinder functions of orders μ and ν respectively.

11.3.29

$$\int^z \left\{ (k^2 - l^2) t - \frac{(\mu^2 - \nu^2)}{t} \right\} \mathcal{C}_\mu(kt) \mathcal{D}_\nu(lt) dt \\ = z \{ k \mathcal{C}_{\mu+1}(kz) \mathcal{D}_\nu(lz) - l \mathcal{C}_\mu(kz) \mathcal{D}_{\nu+1}(lz) \} \\ - (\mu - \nu) \mathcal{C}_\mu(kz) \mathcal{D}_\nu(lz) \quad *$$

11.3.30

$$\int^z t^{-\mu-\nu-1} \mathcal{C}_{\mu+1}(t) \mathcal{D}_{\nu+1}(t) dt \\ = -\frac{z^{-\mu-\nu}}{2(\mu+\nu+1)} \{ \mathcal{C}_\mu(z) \mathcal{D}_\nu(z) + \mathcal{C}_{\mu+1}(z) \mathcal{D}_{\nu+1}(z) \}$$

11.3.31

$$\int^z t^{\mu+\nu+1} \mathcal{C}_\mu(t) \mathcal{D}_\nu(t) dt \\ = \frac{z^{\mu+\nu+2}}{2(\mu+\nu+1)} \{ \mathcal{C}_\mu(z) \mathcal{D}_\nu(z) + \mathcal{C}_{\mu+1}(z) \mathcal{D}_{\nu+1}(z) \}$$

11.3.32

$$\int_0^z t J_{\nu-1}^2(t) dt = 2 \sum_{k=0}^\infty (\nu+2k) J_{\nu+2k}^2(z) \quad (\Re \nu > 0)$$

11.3.33

$$\int_0^z t [J_{\nu-1}^2(t) - J_{\nu+1}^2(t)] dt = 2\nu J_\nu^2(z) \quad (\Re \nu > 0)$$

$$11.3.34 \quad \int_0^z t J_0^2(t) dt = \frac{z^2}{2} [J_0^2(z) + J_1^2(z)]$$

11.3.35

$$\int_0^z J_n(t) J_{n+1}(t) dt = \frac{1}{2} [1 - J_0^2(z)] - \sum_{k=1}^n J_k^2(z) \\ = \sum_{k=n+1}^\infty J_k^2(z)$$

*See page II.

11.3.36

$$\begin{aligned}
 & (\mu + \nu) \int_0^z t^{-1} \mathcal{C}_\mu(t) \mathcal{D}_\nu(t) dt \\
 & - (\mu + \nu + 2n) \int_0^z t^{-1} \mathcal{C}_{\mu+n}(t) \mathcal{D}_{\nu+n}(t) dt \\
 & = \mathcal{C}_\mu(z) \mathcal{D}_\nu(z) + \mathcal{C}_{\mu+n}(z) \mathcal{D}_{\nu+n}(z) + 2 \sum_{k=1}^{n-1} \mathcal{C}_{\mu+k}(z) \mathcal{D}_{\nu+k}(z)
 \end{aligned}$$

Convolution Type Integrals

11.3.37

$$\int_0^z J_\mu(t) J_\nu(z-t) dt = 2 \sum_{k=0}^{\infty} (-1)^k J_{\mu+\nu+2k+1}(z) \quad (\Re \mu > -1, \Re \nu > -1)$$

11.3.38

$$\int_0^z J_\nu(t) J_{1-\nu}(z-t) dt = J_0(z) - \cos z \quad (-1 < \Re \nu < 2)$$

11.3.39

$$\int_0^z J_\nu(t) J_{-\nu}(z-t) dt = \sin z \quad (|\Re \nu| < 1)$$

11.3.40

$$\int_0^z t^{-1} J_\mu(t) J_\nu(z-t) dt = \frac{J_{\mu+\nu}(z)}{\mu} \quad (\Re \mu > 0, \Re \nu > -1)$$

11.3.41

$$\int_0^z \frac{J_\mu(t) J_\nu(z-t) dt}{t(z-t)} = \frac{(\mu + \nu) J_{\mu+\nu}(z)}{\mu \nu z} \quad (\Re \mu > 0, \Re \nu > 0)$$

11.4. Definite Integrals

Orthogonality Properties of Bessel Functions

Let $\mathcal{C}_\nu(z)$ be a cylinder function of order ν . In particular, let

$$11.4.1 \quad \mathcal{C}_\nu(z) = A J_\nu(z) + B Y_\nu(z)$$

where A and B are real constants. Then

11.4.2

$$\begin{aligned}
 & \int_a^b t \mathcal{C}_\nu(\lambda_m t) \mathcal{C}_\nu(\lambda_n t) dt = 0 \quad (m \neq n) \\
 & = \left[\frac{1}{2} t^2 \left\{ \left(1 - \frac{\nu^2}{\lambda_n^2 t^2} \right) \mathcal{C}_\nu'(\lambda_n t) + \mathcal{C}_\nu''(\lambda_n t) \right\} \right]_a^b \\
 & \quad (m = n) (0 < a < b)
 \end{aligned}$$

provided the following two conditions hold:

1. λ_n is a real zero of

$$11.4.3 \quad h_1 \lambda \mathcal{C}_{\nu+1}(\lambda b) - h_2 \mathcal{C}_\nu(\lambda b) = 0$$

2. There must exist numbers k_1 and k_2 (both not zero) so that for all n

$$11.4.4 \quad k_1 \lambda_n \mathcal{C}_{\nu+1}(\lambda_n a) - k_2 \mathcal{C}_\nu(\lambda_n a) = 0$$

In connection with these formulae, see 11.3.29. If $a=0$, the above is valid provided $B=0$. This case is covered by the following result.

11.4.5

$$\begin{aligned}
 & \int_0^1 t J_\nu(\alpha_m t) J_\nu(\alpha_n t) dt = 0 \quad (m \neq n, \nu > -1) \\
 & = \frac{1}{2} [J_\nu'(\alpha_n)]^2 \\
 & \quad (m = n, b = 0, \nu > -1) \\
 & = \frac{1}{2\alpha_n^2} \left[\frac{a^2}{b^2} + \alpha_n^2 - \nu^2 \right] [J_\nu(\alpha_n)]^2 \\
 & \quad (m = n, b \neq 0, \nu \geq -1)
 \end{aligned}$$

$\alpha_1, \alpha_2, \dots$ are the positive zeros of $aJ_\nu(x) + bxJ_\nu'(x) = 0$, where a and b are real constants.

11.4.6

$$\begin{aligned}
 & \int_0^\infty t^{-1} J_{\nu+2n+1}(t) J_{\nu+2m+1}(t) dt = 0 \quad (m \neq n) \\
 & = \frac{1}{2(2n+\nu+1)} \\
 & \quad (m = n) (\nu + n + m > -1)
 \end{aligned}$$

Definite Integrals Over a Finite Range

$$11.4.7 \quad \int_0^{\frac{\pi}{2}} J_{2n}(2z \sin t) dt = \frac{\pi}{2} J_n^2(z)$$

$$11.4.8 \quad \int_0^\pi J_0(2z \sin t) \cos 2nt dt = \pi J_n^2(z)$$

$$11.4.9 \quad \int_0^{\frac{\pi}{2}} Y_0(2z \sin t) \cos 2nt dt = \frac{\pi}{2} J_n(z) Y_n(z)$$

11.4.10

$$\begin{aligned}
 & \int_0^{\frac{\pi}{2}} J_\mu(z \sin t) \sin^{\mu+1} t \cos^{2\nu+1} t dt \\
 & = \frac{2^\nu \Gamma(\nu+1)}{z^{\nu+1}} J_{\mu+\nu+1}(z) \quad (\Re \mu > -1, \Re \nu > -1)
 \end{aligned}$$

11.4.11

$$\begin{aligned}
 & \int_0^{\frac{\pi}{2}} J_\mu(z \sin^2 t) J_\nu(z \cos^2 t) \csc 2t dt \\
 & = \frac{(\mu + \nu)}{4\mu\nu} J_{\mu+\nu}(z) \quad (\Re \mu > 0, \Re \nu > 0)
 \end{aligned}$$

Infinite Integrals

Integrals of the Form $\int_0^\infty e^{-\nu t} t^\mu Z_\nu(t) dt$

11.4.12

$$\int_0^\infty e^{it} t^{\mu-1} J_\nu(t) dt = \frac{e^{\frac{1}{2}i\pi(\mu+\nu)} \Gamma(\mu+\nu) \Gamma(\frac{1}{2}-\mu)}{\Gamma(\frac{1}{2}) 2^\mu \Gamma(\nu-\mu+1)} \left(\Re \mu < \frac{1}{2}, \Re(\mu+\nu) > 0 \right)$$

11.4.13

$$\int_0^\infty e^{-t} t^{\mu-1} I_\nu(t) dt = \frac{\Gamma(\mu+\nu) \Gamma(\frac{1}{2}-\mu)}{\Gamma(\frac{1}{2}) 2^\mu \Gamma(\nu-\mu+1)} \left(\Re \mu < \frac{1}{2}, \Re(\mu+\nu) > 0 \right)$$

11.4.14

$$\int_0^\infty \cos bt K_0(t) dt = \frac{\frac{1}{2}\pi}{(1+b^2)^{\frac{1}{2}}} \quad (|\Im b| < 1)$$

11.4.15

$$\int_0^\infty \sin bt K_0(t) dt = \frac{\operatorname{arc} \sinh b}{(1+b^2)^{\frac{1}{2}}} \quad (|\Im b| < 1)$$

$$11.4.16 \quad \int_0^\infty t^\mu J_\nu(t) dt = \frac{2^\mu \Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)} \left(\Re(\mu+\nu) > -1, \Re \mu < \frac{1}{2} \right)$$

$$11.4.17 \quad \int_0^\infty J_\nu(t) dt = 1 \quad (\Re \nu > -1)$$

11.4.18

$$\int_0^\infty \frac{[1-J_0(t)] dt}{t^\mu} = \frac{\Gamma\left(\frac{\mu-1}{2}\right) \Gamma\left(\frac{3-\mu}{2}\right)}{2^\mu \left\{ \Gamma\left(\frac{\mu+1}{2}\right) \right\}^2} \quad (1 < \Re \mu < 3)$$

11.4.19

$$\int_0^\infty t^\mu Y_\nu(t) dt = \frac{2^\mu}{\pi} \Gamma\left(\frac{\mu+\nu+1}{2}\right) \Gamma\left(\frac{\mu-\nu+1}{2}\right) \times \sin \frac{\pi}{2} (\mu-\nu) \left(\Re(\mu \pm \nu) > -1, \Re \mu < \frac{1}{2} \right)$$

$$11.4.20 \quad \int_0^\infty Y_\nu(t) dt = -\tan \frac{\nu\pi}{2} \quad (|\Re \nu| < 1)$$

$$11.4.21 \quad \int_0^\infty Y_0(t) dt = 0$$

11.4.22

$$\int_0^\infty t^\mu K_\nu(t) dt = 2^{\mu-1} \Gamma\left(\frac{\mu+\nu+1}{2}\right) \Gamma\left(\frac{\mu-\nu+1}{2}\right) \left(\Re(\mu \pm \nu) > -1 \right)$$

11.4.23

$$\int_0^\infty K_0(t) dt = \frac{\pi}{2}$$

11.4.24

$$\int_{-\infty}^\infty e^{-i\omega t} J_n(t) dt = \frac{2(-i)^n T_n(\omega)}{(1-\omega^2)^{\frac{1}{2}}} \quad (\omega^2 < 1) \\ = 0 \quad (\omega^2 > 1)$$

where $T_n(\omega)$ is the Chebyshev polynomial of the first kind (see chapter 22).

11.4.25

$$\int_{-\infty}^\infty t^{-1} e^{-i\omega t} J_n(t) dt \\ = \frac{2i}{n} (-i)^n (1-\omega^2)^{\frac{1}{2}} U_{n-1}(\omega) \quad (\omega^2 < 1) \\ = 0 \quad (\omega^2 > 1)$$

where $U_n(\omega)$ is the Chebyshev polynomial of the second kind (see chapter 22).

11.4.26

$$\int_{-\infty}^\infty t^{-\frac{1}{2}} e^{-i\omega t} J_{n+\frac{1}{2}}(t) dt = (-i)^n (2\pi)^{\frac{1}{2}} P_n(\omega) \quad (\omega^2 < 1) \\ = 0 \quad (\omega^2 > 1)$$

where $P_n(\omega)$ is the Legendre polynomial (see chapter 22).

11.4.27

$$\int_0^\infty e^{-t} t^{\frac{a}{2}-1} J_a[2(zt)^{\frac{1}{2}}] dt = \frac{\gamma(a, z)}{z^{a/2}} \quad (\Re a > 0, \Re z > 0)$$

where $\gamma(a, z)$ is the incomplete gamma function (see chapter 6).

Integrals of the Form $\int_0^\infty e^{-a^2 t^2} t^\mu Z_\nu(bt) dt$

11.4.28

$$\int_0^\infty e^{-a^2 t^2} t^{\mu-1} J_\nu(bt) dt \\ = \frac{\Gamma\left(\frac{1}{2}\nu + \frac{1}{2}\mu\right) \left(\frac{1}{2}\frac{b}{a}\right)^\nu}{2a^\mu \Gamma(\nu+1)} M\left(\frac{1}{2}\nu + \frac{1}{2}\mu, \nu+1, -\frac{b^2}{4a^2}\right) \quad (\Re(\mu+\nu) > 0, \Re a^2 > 0)$$

where the notation $M(a, b, z)$ stands for the confluent hypergeometric function (see chapter 13).

11.4.29

$$\int_0^\infty e^{-a^2 t^2} t^{\nu+1} J_\nu(bt) dt \\ = \frac{b^\nu}{(2a^2)^{\nu+1}} e^{-\frac{b^2}{4a^2}} \quad (\Re \nu > -1, \Re a^2 > 0)$$

11.4.30

$$\int_0^\infty e^{-a^2 t^2} Y_{2\nu}(bt) dt = -\frac{\pi^{\frac{1}{2}}}{2a} e^{-\frac{b^2}{8a^2}} \left[I_\nu \left(\frac{b^2}{8a^2} \right) \tan \nu\pi + \frac{1}{\pi} K_\nu \left(\frac{b^2}{8a^2} \right) \sec \nu\pi \right] \quad \left(|\Re \nu| < \frac{1}{2}, \Re a^2 > 0 \right)$$

11.4.31

$$\int_0^\infty e^{-a^2 t^2} I_\nu(bt) dt = \frac{\pi^{\frac{1}{2}}}{2a} e^{-\frac{b^2}{8a^2}} I_{\frac{1}{2}\nu} \left(\frac{b^2}{8a^2} \right) \quad (\Re \nu > -1, \Re a^2 > 0)$$

11.4.32

$$\int_0^\infty e^{-a^2 t^2} K_\nu(bt) dt = \frac{\pi^{\frac{1}{2}}}{4a} e^{-\frac{b^2}{8a^2}} K_\nu \left(\frac{b^2}{8a^2} \right) \quad (\Re a^2 > 0)$$

Weber-Schafheitlin Type Integrals

11.4.33

$$\begin{aligned} \int_0^\infty \frac{J_\mu(at) J_\nu(bt) dt}{t^\lambda} &= \frac{b^\nu \Gamma \left(\frac{\mu + \nu - \lambda + 1}{2} \right)}{2^\lambda a^{\nu - \lambda + 1} \Gamma(\nu + 1) \Gamma \left(\frac{\mu - \nu + \lambda + 1}{2} \right)} \\ &\times {}_2F_1 \left(\frac{\mu + \nu - \lambda + 1}{2}, \frac{\nu - \mu - \lambda + 1}{2}; \nu + 1; \frac{b^2}{a^2} \right) \\ &(\Re(\mu + \nu - \lambda + 1) > 0, \Re \lambda > -1, 0 < b < a) \end{aligned}$$

11.4.34

$$\begin{aligned} \int_0^\infty \frac{J_\mu(at) J_\nu(bt) dt}{t^\lambda} &= \frac{a^\mu \Gamma \left(\frac{\mu + \nu - \lambda + 1}{2} \right)}{2^\lambda b^{\mu - \lambda + 1} \Gamma(\mu + 1) \Gamma \left(\frac{\nu - \mu + \lambda + 1}{2} \right)} \\ &\times {}_2F_1 \left(\frac{\mu + \nu - \lambda + 1}{2}, \frac{\mu - \nu - \lambda + 1}{2}; \mu + 1; \frac{a^2}{b^2} \right) \\ &(\Re(\mu + \nu - \lambda + 1) > 0, \Re \lambda > -1, 0 < a < b) \end{aligned}$$

For ${}_2F_1$, see chapter 15.

Special Cases of the Discontinuous Weber-Schafheitlin Integral

11.4.35

$$\begin{aligned} \int_0^\infty \frac{J_\mu(at) \sin bt dt}{t} &= \frac{1}{\mu} \sin \left[\mu \arcsin \frac{b}{a} \right] \quad (0 \leq b \leq a) \\ &= \frac{a^\mu \sin \frac{\pi\mu}{2}}{\mu [b + (b^2 - a^2)^{\frac{1}{2}}]^\mu} \quad (b \geq a > 0) \\ &(\Re \mu > -1) \end{aligned}$$

11.4.36

$$\begin{aligned} \int_0^\infty \frac{J_\mu(at) \cos bt dt}{t} &= \frac{1}{\mu} \cos \left[\mu \arcsin \frac{b}{a} \right] \quad (0 \leq b \leq a) \\ &= \frac{a^\mu \cos \frac{\pi\mu}{2}}{\mu [b + (b^2 - a^2)^{\frac{1}{2}}]^\mu} \quad (b \geq a > 0) \\ &(\Re \mu > 0) \end{aligned}$$

11.4.37

$$\begin{aligned} \int_0^\infty J_\mu(at) \cos bt dt &= \frac{\cos \left[\mu \arcsin \frac{b}{a} \right]}{(a^2 - b^2)^{\frac{1}{2}}} \quad (0 \leq b < a) \\ &= \frac{-a^\mu \sin \frac{\pi\mu}{2}}{(b^2 - a^2)^{\frac{1}{2}} [b + (b^2 - a^2)^{\frac{1}{2}}]^\mu} \\ &(b > a > 0) \quad (\Re \mu > -1) \end{aligned}$$

11.4.38

$$\begin{aligned} \int_0^\infty J_\mu(at) \sin bt dt &= \frac{\sin \left[\mu \arcsin \frac{b}{a} \right]}{(a^2 - b^2)^{\frac{1}{2}}} \quad (0 \leq b < a) \\ &= \frac{a^\mu \cos \frac{\pi\mu}{2}}{(b^2 - a^2)^{\frac{1}{2}} [b + (b^2 - a^2)^{\frac{1}{2}}]^\mu} \\ &(b > a > 0) \quad (\Re \mu > -2) \end{aligned}$$

$$\begin{aligned} 11.4.39 \quad \int_0^\infty e^{ibt} J_0(at) dt &= \frac{1}{(a^2 - b^2)^{\frac{1}{2}}} \quad (0 \leq b < a) \\ &= \frac{i}{(b^2 - a^2)^{\frac{1}{2}}} \quad (0 < a < b) \end{aligned}$$

11.4.40

$$\begin{aligned} \int_0^\infty e^{ibt} Y_0(at) dt &= \frac{2i}{\pi(a^2 - b^2)^{\frac{1}{2}}} \arcsin \frac{b}{a} \quad (0 \leq b < a) \\ &= \frac{-1}{(b^2 - a^2)^{\frac{1}{2}}} + \frac{2i}{\pi(b^2 - a^2)^{\frac{1}{2}}} \\ &\times \ln \left\{ \frac{b - (b^2 - a^2)^{\frac{1}{2}}}{a} \right\} \quad (0 < a < b) \end{aligned}$$

11.4.41

$$\begin{aligned} \int_0^\infty t^{\mu - \nu + 1} J_\mu(at) J_\nu(bt) dt &= 0 \quad (0 < b < a) \\ &= \frac{2^{\mu - \nu + 1} a^\mu (b^2 - a^2)^{\nu - \mu - 1}}{b^\nu \Gamma(\nu - \mu)} \\ &(b > a > 0) \end{aligned}$$

$$(\Re \nu > \Re \mu > -1)$$

$$\begin{aligned} 11.4.42 \quad \int_0^\infty J_\mu(at) J_{\mu-1}(bt) dt &= \frac{b^{\mu-1}}{a^\mu} \quad (0 < b < a) \\ &= \frac{1}{2b} \quad (0 < b = a) \\ &= 0 \quad (b > a > 0) \\ &(\Re \mu > 0) \end{aligned}$$

$$\begin{aligned} 11.4.43 \quad \int_0^\infty \frac{J_0(at)}{t} \{1 - J_0(bt)\} dt &= 0 \quad (0 < b \leq a) \\ &= \ln \frac{b}{a} \quad (b \geq a > 0) \end{aligned}$$

Hankel-Nicholson Type Integrals

11.4.44

$$\int_0^\infty \frac{t^{\nu+1} J_\nu(at) dt}{(t^2+z^2)^{\mu+1}} = \frac{a^\mu z^{\nu-\mu}}{2^\mu \Gamma(\mu+1)} K_{\nu-\mu}(az) \\ \left(a > 0, \Re z > 0, -1 < \Re \nu < 2\Re \mu + \frac{3}{2} \right)$$

11.4.45

$$\int_0^\infty \frac{J_\nu(at) dt}{t^\nu(t^2+z^2)} = \frac{\pi}{2z^{\nu+1}} [I_\nu(az) - L_\nu(az)] \\ \left(a > 0, \Re z > 0, \Re \nu > -\frac{5}{2} \right)$$

11.4.46

$$\int_0^\infty \frac{Y_0(at) dt}{t^2+z^2} = -\frac{K_0(az)}{z} \quad (a > 0, \Re z > 0)$$

11.4.47

$$\int_0^\infty \frac{K_\nu(at) dt}{t^\nu(t^2+z^2)} = \frac{\pi^2}{4z^{\nu+1} \cos \nu\pi} [\mathbf{H}_\nu(az) - Y_\nu(az)] \\ (\Re a > 0, \Re z > 0, \Re \nu < \frac{1}{2})$$

11.4.48

$$\int_0^\infty \frac{J_\nu(at) dt}{(t^2+z^2)^{\frac{1}{2}}} = I_{\frac{1}{2}\nu}(\frac{1}{2}az) K_{\frac{1}{2}\nu}(\frac{1}{2}az) \\ (a > 0, \Re z > 0, \Re \nu > -1)$$

11.4.49

$$\int_0^\infty \frac{J_\nu(at) dt}{t^\nu(t^2+z^2)^{\nu+\frac{1}{2}}} = \frac{\left(\frac{2a}{z^2}\right)^\nu \Gamma(\nu+1)}{\Gamma(2\nu+1)} I_\nu(\frac{1}{2}az) K_\nu(\frac{1}{2}az) \\ (a > 0, \Re z > 0, \Re \nu > -\frac{1}{2})$$

Numerical Methods

11.5. Use and Extension of the Tables

$$\int_0^x J_0(t) dt, \int_0^x Y_0(t) dt, \int_0^x I_0(t) dt, \int_x^\infty K_0(t) dt$$

For moderate values of x , use 11.1.2 and 11.1.7-11.1.10 as appropriate. For x sufficiently large, use the asymptotic expansions or the polynomial approximations 11.1.11-11.1.18.

Example 1. Compute $\int_0^{3.05} J_0(t) dt$ to 5D. Using 11.1.2 and interpolating in Tables 9.1 and 9.2, we have

$$\int_0^{3.05} J_0(t) dt = 2[.32019 \ 09 + .31783 \ 69 + .04611 \ 52 \\ + .00283 \ 19 + .00009 \ 72 + .00000 \ 21] \\ = 1.37415$$

Example 2. Compute $\int_0^{3.05} J_0(t) dt$ to 5D by interpolation of Table 11.1 using Taylor's formula. We have

$$\int_0^{x+h} J_0(t) dt = \int_0^x J_0(t) dt + hJ_0(x) - \frac{h^2}{2} J_1(x) \\ + \frac{h^3}{12} [J_2(x) - J_0(x)] + \frac{h^4}{96} [3J_1(x) - J_3(x)] + \dots$$

Then with $x=3.0$ and $h=.05$,

$$\int_0^{3.05} J_0(t) dt = 1.387567 + (.05)(-.260052) \\ - (.00125)(.339059) \\ + (.000010)(.746143) = 1.37415$$

This value is readily checked using $x=3.1$ and $h=-.05$. Now $|J_0(x)| \leq 1$ for all x and $|J_n(x)| < 2^{-\frac{1}{2}n}$, $n \geq 1$ for all x . In Table 11.1, we can always choose $|h| \leq .05$. Thus if all terms of $O(h^4)$ and higher are neglected, then a bound for the absolute error is $2^{\frac{1}{2}} h^4 / 48 < .2 \cdot 10^{-6}$ for all x if $|h| \leq .05$. Similarly, the absolute error for quadratic interpolation does not exceed

$$h^3(2^{\frac{1}{2}} + 2)/24 < .2 \cdot 10^{-4}.$$

Example 3. Interpolation of $\int_0^x J_0(t) dt$ using Simpson's rule. We have

$$\int_0^{x+h} J_0(t) dt = \int_0^x J_0(t) dt + \int_x^{x+h} J_0(t) dt \\ \int_x^{x+h} J_0(t) dt = \frac{h}{6} \left[J_0(x) + 4J_0\left(x + \frac{h}{2}\right) + J_0(x+h) \right] + R$$

$$R = -\frac{h^5}{2880} J_0^{(5)}(\xi), \quad x < \xi < x+h$$

Now

$$J_0^{(4)}(x) = \frac{1}{8} [J_4(x) - 4J_2(x) + 3J_0(x)]$$

$$|J_0^{(4)}(x)| < \frac{6+5\sqrt{2}}{16} < .82$$

and with $|h| \leq .05$, it follows that

$$|R| < .9 \cdot 10^{-10}$$

Thus if $x=3.0$ and $h=.05$

$$\int_0^{3.05} J_0(t) dt = 1.38756 \ 72520 + \frac{(.05)}{6} [-.26005 \ 19549 \\ + 4(-.26841 \ 13883) - .27653 \ 49599] \\ = 1.37414 \ 86481$$

which is correct to 10D. The above procedure gives high accuracy though it may be necessary to interpolate twice in $J_0(x)$ to compute $J_0\left(x+\frac{h}{2}\right)$ and $J_0(x+h)$. A similar technique based on the trapezoidal rule is less accurate, but at most only one interpolation of $J_0(x)$ is required.

Example 4. Compute $\int_0^3 J_0(t)dt$ and $\int_0^3 Y_0(t)dt$ to 5D using the representation in terms of Struve functions and the tables in chapters 9 and 12.

For $x=3$, from **Tables 9.1 and 12.1**

$$\begin{array}{ll} J_0 = -.260052 & J_1 = .339059 \\ Y_0 = .376850 & Y_1 = .324674 \\ H_0 = .574306 & H_1 = 1.020110 \end{array}$$

Using 11.1.7, we have

$$\begin{aligned} \int_0^3 J_0(t)dt &= 3(-.260052) + \frac{3\pi}{2} [(.574306)(.339059) \\ &\quad - (1.020110)(-.260052)] \\ &= 1.38757 \end{aligned}$$

Similarly,

$$\int_0^3 Y_0(t)dt = .19766$$

Using 11.1.8 and **Tables 9.8 and 12.1**, one can compute $\int_0^x I_0(t)dt$ and $\int_0^x K_0(t)dt$.

$$\int_x^\infty \frac{J_0(t)dt}{t}, \int_x^\infty \frac{Y_0(t)dt}{t}, \int_0^\infty \frac{[I_0(t)-1]dt}{t}, \int_x^\infty \frac{K_0(t)dt}{t}$$

For moderate values of x , use 11.1.19–11.1.23. For x sufficiently large, use the asymptotic expansions or the polynomial approximations 11.1.24–11.1.31.

Repeated Integrals of $J_0(x)$

For moderate values of x and r , use 11.2.4. If $r=1$, see **Example 1**. For moderate values of x , use the recurrence formula 11.2.5. If x is large and $x \gg r$, see the discussion below.

Example 5. Compute $f_{r,0}(x) = f_r(x)$ to 5D for $x=2$ and $r=0(1)5$ using 11.2.6. We have

$$rf_{r+1}(x) = xf_r(x) - (r-1)f_{r-1}(x) + xf_{r-2}(x)$$

$$f_{-1}(x) = -J_1(x), f_0(x) = J_0(x), f_1(x) = \int_0^x J_0(t)dt$$

and the terms on this last line are tabulated. Thus for $x=2$,

$$f_{-1} = -.57672 \ 48, f_0 = .22389 \ 08, f_1 = 1.42577 \ 03$$

The recurrence formula gives

$$f_2 = 2(f_1 + f_{-1}) = 1.69809 \ 10$$

Similarly,

$$f_3 = 1.20909 \ 66, f_4 = .62451 \ 73, f_5 = .25448 \ 17$$

When $x \gg r$, it is convenient to use the auxiliary function

$$g_r(x) = (r-1)!x^{-r+1}f_r(x)$$

This satisfies the recurrence relation

$$\begin{aligned} x^2 g_{r+1}(x) &= x^2 g_r - (r-1)^2 g_{r-1}(x) \\ &\quad + (r-1)(r-2)g_{r-2}(x), \quad r \geq 3 \end{aligned}$$

$$\begin{aligned} g_1(x) &= \int_0^x J_0(t)dt, \quad g_2(x) = g_1(x) - J_1(x) \\ g_3(x) &= [x^2 g_2(x) - g_1(x) + xJ_0(x)]/x^2 \end{aligned}$$

Example 6. Compute $g_r(x)$ to 5D for $x=10$ and $r=0(1)6$. We have for $x=10$,

$$J_0 = -.24593 \ 58, J_1 = .04347 \ 27, g_1 = 1.06701 \ 13$$

Thus

$$g_2 = 1.02353 \ 86, g_3 = .98827 \ 49$$

and the forward recurrence formula gives

$$g_4 = .96867 \ 36, g_5 = .94114 \ 12, g_6 = .90474 \ 64$$

For tables of $2^{-r}f_r(x)$, see [11.16].

Repeated Integrals of $K_0(x)$

For moderate values of x , use the recurrence formula 11.2.14 for all r .

Example 7. Compute $Ki_r(x)$ to 5D for $x=2$ and $r=0(1)5$. We have

$$rKi_{r+1}(x) = -xKi_r(x) + (r-1)Ki_{r-1}(x) + xKi_{r-2}(x)$$

$$Ki_{-1}(x) = K_1(x), Ki_0(x) = K_0(x), Ki_1(x) = \int_x^\infty K_0(t)dt$$

and the functions on this last line are tabulated. Thus for $x=2$,

$$K_0 = .11389 \ 39, K_1 = .13986 \ 59, Ki_1 = .09712 \ 06$$

and

$$Ki_2 = -2Ki_1 + 2K_1 = .08549 \ 06$$

Similarly,

$$Ki_3 = .07696 \ 36, Ki_4 = .07043 \ 17, Ki_5 = .06525 \ 22$$

If x/r is not large the formula can still be used provided that the starting values are sufficiently accurate to offset the growth of rounding error.

For tables of $Ki_r(x)$, see [11.11].

$$f_m(x) = x^{-m} \int_0^x t^m K_0(t) dt$$

Now

$$f_0(x) = \int_0^x K_0(t) dt, f_1(x) = [1 - xK_1(x)]/x$$

the latter following from 11.3.27 with $\nu=1$. In 11.3.5, put $a=1$, $b=-1$, $p=0$ and $\nu=0$. Let $\mu=m$. Then

$$f_m(x) = [(m-1)^2 f_{m-2}(x) - x^2 K_1(x) - x(m-1)K_0(x)]/x^2 \quad (m > 1)$$

Using tabular values of f_0 and f_1 , one can compute in succession $f_2, f_3, \dots$ provided that m/x is not large.

Example 8. Compute $f_m(x)$ to 5D for $x=5$ and $m=0(1)6$. We have, retaining two additional decimals

$$K_0 = .00369 \ 11 \quad K_1 = .00404 \ 46$$

$$f_0 = 1.56738 \ 74 \quad f_1 = .19595 \ 54$$

Thus

$$f_2 = .05791 \ 27, f_4 = .01458 \ 93, f_6 = .00685 \ 36$$

Similarly starting with f_1 , we can compute f_3 and f_5 .

If $m > x$, employ the recurrence formula in backward form and write

$$f_{m-2}(x) = [x^2 f_m(x) + x^2 K_1(x) + x(m-1)K_0(x)]/(m-1)^2$$

In the latter expression, replace f_m by g_m . Fix x . Take $r > m$ and assume $g_r = 0$. Compute g_{r-2}, g_{r-4} , etc. Then

$$\lim_{r \rightarrow \infty} g_{r-2k}(x) = f_m(x), \quad m = r - 2k$$

Apart from round-off error, the value of r needed to achieve a stated accuracy for given x and m can be determined a priori. Let

$$\epsilon_r = |g_r - f_r|$$

Then

$$\epsilon_{r-2k} = \frac{x^{2k} \epsilon_r}{(r-1)^2 (r-3)^2 \dots (r-2k+1)^2}$$

$$\epsilon_r \leq [x^2 K_1(x) + x(r-1)K_0(x)]/(r-1)^2$$

since for x fixed, $f_r(x)$ is positive and decreases as r increases.

Example 9. Compute $f_m(x)$ to 5D for $x=3$ and $m=0(2)10$. We have

$$K_0 = .03473 \ 95 \quad K_1 = .04015 \ 64$$

If $r=16$,

$$\epsilon_{16} < .86 \cdot 10^{-2} \quad \epsilon_{10} < 1.4 \cdot 10^{-6}$$

Taking $g_{16}=0$, we compute the following values of $g_{14}, g_{12}, \dots, g_0$ by recurrence. Also recorded are the required values of f_m to 5D.

m	g_m	f_m
14	.00855 42	
12	.01061 09	
10	.01325 05	.01325
8	.01751 39	.01751
6	.02548 09	.02548
4	.04447 31	.04447
2	.11936 90	.11937
0	1.53994 71	1.53995

For tables of $f_m(x)$, see [11.21].

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Tables

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12. Struve Functions and Related Functions

MILTON ABRAMOWITZ¹

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¹ National Bureau of Standards. (Deceased.)

12. Struve Functions and Related Functions

Mathematical Properties

12.1. Struve Function $\mathbf{H}_\nu(z)$

Differential Equation and General Solution

12.1.1

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2)w = \frac{4(\frac{1}{2}z)^{\nu+1}}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})}$$

The general solution is

12.1.2 $w = aJ_\nu(z) + bY_\nu(z) + \mathbf{H}_\nu(z)$ (a, b , constants)

where $z^{-\nu}\mathbf{H}_\nu(z)$ is an entire function of z .

Power Series Expansion

12.1.3

$$\mathbf{H}_\nu(z) = (\frac{1}{2}z)^{\nu+1} \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2}z)^{2k}}{\Gamma(k + \frac{3}{2})\Gamma(k + \nu + \frac{3}{2})}$$

12.1.4 $\mathbf{H}_0(z) = \frac{2}{\pi} \left[z - \frac{z^3}{1^2 \cdot 3^2} + \frac{z^5}{1^2 \cdot 3^2 \cdot 5^2} - \dots \right]$

12.1.5

$$\mathbf{H}_1(z) = \frac{2}{\pi} \left[\frac{z^2}{1^2 \cdot 3} - \frac{z^4}{1^2 \cdot 3^2 \cdot 5} + \frac{z^6}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7} - \dots \right]$$

Integral Representations

If $\Re \nu > -\frac{1}{2}$,

12.1.6

$$\mathbf{H}_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \sin(zt) dt$$

12.1.7 $= \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^{\frac{\pi}{2}} \sin(z \cos \theta) \sin^{2\nu} \theta d\theta$

12.1.8 $= Y_\nu(z)$

$$+ \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{-zt} (1+t^2)^{\nu-\frac{1}{2}} dt$$

($|\arg z| < \frac{\pi}{2}$)

Recurrence Relations

12.1.9 $\mathbf{H}_{\nu-1} + \mathbf{H}_{\nu+1} = \frac{2\nu}{z} \mathbf{H}_\nu + \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{3}{2})}$

12.1.10 $\mathbf{H}_{\nu-1} - \mathbf{H}_{\nu+1} = 2\mathbf{H}'_\nu - \frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{3}{2})}$

12.1.11

$$\mathbf{H}'_0 = (2/\pi) - \mathbf{H}_1$$

12.1.12

$$\frac{d}{dz} (z^\nu \mathbf{H}_\nu) = z^\nu \mathbf{H}_{\nu-1}$$

12.1.13

$$\frac{d}{dz} (z^{-\nu} \mathbf{H}_\nu) = \frac{1}{\sqrt{\pi} 2^\nu \Gamma(\nu + \frac{3}{2})} - z^{-\nu} \mathbf{H}_{\nu+1}$$

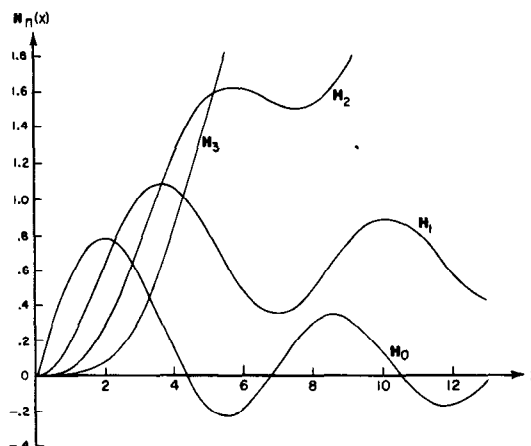


FIGURE 12.1. Struve functions.

$\mathbf{H}_n(x)$, $n=0(1)3$

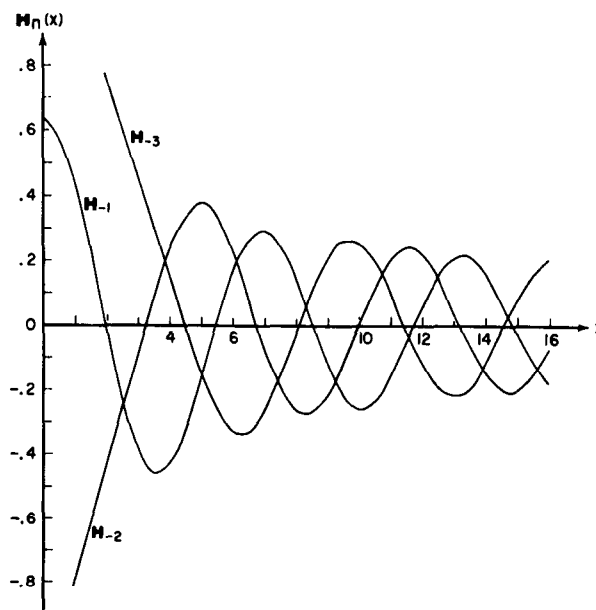


FIGURE 12.2. Struve functions.

$\mathbf{H}_n(x)$, $-n=1(1)3$

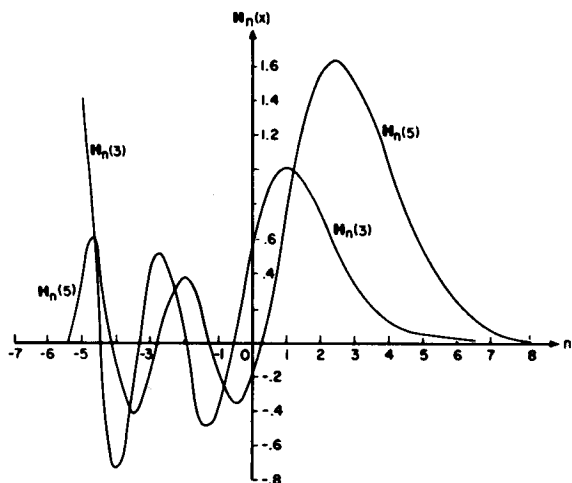


FIGURE 12.3. Struve functions.

$$\mathbf{H}_n(x), x=3, 5$$

Special Properties

$$12.1.14 \quad \mathbf{H}_\nu(x) \geq 0 \quad (x > 0 \text{ and } \nu \geq \tfrac{1}{2})$$

$$12.1.15$$

$$\mathbf{H}_{-(n+\frac{1}{2})}(z) = (-1)^n J_{n+\frac{1}{2}}(z) \quad (n \text{ an integer} \geq 0)$$

$$12.1.16 \quad \mathbf{H}_1(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} (1 - \cos z)$$

$$12.1.17$$

$$\mathbf{H}_1(z) = \left(\frac{z}{2\pi}\right)^{\frac{1}{2}} \left(1 + \frac{2}{z^2}\right) - \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left(\sin z + \frac{\cos z}{z}\right)$$

$$12.1.18 \quad \mathbf{H}_\nu(ze^{m\pi i}) = e^{m(\nu+1)\pi i} \mathbf{H}_\nu(z) \quad (m \text{ an integer})$$

$$12.1.19 \quad \mathbf{H}_0(z) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{J_{2k+1}(z)}{2k+1}$$

$$12.1.20 \quad \mathbf{H}_1(z) = \frac{2}{\pi} - \frac{2}{\pi} J_0(z) + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{J_{2k}(z)}{4k^2 - 1}$$

$$12.1.21 \quad \mathbf{H}_\nu(z) = \frac{2(z/2)^{\nu+1}}{\sqrt{\pi} \Gamma(\nu + \frac{3}{2})} {}_1F_2\left(1; \frac{3}{2} + \nu, \frac{3}{2} - \frac{z^2}{4}\right)$$

Integrals (See chapter 11)

$$12.1.22 \quad \int_0^{\infty} t^{-1} \mathbf{H}_0(t) dt = \frac{\pi}{2}$$

$$12.1.23$$

$$\int_0^{\infty} \mathbf{H}_0(t) dt = \frac{2}{\pi} \left[\frac{z^2}{2} - \frac{z^4}{1^2 \cdot 3^2 \cdot 4} + \frac{z^6}{1^2 \cdot 3^2 \cdot 5^2 \cdot 6} - \dots \right]$$

$$12.1.24 \quad \int_0^{\infty} t^{-\nu} \mathbf{H}_{\nu+1}(t) dt = \frac{z}{2^{\nu} \sqrt{\pi} \Gamma(\nu + \frac{3}{2})} - z^{-\nu} \mathbf{H}_{\nu}(z)$$

Struve's Integral

$$12.1.25$$

$$\frac{4}{\pi} \int_0^{\infty} t^{-2} \mathbf{H}_1(t) dt = \frac{2}{\pi z} \mathbf{H}_1(z) + \frac{2}{\pi} \int_0^{\infty} t^{-1} \mathbf{H}_0(t) dt$$

$$12.1.26$$

$$\frac{2}{\pi} \int_0^{\infty} t^{-1} \mathbf{H}_0(t) dt = 1 - \frac{4}{\pi^2} \left[z - \frac{z^3}{1^2 \cdot 3^2 \cdot 3} + \frac{z^5}{1^2 \cdot 3^2 \cdot 5^2 \cdot 5} - \dots \right]$$

$$12.1.27$$

$$\int_0^{\infty} t^{\mu-\nu-1} \mathbf{H}_{\nu}(t) dt = \frac{\Gamma(\frac{1}{2}\mu) 2^{\mu-\nu-1} \tan(\frac{1}{2}\pi\mu)}{\Gamma(\nu - \frac{1}{2}\mu + 1)} \quad (|\Re \mu| < 1, \Re \nu > \Re \mu - \frac{3}{2})$$

$$\text{If } f_{\nu}(z) = \int_0^{\infty} \mathbf{H}_{\nu}(t) t^{\nu} dt$$

$$12.1.28$$

$$f_{\nu+1} = (2\nu+1)f_{\nu}(z) - z^{\nu+1} \mathbf{H}_{\nu}(z) + \frac{z^{2\nu+2}}{(\nu+1)2^{\nu+1}\Gamma(\frac{1}{2})\Gamma(\nu+\frac{3}{2})} \quad (\Re \nu > -\frac{1}{2})$$

 Asymptotic Expansions for Large $|z|$

$$12.1.29$$

$$\mathbf{H}_{\nu}(z) - Y_{\nu}(z) = \frac{1}{\pi} \sum_{k=0}^{m-1} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\nu+\frac{1}{2}-k)} \left(\frac{z}{2}\right)^{2k-\nu+1} + R_m \quad (|\arg z| < \pi)$$

where $R_m = O(|z|^{-2m-1})$. If ν is real, z positive * and $m + \frac{1}{2} - \nu \geq 0$, the remainder after m terms is of the same sign and numerically less than the first term neglected.

$$12.1.30$$

$$\mathbf{H}_0(z) - Y_0(z) \sim \frac{2}{\pi} \left[\frac{1}{z} - \frac{1}{z^3} + \frac{1^2 \cdot 3^2}{z^5} - \frac{1^2 \cdot 3^2 \cdot 5^2}{z^7} + \dots \right] \quad (|\arg z| < \pi)$$

$$12.1.31$$

$$\mathbf{H}_1(z) - Y_1(z) \sim \frac{2}{\pi} \left[1 + \frac{1}{z^2} - \frac{1^2 \cdot 3}{z^4} + \frac{1^2 \cdot 3^2 \cdot 5}{z^6} - \dots \right] \quad (|\arg z| < \pi)$$

$$12.1.32$$

$$\int_0^{\infty} [\mathbf{H}_0(t) - Y_0(t)] dt = \frac{2}{\pi} [\ln(2z) + \gamma] \sim \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (2k)! (2k-1)!}{(k!)^2 (2z)^{2k}} \quad (|\arg z| < \pi)$$

where $\gamma = .57721\ 56649 \dots$ is Euler's constant.

$$12.1.33$$

$$\int_0^{\infty} t^{-1} [\mathbf{H}_0(t) - Y_0(t)] dt \sim \frac{2}{\pi z} \sum_{k=0}^{\infty} \frac{(-1)^k [(2k)!]^2}{(k!)^2 (2k+1) (2z)^{2k}} \quad (|\arg z| < \pi)$$

*See page 11.

Asymptotic Expansions for Large Orders

12.1.34

$$\mathbf{H}_\nu(z) - Y_\nu(z) \sim \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \sum_{k=0}^{\infty} \frac{k!b_k}{z^{k+1}} \quad (|\arg z| < \frac{1}{2}\pi, |\nu| < |z|)$$

$$b_0=1, b_1=2\nu/z, b_2=6(\nu/z)^2-\frac{1}{2}, b_3=20(\nu/z)^3-4(\nu/z)$$

12.1.35

$$\mathbf{H}_\nu(z) + iJ_\nu(z) \sim \frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \sum_{k=0}^{\infty} \frac{k!b_k}{z^{k+1}} \quad (|\nu| > |z|)$$

12.2. Modified Struve Function $\mathbf{L}_\nu(z)$

Power Series Expansion

$$\begin{aligned} 12.2.1 \quad \mathbf{L}_\nu(z) &= -ie^{-\frac{i\pi\nu}{2}} \mathbf{H}_\nu(iz) \\ &= (\frac{1}{2}z)^{\nu+1} \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+\nu+\frac{3}{2})} \end{aligned}$$

Integral Representations

$$12.2.2 \quad \mathbf{L}_\nu(z) = \frac{2(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \int_0^{\frac{\pi}{2}} \sinh(z \cos \theta) \sin^{2\nu} \theta d\theta \quad (\Re \nu > -\frac{1}{2})$$

12.2.3

$$I_{-\nu}(x) - \mathbf{L}_\nu(x) = \frac{2(x/2)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \int_0^\infty \sin(tx) (1+t^2)^{\nu-1} dt \quad (\Re \nu < \frac{1}{2}, x > 0)$$

Recurrence Relations

$$12.2.4 \quad \mathbf{L}_{\nu-1} - \mathbf{L}_{\nu+1} = \frac{2\nu}{z} \mathbf{L}_\nu + \frac{(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})}$$

$$12.2.5 \quad \mathbf{L}_{\nu-1} + \mathbf{L}_{\nu+1} = 2\mathbf{L}'_\nu - \frac{(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})}$$

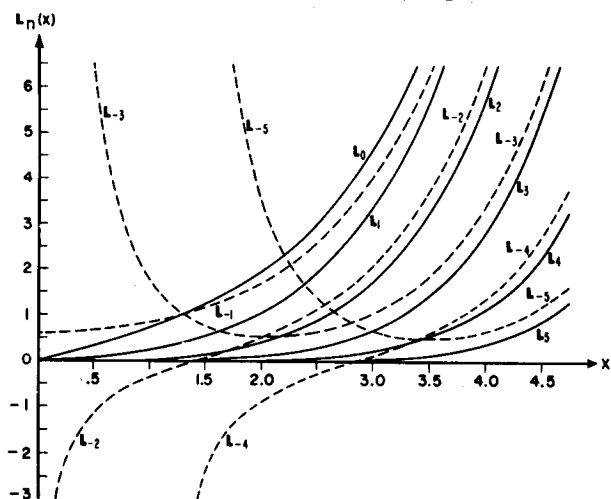


FIGURE 12.4. Modified Struve functions.

$$\mathbf{L}_n(x), \pm n = 0(1)5$$

Asymptotic Expansion for Large $|z|$ 12.2.6 $\mathbf{L}_\nu(z) - I_{-\nu}(z)$

$$\sim \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \Gamma(k+\frac{1}{2})}{\Gamma(\nu+\frac{1}{2}-k) \left(\frac{z}{2}\right)^{2k-\nu+1}} \quad (|\arg z| < \frac{1}{2}\pi)$$

Integrals

12.2.7

$$\int_0^z \mathbf{L}_0(t) dt = \frac{2}{\pi} \left[\frac{z^2}{2} + \frac{z^4}{1^2 \cdot 3^2 \cdot 4} + \frac{z^6}{1^2 \cdot 3^2 \cdot 5^2 \cdot 6} + \dots \right]$$

$$12.2.8 \quad \int_0^z [I_0(t) - \mathbf{L}_0(t)] dt = \frac{2}{\pi} [\ln(2z) + \gamma]$$

$$\sim -\frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(2k)! (2k-1)!}{(k!)^2 (2z)^{2k}} \quad (|\arg z| < \frac{1}{2}\pi)$$

$$12.2.9 \quad \int_0^z \mathbf{L}_1(t) dt = \mathbf{L}_0(z) - \frac{2}{\pi} z$$

Relation to Modified Spherical Bessel Function

$$12.2.10 \quad \mathbf{L}_{-(n+\frac{1}{2})}(z) = I_{(n+\frac{1}{2})}(z) \quad (n \text{ an integer } \geq 0)$$

12.3. Anger and Weber Functions

Anger's Function

$$12.3.1 \quad \mathbf{J}_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(\nu\theta - z \sin \theta) d\theta$$

$$12.3.2 \quad \mathbf{J}_n(z) = J_n(z) \quad (n \text{ an integer})$$

Weber's Function

$$12.3.3 \quad \mathbf{E}_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin(\nu\theta - z \sin \theta) d\theta$$

Relations Between Anger's and Weber's Function

$$12.3.4 \quad \sin(\nu\pi) \mathbf{J}_\nu(z) = \cos(\nu\pi) \mathbf{E}_\nu(z) - \mathbf{E}_{-\nu}(z)$$

$$12.3.5 \quad \sin(\nu\pi) \mathbf{E}_\nu(z) = \mathbf{J}_{-\nu}(z) - \cos(\nu\pi) \mathbf{J}_\nu(z)$$

Relations Between Weber's Function and Struve's Function

If n is a positive integer or zero,

$$12.3.6 \quad \mathbf{E}_n(z) = \frac{1}{\pi} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{\Gamma(k+\frac{1}{2}) (\frac{1}{2}z)^{n-2k-1}}{\Gamma(n+\frac{1}{2}-k)} - \mathbf{H}_n(z) \quad *$$

12.3.7

$$\mathbf{E}_{-n}(z) = \frac{(-1)^{n+1}}{\pi} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{\Gamma(n-k-\frac{1}{2}) (\frac{1}{2}z)^{-n+2k+1}}{\Gamma(k+\frac{3}{2})} - \mathbf{H}_{-n}(z) \quad *$$

*See page II.

$$12.3.8 \quad \mathbf{E}_0(z) = -\mathbf{H}_0(z)$$

$$12.3.9 \quad \mathbf{E}_1(z) = \frac{2}{\pi} - \mathbf{H}_1(z)$$

$$12.3.10 \quad \mathbf{E}_2(z) = \frac{2z}{3\pi} - \mathbf{H}_2(z)$$

Numerical Methods

12.4. Use and Extension of the Tables

Example 1. Compute $\mathbf{L}_0(2)$ to 6D. From **Table 12.1** $I_0(2) - \mathbf{L}_0(2) = .342152$; from **Table 9.11** we have $I_0(2) = 2.279585$ so that $\mathbf{L}_0(2) = 1.937433$.

Example 2. Compute $\mathbf{H}_0(10)$ to 6D. From **Table 12.2** for $x^{-1} = .1$, $\mathbf{H}_0(10) - Y_0(10) = .063072$; from **Table 9.1** we have $Y_0(10) = .055671$. Thus, $\mathbf{H}_0(10) = .118743$.

Example 3. Compute $\int_0^x \mathbf{H}_0(t) dt$ for $x=6$ to 5D. Using **Tables 12.2**, **11.1** and **4.2**, we have

$$\int_0^6 \mathbf{H}_0(t) dt = \int_0^6 Y_0(t) dt + \frac{2}{\pi} \ln 6 + f_1(6)$$

$$= -.125951 + (.636620)(1.791759) \\ + .816764$$

$$= 1.83148$$

Example 4. Compute $\mathbf{H}_n(x)$ for $x=4$, $-n=0(1)8$ to 6S. From **Table 12.1** we have $\mathbf{H}_0(4) = .1350146$, $\mathbf{H}_1(4) = 1.0697267$. Using **12.1.9** we find

$$\begin{array}{ll} \mathbf{H}_{-1}(4) = -.433107 & \mathbf{H}_{-5}(4) = .689652 \\ \mathbf{H}_{-2}(4) = .240694 & \mathbf{H}_{-6}(4) = -1.21906 \\ \mathbf{H}_{-3}(4) = .152624 & \mathbf{H}_{-7}(4) = 2.82066 \\ \mathbf{H}_{-4}(4) = -.439789 & \mathbf{H}_{-8}(4) = -8.24933 \end{array}$$

Example 5. Compute $\mathbf{H}_n(x)$ for $x=4$, $n=0(1)10$ to 7S. Starting with the values of $\mathbf{H}_0(4)$ and $\mathbf{H}_1(4)$ and using **12.1.9** with forward recurrence, we get

$$\begin{array}{ll} \mathbf{H}_0(4) = .1350146 & \mathbf{H}_6(4) = .0543354 \\ \mathbf{H}_1(4) = 1.0697267 & \mathbf{H}_7(4) = .0151037 \\ \mathbf{H}_2(4) = 1.2486751 & \mathbf{H}_8(4) = .0036733 \\ \mathbf{H}_3(4) = .8580095 & \mathbf{H}_9(4) = .0008002 \\ \mathbf{H}_4(4) = .4263741 & \mathbf{H}_{10}(4) = .0001825 \\ \mathbf{H}_5(4) = .1671987 & \end{array}$$

We note that for $n > 6$ there is a rapid loss of significant figures. On the other hand using **12.1.3** for $x=4$ we find $\mathbf{H}_9(4) = .0007935729$, $\mathbf{H}_{10}(4) = .00015447630$ and backward recurrence with **12.1.9** gives

$$\begin{array}{ll} \mathbf{H}_8(4) = .003671495 & \mathbf{H}_3(4) = .8580094 \\ \mathbf{H}_7(4) = .01510315 & \mathbf{H}_2(4) = 1.2486751 \\ \mathbf{H}_6(4) = .05433519 & \mathbf{H}_1(4) = 1.0697267 \\ \mathbf{H}_5(4) = .1671987 & \mathbf{H}_0(4) = .1350146 \\ \mathbf{H}_4(4) = .4263743 & \end{array}$$

Example 6. Compute $\mathbf{L}_n(.5)$ for $n=0(1)5$ to 8S. From **12.2.1** we find $\mathbf{L}_5(.5) = 9.6307462 \times 10^{-7}$, $\mathbf{L}_4(.5) = 2.1212342 \times 10^{-5}$. Then, with **12.2.4** we get

$$\begin{array}{ll} \mathbf{L}_3(.5) = 3.8246503 \times 10^{-4} & \mathbf{L}_1(.5) = .053942181 \\ \mathbf{L}_2(.5) = 5.3686734 \times 10^{-3} & \mathbf{L}_0(.5) = .32724068 \end{array}$$

Example 7. Compute $\mathbf{L}_n(.5)$ for $-n=0(1)5$ to 6S. From **Tables 12.1** and **9.8** we find $\mathbf{L}_0(.5) = .327240$, $\mathbf{L}_1(.5) = .053942$. Then employing **12.2.4** with backward recurrence we get

$$\begin{array}{ll} \mathbf{L}_{-1}(.5) = .690562 & \mathbf{L}_{-4}(.5) = -75.1418 \\ \mathbf{L}_{-2}(.5) = -1.16177 & \mathbf{L}_{-5}(.5) = 1056.92 \\ \mathbf{L}_{-3}(.5) = 7.43824 & \end{array}$$

Example 8. Compute $\mathbf{L}_n(x)$ for $x=6$ and $-n=0(1)6$ to 8S. From **Tables 12.2** and **9.8** we find $\mathbf{L}_0(6) = 67.124454$, $\mathbf{L}_1(6) = 60.725011$. Using **12.2.4** we get

$$\begin{array}{ll} \mathbf{L}_{-1}(6) = 61.361631 & \mathbf{L}_{-4}(6) = 16.626028 \\ \mathbf{L}_{-2}(6) = 46.776680 & \mathbf{L}_{-5}(6) = 7.984089 \\ \mathbf{L}_{-3}(6) = 30.159494 & \mathbf{L}_{-6}(6) = 3.32780 \end{array}$$

We note that there is no essential loss of accuracy until $n = -6$. However, if further values were necessary the recurrence procedure becomes unstable. To avoid the instability use the methods described in **Examples 5** and **6**.

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Tables

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Table 12.2

STRUVE FUNCTIONS FOR LARGE ARGUMENTS

x^{-1}	$H_0(x) - Y_0(x)$	$H_1(x) - Y_1(x)$	$f_1(x)$	$I_0(x) - L_0(x)$	$I_1(x) - L_1(x)$	$f_2(x)$	$f_3(x)$	$\langle x \rangle$
0.20	0.123301	0.659949	0.819924	0.133955	0.607426	0.793280	0.125868	5
0.19	0.117449	0.657819	0.818935	0.126683	0.610467	0.794902	0.119694	5
0.18	0.111556	0.655774	0.817981	0.119468	0.613348	0.796448	0.113505	6
0.17	0.105625	0.653818	0.817062	0.112319	0.616060	0.797910	0.107299	6
0.16	0.099655	0.651952	0.816182	0.105242	0.618598	0.799279	0.101079	6
0.15	0.093647	0.650180	0.815341	0.098241	0.620955	0.800551	0.094843	7
0.14	0.087602	0.648504	0.814541	0.091318	0.623129	0.801721	0.088593	7
0.13	0.081521	0.646927	0.813785	0.084474	0.625119	0.802787	0.082328	8
0.12	0.075404	0.645452	0.813074	0.077706	0.626927	0.803750	0.076051	8
0.11	0.069254	0.644081	0.812411	0.071010	0.628558	0.804611	0.069761	9
0.10	0.063072	0.642817	0.811796	0.064379	0.630018	0.805374	0.063460	10
0.09	0.056860	0.641663	0.811232	0.057805	0.631315	0.806047	0.057147	11
0.08	0.050620	0.640622	0.810722	0.051279	0.632457	0.806634	0.050824	13
0.07	0.044354	0.639696	0.810266	0.044793	0.633450	0.807140	0.044492	14
0.06	0.038064	0.638888	0.809866	0.038340	0.634302	0.807572	0.038152	17
0.05	0.031753	0.638200	0.809525	0.031912	0.635016	0.807933	0.031805	20
0.04	0.025425	0.637634	0.809244	0.025506	0.635596	0.808225	0.025451	25
0.03	0.019082	0.637191	0.809023	0.019116	0.636045	0.808450	0.019093	33
0.02	0.012727	0.636874	0.808865	0.012738	0.636365	0.808611	0.012731	50
0.01	0.006366	0.636683	0.808770	0.006367	0.636556	0.808706	0.006366	100
0.00	0.000000	0.636620	0.808738	0.000000	0.636620	0.808738	0.000000	∞
	$\begin{bmatrix} (-6)5 \\ 3 \end{bmatrix}$	$\begin{bmatrix} (-5)2 \\ 3 \end{bmatrix}$	$\begin{bmatrix} (-6)8 \\ 3 \end{bmatrix}$	$\begin{bmatrix} (-5)1 \\ 3 \end{bmatrix}$	$\begin{bmatrix} (-5)2 \\ 3 \end{bmatrix}$	$\begin{bmatrix} (-5)1 \\ 3 \end{bmatrix}$	$\begin{bmatrix} (-6)2 \\ 3 \end{bmatrix}$	

$$\int_0^x [H_0(t) - Y_0(t)] dt = \frac{2}{\pi} \ln x + f_1(x)$$

$$\int_0^x [L_0(t) - I_0(t)] dt = \frac{2}{\pi} \ln x + f_2(x)$$

$$\int_x^\infty \left[\frac{H_0(t) - Y_0(t)}{t} \right] dt = f_3(x)$$

$\langle x \rangle$ = nearest integer to x .

Starting with $H_0(x)$ and $H_1(x)$, recurrence formula 12.1.9 may be used to generate $H_n(x)$ for $n < 0$. As long as $n < x/2$ (approx.), $H_n(x)$ may be generated by forward recurrence. When $n > x/2$, forward recurrence is unstable. To avoid the instability, choose $n > x$, compute $H_k(x)$ and $H_{k+1}(x)$ with 12.1.3, and then use backward recurrence with 12.1.9.

If $n > 0$, $L_n(x)$ must be generated by backward recurrence. If $n < 0$, $L_n(x)$ may be generated by backward recurrence as long as $L_n(x)$ increases. If $n < 0$ and $L_n(x)$ is decreasing, forward recurrence should be used.

See Examples 4–8.

13. Confluent Hypergeometric Functions

LUCY JOAN SLATER ¹

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The tables were calculated by the author on the electronic calculator EDSACI in the Mathematical Laboratory of Cambridge University, by kind permission of its director, Dr. M. V. Wilkes. The table of $M(a, b, x)$ was recomputed by Alfred E. Beam for uniformity to eight significant figures.

¹ University Mathematical Laboratory, Cambridge. (Prepared under contract with the National Bureau of Standards.)

13. Confluent Hypergeometric Functions

Mathematical Properties

13.1. Definitions of Kummer and Whittaker Functions

Kummer's Equation

$$13.1.1 \quad z \frac{d^2 w}{dz^2} + (b-z) \frac{dw}{dz} - aw = 0$$

It has a regular singularity at $z=0$ and an irregular singularity at ∞ .

Independent solutions are

Kummer's Function

13.1.2

$$M(a, b, z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \dots + \frac{(a)_n z^n}{(b)_n n!} + \dots$$

where

$$(a)_n = a(a+1)(a+2) \dots (a+n-1), (a)_0 = 1,$$

and

13.1.3

$$U(a, b, z) = \frac{\pi}{\sin \pi b} \left\{ \frac{M(a, b, z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b} \frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)} \right\}$$

Parameters
(m, n positive integers)

$$b \neq -n \quad a \neq -m$$

$M(a, b, z)$
a convergent series for
all values of a, b and z
a polynomial of degree m
in z

$$b \neq -n \quad a = -m$$

$$b = -n \quad a \neq -m$$

$$b = -n \quad a = -m,$$

$$m > n$$

a simple pole at $b = -n$

$$b = -n \quad a = -m, \quad \text{undefined}$$

$$m \leq n$$

$U(a, b, z)$ is defined even when $b \rightarrow \pm n$

As $|z| \rightarrow \infty$,

13.1.4

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} [1 + O(|z|^{-1})] \quad (\Re z > 0)$$

and

13.1.5

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^{-a} [1 + O(|z|^{-1})] \quad (\Re z < 0)$$

$U(a, b, z)$ is a many-valued function. Its principal branch is given by $-\pi < \arg z \leq \pi$.

Logarithmic Solution

13.1.6

$$U(a, n+1, z) = \frac{(-1)^{n+1}}{n! \Gamma(a-n)} \left[M(a, n+1, z) \ln z + \sum_{r=0}^n \frac{(a)_r z^r}{(n+1)_r r!} \{ \psi(a+r) - \psi(1+r) - \psi(1+n+r) \} \right] + \frac{(n-1)!}{\Gamma(a)} z^{-n} M(a-n, 1-n, z)_n$$

for $n=0, 1, 2, \dots$, where the last function is the sum to n terms. It is to be interpreted as zero when $n=0$, and $\psi(a) = \Gamma'(a)/\Gamma(a)$.

$$13.1.7 \quad U(a, 1-n, z) = z^n U(a+n, 1+n, z)$$

As $\Re z \rightarrow \infty$

$$13.1.8 \quad U(a, b, z) = z^{-a} [1 + O(|z|^{-1})]$$

Analytic Continuation

13.1.9

$$U(a, b, z e^{\pm \pi i}) = \frac{\pi}{\sin \pi b} e^{-z} \left\{ \frac{M(b-a, b, z)}{\Gamma(1+a-b)\Gamma(b)} - \frac{e^{\pm \pi i(1-b)} z^{1-b} M(1-a, 2-b, z)}{\Gamma(a)\Gamma(2-b)} \right\}$$

where either upper or lower signs are to be taken throughout.

13.1.10

$$U(a, b, z e^{2\pi i n}) = [1 - e^{-2\pi i b n}] \frac{\Gamma(1-b)}{\Gamma(1+a-b)} M(a, b, z) + e^{-2\pi i b n} U(a, b, z)$$

Alternative Notations

$${}_1F_1(a; b; z) \text{ or } \Phi(a; b; z) \text{ for } M(a, b, z)$$

$$z^{-a} {}_2F_0(a, 1+a-b; -; -1/z) \text{ or } \Psi(a; b; z) \text{ for } U(a, b, z)$$

Complete Solution

$$13.1.11 \quad y = AM(a, b, z) + BU(a, b, z)$$

where A and B are arbitrary constants, $b \neq -n$.

Eight Solutions

$$13.1.12 \quad y_1 = M(a, b, z)$$

$$13.1.13 \quad y_2 = z^{1-b} M(1+a-b, 2-b, z)$$

$$13.1.14 \quad y_3 = e^z M(b-a, b, -z)$$

$$13.1.15 \quad y_4 = z^{1-b} e^z M(1-a, 2-b, -z)$$

$$13.1.16 \quad y_5 = U(a, b, z)$$

$$13.1.17 \quad y_6 = z^{1-b} U(1+a-b, 2-b, z)$$

$$13.1.18 \quad y_7 = e^z U(b-a, b, -z)$$

$$13.1.19 \quad y_8 = z^{1-b} e^z U(1-a, 2-b, -z)$$

Wronskians

$$\begin{aligned} \text{If } W\{m, n\} &= y_m y'_n - y_n y'_m \text{ and} \\ \epsilon &= \operatorname{sgn}(\mathcal{J}z) = 1 \text{ if } \mathcal{J}z > 0, \\ &= -1 \text{ if } \mathcal{J}z \leq 0 \end{aligned}$$

13.1.20

$$W\{1, 2\} = W\{3, 4\} = W\{1, 4\} = -W\{2, 3\} = (1-b)z^{-b} e^z$$

13.1.21

$$W\{1, 3\} = W\{2, 4\} = W\{5, 6\} = W\{7, 8\} = 0$$

$$13.1.22 \quad W\{1, 5\} = -\Gamma(b)z^{-b} e^z / \Gamma(a)$$

$$13.1.23 \quad W\{1, 7\} = \Gamma(b) e^{\pi i b} z^{-b} e^z / \Gamma(b-a)$$

$$13.1.24 \quad W\{2, 5\} = -\Gamma(2-b)z^{-b} e^z / \Gamma(1+a-b)$$

$$13.1.25 \quad W\{2, 7\} = -\Gamma(2-b)z^{-b} e^z / \Gamma(1-a)$$

$$13.1.26 \quad W\{5, 7\} = e^{\pi i(b-a)} z^{-b} e^z$$

Kummer Transformations

$$13.1.27 \quad M(a, b, z) = e^z M(b-a, b, -z)$$

13.1.28

$$z^{1-b} M(1+a-b, 2-b, z) = z^{1-b} e^z M(1-a, 2-b, -z)$$

$$13.1.29 \quad U(a, b, z) = z^{1-b} U(1+a-b, 2-b, z)$$

13.1.30

$$e^z U(b-a, b, -z) = e^{\pi i(1-b)} e^z z^{1-b} U(1-a, 2-b, -z)$$

Whittaker's Equation

$$13.1.31 \quad \frac{d^2 w}{dz^2} + \left[-\frac{1}{4} + \frac{\kappa}{z} + \frac{(\frac{1}{2} - \mu^2)}{z^2} \right] w = 0$$

Solutions:

Whittaker's Functions

$$13.1.32 \quad M_{\kappa, \mu}(z) = e^{-\frac{1}{2}z} z^{\frac{1}{2}+\mu} M(\frac{1}{2}+\mu-\kappa, 1+2\mu, z)$$

13.1.33

$$\begin{aligned} W_{\kappa, \mu}(z) &= e^{-\frac{1}{2}z} z^{\frac{1}{2}+\mu} U(\frac{1}{2}+\mu-\kappa, 1+2\mu, z) \\ & \quad (-\pi < \arg z \leq \pi, \kappa = \frac{1}{2}b-a, \mu = \frac{1}{2}b-\frac{1}{2}) \end{aligned}$$

13.1.34

$$W_{\kappa, \mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\kappa)} M_{\kappa, \mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\kappa)} M_{\kappa, -\mu}(z)$$

General Confluent Equation

13.1.35

$$\begin{aligned} w'' + \left[\frac{2A}{Z} + 2f' + \frac{bh'}{h} - h' - \frac{h''}{h'} \right] w' \\ + \left[\left(\frac{bh'}{h} - h' - \frac{h''}{h'} \right) \left(\frac{A}{Z} + f' \right) + \frac{A(A-1)}{Z^2} \right. \\ \left. + \frac{2Af'}{Z} + f'' + f'^2 - \frac{ah'^2}{h} \right] w = 0 \end{aligned}$$

Solutions:

$$13.1.36 \quad Z^{-A} e^{-f(Z)} M(a, b, h(Z))$$

$$13.1.37 \quad Z^{-A} e^{-f(Z)} U(a, b, h(Z))$$

13.2. Integral Representations

$$\Re b > \Re a > 0$$

13.2.1

$$\begin{aligned} \frac{\Gamma(b-a)\Gamma(a)}{\Gamma(b)} M(a, b, z) \\ = \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt \end{aligned}$$

13.2.2

$$= 2^{1-b} e^{\frac{1}{2}z} \int_{-1}^{+1} e^{-\frac{1}{2}zt} (1+t)^{b-a-1} (1-t)^{a-1} dt$$

13.2.3

$$= 2^{1-b} e^{\frac{1}{2}z} \int_0^\pi e^{-\frac{1}{2}z \cos \theta} \sin^{b-1} \theta \cot^{b-2a}(\frac{1}{2}\theta) d\theta$$

13.2.4

$$\begin{aligned} = e^{-Az} \int_A^B e^{zt} (t-A)^{a-1} (B-t)^{b-a-1} dt \\ (A=B-1) \\ \Re a > 0, \Re z > 0 \end{aligned}$$

13.2.5

$$\Gamma(a) U(a, b, z) = \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1} dt$$

13.2.6

$$= e^z \int_1^\infty e^{-zt} (t-1)^{a-1} t^{b-a-1} dt$$

13.2.7

$$= 2^{1-b} e^{\frac{1}{2}z} \int_0^\infty e^{-\frac{1}{2}z \cosh \theta} \sinh^{b-1} \theta \coth^{b-2a}(\frac{1}{2}\theta) d\theta \quad *$$

*See page II.

13.2.8 $\Gamma(a)U(a, b, z)$

$$= e^{Az} \int_A^\infty e^{-zt} (t-A)^{a-1} (t+B)^{b-a-1} dt$$

$$(A=1-B)$$

Similar integrals for $M_{\kappa, \mu}(z)$ and $W_{\kappa, \mu}(z)$ can be deduced with the help of 13.1.32 and 13.1.33.

Barnes-type Contour Integrals

13.2.9

$$\frac{\Gamma(a)}{\Gamma(b)} M(a, b, z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(-s)\Gamma(a+s)}{\Gamma(b+s)} (-z)^s ds$$

for $|\arg(-z)| < \frac{1}{2}\pi$, $a, b \neq 0, -1, -2, \dots$. The contour must separate the poles of $\Gamma(-s)$ from those of $\Gamma(a+s)$; c is finite.

13.2.10

$$\Gamma(a)\Gamma(1+a-b)z^a U(a, b, z)$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(-s)\Gamma(a+s)\Gamma(1+a-b+s)z^{-s} ds$$

for $|\arg z| < \frac{3\pi}{2}$, $a \neq 0, -1, -2, \dots$, $b-a \neq 1, 2, 3, \dots$. The contour must separate the poles of $\Gamma(-s)$ from those of $\Gamma(a+s)$ and $\Gamma(1+a-b+s)$.

13.3. Connections With Bessel Functions

(see chapters 9 and 10)

Bessel Functions as Limiting Cases

If b and z are fixed,

$$13.3.1 \quad \lim_{a \rightarrow \infty} \{M(a, b, z/a)/\Gamma(b)\} = z^{\frac{1}{2}b-1} I_{b-1}(2\sqrt{z})$$

$$13.3.2 \quad \lim_{a \rightarrow \infty} \{M(a, b, -z/a)/\Gamma(b)\} = z^{\frac{1}{2}b-1} J_{b-1}(2\sqrt{z})$$

$$13.3.3 \quad \lim_{a \rightarrow \infty} \{\Gamma(1+a-b)U(a, b, z/a)\} = 2z^{\frac{1}{2}b-1} K_{b-1}(2\sqrt{z})$$

13.3.4

$$\lim_{a \rightarrow \infty} \{\Gamma(1+a-b)U(a, b, -z/a)\}$$

$$= -\pi i e^{\pi i b} z^{\frac{1}{2}b-1} H_{b-1}^{(1)}(2\sqrt{z}) \quad (\Im z > 0)$$

$$13.3.5 \quad = \pi i e^{-\pi i b} z^{\frac{1}{2}b-1} H_{b-1}^{(2)}(2\sqrt{z}) \quad (\Im z < 0)$$

Expansions in Series

13.3.6

$$M(a, b, z) = e^{\frac{1}{2}z} \Gamma(b-a-\frac{1}{2}) (\frac{1}{2}z)^{a-b+\frac{1}{2}}$$

$$* \sum_{n=0}^{\infty} \frac{(2b-2a-1)_n (b-2a)_n (b-a-\frac{1}{2}+n)}{n! (b)_n}$$

$$(-1)^n I_{b-a-\frac{1}{2}+n}(\frac{1}{2}z) \quad (b \neq 0, -1, -2, \dots)$$

13.3.7

$$\frac{M(a, b, z)}{\Gamma(b)} = e^{\frac{1}{2}z} (\frac{1}{2}bz - az)^{\frac{1}{2}-b}$$

$$\cdot \sum_{n=0}^{\infty} A_n (\frac{1}{2}z)^n (b-2a)^{-n} J_{b-1+n}(\sqrt{2zb-4za})$$

where

$$A_0=1, A_1=0, A_2=\frac{1}{2}b,$$

$$(n+1)A_{n+1} = (n+b-1)A_{n-1} + (2a-b)A_{n-2},$$

$$(a \text{ real})$$

13.3.8

$$\frac{M(a, b, z)}{\Gamma(b)}$$

$$= e^{hz} \sum_{n=0}^{\infty} C_n z^n (-az)^{\frac{1}{2}(1-b-n)} J_{b-1+n}(2\sqrt{-az})$$

where

$$C_0=1, C_1=-bh, C_2=-\frac{1}{2}(2h-1)a + \frac{1}{2}b(b+1)h^2,$$

$$(n+1)C_{n+1} = [(1-2h)n - bh]C_n$$

$$+ [(1-2h)a - h(h-1)(b+n-1)]C_{n-1}$$

$$- h(h-1)aC_{n-2} \quad (h \text{ real})$$

$$13.3.9 \quad M(a, b, z) = \sum_{n=0}^{\infty} C_n(a, b) I_n(z)$$

where

$$C_0=1, C_1(a, b)=2a/b,$$

$$C_{n+1}(a, b)=2aC_n(a+1, b+1)/b - C_{n-1}(a, b)$$

13.4. Recurrence Relations and Differential Properties

13.4.1

$$(b-a)M(a-1, b, z) + (2a-b+z)M(a, b, z)$$

$$- aM(a+1, b, z) = 0$$

13.4.2

$$b(b-1)M(a, b-1, z) + b(1-b-z)M(a, b, z)$$

$$+ z(b-a)M(a, b+1, z) = 0$$

13.4.3

$$(1+a-b)M(a, b, z) - aM(a+1, b, z)$$

$$+ (b-1)M(a, b-1, z) = 0$$

13.4.4

$$bM(a, b, z) - bM(a-1, b, z) - zM(a, b+1, z) = 0$$

13.4.5

$$b(a+z)M(a, b, z) + z(a-b)M(a, b+1, z)$$

$$- abM(a+1, b, z) = 0$$

13.4.6

$$(a-1+z)M(a, b, z) + (b-a)M(a-1, b, z) \\ + (1-b)M(a, b-1, z) = 0$$

13.4.7

$$b(1-b+z)M(a, b, z) + b(b-1)M(a-1, b-1, z) \\ - azM(a+1, b+1, z) = 0$$

$$\mathbf{13.4.8} \quad M'(a, b, z) = \frac{a}{b} M(a+1, b+1, z)$$

$$\mathbf{13.4.9} \quad \frac{d^n}{dz^n} \{M(a, b, z)\} = \frac{(a)_n}{(b)_n} M(a+n, b+n, z)$$

$$\mathbf{13.4.10} \quad aM(a+1, b, z) = aM(a, b, z) + zM'(a, b, z)$$

13.4.11

$$(b-a)M(a-1, b, z) = (b-a-z)M(a, b, z) \\ + zM'(a, b, z)$$

13.4.12

$$(b-a)M(a, b+1, z) = bM(a, b, z) - bM'(a, b, z)$$

13.4.13

$$(b-1)M(a, b-1, z) = (b-1)M(a, b, z) \\ + zM'(a, b, z)$$

13.4.14

$$(b-1)M(a-1, b-1, z) = (b-1-z)M(a, b, z) \\ + zM'(a, b, z)$$

13.4.15

$$U(a-1, b, z) + (b-2a-z)U(a, b, z) \\ + a(1+a-b)U(a+1, b, z) = 0$$

13.4.16

$$(b-a-1)U(a, b-1, z) + (1-b-z)U(a, b, z) \\ + zU(a, b+1, z) = 0$$

13.4.17

$$U(a, b, z) - aU(a+1, b, z) - U(a, b-1, z) = 0$$

13.4.18

$$(b-a)U(a, b, z) + U(a-1, b, z) \\ - zU(a, b+1, z) = 0$$

13.4.19

$$(a+z)U(a, b, z) - zU(a, b+1, z) \\ + a(b-a-1)U(a+1, b, z) = 0$$

13.4.20

$$(a+z-1)U(a, b, z) - U(a-1, b, z) \\ + (1+a-b)U(a, b-1, z) = 0$$

$$\mathbf{13.4.21} \quad U'(a, b, z) = -aU(a+1, b+1, z)$$

13.4.22

$$\frac{d^n}{dz^n} \{U(a, b, z)\} = (-1)^n (a)_n U(a+n, b+n, z)$$

13.4.23

$$a(1+a-b)U(a+1, b, z) = aU(a, b, z) \\ + zU'(a, b, z)$$

13.4.24

$$(1+a-b)U(a, b-1, z) = (1-b)U(a, b, z) \\ - zU'(a, b, z)$$

$$\mathbf{13.4.25} \quad U(a, b+1, z) = U(a, b, z) - U'(a, b, z)$$

13.4.26

$$U(a-1, b, z) = (a-b+z)U(a, b, z) - zU'(a, b, z)$$

13.4.27

$$U(a-1, b-1, z) = (1-b+z)U(a, b, z) \\ - zU'(a, b, z)$$

$$\mathbf{13.4.28} \quad 2\mu M_{\kappa-\frac{1}{2}, \mu-\frac{1}{2}}(z) - z^{\frac{1}{2}} M_{\kappa, \mu}(z) = 2\mu M_{\kappa+\frac{1}{2}, \mu-\frac{1}{2}}(z)$$

13.4.29

$$(1+2\mu+2\kappa)M_{\kappa+1, \mu}(z) - (1+2\mu-2\kappa)M_{\kappa-1, \mu}(z) \\ = 2(2\kappa-z)M_{\kappa, \mu}(z)$$

13.4.30

$$W_{\kappa+\frac{1}{2}, \mu}(z) - z^{\frac{1}{2}} W_{\kappa, \mu+\frac{1}{2}}(z) + (\kappa+\mu)W_{\kappa-\frac{1}{2}, \mu}(z) = 0$$

13.4.31

$$(2\kappa-z)W_{\kappa, \mu}(z) + W_{\kappa+1, \mu}(z) \\ = (\mu-\kappa+\frac{1}{2})(\mu+\kappa-\frac{1}{2})W_{\kappa-1, \mu}(z)$$

13.4.32

$$zM'_{\kappa, \mu}(z) = (\frac{1}{2}z-\kappa)M_{\kappa, \mu}(z) + (\frac{1}{2}+\mu+\kappa)M_{\kappa+1, \mu}(z)$$

$$\mathbf{13.4.33} \quad zW'_{\kappa, \mu}(z) = (\frac{1}{2}z-\kappa)W_{\kappa, \mu}(z) - W_{\kappa+1, \mu}(z)$$

13.5. Asymptotic Expansions and Limiting Forms

For $|z|$ large, (a, b fixed)

13.5.1

$$\frac{M(a, b, z)}{\Gamma(b)} = \frac{e^{\pm i\pi a} z^{-a}}{\Gamma(b-a)} \left\{ \sum_{n=0}^{R-1} \frac{(a)_n (1+a-b)_n}{n!} (-z)^{-n} + O(|z|^{-R}) \right\} \\ + \frac{e^{\pm i\pi a} z^{-a}}{\Gamma(a)} \left\{ \sum_{n=0}^{S-1} \frac{(b-a)_n (1-a)_n}{n!} z^{-n} + O(|z|^{-S}) \right\}$$

the upper sign being taken if $-\frac{1}{2}\pi < \arg z < \frac{3}{2}\pi$, the lower sign if $-\frac{3}{2}\pi < \arg z \leq -\frac{1}{2}\pi$.

13.5.2

$$U(a, b, z) = z^{-a} \left\{ \sum_{n=0}^{R-1} \frac{(a)_n (1+a-b)_n}{n!} (-z)^{-n} + O(|z|^{-R}) \right\} \quad (-\frac{3}{2}\pi < \arg z < \frac{3}{2}\pi)$$

Converging Factors for the Remainders

13.5.3

$$O(|z|^{-R}) = \frac{(a)_R (1+a-b)_R}{R!} (-z)^{-R} \\ \left[\frac{1}{2} + \frac{(\frac{1}{2} + \frac{1}{2}b - \frac{1}{2}a + \frac{1}{2}z - \frac{1}{2}R)}{z} + O(|z|^{-2}) \right]$$

and

13.5.4

$$O(|z|^{-S}) = \frac{(b-a)_S (1-a)_S}{S!} z^{-S} \\ \left[\frac{3}{2} - b + 2a + z - S + O(|z|^{-1}) \right]$$

where the R 'th and S 'th terms are the smallest in the expansions 13.5.1 and 13.5.2.

For small z (a, b fixed)

$$13.5.5 \quad \text{As } |z| \rightarrow 0, M(a, b, 0) = 1, b \neq -n$$

$$13.5.6 \quad U(a, b, z) = \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + O(|z|^{\Re b-2}) \\ (\Re b \geq 2, b \neq 2)$$

$$13.5.7 \quad = \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + O(|\ln z|) \quad (b=2)$$

$$13.5.8 \quad = \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + O(1) \quad (1 < \Re b < 2)$$

$$13.5.9 \quad = -\frac{1}{\Gamma(a)} [\ln z + \psi(a)] + O(|z \ln z|) \quad (b=1)$$

$$13.5.10 \quad U(a, b, z) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} + O(|z|^{1-\Re b}) \quad (0 < \Re b < 1)$$

$$13.5.11 \quad = \frac{1}{\Gamma(1+a)} + O(|z \ln z|) \quad (b=0)$$

$$13.5.12 \quad = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} + O(|z|) \quad (\Re b \leq 0, b \neq 0)$$

For large a (b, z fixed)

13.5.13

$$M(a, b, z) = \Gamma(b) e^{i\pi(\frac{1}{2}bz - az)^{1-1/b}} J_{b-1}(\sqrt{(2bz-4az)}) \\ [1 + O(|\frac{1}{2}b - a|^{-\sigma})]$$

where

$$|z| = \left| \frac{1}{2}b - a \right|^{\rho} \text{ and } \sigma = \min(1-\rho, \frac{1}{2} - \frac{3}{2}\rho), 0 \leq \rho < \frac{1}{2}.$$

13.5.14

$$M(a, b, x) = \Gamma(b) e^{i\pi(\frac{1}{2}bx - ax)^{1-1/b}} \pi^{-1/2} \\ \cos(\sqrt{(2bx-4ax)} - \frac{1}{2}b\pi + \frac{1}{4}\pi) \\ [1 + O(|\frac{1}{2}b - a|^{-1})]$$

as $a \rightarrow -\infty$ for b bounded, x real.

13.5.15

$$U(a, b, z) = \Gamma(\frac{1}{2}b - a + \frac{1}{2}) e^{i\pi z^{1-1/b}} [\cos(a\pi) J_{b-1}(\sqrt{(2bz-4az)}) \\ - \sin(a\pi) Y_{b-1}(\sqrt{(2bz-4az)})] [1 + O(|\frac{1}{2}b - a|^{-\sigma})]$$

where σ is defined in 13.5.13.

13.5.16

$$U(a, b, x) = \Gamma(\frac{1}{2}b - a + \frac{1}{2}) \pi^{-1/2} e^{i\pi x^{1-1/b}} \\ \cos(\sqrt{(2bx-4ax)} - \frac{1}{2}b\pi + a\pi + \frac{1}{4}\pi) \\ [1 + O(|\frac{1}{2}b - a|^{-1})]$$

as $a \rightarrow -\infty$ for b bounded, x real.

For large real a, b, z

If $\cosh^2 \theta = x/(2b-4a)$ so that $x > 2b-a > 1$,

13.5.17

$$M(a, b, x) = \Gamma(b) \sin(a\pi) \\ \exp[(b-2a)(\frac{1}{2} \sinh 2\theta - \theta + \cosh^2 \theta)] \\ [(b-2a) \cosh \theta]^{1-b} [\pi(\frac{1}{2}b - a) \sinh 2\theta]^{-1/2} \\ [1 + O(|\frac{1}{2}b - a|^{-1})]$$

13.5.18

$$U(a, b, x) = \exp[(b-2a)(\frac{1}{2} \sinh 2\theta - \theta + \cosh^2 \theta)] \\ [(b-2a) \cosh \theta]^{1-b} [(\frac{1}{2}b - a) \sinh 2\theta]^{-1/2} \\ [1 + O(|\frac{1}{2}b - a|^{-1})]$$

If $x = (2b - 4a)[1 + t/(b - 2a)^{1/2}]$, so that

$$x \sim 2b - 4a$$

13.5.19

$$M(a, b, x) = e^{1/2} (b - 2a)^{-1/2} \Gamma(b) [\text{Ai}(t) \cos(a\pi) + \text{Bi}(t) \sin(a\pi) + O(|\frac{1}{2}b - a|^{-1})]$$

13.5.20

$$U(a, b, x) = e^{1/2} x^{a-1/2} \Gamma(\frac{1}{2}) \pi^{-1} x^{b-1} \{1 - t \Gamma(\frac{5}{6}) (bx - 2ax)^{-1/2} 3^{1/2} \pi^{-1} + O(|\frac{1}{2}b - a|^{-1})\}$$

If $\cos^2 \theta = x/(2b - 4a)$ so that $2b - 4a > x > 0$,

13.5.21

$$M(a, b, x) = \Gamma(b) \exp \{ (b - 2a) \cos^2 \theta \} [(b - 2a) \cos \theta]^{1-b} [\pi(\frac{1}{2}b - a) \sin 2\theta]^{-1/2} [\sin(a\pi) + \sin \{ (\frac{1}{2}b - a)(2\theta - \sin 2\theta) + \frac{1}{4}\pi \} + O(|\frac{1}{2}b - a|^{-1})]$$

13.5.22

$$U(a, b, x) = \exp [(b - 2a) \cos^2 \theta] [(b - 2a) \cos \theta]^{1-b} [(\frac{1}{2}b - a) \sin 2\theta]^{-1/2} \{ \sin [(\frac{1}{2}b - a)(2\theta - \sin 2\theta) + \frac{1}{4}\pi] + O(|\frac{1}{2}b - a|^{-1}) \}$$

13.6. Special Cases

	$M(a, b, z)$			Relation	Function
	a	b	z		
13.6.1	$\nu + \frac{1}{2}$	$2\nu + 1$	$2iz$	$\Gamma(1 + \nu) e^{iz} (\frac{1}{2}z)^{-\nu} J_{\nu}(z)$	Bessel
13.6.2	$-\nu + \frac{1}{2}$	$-2\nu + 1$	$2iz$	$\Gamma(1 - \nu) e^{iz} (\frac{1}{2}z)^{-\nu} [\cos(\nu\pi) J_{\nu}(z) - \sin(\nu\pi) Y_{\nu}(z)]$	Bessel
13.6.3	$\nu + \frac{1}{2}$	$2\nu + 1$	$2z$	$\Gamma(1 + \nu) e^{iz} (\frac{1}{2}z)^{-\nu} I_{\nu}(z)$	Modified Bessel
13.6.4	$n + 1$	$2n + 2$	$2iz$	$\Gamma(\frac{3}{2} + n) e^{iz} (\frac{1}{2}z)^{-n-1/2} J_{n+1/2}(z)$	Spherical Bessel
13.6.5	$-n$	$-2n$	$2iz$	$\Gamma(\frac{1}{2} - n) e^{iz} (\frac{1}{2}z)^{n+1/2} J_{n-1/2}(z)$	Spherical Bessel
13.6.6	$n + 1$	$2n + 2$	$2z$	$\Gamma(\frac{3}{2} + n) e^{iz} (\frac{1}{2}z)^{-n-1/2} I_{n+1/2}(z) \quad *$	Spherical Bessel
13.6.7	$n + \frac{1}{2}$	$2n + 1$	$-2\sqrt{iz}$	$\Gamma(1 + n) e^{-2iz} (\frac{1}{2}iz\pi)^{-n} (\text{ber}_n x + i \text{bei}_n x)$	Kelvin
13.6.8	$L + 1 - i\eta$	$2L + 2$	$2ix$	$e^{ix} F_L(\eta, x) x^{L-1} / C_L(\eta)$	Coulomb Wave
13.6.9	$-n$	$\alpha + 1$	x	$\frac{n!}{(\alpha + 1)_n} L_n^{(\alpha)}(x)$	Laguerre
13.6.10	a	$a + 1$	$-x$	$ax^{-a} \gamma(a, x)$	Incomplete Gamma
13.6.11	$-n$	$1 + \nu - n$	x	$\frac{(n!)^{1/2} x^{1/2}}{(1 + \nu - n)_n} \rho_n(\nu, x)$	Poisson-Charlier
13.6.12	a	a	z	e^z	Exponential
13.6.13	1	2	$-2iz$	$\frac{e^{-iz}}{z} \sin z$	Trigonometric
13.6.14	1	2	$2z$	$\frac{e^z}{z} \sinh z$	Hyperbolic
13.6.15	$-\frac{1}{2}\nu$	$\frac{1}{2}$	$\frac{1}{2}z^2$	$2^{-1/2} \exp(\frac{1}{2}z^2) E_{\nu}^{(0)}(z)$	Weber
13.6.16	$\frac{1}{2} - \frac{1}{2}\nu$	$\frac{3}{2}$	$\frac{1}{2}z^2$	$\frac{\exp(\frac{1}{2}z^2)}{2z} E_{\nu}^{(1)}(z)$	or Parabolic Cylinder
13.6.17	$-n$	$\frac{1}{2}$	$\frac{1}{2}z^2$	$\frac{n!}{(2n)!} (-\frac{1}{2})^{-n} He_{2n}(x)$	Hermite
13.6.18	$-n$	$\frac{3}{2}$	$\frac{1}{2}z^2$	$\frac{n!}{(2n+1)!} (-\frac{1}{2})^{-n} \frac{1}{x} He_{2n+1}(x) \quad *$	Hermite
13.6.19	$\frac{1}{2}$	$\frac{3}{2}$	$-x^2$	$\frac{\pi^{1/2}}{2x} \text{erf } x$	Error Integral
13.6.20	$\frac{1}{2}m + \frac{1}{2}$	$1 + n$	r^2	$\frac{n! r^{-2n+m-1}}{\Gamma(\frac{1}{2}m + \frac{1}{2})} e^{r^2} T(m, n, r) \quad *$	Toronto

*See page II.

13.6. Special Cases—Continued

	$U(a, b, z)$			Relation	Function
	a	b	z		
13.6.21	$\nu + \frac{1}{2}$	$2\nu + 1$	$2z$	$\pi^{-1/2} e^z (2z)^{-\nu} K_\nu(z)$	Modified Bessel
13.6.22	$\nu + \frac{1}{2}$	$2\nu + 1$	$-2iz$	$\frac{1}{2} \pi^{1/2} e^{i[\pi(\nu + \frac{1}{2}) - z]} (2z)^{-\nu} H_\nu^{(1)}(z)^*$	Hankel
13.6.23	$\nu + \frac{1}{2}$	$2\nu + 1$	$2iz$	$\frac{1}{2} \pi^{1/2} e^{-i[\pi(\nu + \frac{1}{2}) - z]} (2z)^{-\nu} H_\nu^{(2)}(z)^*$	Hankel
13.6.24	$n + 1$	$2n + 2$	$2z$	$\pi^{-1/2} e^z (2z)^{-n-1} K_{n+1/2}(z)$	Spherical Bessel
13.6.25	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3} z^{3/2}$	$\pi^{1/2} z^{-1} \exp(\frac{2}{3} z^{3/2}) 2^{-2/3} 3^{5/6} \text{Ai}(z)$	Airy
13.6.26	$n + \frac{1}{2}$	$2n + 1$	$\sqrt{ix}$	$i^n \pi^{-1/2} e^{\sqrt{ix}} (2\sqrt{ix})^{-n} [\text{ker}_n x + i \text{kei}_n x]$	Kelvin
13.6.27	$-n$	$\alpha + 1$	x	$(-1)^n n! L_n^{(\alpha)}(x)$	Laguerre
13.6.28	$1 - a$	$1 - a$	z	$e^z \Gamma(a, z)$	Incomplete Gamma
13.6.29	1	1	$-x$	$-e^{-x} \text{Ei}(x)$	Exponential Integral
13.6.30	1	1	x	$e^x E_1(x)$	Exponential Integral
13.6.31	1	1	$-\ln x$	$-\frac{1}{x} \text{li}(x)$	Logarithmic Integral
13.6.32	$\frac{1}{2}m - n$	$1 + m$	x	$\Gamma(1 + n - \frac{1}{2}m) e^{x - \pi i(\frac{1}{2}m - n)} \omega_{n, m}(x)$	Cunningham
13.6.33	$-\frac{1}{2}\nu$	0	$2x$	$\Gamma(1 + \frac{1}{2}\nu) e^{x^2} k_\nu(x)$ for $x > 0$	Bateman
13.6.34	1	1	ix	$e^{ix} [-\frac{1}{2}\pi i + i \text{Si}(x) - \text{Ci}(x)]$	Sine and Cosine Integral
13.6.35	1	1	$-ix$	$e^{-ix} [\frac{1}{2}\pi i - i \text{Si}(x) - \text{Ci}(x)]$	Sine and Cosine Integral
13.6.36	$-\frac{1}{2}\nu$	$\frac{1}{2}$	$\frac{1}{2}z^2$	$2^{-1/2} e^{z^2/4} D_\nu(z)$	Weber
13.6.37	$\frac{1}{2} - \frac{1}{2}\nu$	$\frac{3}{2}$	$\frac{1}{2}z^2$	$2^{1/2} e^{z^2/4} D_\nu(z)/z^*$	Parabolic Cylinder
13.6.38	$\frac{1}{2} - \frac{1}{2}n$	$\frac{3}{2}$	x^2	$2^{-n} H_n(x)/x^*$	Hermite
13.6.39	$\frac{1}{2}$	$\frac{1}{2}$	x^2	$\sqrt{\pi} \exp(x^2) \text{erfc } x$	Error Integral

13.7. Zeros and Turning Values

If $j_{b-1, r}$ is the r 'th positive zero of $J_{b-1}(x)$, then a first approximation X_0 to the r 'th positive zero of $M(a, b, x)$ is

$$13.7.1 \quad X_0 = j_{b-1, r}^2 \{1/(2b-4a) + O(1/(\frac{1}{2}b-a)^2)\}$$

$$13.7.2 \quad X_0 \approx \frac{\pi^2(r + \frac{1}{2}b - \frac{3}{4})^2}{2b-4a}$$

A closer approximation is given by

$$13.7.3 \quad X_1 = X_0 - M(a, b, X_0)/M'(a, b, X_0)$$

For the derivative,

13.7.4

$$M'(a, b, X_1) =$$

$$M'(a, b, X_0) \{1 + (b - X_0) \frac{M(a, b, X_0)}{M'(a, b, X_0)}\}$$

If X'_0 is the first approximation to a turning value of $M(a, b, x)$, that is, to a zero of $M'(a, b, x)$ then a better approximation is

13.7.5

$$X'_1 = X'_0 - \frac{X'_0 M'(a, b, X'_0)}{a M(a, b, X'_0)}$$

*See page II.

The self-adjoint equation 13.1.1 can also be written

$$13.7.6 \quad \frac{d}{dz}[z^b e^{-z} \frac{dw}{dz}] = az^{b-1} e^{-z} w$$

The Sonine-Polya Theorem

The maxima and minima of $|w|$ form an increasing or decreasing sequence according as

$$-ax^{2b-1}e^{-2x}$$

is an increasing or decreasing function of x , that is, they form an increasing sequence for $M(a, b, x)$ if $a > 0$, $x < b - \frac{1}{2}$ or if $a < 0$, $x > b - \frac{1}{2}$, and a decreasing sequence if $a > 0$ and $x > b - \frac{1}{2}$ or if $a < 0$ and $x < b - \frac{1}{2}$.

The turning values of $|w|$ lie near the curves

13.7.7

$$w = \pm \Gamma(b) \pi^{-1/2} e^{x/2} (\frac{1}{2}bx - ax)^{1/2} \{1 - x/(2b - 4a)\}^{-1/4}$$

Numerical Methods

13.8. Use and Extension of the Tables

Calculation of $M(a, b, x)$

Kummer's Transformation

Example 1. Compute $M(.3, .2, -.1)$ to 7S. Using 13.1.27 and Tables 4.4 and 13.1 we have $a = .3$, $b = .2$ so that

$$M(.3, .2, -.1) = e^{-.1} M(-.1, .2, .1) \\ = .85784 \ 90.$$

Thus 13.1.27 can be used to extend Table 13.1 to negative values of x . Kummer's transformation should also be used when a and b are large and nearly equal, for x large or small.

Example 2. Compute $M(17, 16, 1)$ to 7S. Here $a = 17$, $b = 16$, and

$$M(17, 16, 1) = e^1 M(-1, 16, -1) \\ = 2.71828 \ 18 \times 1.06250 \ 00 \\ = 2.88817 \ 44.$$

Recurrence Relations

Example 3. Compute $M(-1.3, 1.2, .1)$ to 7S. Using 13.4.1 and Table 13.1 we have $a = -.3$, $b = .2$ so that

$$M(-1.3, .2, .1) = 2[.7 M(-.3, .2, .1) - .3 M(.7, .2, .1)] \\ = .35821 \ 23.$$

By 13.4.5 when $a = -1.3$ and $b = .2$,

$$M(-1.3, 1.2, .1) = [.26 M(-.3, .2, .1) \\ - .24 M(-1.3, .2, .1)] / .15 \\ = .89241 \ 08.$$

Similarly when $a = -.3$ and $b = .2$

$$M(-.3, 1.2, .1) = .97459 \ 52.$$

Check, by 13.4.6,

$$M(-1.3, 1.2, .1) = [.2 M(-.3, .2, .1) \\ + 1.2 M(-.3, 1.2, .1)] / 1.5 \\ = .89241 \ 08.$$

In this way 13.4.1–13.4.7 can be used together with 13.1.27 to extend Table 13.1 to the range

$$-10 \leq a \leq 10, -10 \leq b \leq 10, -10 \leq x \leq 10.$$

This extension of ten units in any direction is possible with the loss of about 1S. All the recurrence relations are stable except i) if $a < 0$, $b < 0$ and $|a| > |b|$, $x > 0$, or ii) $b < a$, $b < 0$, $|b - a| > |b|$, $x < 0$, when the oscillations may become large, especially if $|x|$ also is large.

Neither interpolation nor the use of recurrence relations should be attempted in the strips $b = -n \pm .1$ where the function is very large numerically. In particular $M(a, b, x)$ cannot be evaluated in the neighborhood of the points $a = -m$, $b = -n$, $m \leq n$, as near these points small changes in a , b or x can produce very large changes in the numerical value of $M(a, b, x)$.

Example 4. At the point $(-1, -1, x)$, $M(a, b, x)$ is undefined.

When $a = -1$, $M(-1, b, x) = 1 - \frac{x}{b}$ for all x .

Hence $\lim_{b \rightarrow -1} M(-1, b, x) = 1 + x$. But $M(b, b, x) = e^x$ for all x , when $a = b$. Hence $\lim_{b \rightarrow -1} M(b, b, x) = e^x$.

In the first case $b \rightarrow -1$ along the line $a = -1$, and in the second case $b \rightarrow -1$ along the line $a = b$.

Derivatives

Example 5. To evaluate $M'(-.7, -.6, .5)$ to 7S. By 13.4.8, when $a = -.7$ and $b = -.6$, we have

$$M'(-.7, -.6, .5) = \frac{-.7}{-.6} M(.3, .4, .5) \\ = 1.724128.$$

Asymptotic Formulas

For $x \geq 10$, a and b small, $M(a, b, x)$ should be evaluated by 13.5.1 using converging factors 13.5.3 and 13.5.4 to improve the accuracy if necessary.

Example 6. Calculate $M(.9, .1, 10)$ to 7S, using 13.5.1.

$$\begin{aligned} M(.9, .1, 10) &= \frac{\Gamma(.1)}{\Gamma(-.8)} e^{.9\pi i 10^{-.9}} \sum_{n=0}^N \frac{(.9)_n (1.8)_n}{n! (-10)^n} \\ &\quad + \frac{\Gamma(.1)}{\Gamma(.9)} e^{.10 10^{-.8}} \sum_{n=0}^N \frac{(-.8)_n (.1)_n}{n! 10^n} + O(10^{-N}) \\ &= -.198(.869) + 1237253(.99190 \ 285) + O(1) \\ &= 1227235.23 - .17 + O(1) \\ &= 1227235 + O(1) \end{aligned}$$

Check, from Table 13.1, $M(.9, .1, 10) = 1227235$. To evaluate $M(a, b, x)$ with a large, x small and b small or large 13.5.13-14 should be used.

Example 7. Compute $M(-52.5, .1, 1)$ to 3S, using 13.5.14.

$$\begin{aligned} M(-52.5, .1, 1) &= \Gamma(.1) e^{.5(.05 + 52.5).25 - .05} \\ &\quad .5642 \cos [(.2 - 4(-52.5)).5 - .05\pi + .25\pi] \\ &\quad [1 + O((.05 + 52.5)^{-.5})] = -16.34 + O(.2) \end{aligned}$$

By direct application of a recurrence relation, $M(-52.5, .1, 1)$ has been calculated as -16.447 . To evaluate $M(a, b, x)$ with x, a and/or b large, 13.5.17, 19 or 21 should be tried.

Example 8. Compute $M(-52.5, .1, 1)$ using 13.5.21 to 3S, $\cos \theta = \sqrt{1/210.2}$.

$$\begin{aligned} M(-52.5, .1, 1) &= \Gamma(.1) e^{.105.1 \cos^2 \theta} [105.1 \cos \theta]^{1-.1.5641} \\ &\quad 52.55^{-.1} \sin 2\theta^{-.1} [\sin (-52.5\pi) \\ &\quad + \sin \{52.55(2\theta - \sin 2\theta) + \frac{1}{2}\pi\} \\ &\quad + O((52.55)^{-.1})] = -16.47 + O(.02) \end{aligned}$$

A full range of asymptotic formulas to cover all possible cases is not yet known.

Calculation of $U(a, b, x)$

For $-10 \leq x \leq 10$, $-10 \leq a \leq 10$, $-10 \leq b \leq 10$ this is possible by 13.1.3, using Table 13.1 and the recurrence relations 13.4.15-20.

Example 9. Compute $U(1.1, .2, 1)$ to 5S. Using Tables 13.1, 4.12 and 6.1 and 13.1.3, we have

$$\begin{aligned} U(1, .2, 1) &= \\ &\quad \frac{\pi}{\sin(2\pi)} \left\{ \frac{M(1, .2, 1)}{\Gamma(.9)\Gamma(.2)} - \frac{M(.9, 1.8, 1)}{\Gamma(.1)\Gamma(1.8)} \right\}. \end{aligned}$$

But $M(.9, 1.8, 1) = .8[M(.9, .8, 1) - M(-.1, .8, 1)]$
 $= 1.72329$, using 13.4.4.

Hence

$$\begin{aligned} U(1, .2, 1) &= 5.344799(.371765 - .194486) \\ &= .94752. \end{aligned}$$

Similarly

$$U(-.9, .2, 1) = .91272.$$

Hence by 13.4.15

$$\begin{aligned} U(1.1, .2, 1) &= [U(1, .2, 1) - U(-.9, .2, 1)]/.09 \\ &= .38664. \end{aligned}$$

Example 10. To compute $U'(-.9, -.8, 1)$ to 5S. By 13.4.21

$$\begin{aligned} U'(-.9, -.8, 1) &= .9U(1, .2, 1) \\ &= (.9)(.94752) \\ &= .85276. \end{aligned}$$

Asymptotic Formulas

Example 11. To compute $U(1, .1, 100)$ to 5S. By 13.5.2

$$\begin{aligned} U(1, .1, 100) &= \frac{1}{100} \left\{ 1 - \frac{1.9}{100} + \frac{1.9}{100} \frac{2.9}{100} \right. \\ &\quad \left. - \frac{1.9}{100} \frac{2.9}{100} \frac{3.9}{100} + O(10^{-9}) \right\}. \\ &= .01 \{ 1 - .019 + .000551 - .000021 \\ &\quad + O(10^{-9}) \}, \\ &= .00981 \ 53. \end{aligned}$$

Example 12. To evaluate $U(1, .2, .01)$. For x small, 13.5.6-12 should be used.

$$\begin{aligned} U(1, .2, .01) &= \frac{\Gamma(1-.2)}{\Gamma(1.1-.2)} + O((.01)^{1-.2}) \\ &= \frac{\Gamma(.8)}{\Gamma(.9)} + O((.01)^{.8}) \\ &= 1.09 \text{ to 3S, by 13.5.10.} \end{aligned}$$

To evaluate $U(a, b, x)$ with a large, x small and b small or large 13.5.15 or 16 should be used.

To evaluate $U(a, b, x)$ with x, a and/or b large 13.5.18, 20 or 22 should be tried. In all these cases the size of the remainder term is the guide to the number of significant figures obtainable.

Calculation of the Whittaker Functions

Example 13. Compute $M_{0,-.4}(1)$ and $W_{0,-.4}(1)$ to 5S. By formulas 13.1.32 and 13.1.33 and Tables 13.1, 4.4

$$\begin{aligned} M_{0,-.4}(1) &= e^{-.5} M(1, .2, 1) = 1.10622, \\ W_{0,-.4}(1) &= e^{-.5} U(1, .2, 1) = .57469. \end{aligned}$$

Thus the values of $M_{\kappa,\mu}(x)$ and $W_{\kappa,\mu}(x)$ can always be found if the values of $M(a, b, x)$ and $U(a, b, x)$ are known.

13.9. Calculation of Zeros and Turning Points

Example 14. Compute the smallest positive zero of $M(-4, .6, x)$. This is outside the range of Table 13.2. Using 13.7.2 we have, as a first approximation

$$X_0 = \frac{(.55\pi)^2}{17.2} = .174.$$

Using 13.7.3 we have

$$X_1 = X_0 - M(-4, .6, X_0)/M'(-4, .6, X_0).$$

But, by 13.4.8,

$$M'(-4, .6, X_0) = -(.15)^{-1}M(-3, 1.6, X_0)$$

Hence

$$\begin{aligned} X_1 &= X_0 + .15M(-4, .6, X_0)/M(-3, 1.6, X_0), \\ &= .174 + (.15)(.030004) \\ &= .17850 \text{ as a second approximation.} \end{aligned}$$

If we repeat this calculation, we find that

$$X_2 = X_1 + .0000299 = .1785299 \text{ to 7S.}$$

Calculation of Maxima and Minima

Example 15. Compute the value of x at which $M(-1.8, -.2, x)$ has a turning value. Using 13.4.8 and Table 13.2, we find that $M'(-1.8, -.2, x) = 9M(-.8, .8, x) = 0$ when $x = .9429159$. Also $M''(-1.8, -.2, x) = 9M'(-.8, .8, x) = -9M(.2, 1.8, x)$ and $M(.2, 1.8, .9429159) > 0$. Hence $M(-1.8, -.2, x)$ has a maximum in x when $x = .9429159$.

Example 16. Compute the smallest positive value of x for which $M(-3, .6, x)$ has a turning value, X'_1 . This is outside the range of Table 13.2. Using 13.4.8 we have

$$M'(-3, .6, x) = -3M(-2, 1.6, x)/.6.$$

By 13.7.2 for $M(-2, 1.6, x)$,

$$X_0 = (1.05\pi)^2/(11.2) = .9715.$$

This is a first approximation to X'_0 for $M(-3, .6, x)$. Using 13.7.5 and 13.4.8 we find a second approximation

$$\begin{aligned} X'_1 &= X'_0 \left[1 - \frac{M'(-3, .6, X'_0)}{-3M(-3, .6, X'_0)} \right] \\ &= X'_0 [1 - M(-2, 1.6, X'_0)/.6M(-3, .6, X'_0)] \\ &= .9715 \times 1.0163 = .9873 \text{ to 4S.} \end{aligned}$$

This process can be repeated to give as many significant figures as are required.

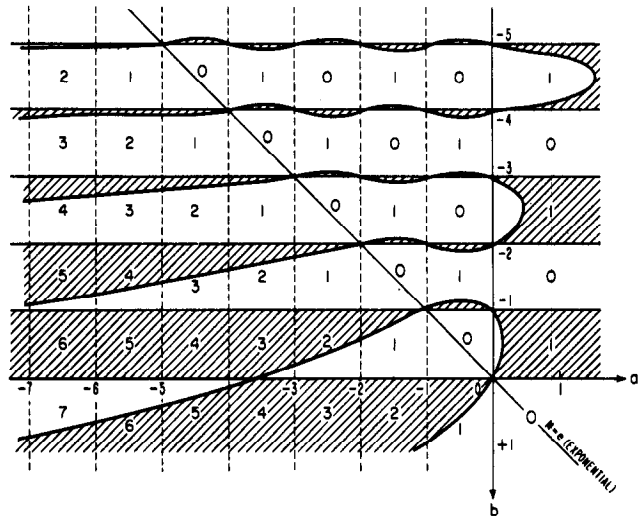


FIGURE 13.1.

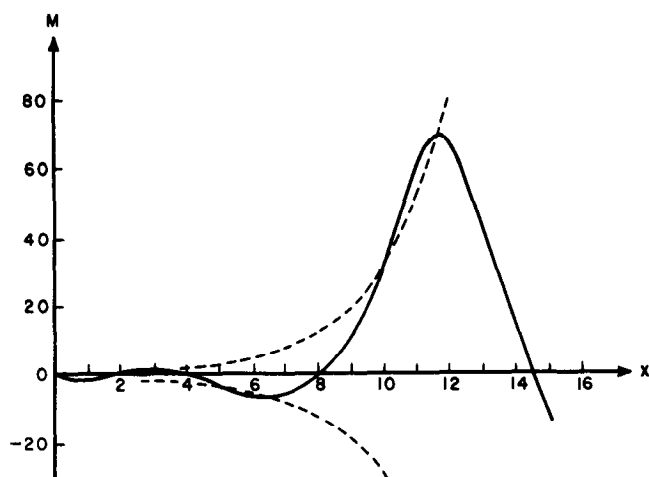
Figure 13.1 shows the curves on which $M(a, b, x) = 0$ in the a, b plane when $x = 1$. The function is positive in the unshaded areas, and negative in the shaded areas. The number in each square gives the number of real positive zeros of $M(a, b, x)$ as a function of x in that square. The vertical boundaries to the left are to be included in each square.

13.10. Graphing $M(a, b, x)$

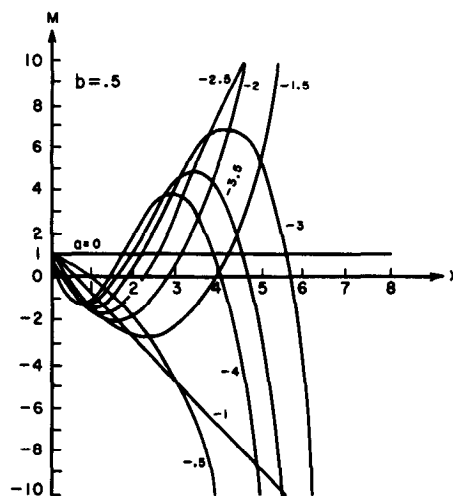
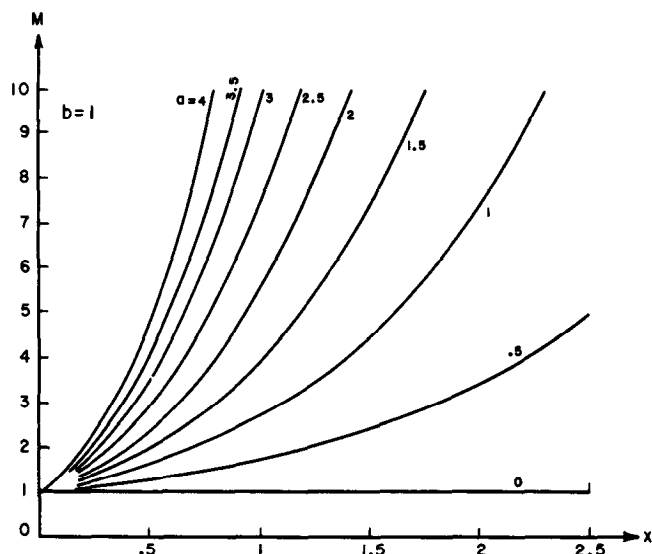
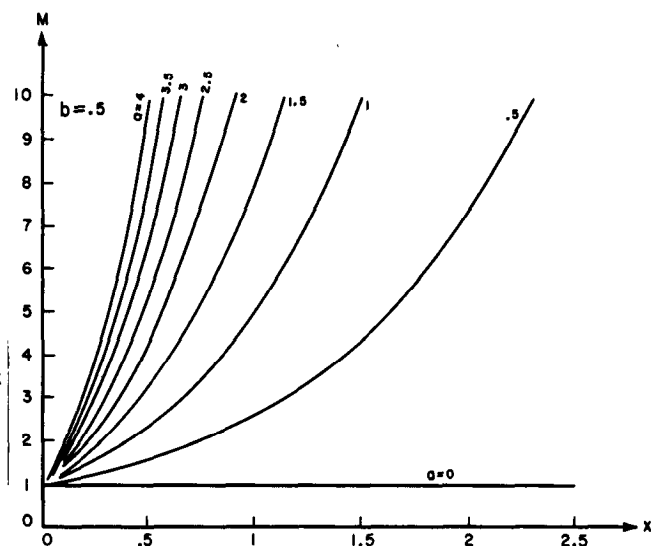
Example 17. Sketch $M(-4.5, 1, x)$. Firstly, from **Figure 13.1** we see that the function has five real positive zeros. From 13.5.1, we find that $M \rightarrow -\infty$, $M' \rightarrow -\infty$ as $x \rightarrow +\infty$ and that $M \rightarrow +\infty$, $M' \rightarrow +\infty$ as $x \rightarrow -\infty$. By 13.7.2 we have as first approximations to the zeros, .3, 1.5, 3.7, 6.9, 10.6, and by 13.7.2 and 13.4.8 we find as first approximations to the turning values .9, 2.8, 5.8, 9.9. From 13.7.7, we see that these must lie near the curves

$$y = \pm e^{1/2}(5x)^{-1/2}(1-x/11)^{-1/2}\pi^{-1/2}.$$

From these facts we can form a rough graph of the behavior of the function, **Figure 13.2**.

FIGURE 13.2. $M(-4.5, 1, x)$.

(From F. G. Tricomi, *Funzioni ipergeometriche confluenti*, Edizioni Cremonese, Rome, Italy, 1954, with permission.)

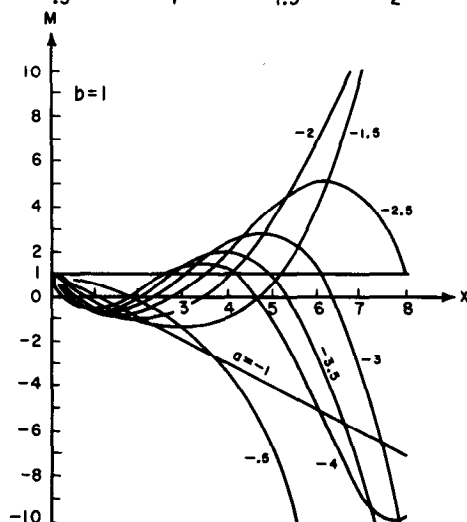
FIGURE 13.4. $M(a, .5, x)$.

(From E. Jahnke and F. Emde, *Tables of functions*, Dover Publications, Inc., New York, N.Y., 1945, with permission.)

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FIGURE 13.3. $M(a, 1, x)$.

(From E. Jahnke and F. Emde, *Tables of functions*, Dover Publications, Inc., New York, N.Y., 1945, with permission.)

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Tables

14. Coulomb Wave Functions

MILTON ABRAMOWITZ¹

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¹ National Bureau of Standards (deceased).

14. Coulomb Wave Functions

Mathematical Properties

14.1. Differential Equation, Series Expansions

Differential Equation

14.1.1

$$\frac{d^2 w}{d\rho^2} + \left[1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2}\right] w = 0$$

($\rho > 0$, $-\infty < \eta < \infty$, L a non-negative integer)

The Coulomb wave equation has a regular singularity at $\rho=0$ with indices $L+1$ and $-L$; it has an irregular singularity at $\rho=\infty$.

General Solution

14.1.2

$$w = C_1 F_L(\eta, \rho) + C_2 G_L(\eta, \rho) \quad (C_1, C_2 \text{ constants})$$

where $F_L(\eta, \rho)$ is the regular Coulomb wave function and $G_L(\eta, \rho)$ is the irregular (logarithmic) Coulomb wave function.

Regular Coulomb Wave Function $F_L(\eta, \rho)$

14.1.3

$$F_L(\eta, \rho) = C_L(\eta) \rho^{L+1} e^{-i\pi/2} M(L+1-i\eta, 2L+2, 2i\rho)$$

14.1.4

$$= C_L(\eta) \rho^{L+1} \Phi_L(\eta, \rho)$$

14.1.5

$$\Phi_L(\eta, \rho) = \sum_{k=L+1}^{\infty} A_k^L(\eta) \rho^{k-L-1}$$

14.1.6

$$A_{L+1}^L = 1, \quad A_{L+2}^L = \frac{\eta}{L+1},$$

$$(k+L)(k-L-1)A_k^L = 2\eta A_{k-1}^L - A_{k-2}^L \quad (k > L+2)$$

$$14.1.7 \quad C_L(\eta) = \frac{2^L e^{-\frac{\pi\eta}{2}} |\Gamma(L+1+i\eta)|}{\Gamma(2L+2)}$$

(See chapter 6.)

$$14.1.8 \quad C_0^2(\eta) = 2\pi\eta(e^{2\pi\eta} - 1)^{-1}$$

$$14.1.9 \quad C_L^2(\eta) = \frac{p_L(\eta) C_0^2(\eta)}{2\eta(2L+1)}$$

$$14.1.10 \quad C_L(\eta) = \frac{(L^2 + \eta^2)^{\frac{1}{2}}}{L(2L+1)} C_{L-1}(\eta)$$

$$14.1.11 \quad \frac{p_L(\eta)}{2\eta} = \frac{(1+\eta^2)(4+\eta^2) \dots (L^2+\eta^2) 2^{2L}}{(2L+1)[(2L)!]^2}$$

$$14.1.12 \quad F'_L = \frac{d}{d\rho} F_L(\eta, \rho) = C_L(\eta) \rho^L \Phi_L^*(\eta, \rho)$$

$$14.1.13 \quad \Phi_L^*(\eta, \rho) = \sum_{k=L+1}^{\infty} k A_k^L(\eta) \rho^{k-L-1}$$

Irregular Coulomb Wave Function $G_L(\eta, \rho)$

14.1.14

$$G_L(\eta, \rho) = \frac{2\eta}{C_0^2(\eta)} F_L(\eta, \rho) [\ln 2\rho + \frac{q_L(\eta)}{p_L(\eta)}] + \theta_L(\eta, \rho)$$

$$14.1.15 \quad \theta_L(\eta, \rho) = D_L(\eta) \rho^{-L} \psi_L(\eta, \rho)$$

$$14.1.16 \quad D_L(\eta) C_L(\eta) = \frac{1}{2L+1}$$

$$14.1.17 \quad \psi_L(\eta, \rho) = \sum_{k=-L}^{\infty} a_k^L(\eta) \rho^{k+L}$$

14.1.18

$$a_L^L = 1, \quad a_{L+1}^L = 0,$$

$$(k-L-1)(k+L)a_k^L = 2\eta a_{k-1}^L - a_{k-2}^L - (2k-1)p_L(\eta)A_k^L$$

14.1.19

$$\frac{q_L(\eta)}{p_L(\eta)} = \sum_{s=1}^L \frac{s}{s^2 + \eta^2} - \sum_{s=1}^{2L+1} \frac{1}{s} + \mathcal{O}\left\{\frac{\Gamma'(1+i\eta)}{\Gamma(1+i\eta)}\right\} + 2\gamma + \frac{r_L(\eta)}{p_L(\eta)}$$

(See Table 6.8.)

14.1.20

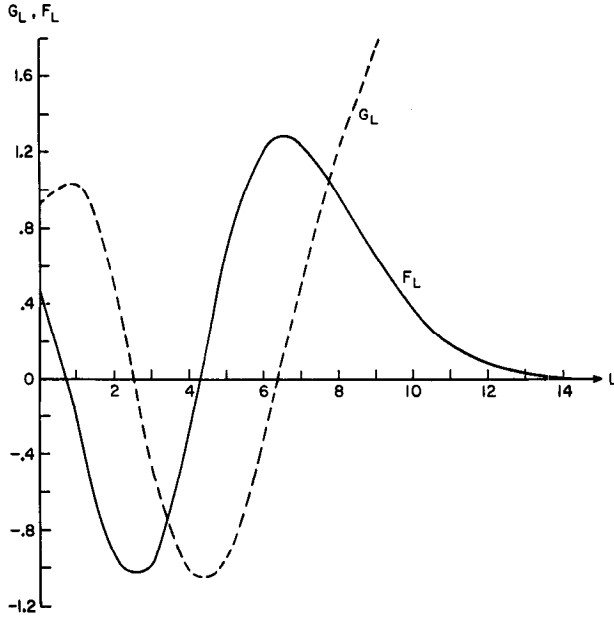
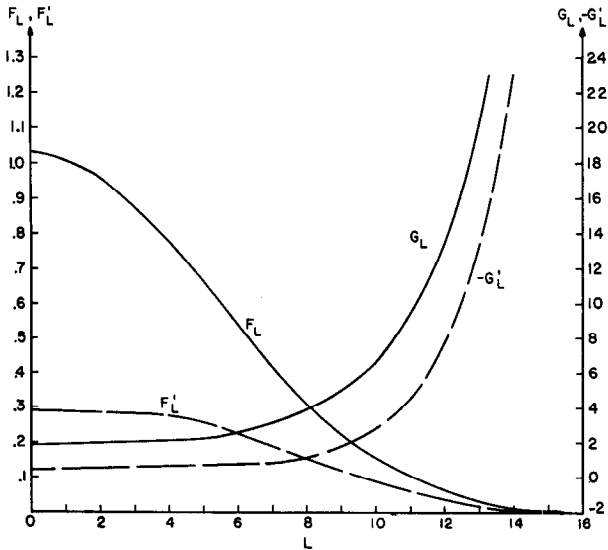
$$r_L(\eta) = \frac{(-1)^{L+1}}{(2L)!} \mathcal{J} \left\{ \frac{1}{2L+1} + \frac{2(i\eta-L)}{2L(1!)} + \frac{2^2(i\eta-L)(i\eta-L+1)}{(2L-1)(2!)} + \dots + \frac{2^{2L}(i\eta-L)(i\eta-L+1) \dots (i\eta+L-1)}{(2L)!} \right\}$$

14.1.21

$$G'_L = \frac{dG_L}{d\rho} = \frac{2\eta}{C_0^2(\eta)} \{ F'_L [\ln 2\rho + \frac{q_L(\eta)}{p_L(\eta)}] + \rho^{-1} F_L(\eta, \rho) \} + \theta'_L(\eta, \rho)$$

$$14.1.22 \quad \theta'_L = \frac{d}{d\rho} \theta_L(\eta, \rho) = D_L(\eta) \rho^{-L-1} \psi_L^*(\eta, \rho)$$

$$14.1.23 \quad \psi_L^*(\eta, \rho) = \sum_{k=-L}^{\infty} k a_k^L(\eta) \rho^{k+L}$$

FIGURE 14.1. $F_L(\eta, \rho)$, $G_L(\eta, \rho)$. $\eta=1, \rho=10$ FIGURE 14.2. F_L , F'_L , G_L and G'_L . $\eta=10, \rho=20$

14.2. Recurrence and Wronskian Relations

Recurrence Relations

If $u_L = F_L(\eta, \rho)$ or $G_L(\eta, \rho)$,

$$14.2.1 \quad L \frac{du_L}{d\rho} = (L^2 + \eta^2) u_{L-1} - \left(\frac{L^2}{\rho} + \eta\right) u_L$$

$$14.2.2$$

$$(L+1) \frac{du_L}{d\rho} = \left[\frac{(L+1)^2}{\rho} + \eta\right] u_L - [(L+1)^2 + \eta^2] u_{L+1}$$

$$14.2.3$$

$$L[(L+1)^2 + \eta^2] u_{L+1} = (2L+1) \left[\eta + \frac{L(L+1)}{\rho}\right] u_L - (L+1)[L^2 + \eta^2] u_{L-1}$$

Wronskian Relations

$$14.2.4 \quad F'_L G_L - F_L G'_L = 1$$

$$14.2.5 \quad F_{L-1} G_L - F_L G_{L-1} = L(L^2 + \eta^2)^{-1/2}$$

14.3. Integral Representations

$$14.3.1$$

$$F_L + iG_L = \frac{ie^{-i\rho}\rho^{-L}}{(2L+1)!C_L(\eta)} \int_0^\infty e^{-t} t^{L-\eta} (t+2i\rho)^{L+i\eta} dt$$

$$14.3.2$$

$$F_L - iG_L =$$

$$\frac{e^{-\pi\eta}\rho^{L+1}}{(2L+1)!C_L(\eta)} \int_{-1}^{-i\infty} e^{-t\rho t} (1-t)^{L-i\eta} (1+t)^{L+i\eta} dt$$

$$14.3.3$$

$$F_L + iG_L = \frac{e^{-\pi\eta}\rho^{L+1}}{(2L+1)!C_L(\eta)} \int_0^\infty \{ (1 - \tanh^2 t)^{L+1} \exp[-i(\rho \tanh t - 2\eta t)] + i(1+t^2)^L \exp[-\rho t + 2\eta \arctan t] \} dt$$

14.4. Bessel Function Expansions

Expansion in Terms of Bessel-Clifford Functions

$$14.4.1$$

$$F_L(\eta, \rho) = C_L(\eta) \frac{(2L+1)!}{(2\eta)^{2L+1}} \rho^{-L} \sum_{k=2L+1}^{\infty} b_k t^{k/2} I_k(2\sqrt{t})$$

$$(t=2\eta\rho, \eta>0)$$

$$14.4.2$$

$$G_L(\eta, \rho) \sim D_L(\eta) \lambda_L(\eta) \rho^{-L} \sum_{k=2L+1}^{\infty} (-1)^k b_k t^{k/2} K_k(2\sqrt{t})$$

14.4.3

$$b_{2L+1}=1, \quad b_{2L+2}=0, \\ 4\eta^2(k-2L)b_{k+1}+kb_{k-1}+b_{k-2}=0 \quad (k>2L+2)$$

14.4.4

$$\lambda_L(\eta) \sum_{k=2L+1}^{\infty} (-1)^k (k-1)! b_k = 2$$

(See chapter 9.)

Expansion in Terms of Spherical Bessel Functions

14.4.5

$$F_L(\eta, \rho) = 1 \cdot 3 \cdot 5 \dots (2L+1) \rho C_L(\eta) \sum_{k=L}^{\infty} b_k \sqrt{\frac{\pi}{2\rho}} J_{k+\frac{1}{2}}(\rho)$$

14.4.6

$$b_L=1, \quad b_{L+1}=\frac{2L+3}{L+1} \eta$$

$$b_k = \frac{(2k+1)}{k(k+1)-L(L+1)} \\ \left\{ 2\eta b_{k-1} - \frac{(k-1)(k-2)-L(L+1)}{2k-3} b_{k-2} \right\} \\ (k>L+1)$$

14.4.7

$$F'_L(\eta, \rho) = 1 \cdot 3 \cdot 5 \dots (2L+1) \rho C_L(\eta) \\ \left\{ \frac{(L+1)}{(2L+1)} b_L \sqrt{\frac{\pi}{2\rho}} J_{L-\frac{1}{2}}(\rho) + \frac{(L+2)}{(2L+3)} b_{L+1} \right. \\ \left. \cdot \sqrt{\frac{\pi}{2\rho}} J_{L+\frac{1}{2}}(\rho) + \sum_{k=L+1}^{\infty} b'_k \sqrt{\frac{\pi}{2\rho}} J_{k+\frac{1}{2}}(\rho) \right\}$$

$$14.4.8 \quad b'_k = \frac{(k+2)}{(2k+3)} b_{k+1} - \frac{(k-1)}{(2k-1)} b_{k-1}$$

Expansion in Terms of Airy Functions

$$x = (2\eta - \rho)/(2\eta)^{1/3} \quad \mu = (2\eta)^{2/3}, \quad \eta > 0 \\ |\rho - 2\eta| < 2\eta$$

14.4.9

$$F_0(\eta, \rho) = \pi^{1/2} (2\eta)^{1/2} \left\{ \frac{\text{Ai}(x)}{\text{Bi}(x)} \left[1 + \frac{g_1}{\mu} + \frac{g_2}{\mu^2} + \dots \right] \right. \\ \left. + \frac{\text{Ai}'(x)}{\text{Bi}'(x)} \left[\frac{f_1}{\mu} + \frac{f_2}{\mu^2} + \dots \right] \right\}$$

14.4.10

$$F'_0(\eta, \rho) = -\pi^{1/2} (2\eta)^{-1/2} \left\{ \frac{\text{Ai}(x)}{\text{Bi}(x)} \left[\frac{g'_1 + x f_1}{\mu} \right. \right. \\ \left. \left. + \frac{g'_2 + x f_2}{\mu^2} + \dots \right] + \frac{\text{Ai}'(x)}{\text{Bi}'(x)} \left[1 + \frac{(g_1 + f'_1)}{\mu} \right. \right. \\ \left. \left. + \frac{(g_2 + f'_2)}{\mu^2} + \dots \right] \right\}$$

$$f_1 = (1/5)x^2$$

$$f_2 = \frac{1}{35} (2x^3 + 6)$$

$$f_3 = \frac{1}{63000} (84x^7 + 1480x^4 + 2320x)$$

$$g_1 = -(1/5)x$$

$$g_2 = \frac{1}{350} (7x^5 - 30x^2)$$

$$g_3 = \frac{1}{63000} (1056x^6 - 1160x^3 - 2240)$$

(See chapter 10.)

14.5. Asymptotic Expansions

Asymptotic Expansion for Large Values of ρ

$$14.5.1 \quad F_L = g \cos \theta_L + f \sin \theta_L$$

$$14.5.2 \quad G_L = f \cos \theta_L - g \sin \theta_L$$

$$14.5.3 \quad F'_L = g^* \cos \theta_L + f^* \sin \theta_L$$

$$14.5.4 \quad G'_L = f^* \cos \theta_L - g^* \sin \theta_L, \quad gf^* - fg^* = 1$$

$$14.5.5 \quad \theta_L = \rho - \eta \ln 2\rho - L \frac{\pi}{2} + \sigma_L$$

$$14.5.6 \quad \sigma_L = \arg \Gamma(L+1+i\eta)$$

(See 6.1.27, 6.1.44.)

$$14.5.7 \quad \sigma_{L+1} = \sigma_L + \arctan \frac{\eta}{L+1}$$

(See Tables 4.14, 6.7.)

$$14.5.8 \quad f \sim \sum_{k=0}^{\infty} f_k, \quad g \sim \sum_{k=0}^{\infty} g_k, \quad f^* \sim \sum_{k=0}^{\infty} f_k^*, \quad g^* \sim \sum_{k=0}^{\infty} g_k^*$$

where

$$f_0 = 1, \quad g_0 = 0, \quad f_0^* = 0, \quad g_0^* = 1 - \eta/\rho$$

$$f_{k+1} = a_k f_k - b_k g_k$$

$$g_{k+1} = a_k g_k + b_k f_k$$

$$f_{k+1}^* = a_k f_k^* - b_k g_k^* - f_{k+1}/\rho$$

$$g_{k+1}^* = a_k g_k^* + b_k f_k^* - g_{k+1}/\rho$$

$$a_k = \frac{(2k+1)\eta}{(2k+2)\rho}, \quad b_k = \frac{L(L+1)-k(k+1)+\eta^2}{(2k+2)\rho}$$

14.5.9

$$f+ig \sim 1 + \frac{(i\eta-L)(i\eta+L+1)}{1!(2i\rho)} + \frac{(i\eta-L)(i\eta-L+1)(i\eta+L+1)(i\eta+L+2)}{2!(2i\rho)^2} \\ + \frac{(i\eta-L)(i\eta-L+1)(i\eta-L+2)(i\eta+L+1)(i\eta+L+2)(i\eta+L+3)}{3!(2i\rho)^3} + \dots$$

Asymptotic Expansion for $L=0, \rho=2\eta \gg 0$

$$14.5.10 \quad \frac{F_0(2\eta)}{G_0(2\eta)/\sqrt{3}} \sim \frac{\Gamma(1/3)\beta^{1/4}}{2\sqrt{\pi}} \left\{ 1 \mp \frac{2}{35} \frac{\Gamma(2/3)}{\Gamma(1/3)} \frac{1}{\beta^4} - \frac{32}{8100} \frac{1}{\beta^6} \mp \frac{92672}{7371 \cdot 10^4} \frac{\Gamma(2/3)}{\Gamma(1/3)} \frac{1}{\beta^{10}} - \dots \right\}$$

14.5.11

$$\frac{F'_0(2\eta)}{G'_0(2\eta)/\sqrt{3}} \sim \frac{\Gamma(2/3)}{2\sqrt{\pi}\beta^{3/2}} \left\{ \pm 1 + \frac{1}{15} \frac{\Gamma(1/3)}{\Gamma(2/3)} \frac{1}{\beta^2} \pm \frac{8}{56700} \frac{1}{\beta^6} + \frac{11488}{18711 \cdot 10^3} \frac{\Gamma(1/3)}{\Gamma(2/3)} \frac{1}{\beta^8} \pm \dots \right\}$$

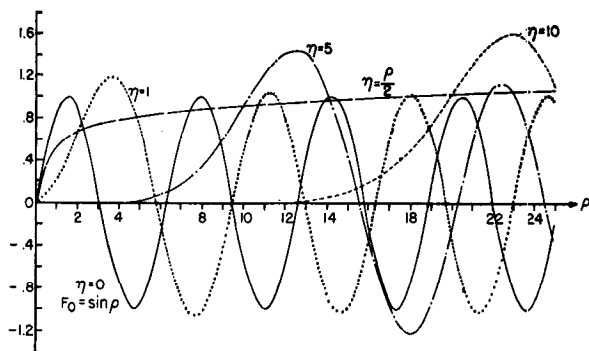
$$\beta = (2\eta/3)^{1/2}, \Gamma(1/3) = 2.6789 38534 \dots, \Gamma(2/3) = 1.3541 17939 \dots$$

14.5.12

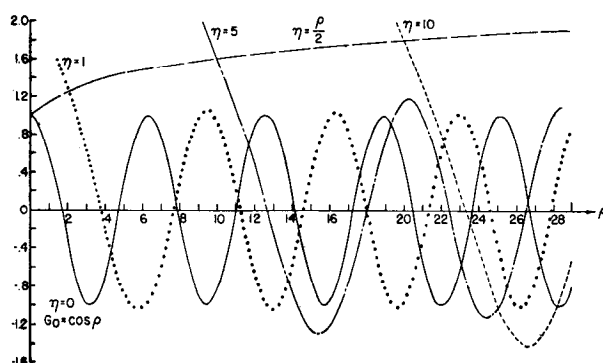
$$14.5.12 \quad \frac{F_0(2\eta)}{G_0(2\eta)} \sim \left\{ \begin{array}{l} .70633 \ 26373 \\ 1.22340 \ 4016 \end{array} \right\} \eta^{1/6} \left\{ 1 \mp \frac{.04959 \ 570165}{\eta^{3/2}} - \frac{.00888 \ 88888 \ 89}{\eta^2} \right. \\ \left. \mp \frac{.00245 \ 51991 \ 81}{\eta^{9/2}} - \frac{.00091 \ 08958 \ 061}{\eta^4} \mp \frac{.00025 \ 34684 \ 115}{\eta^{13/2}} - \dots \right\}$$

14.5.13

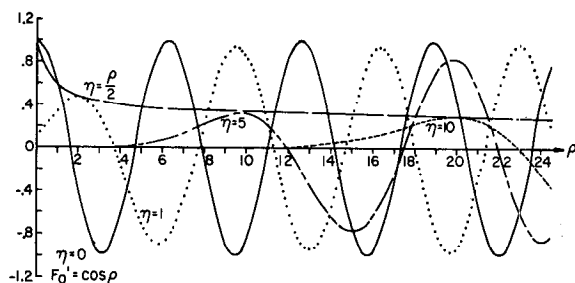
$$14.5.13 \quad \frac{F'_0(2\eta)}{G'_0(2\eta)} \sim \left\{ \begin{array}{l} .40869 \ 57323 \\ -.70788 \ 17734 \end{array} \right\} \eta^{-1/6} \left\{ 1 \pm \frac{.17282 \ 60369}{\eta^{3/2}} + \frac{.00031 \ 74603 \ 174}{\eta^2} \right. \\ \left. \pm \frac{.00358 \ 12148 \ 50}{\eta^{9/2}} + \frac{.00031 \ 17824 \ 680}{\eta^4} \pm \frac{.00090 \ 73966 \ 427}{\eta^{13/2}} + \dots \right\}$$

FIGURE 14.3. $F_0(\eta, \rho)$.

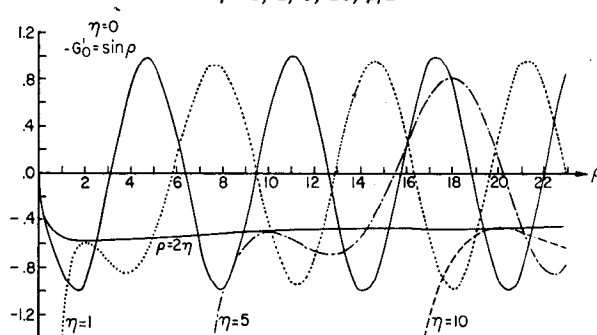
$$\eta = 0, 1, 5, 10, \rho/2$$

FIGURE 14.5. $G_0(\eta, \rho)$.

$$\eta = 0, 1, 5, 10, \rho/2$$

FIGURE 14.4. $F'_0(\eta, \rho)$.

$$\eta = 0, 1, 5, 10, \rho/2$$

FIGURE 14.6. $G'_0(\eta, \rho)$.

$$\eta = 0, 1, 5, 10, \rho/2$$

14.6. Special Values and Asymptotic Behavior

14.6.1 $L > 0, \rho = 0$

$$F_L = 0, F'_L = 0$$

$$G_L = \infty, G'_L = -\infty$$

14.6.2 $L = 0, \rho = 0$

$$F_0 = 0, F'_0 = C_0(\eta)$$

$$G_0 = 1/C_0(\eta), G'_0 = -\infty$$

14.6.3 $L \rightarrow \infty$

$$F_L \sim C_L(\eta) \rho^{L+1}, G_L \sim D_L(\eta) \rho^{-L}$$

14.6.4 $L = 0, \eta = 0$

$$F_0 = \sin \rho, F'_0 = \cos \rho$$

$$G_0 = \cos \rho, G'_0 = -\sin \rho$$

14.6.5 $\rho \rightarrow \infty$

$$G_L + iF_L \sim \exp i[\rho - \eta \ln 2\rho - \frac{L\pi}{2} + \sigma_L]$$

14.6.6 $L \geq 0, \eta = 0$

$$F_L = (\frac{1}{2}\pi\rho)^{\frac{1}{2}} J_{L+\frac{1}{2}}(\rho)$$

$$G_L = (-1)^L (\frac{1}{2}\pi\rho)^{\frac{1}{2}} J_{-(L+\frac{1}{2})}(\rho)$$

14.6.7 $L \geq 0, 2\eta \gg \rho$

$$F_L \sim \frac{(2L+1)! C_L(\eta)}{(2\eta)^{L+1}} (2\eta\rho)^{\frac{1}{2}} I_{2L+1}[2(2\eta\rho)^{\frac{1}{2}}]$$

$$G_L \sim \frac{2(2\eta)^L}{(2L+1)! C_L(\eta)} (2\eta\rho)^{\frac{1}{2}} K_{2L+1}[2(2\eta\rho)^{\frac{1}{2}}]$$

14.6.8 $L = 0, 2\eta \gg \rho$

$$F_0 \sim e^{-\pi\eta} (\pi\rho)^{\frac{1}{2}} I_1[2(2\eta\rho)^{\frac{1}{2}}]$$

$$F'_0 \sim e^{-\pi\eta} (2\pi\eta)^{\frac{1}{2}} I_0[2(2\eta\rho)^{\frac{1}{2}}]$$

$$G_0 \sim 2e^{\pi\eta} \left(\frac{\rho}{\pi}\right)^{\frac{1}{2}} K_1[2(2\eta\rho)^{\frac{1}{2}}]$$

$$G'_0 \sim -2 \left(\frac{2\eta}{\pi}\right)^{\frac{1}{2}} e^{\pi\eta} K_0[2(2\eta\rho)^{\frac{1}{2}}]$$

14.6.9 $L = 0, 2\eta \gg \rho$

$$F_0 \sim \frac{1}{2} \beta e^{\alpha}; F'_0 \sim \frac{1}{2} \beta^{-1} e^{-\alpha}$$

$$G_0 \sim \beta e^{-\alpha}; G'_0 \sim -\beta^{-1} e^{-\alpha}$$

$$\alpha = 2\sqrt{2\eta\rho} - \pi\eta$$

$$\beta = (\rho/2\eta)^{\frac{1}{2}}$$

14.6.10

$$L = 0, 2\eta \gg \rho$$

$$F_0 \sim \frac{1}{2} \beta e^{\alpha}; F'_0 \sim \left\{ \beta^{-2} + \frac{1}{8\eta} t^{-2} \beta^4 \right\} F_0$$

$$G_0 \sim \beta e^{-\alpha}; G'_0 \sim \left\{ -\beta^{-2} + \frac{1}{8\eta} t^{-2} \beta^4 \right\} G_0$$

$$t = \rho/2\eta$$

$$\alpha = 2\eta \{ [t(1-t)]^{\frac{1}{2}} + \arcsin t^{\frac{1}{2}} - \frac{1}{2}\pi \}$$

$$\beta = \{ t/(1-t) \}^{\frac{1}{2}}$$

14.6.11

$$L = 0, \rho \gg 2\eta$$

$$F_0 = \alpha \sin \beta; F'_0 = -t^2(bF_0 - aG_0)$$

$$G_0 = \alpha \cos \beta; G'_0 = -t^2(aF_0 + bG_0)$$

$$t = \frac{2\eta}{\rho}$$

$$\alpha = \left(\frac{1}{1-t}\right)^{\frac{1}{2}} \exp \left[-\frac{8t^3 - 3t^4}{64(2\eta)^2(1-t)^3} \right]$$

$$\beta = \frac{\pi}{4} + 2\eta \left\{ \frac{(1-t)^{\frac{1}{2}}}{t} + \frac{1}{2} \ln \left[\frac{1-(1-t)^{\frac{1}{2}}}{1+(1-t)^{\frac{1}{2}}} \right] \right\}$$

$$a = t^{-2}(1-t)^{\frac{1}{2}}, b = [8\eta(1-t)]^{-1}$$

14.6.12

$$\eta \gg 0, 2\eta \sim \rho$$

$$\frac{F_L(\eta, \rho)}{G_L(\eta, \rho)} \sim \sqrt{\pi} \left\{ \frac{\rho_L}{1 + \frac{L(L+1)}{\rho_L^2}} \right\}^{1/6} \left\{ \begin{matrix} \text{Ai}(x) \\ \text{Bi}(x) \end{matrix} \right\}$$

$$\rho_L = \eta + [\eta^2 + L(L+1)]^{1/2}$$

$$x = (\rho_L - \rho) \left[\frac{1}{\rho_L} + \frac{L(L+1)}{\rho_L^3} \right]^{1/3}$$

14.6.13

$$\eta \gg 0, 2\eta \sim \rho$$

$$x = (2\eta - \rho)(2\eta)^{-1/3}$$

$$[G_0 + iF_0] \sim \pi^{1/2} (2\eta)^{1/6} [\text{Bi}(x) + i\text{Ai}(x)]$$

$$[G'_0 + iF'_0] \sim -\pi^{1/2} (2\eta)^{-1/6} [\text{Bi}'(x) + i\text{Ai}'(x)]$$

14.6.14

$$\eta \gg 0$$

$$\rho_L = \eta + [\eta^2 + L(L+1)]^{1/2}$$

$$\frac{F_L(\rho_L)}{G_L(\rho_L)/\sqrt{3}} \sim \frac{\Gamma(1/3)}{2\sqrt{\pi}} \left(\frac{\rho_L}{3}\right)^{1/6} \left\{ 1 + \frac{L(L+1)}{\rho_L^2} \right\}^{-1/6}$$

$$\frac{F'_L(\rho_L)}{G'_L(\rho_L)/\sqrt{3}} \sim \pm \frac{\Gamma(2/3)}{2\sqrt{\pi}} \left(\frac{\rho_L}{3}\right)^{-1/6} \left\{ 1 + \frac{L(L+1)}{\rho_L^2} \right\}^{1/6}$$

$$14.6.15 \quad \rho = 2\eta \gg 0$$

$$F_0 \sim \frac{\Gamma(1/3)}{2\sqrt{\pi}} \left(\frac{2\eta}{3}\right)^{1/6}$$

$$G_0/\sqrt{3} \sim \frac{\Gamma(2/3)}{2\sqrt{\pi}(2\eta/3)^{1/6}}$$

$$F'_0 \sim \frac{\Gamma(2/3)}{2\sqrt{\pi}(2\eta/3)^{1/6}}$$

$$-G'_0/\sqrt{3} \sim \frac{\Gamma(2/3)}{2\sqrt{\pi}(2\eta/3)^{1/6}}$$

$$14.6.16 \quad \eta \rightarrow \infty$$

$$\sigma_0(\eta) \sim \left[\frac{\pi}{4} + \eta(\ln \eta - 1)\right]$$

$$C_0(\eta) \sim (2\pi\eta)^{1/2} e^{-\pi\eta},$$

(Equality to 8S for $\eta > 3$.)

$$14.6.17 \quad \eta \rightarrow 0$$

$$\sigma_0(\eta) \sim -\gamma\eta \quad (\gamma = \text{Euler's constant})$$

$$C_L(\eta) \sim \frac{2^L L!}{(2L+1)!}$$

$$14.6.18 \quad L \rightarrow \infty$$

$$C_L(\eta) \sim \frac{2^L L!}{(2L+1)!} e^{-\pi\eta/2}$$

Numerical Methods

14.7. Use and Extension of the Tables

In general the tables as presented are not simply interpolable. However, values for $L > 0$ may be obtained with the help of the recurrence relations. The values of $G_L(\eta, \rho)$ may be obtained by applying the recurrence relations in increasing order of L . Forward recurrence may be used for $F_L(\eta, \rho)$ as long as the instability does not produce errors in excess of the accuracy needed. In this case the backwards recurrence scheme (see **Example 1**) should be used.

Example 1. Compute $F_L(\eta, \rho)$ and $F'_L(\eta, \rho)$ for $\eta=2$, $\rho=5$, $L=0(1)5$. Starting with $F_{10}^*=1$, $F_{11}^*=0$, where $F_L^*=cF_L$, we compute from 14.2.3 in decreasing order of L :

L	(1) F_L^*	(2) F_L	(3) F_L	(4) F'_L
11	0.			
10	1.			
9	4.49284			
8	17.5225			
7	61.3603			
6	191.238			
5	523.472	.090791	.091	.1043
4	1238.53	.21481	.215	.2030
3	2486.72	.43130	.4313	.3205
2	4158.46	.72124	.72125	.3952
1	5727.97	.99346	.99347	.3709
0	6591.81	1.1433	1.1433	.29380

$$F_0/F_0^* = 1.7344 \times 10^{-4} = c^{-1}.$$

The values in the second column are obtained from those in the first by multiplying by the normalization constant, F_0/F_0^* where F_0 is the known value obtained from **Table 14.1**.

Repetition starting with $F_{15}^*=1$ and $F_{16}^*=0$ yields the same results.

In column 3, the results have been given when 14.2.3 is used in increasing order of L .

F'_L (column 4) follows from 14.2.2.

Example 2. Compute $G_L(\eta, \rho)$ and $G'_L(\eta, \rho)$ for $\eta=2$, $\rho=5$, $L=1(1)5$.

Using 14.2.2 and $G_0(2, 5) = .79445$, $G'_0 = -.67049$ from **Table 14.1** we find $G_1(2, 5) = 1.0815$. Then by forward recurrence using 14.2.3 we find:

L	G_L	$-G'_L$ *
1	1.0815	.60286
2	1.4969	.56619
3	2.0487	.79597
4	3.0941	1.7318
5	5.6298	4.5493

The values of G'_L are obtained with 14.2.1.

Example 3. Compute $G_0(\eta, \rho)$ for $\eta=2$, $\rho=2.5$.

From **Table 14.1**, $G_0(2, 2) = 3.5124$, $G'_0(2, 2) = -2.5554$. Successive differentiation of 14.1.1 for $L=0$ gives

$$\rho \frac{d^{k+2}w}{d\rho^{k+2}} = (2\eta - \rho) \frac{d^k w}{d\rho^k} - k \left\{ \frac{d^{k+1}w}{d\rho^{k+1}} + \frac{d^{k-1}w}{d\rho^{k-1}} \right\}$$

Taylor's expansion is $w(\rho + \Delta\rho) = w(\rho) + (\Delta\rho)w' + \frac{(\Delta\rho)^2}{2!}w'' + \dots$. With $w = G_0(\eta, \rho)$ and $\Delta\rho = .5$ we get:

k	$\frac{d^k G_0}{d\rho^k}$	$\frac{(\Delta\rho)^k}{k!} \frac{d^k G_0}{d\rho^k}$
0	3.5124	3.5124
1	-2.5554	-1.2777
2	3.5124	.43905
3	-6.0678	-.12641
4	12.136	.03160
5	-29.540	-.00769
6	83.352	.00181
7	-268.26	-.00042

$$G_0(2, 2.5) = 2.5726$$

As a check the result is obtained with $\eta=2$, $\rho=3$, $\Delta\rho = -.5$. The derivative $G'_0(\eta, \rho)$ may be obtained using Taylor's formula with $w = G'_0(\eta, \rho)$.

*See page 11.

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Texts

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Tables

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15. Hypergeometric Functions

FRITZ OBERHETTINGER¹

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¹ National Bureau of Standards. (Presently, Oregon State University, Corvallis, Oregon.)

15. Hypergeometric Functions

Mathematical Properties

15.1. Gauss Series, Special Elementary Cases, Special Values of the Argument

Gauss Series

The circle of convergence of the Gauss hypergeometric series

15.1.1

$$F(a, b; c; z) = {}_2F_1(a, b; c; z)$$

$$= F(b, a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

is the unit circle $|z|=1$. The behavior of this series on its circle of convergence is:

- (a) Divergence when $\Re(c-a-b) \leq -1$.
- (b) Absolute convergence when $\Re(c-a-b) > 0$.
- (c) Conditional convergence when $-1 < \Re(c-a-b) \leq 0$; the point $z=1$ is excluded. The Gauss series reduces to a polynomial of degree n in z when a or b is equal to $-n$, ($n=0, 1, 2, \dots$). (For these cases see also 15.4.) The series 15.1.1 is not defined when c is equal to $-m$, ($m=0, 1, 2, \dots$), provided a or b is not a negative integer n with $n < m$. For $c = -m$

15.1.2

$$\lim_{c \rightarrow -m} \frac{1}{\Gamma(c)} F(a, b; c; z) =$$

$$\frac{(a)_{m+1} (b)_{m+1}}{(m+1)!} z^{m+1} F(a+m+1, b+m+1; m+2; z)$$

Special Elementary Cases of Gauss Series

(For cases involving higher functions see 15.4.)

$$15.1.3 \quad F(1, 1; 2; z) = -z^{-1} \ln(1-z) *$$

$$15.1.4 \quad F\left(\frac{1}{2}, 1; \frac{3}{2}; z^2\right) = \frac{1}{2} z^{-1} \ln\left(\frac{1+z}{1-z}\right)$$

$$15.1.5 \quad F\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) = z^{-1} \arctan z$$

15.1.6

$$F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right) = (1-z^2)^{-1/2} F(1, 1; \frac{3}{2}; z^2) = z^{-1} \arcsin z$$

15.1.7

$$F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -z^2\right) = (1+z^2)^{-1/2} F(1, 1; \frac{3}{2}; -z^2)$$

$$= z^{-1} \ln[z + (1+z^2)^{1/2}]$$

$$15.1.8 \quad F(a, b; b; z) = (1-z)^{-a}$$

$$15.1.9 \quad F(a, \frac{1}{2}+a; \frac{1}{2}; z^2) = \frac{1}{2} [(1+z)^{-2a} + (1-z)^{-2a}]$$

15.1.10

$$F(a, \frac{1}{2}+a; \frac{3}{2}; z^2) =$$

$$\frac{1}{2} z^{-1} (1-2a)^{-1} [(1+z)^{1-2a} - (1-z)^{1-2a}]$$

15.1.11

$$F(-a, a; \frac{1}{2}; -z^2) = \frac{1}{2} \{ [(1+z^2)^{1/2} + z]^{2a} + [(1+z^2)^{1/2} - z]^{2a} \}$$

15.1.12

$$F(a, 1-a; \frac{1}{2}; -z^2) =$$

$$\frac{1}{2} (1+z^2)^{-1/2} \{ [(1+z^2)^{1/2} + z]^{2a-1} + [(1+z^2)^{1/2} - z]^{2a-1} \}$$

15.1.13

$$F(a, \frac{1}{2}+a; 1+2a; z) = 2^{2a} [1 + (1-z)^{1/2}]^{-2a}$$

$$= (1-z)^{1/2} F(1+a, \frac{1}{2}+a; 1+2a; z)$$

15.1.14

$$F(a, \frac{1}{2}+a; 2a; z) = 2^{2a-1} (1-z)^{-1/2} [1 + (1-z)^{1/2}]^{1-2a}$$

$$15.1.15 \quad F(a, 1-a; \frac{3}{2}; \sin^2 z) = \frac{\sin[(2a-1)z]}{(2a-1) \sin z}$$

$$15.1.16 \quad F(a, 2-a; \frac{3}{2}; \sin^2 z) = \frac{\sin[(2a-2)z]}{(a-1) \sin(2z)}$$

$$15.1.17 \quad F(-a, a; \frac{1}{2}; \sin^2 z) = \cos(2az)$$

$$15.1.18 \quad F(a, 1-a; \frac{1}{2}; \sin^2 z) = \frac{\cos[(2a-1)z]}{\cos z}$$

$$15.1.19 \quad F(a, \frac{1}{2}+a; \frac{1}{2}; -\tan^2 z) = \cos^{2a} z \cos(2az)$$

Special Values of the Argument

15.1.20

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

$$(c \neq 0, -1, -2, \dots, \Re(c-a-b) > 0)$$

*See page II.

15.1.21

$$F(a, b; a-b+1; -1) = 2^{-a} \pi^{\frac{1}{2}} \frac{\Gamma(1+a-b)}{\Gamma(1+\frac{1}{2}a-b)\Gamma(\frac{1}{2}+\frac{1}{2}a)} \\ (1+a-b \neq 0, -1, -2, \dots)$$

15.1.22

$$F(a, b; a-b+2; -1) = 2^{-a} \pi^{\frac{1}{2}} (b-1)^{-1} \Gamma(a-b+2) \\ \left[\frac{1}{\Gamma(\frac{1}{2}a)\Gamma(\frac{3}{2}+\frac{1}{2}a-b)} - \frac{1}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(1+\frac{1}{2}a-b)} \right] \\ (a-b+2 \neq 0, -1, -2, \dots)$$

$$15.1.23 \quad F(1, a; a+1; -1) = \frac{1}{2} a [\psi(\frac{1}{2}+\frac{1}{2}a) - \psi(\frac{1}{2}a)]$$

15.1.24

$$F(a, b; \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}; \frac{1}{2}) = \pi^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)} \\ (\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2} \neq 0, -1, -2, \dots)$$

15.1.25

$$F(a, b; \frac{1}{2}a+\frac{1}{2}b+1; \frac{1}{2}) = 2\pi^{\frac{1}{2}} (a-b)^{-1} \Gamma(1+\frac{1}{2}a+\frac{1}{2}b) \\ \{ [\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)]^{-1} - [\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}b)]^{-1} \} \\ (\frac{1}{2}(a+b)+1 \neq 0, -1, -2, \dots)$$

15.1.26

$$F(a, 1-a; b; \frac{1}{2}) = \\ 2^{1-b} \pi^{\frac{1}{2}} \Gamma(b) [\Gamma(\frac{1}{2}a+\frac{1}{2}b) \Gamma(\frac{1}{2}+\frac{1}{2}b-\frac{1}{2}a)]^{-1} \\ (b \neq 0, -1, -2, \dots)$$

15.1.27

$$F(1, 1; a+1; \frac{1}{2}) = a [\psi(\frac{1}{2}+\frac{1}{2}a) - \psi(\frac{1}{2}a)] \\ (a \neq -1, -2, -3, \dots)$$

15.1.28

$$F(a, a; a+1; \frac{1}{2}) = 2^{a-1} a [\psi(\frac{1}{2}+\frac{1}{2}a) - \psi(\frac{1}{2}a)] \\ (a \neq -1, -2, -3, \dots)$$

15.1.29

$$F(a, \frac{1}{2}+a; \frac{3}{2}-2a; -\frac{1}{3}) = (\frac{3}{2})^{-2a} \frac{\Gamma(\frac{4}{3})\Gamma(\frac{3}{2}-2a)}{\Gamma(\frac{2}{3})\Gamma(\frac{4}{3}-2a)} \\ (\frac{3}{2}-2a \neq 0, -1, -2, \dots)$$

15.1.30

$$F(a, \frac{1}{2}+a; \frac{5}{6}+a; \frac{1}{6}) = (\frac{3}{2})^a \pi^{\frac{1}{2}} \frac{\Gamma(\frac{5}{6}+\frac{2}{3}a)}{\Gamma(\frac{1}{2}+\frac{1}{3}a)\Gamma(\frac{5}{6}+\frac{1}{3}a)} \\ (\frac{5}{6}+\frac{2}{3}a \neq 0, -1, -2, \dots)$$

15.1.31

$$F(a, \frac{1}{8}a+\frac{1}{8}; \frac{3}{8}a+\frac{3}{8}; e^{i\pi/3}) \\ = 2^{\frac{2}{3}a+\frac{2}{3}} \pi^{\frac{1}{2}} 3^{-\frac{1}{2}(a+1)} e^{i\pi a/6} \frac{\Gamma(\frac{1}{3}a+\frac{5}{6})}{\Gamma(\frac{1}{3}a+\frac{2}{3})\Gamma(\frac{2}{3})} \\ (\frac{1}{3}a \neq -\frac{5}{6}, -\frac{11}{6}, -\frac{17}{6}, \dots)$$

15.2. Differentiation Formulas and Gauss' Relations for Contiguous Functions

Differentiation Formulas

$$15.2.1 \quad \frac{d}{dz} F(a, b; c; z) = \frac{ab}{c} F(a+1, b+1; c+1; z)$$

15.2.2

$$\frac{d^n}{dz^n} F(a, b; c; z) = \frac{(a)_n (b)_n}{(c)_n} F(a+n, b+n; c+n; z)$$

15.2.3

$$\frac{d^n}{dz^n} [z^{a+n-1} F(a, b; c; z)] = (a)_n z^{a-1} F(a+n, b; c; z)$$

15.2.4

$$\frac{d^n}{dz^n} [z^{c-1} F(a, b; c; z)] = (c-n)_n z^{c-n-1} F(a, b; c-n; z)$$

15.2.5

$$\frac{d^n}{dz^n} [z^{c-a+n-1} (1-z)^{a+b-c} F(a, b; c; z)] \\ = (c-a)_n z^{c-a-1} (1-z)^{a+b-c-n} F(a-n, b; c; z)$$

15.2.6

$$\frac{d^n}{dz^n} [(1-z)^{a+b-c} F(a, b; c; z)] \\ = \frac{(c-a)_n (c-b)_n}{(c)_n} (1-z)^{a+b-c-n} F(a, b; c+n; z)$$

15.2.7

$$\frac{d^n}{dz^n} [(1-z)^{a+n-1} F(a, b; c; z)] \\ = \frac{(-1)^n (a)_n (c-b)_n}{(c)_n} (1-z)^{a-1} F(a+n, b; c+n; z)$$

15.2.8

$$\frac{d^n}{dz^n} [z^{c-1} (1-z)^{b-c+n} F(a, b; c; z)] \\ = (c-n)_n z^{c-n-1} (1-z)^{b-c} F(a-n, b; c-n; z)$$

15.2.9

$$\frac{d^n}{dz^n} [z^{c-1} (1-z)^{a+b-c} F(a, b; c; z)] \\ = (c-n)_n z^{c-n-1} (1-z)^{a+b-c-n} F(a-n, b-n; c-n; z)$$

Gauss' Relations for Contiguous Functions

The six functions $F(a \pm 1, b; c; z)$, $F(a, b \pm 1; c; z)$, $F(a, b; c \pm 1; z)$ are called contiguous to $F(a, b; c; z)$. Relations between $F(a, b; c; z)$ and

any two contiguous functions have been given by Gauss. By repeated application of these relations the function $F(a+m, b+n; c+l; z)$ with integral $m, n, l (c+l \neq 0, -1, -2, \dots)$ can be expressed as a linear combination of $F(a, b; c; z)$ and one of its contiguous functions with coefficients which are rational functions of a, b, c, z .

15.2.10

$$(c-a)F(a-1, b; c; z) + (2a-c-az+bz)F(a, b; c; z) + a(z-1)F(a+1, b; c; z) = 0$$

15.2.11

$$(c-b)F(a, b-1; c; z) + (2b-c-bz+az)F(a, b; c; z) + b(z-1)F(a, b+1; c; z) = 0$$

15.2.12

$$c(c-1)(z-1)F(a, b; c-1; z) + c[c-1-(2c-a-b-1)z]F(a, b; c; z) + (c-a)(c-b)zF(a, b; c+1; z) = 0$$

15.2.13

$$[c-2a-(b-a)z]F(a, b; c; z) + a(1-z)F(a+1, b; c; z) - (c-a)F(a-1, b; c; z) = 0$$

15.2.14

$$(b-a)F(a, b; c; z) + aF(a+1, b; c; z) - bF(a, b+1; c; z) = 0$$

15.2.15

$$(c-a-b)F(a, b; c; z) + a(1-z)F(a+1, b; c; z) - (c-b)F(a, b-1; c; z) = 0$$

15.2.16

$$c[a-(c-b)z]F(a, b; c; z) - ac(1-z)F(a+1, b; c; z) + (c-a)(c-b)zF(a, b; c+1; z) = 0$$

15.2.17

$$(c-a-1)F(a, b; c; z) + aF(a+1, b; c; z) - (c-1)F(a, b; c-1; z) = 0$$

15.2.18

$$(c-a-b)F(a, b; c; z) - (c-a)F(a-1, b; c; z) + b(1-z)F(a, b+1; c; z) = 0$$

15.2.19

$$(b-a)(1-z)F(a, b; c; z) - (c-a)F(a-1, b; c; z) + (c-b)F(a, b-1; c; z) = 0$$

15.2.20

$$c(1-z)F(a, b; c; z) - cF(a-1, b; c; z) + (c-b)zF(a, b; c+1; z) = 0$$

15.2.21

$$[a-1-(c-b-1)z]F(a, b; c; z) + (c-a)F(a-1, b; c; z) - (c-1)(1-z)F(a, b; c-1; z) = 0$$

15.2.22

$$[c-2b+(b-a)z]F(a, b; c; z) + b(1-z)F(a, b+1; c; z) - (c-b)F(a, b-1; c; z) = 0$$

15.2.23

$$c[b-(c-a)z]F(a, b; c; z) - bc(1-z)F(a, b+1; c; z) + (c-a)(c-b)zF(a, b; c+1; z) = 0$$

15.2.24

$$(c-b-1)F(a, b; c; z) + bF(a, b+1; c; z) - (c-1)F(a, b; c-1; z) = 0$$

15.2.25

$$c(1-z)F(a, b; c; z) - cF(a, b-1; c; z) + (c-a)zF(a, b; c+1; z) = 0$$

15.2.26

$$[b-1-(c-a-1)z]F(a, b; c; z) + (c-b)F(a, b-1; c; z) - (c-1)(1-z)F(a, b; c-1; z) = 0$$

15.2.27

$$c[c-1-(2c-a-b-1)z]F(a, b; c; z) + (c-a)(c-b)zF(a, b; c+1; z) - c(c-1)(1-z)F(a, b; c-1; z) = 0$$

15.3. Integral Representations and Transformation Formulas

Integral Representations

15.3.1

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt \quad (\Re c > \Re b > 0)$$

The integral represents a one valued analytic function in the z -plane cut along the real axis from 1 to ∞ and hence 15.3.1 gives the analytic continuation of 15.1.1, $F(a, b; c; z)$. Another integral representation is in the form of a Mellin-Barnes integral

$$15.3.2 \quad F(a, b; c; z) = \frac{\Gamma(c)}{2\pi i \Gamma(a) \Gamma(b)} \int_{-\infty}^{\infty} \frac{\Gamma(a+s) \Gamma(b+s) \Gamma(-s)}{\Gamma(c+s)} (-z)^s ds$$

$$= \frac{1}{2} i \frac{\Gamma(c)}{\Gamma(a) \Gamma(b)} \int_{-\infty}^{\infty} \frac{\Gamma(a+s) \Gamma(b+s)}{\Gamma(1+s) \Gamma(c+s)} \csc(\pi s) (-z)^s ds$$

Here $-\pi < \arg(-z) < \pi$ and the path of integration is chosen such that the poles of $\Gamma(a+s)$ and $\Gamma(b+s)$ i.e. the points $s = -a - n$ and $s = -b - m$ ($n, m = 0, 1, 2, \dots$) respectively, are at its left side and the poles of $\csc(\pi s)$ or $\Gamma(-s)$ i.e. $s = 0, 1, 2, \dots$ are at its right side. The cases in which $-a, -b$ or $-c$ are non-negative integers or $a-b$ equal to an integer are excluded.

Linear Transformation Formulas

From 15.3.1 and 15.3.2 a number of transformation formulas for $F(a, b; c; z)$ can be derived.

$$15.3.3 \quad F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z)$$

$$15.3.4 \quad = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right)$$

$$15.3.5 \quad = (1-z)^{-b} F\left(b, c-a; c; \frac{z}{z-1}\right)$$

$$15.3.6 \quad = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} F(a, b; a+b-c+1; 1-z)$$

$$+ (1-z)^{c-a-b} \frac{\Gamma(c) \Gamma(a+b-c)}{\Gamma(a) \Gamma(b)} F(c-a, c-b; c-a-b+1; 1-z) \quad (|\arg(1-z)| < \pi)$$

$$15.3.7 \quad = \frac{\Gamma(c) \Gamma(b-a)}{\Gamma(b) \Gamma(c-a)} (-z)^{-a} F\left(a, 1-c+a; 1-b+a; \frac{1}{z}\right)$$

$$+ \frac{\Gamma(c) \Gamma(a-b)}{\Gamma(a) \Gamma(c-b)} (-z)^{-b} F\left(b, 1-c+b; 1-a+b; \frac{1}{z}\right) \quad (|\arg(-z)| < \pi)$$

$$15.3.8 \quad = (1-z)^{-a} \frac{\Gamma(c) \Gamma(b-a)}{\Gamma(b) \Gamma(c-a)} F\left(a, c-b; a-b+1; \frac{1}{1-z}\right)$$

$$+ (1-z)^{-b} \frac{\Gamma(c) \Gamma(a-b)}{\Gamma(a) \Gamma(c-b)} F\left(b, c-a; b-a+1; \frac{1}{1-z}\right) \quad (|\arg(1-z)| < \pi)$$

$$15.3.9 \quad = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} z^{-a} F\left(a, a-c+1; a+b-c+1; 1-\frac{1}{z}\right)$$

$$+ \frac{\Gamma(c) \Gamma(a+b-c)}{\Gamma(a) \Gamma(b)} (1-z)^{c-a-b} z^{a-c} F\left(c-a, 1-a; c-a-b+1; 1-\frac{1}{z}\right) \quad (|\arg z| < \pi, |\arg(1-z)| < \pi)$$

Each term of 15.3.6 has a pole when $c = a + b \pm m$, ($m = 0, 1, 2, \dots$); this case is covered by

$$15.3.10 \quad F(a, b; a+b; z) = \frac{\Gamma(a+b)}{\Gamma(a) \Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} [2\psi(n+1) - \psi(a+n) - \psi(b+n) - \ln(1-z)] (1-z)^n$$

$$(|\arg(1-z)| < \pi, |1-z| < 1)$$

Furthermore for $m = 1, 2, 3, \dots$

$$15.3.11 \quad F(a, b; a+b+m; z) = \frac{\Gamma(m) \Gamma(a+b+m)}{\Gamma(a+m) \Gamma(b+m)} \sum_{n=0}^{m-1} \frac{(a)_n (b)_n}{n! (1-m)_n} (1-z)^n$$

$$- \frac{\Gamma(a+b+m)}{\Gamma(a) \Gamma(b)} (z-1)^m \sum_{n=0}^{\infty} \frac{(a+m)_n (b+m)_n}{n! (n+m)!} (1-z)^n [\ln(1-z) - \psi(n+1)$$

$$- \psi(n+m+1) + \psi(a+n+m) + \psi(b+n+m)] \quad (|\arg(1-z)| < \pi, |1-z| < 1)$$

$$\begin{aligned}
 15.3.12 \quad F(a, b; a+b-m; z) &= \frac{\Gamma(m)\Gamma(a+b-m)}{\Gamma(a)\Gamma(b)} (1-z)^{-m} \sum_{n=0}^{m-1} \frac{(a-m)_n(b-m)_n}{n!(1-m)_n} (1-z)^n \\
 &\quad - \frac{(-1)^m \Gamma(a+b-m)}{\Gamma(a-m)\Gamma(b-m)} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(n+m)!} (1-z)^n [\ln(1-z) - \psi(n+1) \\
 &\quad \quad - \psi(n+m+1) + \psi(a+n) + \psi(b+n)] \\
 &\quad \quad (|\arg(1-z)| < \pi, |1-z| < 1)
 \end{aligned}$$

Similarly each term of 15.3.7 has a pole when $b=a \pm m$ or $b-a = \pm m$ and the case is covered by

$$\begin{aligned}
 15.3.13 \quad F(a, a; c; z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} (-z)^{-a} \sum_{n=0}^{\infty} \frac{(a)_n(1-c+a)_n}{(n!)^2} z^{-n} [\ln(-z) + 2\psi(n+1) - \psi(a+n) - \psi(c-a-n)] \\
 &\quad (|\arg(-z)| < \pi, |z| > 1, (c-a) \neq 0, \pm 1, \pm 2, \dots)
 \end{aligned}$$

The case $b-a=m$, ($m=1, 2, 3, \dots$) is covered by

$$\begin{aligned}
 15.3.14 \quad F(a, a+m; c; z) &= F(a+m, a; c; z) \\
 &= \frac{\Gamma(c)(-z)^{-a-m}}{\Gamma(a+m)\Gamma(c-a)} \sum_{n=0}^{\infty} \frac{(a)_{n+m}(1-c+a)_{n+m}}{n!(n+m)!} z^{-n} [\ln(-z) + \psi(1+m+n) + \psi(1+n) \\
 &\quad - \psi(a+m+n) - \psi(c-a-m-n)] + (-z)^{-a} \frac{\Gamma(c)}{\Gamma(a+m)} \sum_{n=0}^{m-1} \frac{\Gamma(m-n)(a)_n}{n!\Gamma(c-a-n)} z^{-n} \\
 &\quad (|\arg(-z)| < \pi, |z| > 1, (c-a) \neq 0, \pm 1, \pm 2, \dots)
 \end{aligned}$$

The case $c-a=0, -1, -2, \dots$ becomes elementary, 15.3.3, and the case $c-a=1, 2, 3, \dots$ can be obtained from 15.3.14, by a limiting process (see [15.2]).

Quadratic Transformation Formulas

If, and only if the numbers $\pm(1-c)$, $\pm(a-b)$, $\pm(a+b-c)$ are such, that two of them are equal or one of them is equal to $\frac{1}{2}$, then there exists a quadratic transformation. The basic formulas are due to Kummer [15.7] and a complete list is due to Goursat [15.3]. See also [15.2].

$$15.3.15 \quad F(a, b; 2b; z) = (1-z)^{-1/2} F\left(\frac{1}{2}a, b-\frac{1}{2}a; b+\frac{1}{2}; \frac{z^2}{4z-4}\right)$$

$$15.3.16 \quad = (1-\frac{1}{2}z)^{-a} F\left(\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}a; b+\frac{1}{2}; z^2(2-z)^{-2}\right)$$

$$15.3.17 \quad = \left(\frac{1}{2}+\frac{1}{2}\sqrt{1-z}\right)^{-2a} F\left[a, a-b+\frac{1}{2}; b+\frac{1}{2}; \left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^2\right]$$

$$15.3.18 \quad = (1-z)^{-1/2} F\left(a, 2b-a; b+\frac{1}{2}; -\frac{(1-\sqrt{1-z})^2}{4\sqrt{1-z}}\right)$$

$$15.3.19 \quad F(a, a+\frac{1}{2}; c; z) = \left(\frac{1}{2}+\frac{1}{2}\sqrt{1-z}\right)^{-2a} F\left(2a, 2a-c+1; c; \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)$$

$$15.3.20 \quad = (1 \pm \sqrt{z})^{-2a} F\left(2a, c-\frac{1}{2}; 2c-1; \pm \frac{2\sqrt{z}}{1 \pm \sqrt{z}}\right)$$

$$15.3.21 \quad = (1-z)^{-a} F\left(2a, 2c-2a-1; c; \frac{\sqrt{1-z}-1}{2\sqrt{1-z}}\right)$$

$$15.3.22 \quad F(a, b; a+b+\frac{1}{2}; z) = F(2a, 2b; a+b+\frac{1}{2}; \frac{1}{2}-\frac{1}{2}\sqrt{1-z})$$

$$15.3.23 \quad = \left(\frac{1}{2}+\frac{1}{2}\sqrt{1-z}\right)^{-2a} F\left(2a, a-b+\frac{1}{2}; a+b+\frac{1}{2}; \frac{\sqrt{1-z}-1}{\sqrt{1-z}+1}\right)$$

- 15.3.24 $F(a, b; a+b-\frac{1}{2}; z) = (1-z)^{-\frac{1}{2}} F(2a-1, 2b-1; a+b-\frac{1}{2}; \frac{1}{2}-\frac{1}{2}\sqrt{1-z})$
- 15.3.25 $= (1-z)^{-\frac{1}{2}} (\frac{1}{2}+\frac{1}{2}\sqrt{1-z})^{1-2a} F\left(2a-1, a-b+\frac{1}{2}; a+b-\frac{1}{2}; \frac{\sqrt{1-z}-1}{\sqrt{1-z}+1}\right)$
- 15.3.26 $F(a, b; a-b+1; z) = (1+z)^{-a} F(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; a-b+1; 4z(1+z)^{-2})$
- 15.3.27 $= (1\pm\sqrt{z})^{-2a} F(a, a-b+\frac{1}{2}; 2a-2b+1; \pm 4\sqrt{z}(1\pm\sqrt{z})^{-2})$
- 15.3.28 $= (1-z)^{-a} F(\frac{1}{2}a, \frac{1}{2}a-b+\frac{1}{2}; a-b+1; -4z(1-z)^{-2})$
- 15.3.29 $F(a, b; \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}; z) = (1-2z)^{-a} F\left(\frac{1}{2}a, \frac{1}{2}a+\frac{1}{2}; \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}; \frac{4z^2-4z}{(1-2z)^2}\right)$
- 15.3.30 $= F(\frac{1}{2}a, \frac{1}{2}b; \frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}; 4z-4z^2)$
- 15.3.31 $F(a, 1-a; c; z) = (1-z)^{c-1} F(\frac{1}{2}c-\frac{1}{2}a, \frac{1}{2}c+\frac{1}{2}a-\frac{1}{2}; c; 4z-4z^2)$
- 15.3.32 $= (1-z)^{c-1} (1-2z)^{a-c} F(\frac{1}{2}c-\frac{1}{2}a, \frac{1}{2}c-\frac{1}{2}a+\frac{1}{2}; c; (4z^2-4z)(1-2z)^{-2})$

Cubic transformations are listed in [15.2] and [15.3].

In the formulas above, the square roots are defined so that their value is real and positive when $0 \leq z < 1$. All formulas are valid in the neighborhood of $z=0$.

15.4. Special Cases of $F(a, b; c; z)$

Polynomials

When a or b is equal to a negative integer, then

$$15.4.1 \quad F(-m, b; c; z) = \sum_{n=0}^m \frac{(-m)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

This formula is also valid when $c = -m-l$; $m, l=0, 1, 2, \dots$

$$15.4.2 \quad F(-m, b; -m-l; z) = \sum_{n=0}^m \frac{(-m)_n (b)_n}{(-m-l)_n} \frac{z^n}{n!}$$

Some particular cases are

$$15.4.3 \quad F(-n, n; \frac{1}{2}; x) = T_n(1-2x)$$

$$15.4.4 \quad F(-n, n+1; 1; x) = P_n(1-2x)$$

$$15.4.5 \quad F\left(-n, n+2\alpha; \alpha+\frac{1}{2}; x\right) = \frac{n!}{(2\alpha)_n} C_n^{(\alpha)}(1-2x)$$

$$15.4.6 \quad F(-n, \alpha+1+\beta+n; \alpha+1; x) = \frac{n!}{(\alpha+1)_n} P_n^{(\alpha, \beta)}(1-2x)$$

Here T_n , P_n , $C_n^{(\alpha)}$, $P_n^{(\alpha, \beta)}$ denote Chebyshev, Legendre's, Gegenbauer's and Jacobi's polynomials respectively (see chapter 22).

Legendre Functions

Legendre functions are connected with those special cases of the hypergeometric function for which a quadratic transformation exists (see 15.3).

$$15.4.7 \quad F(a, b; 2b; z) = 2^{2b-1} \Gamma(\frac{1}{2}+b) z^{\frac{1}{2}-b} (1-z)^{\frac{1}{2}(b-a-\frac{1}{2})} P_{a-b-\frac{1}{2}}^{\frac{1}{2}-b}\left[\left(1-\frac{z}{2}\right)(1-z)^{-\frac{1}{2}}\right]$$

$$15.4.8 \quad = 2^{2b} \pi^{-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}+b)}{\Gamma(2b-a)} z^{-b} (1-z)^{\frac{1}{2}(b-a)} e^{i\pi(a-b)} P_{b-\frac{1}{2}}^{b-a}\left(\frac{2-z}{z}-1\right)$$

$$15.4.9 \quad F(a, b; 2b; -z) = 2^{2b} \pi^{-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}+b)}{\Gamma(a)} z^{-b} (1+z)^{\frac{1}{2}(b-a)} e^{-i\pi(a-b)} P_{b-\frac{1}{2}}^{a-b}\left(1+\frac{2}{z}\right) (|\arg z| < \pi, |\arg(1\pm z)| < \pi)$$

$$15.4.10 \quad F(a, a + \tfrac{1}{2}; c; z) = 2^{c-1} \Gamma(c) z^{\frac{1}{2}-ic} (1-z)^{\frac{1}{2}-a-\frac{1}{2}} P_{2a-\frac{1}{2}}^{1-c} [(1-z)^{-\frac{1}{2}}] \\ (|\arg z| < \pi, |\arg(1-z)| < \pi, z \text{ not between } 0 \text{ and } -\infty)$$

$$15.4.11 \quad F(a, a + \tfrac{1}{2}; c; x) = 2^{c-1} \Gamma(c) (-x)^{\frac{1}{2}-ic} (1-x)^{\frac{1}{2}-a-\frac{1}{2}} P_{2a-\frac{1}{2}}^{1-c} [(1-x)^{-\frac{1}{2}}] \quad (-\infty < x < 0)$$

$$15.4.12 \quad F(a, b; a+b+\tfrac{1}{2}; z) = 2^{a+b-\frac{1}{2}} \Gamma(\tfrac{1}{2}+a+b) (-z)^{\frac{1}{2}(1-a-b)} P_{a-b-\frac{1}{2}}^{\frac{1}{2}-a-b} [(1-z)^{\frac{1}{2}}] \\ (|\arg(-z)| < \pi, z \text{ not between } 0 \text{ and } 1)$$

$$15.4.13 \quad F(a, b; a+b+\tfrac{1}{2}; x) = 2^{a+b-\frac{1}{2}} \Gamma(\tfrac{1}{2}+a+b) x^{\frac{1}{2}(1-a-b)} P_{a-b-\frac{1}{2}}^{\frac{1}{2}-a-b} [(1-x)^{\frac{1}{2}}] \quad (0 < x < 1)$$

$$15.4.14 \quad F(a, b; a-b+1; z) = \Gamma(a-b+1) z^{\frac{1}{2}-ia} (1-z)^{-b} P_{-b}^{\frac{1}{2}-a} \left(\frac{1+z}{1-z} \right) \\ (|\arg(1-z)| < \pi, z \text{ not between } 0 \text{ and } -\infty)$$

$$15.4.15 \quad F(a, b; a-b+1; x) = \Gamma(a-b+1) (1-x)^{-b} (-x)^{\frac{1}{2}-ia} P_{-b}^{\frac{1}{2}-a} \left(\frac{1+x}{1-x} \right) \quad (-\infty < x < 0)$$

$$15.4.16 \quad F(a, 1-a; c; z) = \Gamma(c) (-z)^{\frac{1}{2}-ic} (1-z)^{\frac{1}{2}-a-\frac{1}{2}} P_{-a}^{1-c} (1-2z) \\ (|\arg(-z)| < \pi, |\arg(1-z)| < \pi, z \text{ not between } 0 \text{ and } 1)$$

$$15.4.17 \quad F(a, 1-a; c; x) = \Gamma(c) x^{\frac{1}{2}-ic} (1-x)^{\frac{1}{2}-a-\frac{1}{2}} P_{-a}^{1-c} (1-2x) \quad (0 < x < 1)$$

$$15.4.18 \quad F(a, b; \tfrac{1}{2}a + \tfrac{1}{2}b + \tfrac{1}{2}; z) = \Gamma(\tfrac{1}{2} + \tfrac{1}{2}a + \tfrac{1}{2}b) [z(z-1)]^{\frac{1}{2}(1-a-b)} P_{\frac{1}{2}(a-b-1)}^{\frac{1}{2}(1-a-b)} (1-2z) \\ (|\arg z| < \pi, |\arg(z-1)| < \pi, z \text{ not between } 0 \text{ and } 1)$$

$$15.4.19 \quad F(a, b; \tfrac{1}{2}a + \tfrac{1}{2}b + \tfrac{1}{2}; x) = \Gamma(\tfrac{1}{2} + \tfrac{1}{2}a + \tfrac{1}{2}b) (x-x^2)^{\frac{1}{2}(1-a-b)} P_{\frac{1}{2}(a-b-1)}^{\frac{1}{2}(1-a-b)} (1-2x) \quad (0 < x < 1)$$

$$15.4.20 \quad F(a, b; a+b-\tfrac{1}{2}; z) = 2^{a+b-\frac{1}{2}} \Gamma(a+b-\tfrac{1}{2}) (-z)^{\frac{1}{2}(1-a-b)} (1-z)^{-\frac{1}{2}} P_{a-b-\frac{1}{2}}^{\frac{1}{2}-a-b} [(1-z)^{\frac{1}{2}}] \\ (|\arg(-z)| < \pi, |\arg(1-z)| < \pi, \Re[(1-z)^{\frac{1}{2}}] > 0, z \text{ not between } 0 \text{ and } 1)$$

$$15.4.21 \quad F(a, b; a+b-\tfrac{1}{2}; x) = 2^{a+b-\frac{1}{2}} \Gamma(a+b-\tfrac{1}{2}) x^{\frac{1}{2}(1-a-b)} (1-x)^{-\frac{1}{2}} P_{a-b-\frac{1}{2}}^{\frac{1}{2}-a-b} [(1-x)^{\frac{1}{2}}] \quad (0 < x < 1)$$

$$15.4.22 \quad F(a, b; \tfrac{1}{2}; z) = \pi^{-1} 2^{a+b-\frac{1}{2}} \Gamma(\tfrac{1}{2}+a) \Gamma(\tfrac{1}{2}+b) (z-1)^{\frac{1}{2}(1-a-b)} [P_{a-b-\frac{1}{2}}^{\frac{1}{2}-a-b}(z^{\frac{1}{2}}) + P_{a-b-\frac{1}{2}}^{\frac{1}{2}-a-b}(-z^{\frac{1}{2}})] \\ (|\arg z| < \pi, |\arg(z-1)| < \pi, z \text{ not between } 0 \text{ and } 1)$$

$$15.4.23 \quad F(a, b; \tfrac{1}{2}; x) = \pi^{-1} 2^{a+b-\frac{1}{2}} \Gamma(\tfrac{1}{2}+a) \Gamma(\tfrac{1}{2}+b) (1-x)^{\frac{1}{2}(1-a-b)} [P_{a-b-\frac{1}{2}}^{\frac{1}{2}-a-b}(x^{\frac{1}{2}}) + P_{a-b-\frac{1}{2}}^{\frac{1}{2}-a-b}(-x^{\frac{1}{2}})] \quad (0 < x < 1)$$

$$15.4.24 \quad F(a, b; \tfrac{1}{2}; -z) = \pi^{-1} 2^{a+b-\frac{1}{2}} \Gamma(\tfrac{1}{2}+a) \Gamma(1-b) (z+1)^{-\frac{1}{2}-ib} e^{\pm i \frac{\pi}{2} (b-a)} \{ P_{a+b-1}^{\frac{1}{2}-a-b} [z^{\frac{1}{2}}(1+z)^{-\frac{1}{2}}] \\ + P_{a+b-1}^{\frac{1}{2}-a-b} [-z^{\frac{1}{2}}(1+z)^{-\frac{1}{2}}] \} \\ (\pm \text{ according as } \Im z \geq 0, z \text{ not between } 0 \text{ and } \infty)$$

$$* 15.4.25 \quad F(a, b; \tfrac{1}{2}; -x) = \pi^{-1} 2^{a+b-\frac{1}{2}} \Gamma(\tfrac{1}{2}+a) \Gamma(1-b) (1+x)^{-\frac{1}{2}-ib} \{ P_{a+b-1}^{\frac{1}{2}-a-b} [x^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}] + P_{a+b-1}^{\frac{1}{2}-a-b} [-x^{1/2}(1+x)^{-\frac{1}{2}}] \} \\ (0 < x < \infty)$$

$$15.4.26 \quad F(a, b; \tfrac{3}{2}; x) = -\pi^{-1} 2^{a+b-\frac{1}{2}} \Gamma(a-\tfrac{1}{2}) \Gamma(b-\tfrac{1}{2}) x^{-\frac{1}{2}} (1-x)^{\frac{1}{2}(1-a-b)} \{ P_{a-b-\frac{1}{2}}^{\frac{1}{2}-a-b}(x^{\frac{1}{2}}) - P_{a-b-\frac{1}{2}}^{\frac{1}{2}-a-b}(-x^{\frac{1}{2}}) \} \quad (0 < x < 1)$$

15.5. The Hypergeometric Differential Equation

The hypergeometric differential equation

$$15.5.1 \quad z(1-z) \frac{d^2 w}{dz^2} + [c - (a+b+1)z] \frac{dw}{dz} - abw = 0$$

has three (regular) singular points $z=0, 1, \infty$. The pairs of exponents at these points are

$$15.5.2 \quad \rho_{1,2}^{(0)}=0, 1-c, \quad \rho_{1,2}^{(1)}=0, c-a-b, \quad \rho_{1,2}^{(\infty)}=a, b$$

respectively. The general theory of differential equations of the Fuchsian type distinguishes between the following cases.

A. None of the numbers $c, c-a-b; a-b$ is equal to an integer. Then two linearly independent solutions of 15.5.1 in the neighborhood of the singular points $0, 1, \infty$ are respectively

$$15.5.3 \quad w_{1(0)}=F(a, b; c; z)=(1-z)^{c-a-b}F(c-a, c-b; c; z)$$

$$15.5.4 \quad w_{2(0)}=z^{1-c}F(a-c+1, b-c+1; 2-c; z)=z^{1-c}(1-z)^{c-a-b}F(1-a, 1-b; 2-c; z)$$

$$15.5.5 \quad w_{1(1)}=F(a, b; a+b+1-c; 1-z)=z^{1-c}F(1+b-c, 1+a-c; a+b+1-c; 1-z)$$

$$15.5.6 \quad w_{2(1)}=(1-z)^{c-a-b}F(c-b, c-a; c-a-b+1; 1-z)=z^{1-c}(1-z)^{c-a-b}F(1-a, 1-b; c-a-b+1; 1-z)$$

$$15.5.7 \quad w_{1(\infty)}=z^{-a}F(a, a-c+1; a-b+1; z^{-1})=z^{b-c}(z-1)^{c-a-b}F(1-b, c-b; a-b+1; z^{-1})$$

$$15.5.8 \quad w_{2(\infty)}=z^{-b}F(b, b-c+1; b-a+1; z^{-1})=z^{a-c}(z-1)^{c-a-b}F(1-a, c-a; b-a+1; z^{-1})$$

The second set of the above expressions is obtained by applying 15.3.3 to the first set.

Another set of representations is obtained by applying 15.3.4 to 15.5.3 through 15.5.8. This gives 15.5.9–15.5.14.

$$15.5.9 \quad w_{1(0)}=(1-z)^{-a}F\left(a, c-b; c; \frac{z}{z-1}\right)=(1-z)^{-b}F\left(b, c-a; c; \frac{z}{z-1}\right)$$

$$15.5.10 \quad w_{2(0)}=z^{1-c}(1-z)^{c-a-1}F\left(a-c+1, 1-b; 2-c; \frac{z}{z-1}\right)=z^{1-c}(1-z)^{c-b-1}F\left(b-c+1, 1-a; 2-c; \frac{z}{z-1}\right)$$

$$15.5.11 \quad w_{1(1)}=z^{-a}F(a, a-c+1; a+b-c+1; 1-z^{-1})=z^{-b}F(b, b-c+1; a+b-c+1; 1-z^{-1})$$

15.5.12

$$w_{2(1)}=z^{a-c}(1-z)^{c-a-b}F(c-a, 1-a; c-a-b+1; 1-z^{-1})=z^{b-c}(1-z)^{c-a-b}F(c-b, 1-b; c-a-b+1; 1-z^{-1})$$

$$15.5.13 \quad w_{1(\infty)}=(z-1)^{-a}F\left(a, c-b; a-b+1; \frac{1}{1-z}\right)=(z-1)^{-b}F\left(b, c-a; b-a+1; \frac{1}{1-z}\right)$$

15.5.14

$$w_{2(\infty)}=z^{1-c}(z-1)^{c-a-1}F\left(a-c+1, 1-b; a-b+1; \frac{1}{1-z}\right)=z^{1-c}(z-1)^{c-b-1}F\left(b-c+1, 1-a; b-a+1; \frac{1}{1-z}\right)$$

15.5.3 to 15.5.14 constitute Kummer's 24 solutions of the hypergeometric equation. The analytic continuation of $w_{1,2(0)}(z)$ can then be obtained by means of 15.3.3 to 15.3.9.

B. One of the numbers $a, b, c-a, c-b$ is an integer. Then one of the hypergeometric series for instance $w_{1,2(0)}$, 15.5.3, 15.5.4 terminates and the corresponding solution is of the form

$$15.5.15 \quad w=z^a(1-z)^bp_n(z)$$

where $p_n(z)$ is a polynomial in z of degree n . This case is referred to as the degenerate case of the hypergeometric differential equation and its solutions are listed and discussed in great detail in [15.2].

C. The number $c-a-b$ is an integer, c nonintegral. Then 15.3.10 to 15.3.12 give the analytic continuation of $w_{1,2(0)}$ into the neighborhood of $z=1$. Similarly 15.3.13 and 15.3.14 give the analytic continuation of $w_{1,2(0)}$ into the neighborhood of $z=\infty$ in case $a-b$ is an integer but not c , subject of course to the further restrictions $c-a=0, \pm 1, \pm 2 \dots$ (For a detailed discussion of all possible cases, see [15.2]).

D. The number $c=1$. Then 15.5.3, 15.5.4 are replaced by

$$15.5.16 \quad w_{1(0)}=F(a, b; 1; z)$$

$$15.5.17 \quad w_{2(0)} = F(a, b; 1; z) \ln z + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} z^n [\psi(a+n) - \psi(a) + \psi(b+n) - \psi(b) - 2\psi(n+1) + 2\psi(1)] \quad (|z| < 1)$$

E. The number $c = m+1$, $m = 1, 2, 3, \dots$. A fundamental system is

$$15.5.18 \quad w_{1(0)} = F(a, b; m+1; z)$$

$$15.5.19 \quad w_{2(0)} = F(a, b; m+1; z) \ln z + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(1+m)_n n!} z^n [\psi(a+n) - \psi(a) + \psi(b+n) - \psi(b) - \psi(m+1+n) + \psi(m+1) - \psi(n+1) + \psi(1)] - \sum_{n=1}^m \frac{(n-1)! (-m)_n}{(1-a)_n (1-b)_n} z^{-n} \quad (|z| < 1 \text{ and } a, b \neq 0, 1, 2, \dots, (m-1))$$

F. The number $c = 1-m$, $m = 1, 2, 3, \dots$. A fundamental system is

$$15.5.20 \quad w_{1(0)} = z^m F(a+m, b+m; 1+m; z)$$

15.5.21

$$w_{2(0)} = z^m F(a+m, b+m; 1+m; z) \ln z + z^m \sum_{n=1}^{\infty} z^n \frac{(a+m)_n (b+m)_n}{(1+m)_n n!} [\psi(a+m+n) - \psi(a+m) + \psi(b+m+n) - \psi(b+m) - \psi(m+1+n) + \psi(m+1) - \psi(n+1) + \psi(1)] - \sum_{n=1}^m \frac{(n-1)! (-m)_n}{(1-a-m)_n (1-b-m)_n} z^{m-n} \quad (|z| < 1 \text{ and } a, b \neq 0, -1, -2, \dots, -(m-1))$$

15.6. Riemann's Differential Equation

The hypergeometric differential equation 15.5.1 with the (regular) singular points 0, 1, ∞ is a special case of Riemann's differential equation with three (regular) singular points a, b, c

15.6.1

$$\frac{d^2 w}{dz^2} + \left[\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c} \right] \frac{dw}{dz} + \left[\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-c)(b-a)}{z-b} + \frac{\gamma\gamma'(c-a)(c-b)}{z-c} \right] \frac{w}{(z-a)(z-b)(z-c)} = 0$$

The pairs of the exponents with respect to the singular points $a; b; c$ are $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ respectively subject to the condition

$$15.6.2 \quad \alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$$

The complete set of solutions of 15.6.1 is denoted by the symbol

$$15.6.3 \quad w = P \left\{ \begin{matrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \right. z \left. \right\}$$

Special Cases of Riemann's P Function

(a) The generalized hypergeometric function

15.6.4

$$w = P \left\{ \begin{matrix} 0 & \infty & 1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \right. z \left. \right\}$$

(b) The hypergeometric function $F(a, b; c; z)$

15.6.5

$$w = P \left\{ \begin{matrix} 0 & \infty & 1 \\ 0 & a & 0 \\ 1-c & b & c-a-b \end{matrix} \right. z \left. \right\}$$

(c) The Legendre functions $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$

15.6.6

$$w = P \left\{ \begin{matrix} 0 & \infty & 1 \\ -\frac{1}{2}\nu & \frac{1}{2}\mu & 0 \\ \frac{1}{2} + \frac{1}{2}\nu & -\frac{1}{2}\mu & \frac{1}{2} \end{matrix} \right. (1-z^2)^{-1} \left. \right\}$$

(d) The confluent hypergeometric function

15.6.7

$$w = P \left\{ \begin{matrix} 0 & \infty & c \\ \frac{1}{2} + u & -c & c-k \\ \frac{1}{2} - u & 0 & k \end{matrix} \right. z \left. \right\}$$

provided $\lim c \rightarrow \infty$.

Transformation Formulas for Riemann's P Function

$$15.6.8 \quad \left(\frac{z-a}{z-b}\right)^k \left(\frac{z-c}{z-b}\right)^l P \left\{ \begin{matrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \middle| z \right\} = P \left\{ \begin{matrix} a & b & c \\ \alpha+k & \beta-k-l & \gamma+l \\ \alpha'+k & \beta'-k-l & \gamma'+l \end{matrix} \middle| z \right\}$$

$$15.6.9 \quad P \left\{ \begin{matrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \middle| z \right\} = P \left\{ \begin{matrix} a_1 & b_1 & c_1 \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \middle| z_1 \right\}$$

where

$$15.6.10 \quad z = \frac{Az_1+B}{Cz_1+D}, \quad a = \frac{Aa_1+B}{Ca_1+D}, \quad b = \frac{Ab_1+B}{Cb_1+D}, \quad c = \frac{Ac_1+B}{Cc_1+D}$$

and A, B, C, D are arbitrary constants such that $AD-BC \neq 0$.

Riemann's P function reduced to the hypergeometric function is

$$15.6.11 \quad P \left\{ \begin{matrix} a & b & c \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix} \middle| z \right\} = \left(\frac{z-a}{z-b}\right)^\alpha \left(\frac{z-c}{z-b}\right)^\gamma P \left\{ \begin{matrix} 0 & \infty & 1 \\ 0 & \alpha+\beta+\gamma & 0 \\ \alpha'-\alpha & \alpha+\beta'+\gamma & \gamma'-\gamma \end{matrix} \middle| \frac{(z-a)(c-b)}{(z-b)(c-a)} \right\}$$

The P function on the right hand side is Gauss' hypergeometric function (see 15.6.5). If it is replaced by Kummer's 24 solutions 15.5.3 to 15.5.14 the complete set of 24 solutions for Riemann's differential equation 15.6.1 is obtained. The first of these solutions is for instance by 15.5.3 and 15.6.5

$$15.6.12 \quad w = \left(\frac{z-a}{z-b}\right)^\alpha \left(\frac{z-c}{z-b}\right)^\gamma F \left[\alpha+\beta+\gamma, \alpha+\beta'+\gamma; 1+\alpha-\alpha'; \frac{(z-a)(c-b)}{(z-b)(c-a)} \right]$$

15.7. Asymptotic Expansions

The behavior of $F(a, b; c; z)$ for large $|z|$ is described by the transformation formulas of 15.3.

For fixed a, b, z and large $|c|$ one has [15.8]

15.7.1

$$F(a, b; c; z) = \sum_{n=0}^m \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} + O(|c|^{-m-1})$$

For fixed a, c, z , ($c \neq 0, -1, -2, \dots$), $0 < |z| < 1$ and large $|b|$ one has [15.2]

15.7.2

$$F(a, b; c; z) = e^{-i\pi a} [\Gamma(c)/\Gamma(c-a)] (bz)^{-a} [1 + O(|bz|^{-1})] \\ + [\Gamma(c)/\Gamma(a)] e^{bz} (bz)^{a-c} [1 + O(|bz|^{-1})] \\ \left(-\frac{3\pi}{2} < \arg(bz) < \frac{1}{2}\pi \right)$$

15.7.3

$$F(a, b; c; z) = e^{i\pi a} [\Gamma(c)/\Gamma(c-a)] (bz)^{-a} [1 + O(|bz|^{-1})] \\ + [\Gamma(c)/\Gamma(a)] e^{bz} (bz)^{a-c} [1 + O(|bz|^{-1})] \\ \left(-\frac{1}{2}\pi < \arg(bz) < \frac{3}{2}\pi \right)$$

For the case when more than one of the parameters are large consult [15.2].

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16. Jacobian Elliptic Functions and Theta Functions

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¹ University of Arizona. (Prepared under contract with the National Bureau of Standards.)

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$\vartheta_s(\epsilon^\circ \backslash \alpha^\circ), \sqrt{\sec \alpha} \vartheta_c(\epsilon_1^\circ \backslash \alpha^\circ)$ $\vartheta_n(\epsilon^\circ \backslash \alpha^\circ), \sqrt{\sec \alpha} \vartheta_d(\epsilon_1^\circ \backslash \alpha^\circ)$ $\alpha = 0^\circ(5^\circ)85^\circ, \epsilon, \epsilon_1 = 0^\circ(5^\circ)90^\circ, \quad 9-10D$	

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$$\frac{d}{du} \ln \vartheta_s(u) = f(\epsilon^\circ \backslash \alpha^\circ)$$

$$\frac{d}{du} \ln \vartheta_c(u) = -f(\epsilon_1^\circ \backslash \alpha^\circ)$$

$$\frac{d}{du} \ln \vartheta_n(u) = g(\epsilon^\circ \backslash \alpha^\circ)$$

$$\frac{d}{du} \ln \vartheta_d(u) = -g(\epsilon_1^\circ \backslash \alpha^\circ)$$

$$\alpha = 0^\circ(5^\circ)85^\circ, \epsilon, \epsilon_1 = 0^\circ(5^\circ)90^\circ, \quad 5-6D$$

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16. Jacobian Elliptic Functions and Theta Functions

Mathematical Properties

Jacobian Elliptic Functions

16.1. Introduction

A doubly periodic meromorphic function is called an *elliptic function*.

Let m, m_1 be numbers such that

$$m + m_1 = 1.$$

We call m the *parameter*, m_1 the *complementary parameter*.

In what follows we shall assume that the parameter m is a real number. Without loss of generality we can then suppose that $0 \leq m \leq 1$ (see 16.10, 16.11).

We define *quarter-periods* K and iK' by

16.1.1

$$K(m) = K = \int_0^{\pi/2} \frac{d\theta}{(1 - m \sin^2 \theta)^{1/2}},$$

$$iK'(m) = iK' = i \int_0^{\pi/2} \frac{d\theta}{(1 - m_1 \sin^2 \theta)^{1/2}}$$

so that K and K' are real numbers. K is called the real, iK' the imaginary quarter-period.

We note that

16.1.2

$$K(m) = K'(m_1) = K'(1 - m).$$

We also note that if any *one* of the numbers $m, m_1, K(m), K'(m), K'(m)/K(m)$ is given, all the rest are determined. Thus K and K' can not both be chosen arbitrarily.

In the Argand diagram denote the points $0, K, K + iK', iK'$ by s, c, d, n respectively. These points are at the vertices of a rectangle. The translations of this rectangle by $\lambda K, \mu iK'$, where λ, μ are given all integral values positive or negative, will lead to the lattice

s	c	s	c
n	d	n	d
s	c	s	c
n	d	n	d

the pattern being repeated indefinitely on all sides.

Let p, q be any two of the letters s, c, d, n . Then p, q determine in the lattice a minimum rectangle whose sides are of length K and K' and whose vertices s, c, d, n are in counterclockwise order.

Definition

The Jacobian elliptic function $pq u$ is defined by the following three properties.

(i) $pq u$ has a simple zero at p and a simple pole at q .

(ii) The step from p to q is a half-period of $pq u$. Those of the numbers $K, iK', K + iK'$ which differ from this step are only quarter-periods.

(iii) The coefficient of the leading term in the expansion of $pq u$ in ascending powers of u about $u=0$ is unity. With regard to (iii) the leading term is $u, 1/u, 1$ according as $u=0$ is a zero, a pole, or an ordinary point.

Thus the functions with a pole or zero at the origin (i.e., the functions in which one letter is s) are odd, and the others are even.

Should we wish to call explicit attention to the value of the parameter, we write $pq(u|m)$ instead of $pq u$.

The Jacobian elliptic functions can also be defined with respect to certain integrals. Thus if

16.1.3

$$u = \int_0^\varphi \frac{d\theta}{(1 - m \sin^2 \theta)^{1/2}},$$

the angle φ is called the *amplitude*

16.1.4

$$\varphi = \operatorname{am} u$$

and we define

16.1.5

$$\operatorname{sn} u = \sin \varphi, \operatorname{cn} u = \cos \varphi,$$

$$\operatorname{dn} u = (1 - m \sin^2 \varphi)^{1/2} = \Delta(\varphi).$$

Similarly all the functions $pq u$ can be expressed in terms of φ . This second set of definitions, although seemingly different, is mathematically equivalent to the definition previously given in terms of a lattice. For further explanation of notations, including the interpretation, of such expressions as $\operatorname{sn}(\varphi \backslash \alpha), \operatorname{cn}(u|m), \operatorname{dn}(u, k)$, see 17.2.

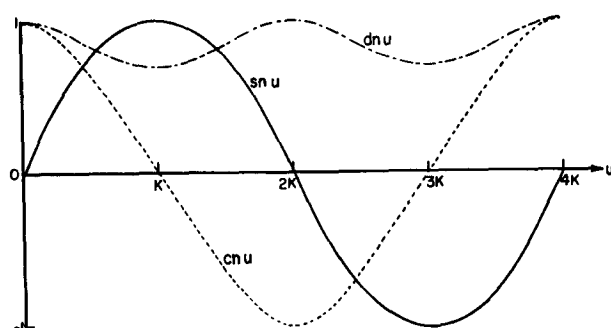
16.2. Classification of the Twelve Jacobian Elliptic Functions

According to Poles and Half-Periods

	Pole iK'	Pole $K+iK'$	Pole K	Pole 0	
Half period iK'	$\text{sn } u$	$\text{cd } u$	$\text{dc } u$	$\text{ns } u$	Periods $2iK'$, $4K+4iK'$, $4K$
Half period $K+iK'$	$\text{cn } u$	$\text{sd } u$	$\text{nc } u$	$\text{ds } u$	Periods $4iK'$, $2K+2iK'$, $4K$
Half period K	$\text{dn } u$	$\text{nd } u$	$\text{sc } u$	$\text{cs } u$	Periods $4iK'$, $4K+4iK'$, $2K$

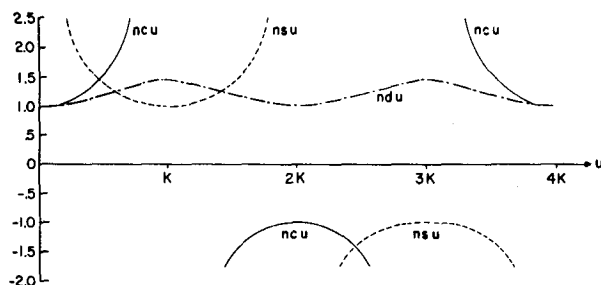
The three functions in a vertical column are *copolar*.

The four functions in a horizontal line are *coperiodic*. Of the periods quoted in the last line of each row only two are independent.

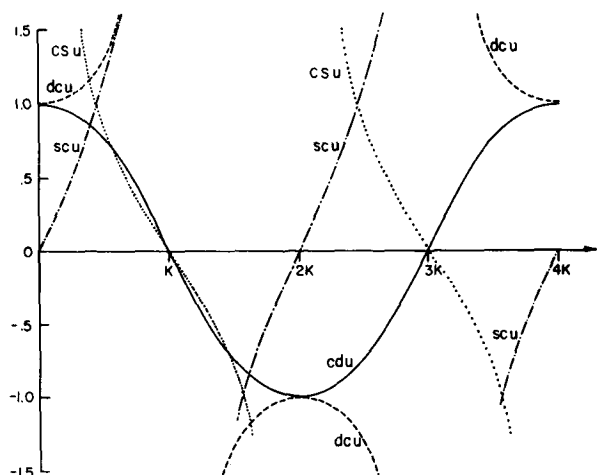
FIGURE 16.1. *Jacobian elliptic functions* $\text{sn } u, \text{cn } u, \text{dn } u$

$$m = \frac{1}{2}$$

The curve for $\text{cn } (u\frac{1}{2})$ is the boundary between those which have an inflexion and those which have not.

FIGURE 16.2. *Jacobian elliptic functions* $\text{ns } u, \text{nc } u, \text{nd } u$

$$m = \frac{1}{2}$$

FIGURE 16.3. *Jacobian elliptic functions* $\text{sc } u, \text{cs } u, \text{cd } u, \text{dc } u$

$$m = \frac{1}{2}$$

16.3. Relation of the Jacobian Functions to the Copolar Trio $\text{sn } u, \text{cn } u, \text{dn } u$

$$16.3.1 \quad \text{cd } u = \frac{\text{cn } u}{\text{dn } u} \quad \text{dc } u = \frac{\text{dn } u}{\text{cn } u} \quad \text{ns } u = \frac{1}{\text{sn } u}$$

$$16.3.2 \quad \text{sd } u = \frac{\text{sn } u}{\text{dn } u} \quad \text{nc } u = \frac{1}{\text{cn } u} \quad \text{ds } u = \frac{\text{dn } u}{\text{sn } u}$$

$$16.3.3 \quad \text{nd } u = \frac{1}{\text{dn } u} \quad \text{sc } u = \frac{\text{sn } u}{\text{cn } u} \quad \text{cs } u = \frac{\text{cn } u}{\text{sn } u}$$

And generally if p, q, r are any three of the letters s, c, d, n ,

$$16.3.4 \quad pq u = \frac{pr u}{qr u}$$

provided that when two letters are the same, e.g., $pp u$, the corresponding function is put equal to unity.

16.4. Calculation of the Jacobian Functions by Use of the Arithmetic-Geometric Mean (A.G.M.)

For the A.G.M. scale see 17.6.

To calculate $\operatorname{sn}(u|m)$, $\operatorname{cn}(u|m)$, and $\operatorname{dn}(u|m)$ form the A.G.M. scale starting with

$$16.4.1 \quad a_0=1, b_0=\sqrt{m_1}, c_0=\sqrt{m},$$

terminating at the step N when c_N is negligible to the accuracy required. Find φ_N in degrees where

$$16.4.2 \quad \varphi_N = 2^N a_N u \frac{180^\circ}{\pi}$$

and then compute successively φ_{N-1} , φ_{N-2} , . . . , φ_1 , φ_0 from the recurrence relation

$$16.4.3 \quad \sin(2\varphi_{n-1}-\varphi_n) = \frac{c_n}{a_n} \sin \varphi_n.$$

Then

16.4.4

$$\operatorname{sn}(u|m) = \sin \varphi_0, \operatorname{cn}(u|m) = \cos \varphi_0$$

$$\operatorname{dn}(u|m) = \frac{\cos \varphi_0}{\cos(\varphi_1 - \varphi_0)}.$$

From these all the other functions can be determined.

16.5. Special Arguments

	u	$\operatorname{sn} u$	$\operatorname{cn} u$	$\operatorname{dn} u$
16.5.1	0	0	1	1
16.5.2	$\frac{1}{2}K$	$\frac{1}{(1+m_1^{1/2})^{1/2}}$	$\frac{m_1^{1/4}}{(1+m_1^{1/2})^{1/2}}$	$m_1^{1/4}$
16.5.3	K	1	0	$m_1^{1/2}$
16.5.4	$\frac{1}{2}(iK')$	$im^{-1/4}$	$\frac{(1+m^{1/2})^{1/2}}{m^{1/4}}$	$(1+m^{1/2})^{1/2}$
* 16.5.5	$\frac{1}{2}(K+iK')$	$2^{-1/2}m^{-1/4}[(1+m^{1/2})^{1/2} + i(1-m^{1/2})^{1/2}]$	$\left(\frac{m_1}{4m}\right)^{1/4}(1-i)$	$\left(\frac{m_1}{4}\right)^{1/4}[(1+m_1^{1/2})^{1/2} - i(1-m_1^{1/2})^{1/2}]$
16.5.6	$K + \frac{1}{2}(iK')$	$m^{-1/4}$	$-i\left(\frac{1-m^{1/2}}{m^{1/2}}\right)^{1/2}$	$(1-m^{1/2})^{1/2}$
16.5.7	iK'	∞	∞	∞
16.5.8	$\frac{1}{2}K + iK'$	$(1-m_1^{1/2})^{-1/2}$	$-i\left(\frac{m_1^{1/2}}{1-m_1^{1/2}}\right)^{1/2}$	$-im_1^{1/4}$
16.5.9	$K + iK'$	$m^{-1/2}$	$-i(m_1/m)^{1/2}$	0

16.6. Jacobian Functions when $m=0$ or 1

		$m=0$	$m=1$
16.6.1	$\operatorname{sn}(u m)$	$\sin u$	$\tanh u$
16.6.2	$\operatorname{cn}(u m)$	$\cos u$	$\operatorname{sech} u$
16.6.3	$\operatorname{dn}(u m)$	1	$\operatorname{sech} u$
16.6.4	$\operatorname{cd}(u m)$	$\cos u$	1
16.6.5	$\operatorname{sd}(u m)$	$\sin u$	$\sinh u$
16.6.6	$\operatorname{nd}(u m)$	1	$\cosh u$
16.6.7	$\operatorname{dc}(u m)$	$\sec u$	1
16.6.8	$\operatorname{nc}(u m)$	$\sec u$	$\cosh u$
16.6.9	$\operatorname{sc}(u m)$	$\tan u$	$\sinh u$
16.6.10	$\operatorname{ns}(u m)$	$\csc u$	$\coth u$
16.6.11	$\operatorname{ds}(u m)$	$\csc u$	$\operatorname{csch} u$
16.6.12	$\operatorname{cs}(u m)$	$\cot u$	$\operatorname{csch} u$
16.6.13	$\operatorname{am}(u m)$	u	$\operatorname{gd} u$

*See page II.

16.7. Principal Terms

When the elliptic function pq is expanded in ascending powers of $(u - K_j)$, where K_j is one of $0, K, iK', K + iK'$, the first term of the expansion is called the principal term and has one of the forms $A, B \times (u - K_j), C \div (u - K_j)$ according as K_j is an ordinary point, a zero, or a pole of pq . The following list gives these forms, where $\times$ means that the factor $(u - K_j)$ has to be supplied and $\div$ means that the divisor $(u - K_j)$ has to be supplied.

$K_j =$	0	K	iK'	$K + iK'$
sn	$1 \times$	1	$m^{-1/2} \div$	$m^{-1/2}$
cn	1	$-m^{1/2} \times$	$-i \div$	$-i \left(\frac{m}{m_1} \right)^{1/2}$
dn	1	$m^{1/2}$	$-i \div$	$-i \left(\frac{m}{m_1} \right)^{1/2} \times$
cd	1	$-1 \times$	$-m^{-1/2}$	$-m^{-1/2} \div$
sd	$1 \times$	$m^{-1/2}$	$i m^{-1/2}$	$-i \frac{(mm_1)^{1/2}}{1} \div$
nd	1	$m^{-1/2}$	$i \times$	$-i m^{1/2} \div$
de	1	$-1 \div$	$m^{1/2}$	$-m^{1/2} \times$
nc	1	$-m^{-1/2} \div$	$i m^{1/2} \times$	$i \left(\frac{m}{m_1} \right)^{1/2}$
sc	$1 \times$	$-m^{-1/2} \div$	i	$i m^{1/2}$
ns	1	$-m^{1/2} \times$	$m^{1/2} \times$	$m^{1/2}$
ds	$1 \div$	$m^{1/2}$	$-i m^{1/2}$	$i (mm_1)^{1/2} \times$
cs	$1 \div$	$-m^{1/2} \times$	$-i$	$-i m^{1/2}$

16.8. Change of Argument

$16.8.1$	sn	$-sn$	u	
$16.8.2$	cn	cn	u	
$16.8.3$	dn	dn	u	
$16.8.4$	cd	cd	u	
$16.8.5$	sd	$-sd$	u	
$16.8.6$	nd	nd	u	
$16.8.7$	de	de	u	
$16.8.8$	nc	nc	u	
$16.8.9$	sc	sc	u	
$16.8.10$	ns	$-ns$	u	
$16.8.11$	ds	$-ds$	u	
$16.8.12$	cs	cs	u	
$16.8.1$	sn	$-sn$	$u + K$	$-K$
$16.8.2$	cn	cn	$u + K$	$-K$
$16.8.3$	dn	dn	$u + K$	$-K$
$16.8.4$	cd	cd	$u + K$	$-K$
$16.8.5$	sd	$-sd$	$u + K$	$-K$
$16.8.6$	nd	nd	$u + K$	$-K$
$16.8.7$	de	de	$u + K$	$-K$
$16.8.8$	nc	nc	$u + K$	$-K$
$16.8.9$	sc	sc	$u + K$	$-K$
$16.8.10$	ns	$-ns$	$u + K$	$-K$
$16.8.11$	ds	$-ds$	$u + K$	$-K$
$16.8.12$	cs	cs	$u + K$	$-K$

16.9. Relations Between the Squares of the Functions

$$16.9.1 \quad -\operatorname{dn}^2 u + m_1 = -m \operatorname{cn}^2 u = m \operatorname{sn}^2 u - m$$

$$16.9.2 \quad -m_1 \operatorname{nd}^2 u + m_1 = -m m_1 \operatorname{sd}^2 u = m \operatorname{cd}^2 u - m$$

$$16.9.3 \quad m_1 \operatorname{sc}^2 u + m_1 = m_1 \operatorname{nc}^2 u = \operatorname{dc}^2 u - m$$

$$16.9.4 \quad \operatorname{cs}^2 u + m_1 = \operatorname{ds}^2 u = \operatorname{ns}^2 u - m$$

In using the above results remember that $m + m_1 = 1$.

If $pq u$, $rt u$ are any two of the twelve functions, one entry expresses $tq^2 u$ in terms of $pq^2 u$ and another expresses $qt^2 u$ in terms of $rt^2 u$. Since $tq^2 u \cdot qt^2 u = 1$, we can obtain from the table the bilinear relation between $pq^2 u$ and $rt^2 u$. Thus for the functions $\operatorname{cd} u$, $\operatorname{sn} u$ we have

$$16.9.5 \quad \operatorname{nd}^2 u = \frac{1 - m \operatorname{cd}^2 u}{m_1}, \quad \operatorname{dn}^2 u = 1 - m \operatorname{sn}^2 u$$

and therefore

$$16.9.6 \quad (1 - m \operatorname{cd}^2 u)(1 - m \operatorname{sn}^2 u) = m_1.$$

16.10. Change of Parameter

Negative Parameter

If m is a positive number, let

$$16.10.1 \quad \mu = \frac{m}{1+m}, \quad \mu_1 = \frac{1}{1+m}, \quad v = \frac{u}{\mu_1^{\frac{1}{2}}} \quad (0 < \mu < 1)$$

$$16.10.2 \quad \operatorname{sn}(u|m) = \mu_1^{\frac{1}{2}} \operatorname{sd}(v|\mu)$$

$$16.10.3 \quad \operatorname{cn}(u|m) = \operatorname{cd}(v|\mu)$$

$$16.10.4 \quad \operatorname{dn}(u|m) = \operatorname{nd}(v|\mu).$$

16.11. Reciprocal Parameter (Jacobi's Real Transformation)

$$16.11.1 \quad m > 0, \quad \mu = m^{-1}, \quad v = um^{1/2}$$

$$16.11.2 \quad \operatorname{sn}(u|m) = \mu^{1/2} \operatorname{sn}(v|\mu)$$

$$16.11.3 \quad \operatorname{cn}(u|m) = \operatorname{dn}(v|\mu)$$

$$16.11.4 \quad \operatorname{dn}(u|m) = \operatorname{cn}(v|\mu)$$

Here if $m > 1$ then $m^{-1} = \mu < 1$.

Thus elliptic functions whose parameter is real can be made to depend on elliptic functions whose parameter lies between 0 and 1.

16.12. Descending Landen Transformation (Gauss' Transformation)

To decrease the parameter, let

$$16.12.1 \quad \mu = \left(\frac{1 - m_1^{1/2}}{1 + m_1^{1/2}} \right)^2, \quad v = \frac{u}{1 + \mu^{1/2}},$$

then

$$16.12.2 \quad \operatorname{sn}(u|m) = \frac{(1 + \mu^{1/2}) \operatorname{sn}(v|\mu)}{1 + \mu^{1/2} \operatorname{sn}^2(v|\mu)}$$

$$16.12.3 \quad \operatorname{cn}(u|m) = \frac{\operatorname{cn}(v|\mu) \operatorname{dn}(v|\mu)}{1 + \mu^{1/2} \operatorname{sn}^2(v|\mu)}$$

$$16.12.4 \quad \operatorname{dn}(u|m) = \frac{\operatorname{dn}^2(v|\mu) - (1 - \mu^{1/2})}{(1 + \mu^{1/2}) - \operatorname{dn}^2(v|\mu)}.$$

Note that successive applications can be made conveniently to find $\operatorname{sn}(u|m)$ in terms of $\operatorname{sn}(v|\mu)$ and $\operatorname{dn}(u|m)$ in terms of $\operatorname{dn}(v|\mu)$, but that the calculation of $\operatorname{cn}(u|m)$ requires all three functions.

16.13. Approximation in Terms of Circular Functions

When the parameter m is so small that we may neglect m^2 and higher powers, we have the approximations

16.13.1

$$\operatorname{sn}(u|m) \approx \sin u - \frac{1}{4} m(u - \sin u \cos u) \cos u$$

16.13.2

$$\operatorname{cn}(u|m) \approx \cos u + \frac{1}{4} m(u - \sin u \cos u) \sin u$$

$$16.13.3 \quad \operatorname{dn}(u|m) \approx 1 - \frac{1}{2} m \sin^2 u$$

$$16.13.4 \quad \operatorname{am}(u|m) \approx u - \frac{1}{4} m(u - \sin u \cos u).$$

One way of calculating the Jacobian functions is to use Landen's descending transformation to reduce the parameter sufficiently for the above formulae to become applicable. See also 16.14.

16.14. Ascending Landen Transformation

To increase the parameter, let

$$16.14.1 \quad \mu = \frac{4m^{1/2}}{(1 + m^{1/2})^2}, \quad \mu_1 = \left(\frac{1 - m^{1/2}}{1 + m^{1/2}} \right)^2, \quad v = \frac{u}{1 + \mu_1^{1/2}}$$

$$16.14.2 \quad \operatorname{sn}(u|m) = (1 + \mu_1^{1/2}) \frac{\operatorname{sn}(v|\mu) \operatorname{cn}(v|\mu)}{\operatorname{dn}(v|\mu)}$$

$$16.14.3 \quad \operatorname{cn}(u|m) = \frac{1 + \mu_1^{1/2}}{\mu} \frac{\operatorname{dn}^2(v|\mu) - \mu_1^{1/2}}{\operatorname{dn}(v|\mu)}$$

$$16.14.4 \quad \operatorname{dn}(u|m) = \frac{1 - \mu_1^{1/2}}{\mu} \frac{\operatorname{dn}^2(v|\mu) + \mu_1^{1/2}}{\operatorname{dn}(v|\mu)}$$

Note that, when successive applications are to be made, it is simplest to calculate $\operatorname{dn}(u|m)$ since this is expressed always in terms of the same function. The calculation of $\operatorname{cn}(u|m)$ leads to that of $\operatorname{dn}(v|\mu)$.

The calculation of $\operatorname{sn}(u|m)$ necessitates the evaluation of all three functions.

16.15. Approximation in Terms of Hyperbolic Functions

When the parameter m is so close to unity that m_1^2 and higher powers of m_1 can be neglected we have the approximations

16.15.1

$$\operatorname{sn}(u|m) \approx \tanh u + \frac{1}{4} m_1 (\sinh u \cosh u - u) \operatorname{sech}^2 u$$

16.15.2

$$\operatorname{cn}(u|m) \approx \operatorname{sech} u$$

$$-\frac{1}{4} m_1 (\sinh u \cosh u - u) \tanh u \operatorname{sech} u$$

16.15.3

$$\operatorname{dn}(u|m) \approx \operatorname{sech} u$$

$$+\frac{1}{4} m_1 (\sinh u \cosh u + u) \tanh u \operatorname{sech} u$$

16.15.4

$$\operatorname{am}(u|m) \approx \operatorname{gd} u + \frac{1}{4} m_1 (\sinh u \cosh u - u) \operatorname{sech} u.$$

Another way of calculating the Jacobian functions is to use Landen's ascending transformation to increase the parameter sufficiently for the above formulae to become applicable. See also 16.13.

16.16. Derivatives

	Function	Derivative
16.16.1	$\operatorname{sn} u$	$\operatorname{cn} u \operatorname{dn} u$
16.16.2	$\operatorname{cn} u$	$-\operatorname{sn} u \operatorname{dn} u$
16.16.3	$\operatorname{dn} u$	$-m \operatorname{sn} u \operatorname{cn} u$
16.16.4	$\operatorname{cd} u$	$-m_1 \operatorname{sd} u \operatorname{nd} u$
16.16.5	$\operatorname{sd} u$	$\operatorname{cd} u \operatorname{nd} u$
16.16.6	$\operatorname{nd} u$	$m \operatorname{sd} u \operatorname{cd} u$
16.16.7	$\operatorname{dc} u$	$m_1 \operatorname{sc} u \operatorname{nc} u$
16.16.8	$\operatorname{nc} u$	$\operatorname{sc} u \operatorname{dc} u$
16.16.9	$\operatorname{sc} u$	$\operatorname{dc} u \operatorname{nc} u$
16.16.10	$\operatorname{ns} u$	$-\operatorname{ds} u \operatorname{cs} u$
16.16.11	$\operatorname{ds} u$	$-\operatorname{cs} u \operatorname{ns} u$
16.16.12	$\operatorname{cs} u$	$-\operatorname{ns} u \operatorname{ds} u$

Note that the derivative is proportional to the product of the two copolar functions.

16.17. Addition Theorems

16.17.1 $\operatorname{sn}(u+v)$

$$= \frac{\operatorname{sn} u \cdot \operatorname{cn} v \cdot \operatorname{dn} v + \operatorname{sn} v \cdot \operatorname{cn} u \cdot \operatorname{dn} u}{1 - m \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}$$

16.17.2 $\operatorname{cn}(u+v)$

$$= \frac{\operatorname{cn} u \cdot \operatorname{cn} v - \operatorname{sn} u \cdot \operatorname{dn} u \cdot \operatorname{sn} v \cdot \operatorname{dn} v}{1 - m \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}$$

16.17.3 $\operatorname{dn}(u+v) = \frac{\operatorname{dn} u \cdot \operatorname{dn} v - m \operatorname{sn} u \cdot \operatorname{cn} u \cdot \operatorname{sn} v \cdot \operatorname{cn} v}{1 - m \operatorname{sn}^2 u \cdot \operatorname{sn}^2 v}$

Addition theorems are derivable one from another and are expressible in a great variety of forms. Thus $\operatorname{ns}(u+v)$ comes from $1/\operatorname{sn}(u+v)$ in the form $(1 - m \operatorname{sn}^2 u \operatorname{sn}^2 v) / (\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u)$ from 16.17.1.

Alternatively $\operatorname{ns}(u+v) = m^{1/2} \operatorname{sn} \{ (iK' - u) - v \}$ which again from 16.17.1 yields the form $(\operatorname{ns} u \operatorname{cs} v \operatorname{ds} v - \operatorname{ns} v \operatorname{cs} u \operatorname{ds} u) / (\operatorname{ns}^2 u - \operatorname{ns}^2 v)$.

The function $\operatorname{pq}(u+v)$ is a rational function of the four functions $\operatorname{pq} u, \operatorname{pq} v, \operatorname{pq}' u, \operatorname{pq}' v$.

16.18. Double Arguments

16.18.1 $\operatorname{sn} 2u$

$$= \frac{2 \operatorname{sn} u \cdot \operatorname{cn} u \cdot \operatorname{dn} u}{1 - m \operatorname{sn}^4 u} = \frac{2 \operatorname{sn} u \cdot \operatorname{cn} u \cdot \operatorname{dn} u}{\operatorname{cn}^2 u + \operatorname{sn}^2 u \cdot \operatorname{dn}^2 u}$$

16.18.2 $\operatorname{cn} 2u$

$$= \frac{\operatorname{cn}^2 u - \operatorname{sn}^2 u \cdot \operatorname{dn}^2 u}{1 - m \operatorname{sn}^4 u} = \frac{\operatorname{cn}^2 u - \operatorname{sn}^2 u \cdot \operatorname{dn}^2 u}{\operatorname{cn}^2 u + \operatorname{sn}^2 u \cdot \operatorname{dn}^2 u}$$

16.18.3 $\operatorname{dn} 2u$

$$= \frac{\operatorname{dn}^2 u - m \operatorname{sn}^2 u \cdot \operatorname{cn}^2 u}{1 - m \operatorname{sn}^4 u} = \frac{\operatorname{dn}^2 u + \operatorname{cn}^2 u (\operatorname{dn}^2 u - 1)}{\operatorname{dn}^2 u - \operatorname{cn}^2 u (\operatorname{dn}^2 u - 1)}$$

16.18.4 $\frac{1 - \operatorname{cn} 2u}{1 + \operatorname{cn} 2u} = \frac{\operatorname{sn}^2 u \cdot \operatorname{dn}^2 u}{\operatorname{cn}^2 u}$

16.18.5 $\frac{1 - \operatorname{dn} 2u}{1 + \operatorname{dn} 2u} = \frac{m \operatorname{sn}^2 u \cdot \operatorname{cn}^2 u}{\operatorname{dn}^2 u}$

16.19. Half Arguments

16.19.1 $\operatorname{sn}^2 \frac{1}{2} u = \frac{1 - \operatorname{cn} u}{1 + \operatorname{dn} u}$

16.19.2 $\operatorname{cn}^2 \frac{1}{2} u = \frac{\operatorname{dn} u + \operatorname{cn} u}{1 + \operatorname{dn} u}$

16.19.3 $\operatorname{dn}^2 \frac{1}{2} u = \frac{m_1 + \operatorname{dn} u + m \operatorname{cn} u}{1 + \operatorname{dn} u}$

16.20. Jacobi's Imaginary Transformation

16.20.1 $\operatorname{sn}(iu|m) = i \operatorname{sc}(u|m_1)$

16.20.2 $\operatorname{cn}(iu|m) = \operatorname{nc}(u|m_1)$

16.20.3 $\operatorname{dn}(iu|m) = \operatorname{dc}(u|m_1)$

16.21. Complex Arguments

With the abbreviations

16.21.1

$$s = \operatorname{sn}(x|m), c = \operatorname{cn}(x|m), d = \operatorname{dn}(x|m), s_1 = \operatorname{sn}(y|m_1), \\ c_1 = \operatorname{cn}(y|m_1), d_1 = \operatorname{dn}(y|m_1)$$

$$16.21.2 \quad \operatorname{sn}(x+iy|m) = \frac{s \cdot d_1 + ic \cdot d \cdot s_1 \cdot c_1}{c_1^2 + ms^2 \cdot s_1^2}$$

$$16.21.3 \quad \operatorname{cn}(x+iy|m) = \frac{c \cdot c_1 - is \cdot d \cdot s_1 \cdot d_1}{c_1^2 + ms^2 \cdot s_1^2}$$

$$16.21.4 \quad \operatorname{dn}(x+iy|m) = \frac{d \cdot c_1 \cdot d_1 - im s \cdot c \cdot s_1}{c_1^2 + ms^2 \cdot s_1^2}$$

16.22. Leading Terms of the Series in Ascending Powers of u

16.22.1

$$\operatorname{sn}(u|m) = u - (1+m) \frac{u^3}{3!} + (1+14m+m^2) \frac{u^5}{5!} \\ - (1+135m+135m^2+m^3) \frac{u^7}{7!} + \dots$$

16.22.2

$$\operatorname{cn}(u|m) = 1 - \frac{u^2}{2!} + (1+4m) \frac{u^4}{4!} \\ - (1+44m+16m^2) \frac{u^6}{6!} + \dots$$

16.22.3

$$\operatorname{dn}(u|m) = 1 - m \frac{u^2}{2!} + m(4+m) \frac{u^4}{4!} \\ - m(16+44m+m^2) \frac{u^6}{6!} + \dots$$

No formulae are known for the general coefficients in these series.

16.23. Series Expansions in Terms of the Nome $q = e^{-\pi K'/K}$ and the Argument $v = \pi u/(2K)$

$$16.23.1 \quad \operatorname{sn}(u|m) = \frac{2\pi}{m^{1/2}K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1-q^{2n+1}} \sin(2n+1)v$$

$$16.23.2 \quad \operatorname{cn}(u|m) = \frac{2\pi}{m^{1/2}K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1+q^{2n+1}} \cos(2n+1)v$$

$$16.23.3 \quad \operatorname{dn}(u|m) = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos 2nv$$

16.23.4

$$\operatorname{cd}(u|m) = \frac{2\pi}{m^{1/2}K} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1/2}}{1-q^{2n+1}} \cos(2n+1)v$$

16.23.5

$$\operatorname{sd}(u|m) = \frac{2\pi}{(mm_1)^{1/2}K} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1/2}}{1+q^{2n+1}} \sin(2n+1)v$$

16.23.6

$$\operatorname{nd}(u|m) = \frac{\pi}{2m_1^{1/2}K} + \frac{2\pi}{m_1^{1/2}K} \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1+q^{2n}} \cos 2nv$$

16.23.7

$$\operatorname{dc}(u|m) = \frac{\pi}{2K} \sec v \\ + \frac{2\pi}{K} \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}}{1-q^{2n+1}} \cos(2n+1)v$$

16.23.8

$$\operatorname{nc}(u|m) = \frac{\pi}{2m_1^{1/2}K} \sec v \\ - \frac{2\pi}{m_1^{1/2}K} \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}}{1+q^{2n+1}} \cos(2n+1)v$$

16.23.9

$$\operatorname{sc}(u|m) = \frac{\pi}{2m_1^{1/2}K} \tan v \\ + \frac{2\pi}{m_1^{1/2}K} \sum_{n=1}^{\infty} \frac{(-1)^n q^{2n}}{1+q^{2n}} \sin 2nv$$

16.23.10

$$\operatorname{ns}(u|m) = \frac{\pi}{2K} \csc v - \frac{2\pi}{K} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1-q^{2n+1}} \sin(2n+1)v$$

16.23.11

$$\operatorname{ds}(u|m) = \frac{\pi}{2K} \csc v - \frac{2\pi}{K} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1+q^{2n+1}} \sin(2n+1)v$$

16.23.12

$$\operatorname{cs}(u|m) = \frac{\pi}{2K} \cot v - \frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^{2n}}{1+q^{2n}} \sin 2nv$$

16.24. Integrals of the Twelve Jacobian Elliptic Functions

$$16.24.1 \quad \int \operatorname{sn} u \, du = m^{-1/2} \ln(\operatorname{dn} u - m^{1/2} \operatorname{cn} u)$$

$$16.24.2 \quad \int \operatorname{cn} u \, du = m^{-1/2} \arccos(\operatorname{dn} u)$$

$$16.24.3 \quad \int \operatorname{dn} u \, du = \arcsin(\operatorname{sn} u)$$

$$16.24.4 \quad \int \operatorname{cd} u \, du = m^{-1/2} \ln(\operatorname{nd} u + m^{1/2} \operatorname{sd} u)$$

$$16.24.5 \quad \int \operatorname{sd} u \, du = (mm_1)^{-1/2} \arcsin(-m^{1/2} \operatorname{cd} u)$$

$$16.24.6 \quad \int \operatorname{nd} u \, du = m_1^{-1/2} \arccos(\operatorname{cd} u)$$

$$16.24.7 \quad \int \operatorname{dc} u \, du = \ln(\operatorname{nc} u + \operatorname{sc} u)$$

$$16.24.8 \quad \int \operatorname{nc} u \, du = m_1^{-1/2} \ln(\operatorname{dc} u + m_1^{1/2} \operatorname{sc} u)$$

$$16.24.9 \quad \int \operatorname{sc} u \, du = m_1^{-1/2} \ln(\operatorname{dc} u + m_1^{1/2} \operatorname{nc} u)$$

$$16.24.10 \quad \int \operatorname{ns} u \, du = \ln(\operatorname{ds} u - \operatorname{cs} u)$$

$$16.24.11 \quad \int \operatorname{ds} u \, du = \ln(\operatorname{ns} u - \operatorname{cs} u)$$

$$16.24.12 \quad \int \operatorname{cs} u \, du = \ln(\operatorname{ns} u - \operatorname{ds} u)$$

In numerical use of the above table certain restrictions must be put on u in order to keep the arguments of the logarithms positive and to avoid

trouble with many-valued inverse circular functions.

16.25. Notation for the Integrals of the Squares of the Twelve Jacobian Elliptic Functions

$$16.25.1 \quad Pq \, u = \int_0^u pq^2 t \, dt \text{ when } q \neq s$$

$$16.25.2 \quad Ps \, u = \int_0^u \left(pq^2 t - \frac{1}{t^2} \right) dt - \frac{1}{u}$$

Examples

$$Cd \, u = \int_0^u cd^2 t \, dt, \quad Ns \, u = \int_0^u \left(ns^2 t - \frac{1}{t^2} \right) dt - \frac{1}{u}$$

16.26. Integrals in Terms of the Elliptic Integral of the Second Kind (see 17.4)

$$16.26.1 \quad mSn \, u = -E(u) + u$$

$$16.26.2 \quad mCn \, u = E(u) - m_1 u \quad \text{Pole } n$$

$$16.26.3 \quad Dn \, u = E(u)$$

$$16.26.4 \quad mCd \, u = -E(u) + u + msn \, u \, cd \, u$$

$$16.26.5 \quad mm_1 Sd \, u = E(u) - m_1 u - msn \, u \, cd \, u \quad \text{Pole } d$$

$$16.26.6 \quad m_1 Nd \, u = E(u) - msn \, u \, cd \, u$$

$$16.26.7 \quad Dc \, u = -E(u) + u + sn \, u \, dc \, u$$

$$16.26.8 \quad m_1 Nc \, u = -E(u) + m_1 u + sn \, u \, dc \, u \quad \text{Pole } c$$

$$16.26.9 \quad m_1 Sc \, u = -E(u) + sn \, u \, dc \, u$$

$$16.26.10 \quad Ns \, u = -E(u) + u - cn \, u \, ds \, u$$

$$16.26.11 \quad Ds \, u = -E(u) + m_1 u - cn \, u \, ds \, u \quad \text{Pole } s$$

$$16.26.12 \quad Cs \, u = -E(u) - cn \, u \, ds \, u$$

All the above may be expressed in terms of Jacobi's zeta function (see 17.4.27).

$$Z(u) = E(u) - \frac{E}{K} u, \text{ where } E = E(K)$$

16.27. Theta Functions; Expansions in Terms of the Nome q

$$16.27.1 \quad \vartheta_1(z, q) = \vartheta_1(z) = 2q^{1/4} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin(2n+1)z$$

$$16.27.2 \quad \vartheta_2(z, q) = \vartheta_2(z) = 2q^{1/4} \sum_{n=0}^{\infty} q^{n(n+1)} \cos(2n+1)z$$

$$16.27.3 \quad \vartheta_3(z, q) = \vartheta_3(z) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nz$$

16.27.4

$$\vartheta_4(z, q) = \vartheta_4(z) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz$$

Theta functions are important because every one of the Jacobian elliptic functions can be expressed as the ratio of two theta functions. See 16.36.

The notation shows these functions as depending on the variable z and the nome q , $|q| < 1$. In this case, here and elsewhere, the convergence is not dependent on the trigonometrical terms. In their relation to the Jacobian elliptic functions, we note that the nome q is given by

$$q = e^{-\pi K'/K},$$

where K and iK' are the quarter periods. Since $q = q(m)$ is determined when the parameter m is given, we can also regard the theta functions as dependent upon m and then we write

$$\vartheta_a(z, q) = \vartheta_a(z|m), \quad a = 1, 2, 3, 4$$

but when no ambiguity is to be feared, we write $\vartheta_a(z)$ simply.

The above notations are those given in Modern Analysis [16.6].

There is a bewildering variety of notations, for example the function $\vartheta_4(z)$ above is sometimes denoted by $\vartheta_0(z)$ or $\vartheta(z)$; see the table given in Modern Analysis [16.6]. Further the argument $u = 2Kz/\pi$ is frequently used so that in consulting books caution should be exercised.

16.28. Relations Between the Squares of the Theta Functions

$$16.28.1 \quad \vartheta_1^2(z) \vartheta_4^2(0) = \vartheta_3^2(z) \vartheta_2^2(0) - \vartheta_2^2(z) \vartheta_3^2(0)$$

$$16.28.2 \quad \vartheta_2^2(z) \vartheta_4^2(0) = \vartheta_4^2(z) \vartheta_2^2(0) - \vartheta_1^2(z) \vartheta_3^2(0)$$

$$16.28.3 \quad \vartheta_3^2(z) \vartheta_4^2(0) = \vartheta_4^2(z) \vartheta_3^2(0) - \vartheta_1^2(z) \vartheta_2^2(0)$$

$$16.28.4 \quad \vartheta_4^2(z) \vartheta_4^2(0) = \vartheta_3^2(z) \vartheta_2^2(0) - \vartheta_2^2(z) \vartheta_3^2(0)$$

$$16.28.5 \quad \vartheta_2^4(0) + \vartheta_4^4(0) = \vartheta_3^4(0)$$

Note also the important relation

$$16.28.6 \quad \vartheta_1'(0) = \vartheta_2(0) \vartheta_3(0) \vartheta_4(0) \text{ or } \vartheta_1' = \vartheta_2 \vartheta_3 \vartheta_4$$

16.29. Logarithmic Derivatives of the Theta Functions

$$16.29.1 \quad \frac{\vartheta_1'(u)}{\vartheta_1(u)} = \cot u + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin 2nu$$

16.29.2

$$\frac{\vartheta_2'(u)}{\vartheta_2(u)} = -\tan u + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^{2n}}{1-q^{2n}} \sin 2nu$$

$$16.29.3 \quad \frac{\vartheta_3'(u)}{\vartheta_3(u)} = 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1-q^{2n}} \sin 2nu$$

$$16.29.4 \quad \frac{\vartheta_4'(u)}{\vartheta_4(u)} = 4 \sum_{n=1}^{\infty} \frac{q^n}{1-q^{2n}} \sin 2nu$$

16.30. Logarithms of Theta Functions of Sum and Difference

16.30.1

$$\ln \frac{\vartheta_1(\alpha+\beta)}{\vartheta_1(\alpha-\beta)} = \ln \frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} + 4 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^{2n}}{1-q^{2n}} \sin 2n\alpha \sin 2n\beta$$

16.30.2

$$\ln \frac{\vartheta_2(\alpha+\beta)}{\vartheta_2(\alpha-\beta)} = \ln \frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{q^{2n}}{1-q^{2n}} \sin 2n\alpha \sin 2n\beta$$

16.30.3

$$\ln \frac{\vartheta_3(\alpha+\beta)}{\vartheta_3(\alpha-\beta)} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{q^n}{1-q^{2n}} \sin 2n\alpha \sin 2n\beta$$

16.30.4

$$\ln \frac{\vartheta_4(\alpha+\beta)}{\vartheta_4(\alpha-\beta)} = 4 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1-q^{2n}} \sin 2n\alpha \sin 2n\beta$$

The corresponding expressions when $\beta = i\gamma$ are easily deduced by use of the formulae 4.3.55 and 4.3.56.

16.31. Jacobi's Notation for Theta Functions

$$16.31.1 \quad \Theta(u|m) = \Theta(u) = \vartheta_4(v), \quad v = \frac{\pi u}{2K}$$

$$16.31.2 \quad \Theta_1(u|m) = \Theta_1(u) = \vartheta_3(v) = \Theta(u+K)$$

$$16.31.3 \quad H(u|m) = H(u) = \vartheta_1(v)$$

$$16.31.4 \quad H_1(u|m) = H_1(u) = \vartheta_2(v) = H(u+K)$$

16.32. Calculation of Jacobi's Theta Function $\Theta(u|m)$ by Use of the Arithmetic-Geometric Mean

Form the A.G.M. scale starting with

$$16.32.1 \quad a_0 = 1, \quad b_0 = \sqrt{m_1}, \quad c_0 = \sqrt{m}$$

terminating with the N th step when c_N is negligible to the accuracy required. Find φ_N in degrees, where

$$16.32.2 \quad \varphi_N = 2^N a_N u \frac{180^\circ}{\pi}$$

and then compute successively $\varphi_{N-1}, \varphi_{N-2}, \dots, \varphi_1, \varphi_0$ from the recurrence relation

$$16.32.3 \quad \sin(2\varphi_{n-1} - \varphi_n) = \frac{c_n}{a_n} \sin \varphi_n.$$

Then

16.32.4

$$\begin{aligned} \ln \Theta(u|m) &= \frac{1}{2} \ln \frac{2m_1^{1/2} K(m)}{\pi} + \frac{1}{2} \ln \frac{\cos(\varphi_1 - \varphi_0)}{\cos \varphi_0} \\ &+ \frac{1}{4} \ln \sec(2\varphi_0 - \varphi_1) + \frac{1}{8} \ln \sec(2\varphi_1 - \varphi_2) + \dots \\ &+ \frac{1}{2^{N+1}} \ln \sec(2\varphi_{N-1} - \varphi_N) \end{aligned}$$

16.33. Addition of Quarter-Periods to Jacobi's Eta and Theta Functions

u	$-u$	$u+K$	$u+2K$	$u+iK'$	$u+2iK'$	$u+K+iK'$	$u+2K+2iK'$
16.33.1 $H(u)$	$-H(u)$	$H_1(u)$	$-H(u)$	$iM(u)\Theta(u)$	$-N(u)H(u)$	$M(u)\Theta_1(u)$	$N(u)H(u)$
16.33.2 $H_1(u)$	$H_1(u)$	$-H(u)$	$-H_1(u)$	$M(u)\Theta_1(u)$	$N(u)H_1(u)$	$-iM(u)\Theta(u)$	$-N(u)H_1(u)$
16.33.3 $\Theta_1(u)$	$\Theta_1(u)$	$\Theta(u)$	$\Theta_1(u)$	$M(u)H_1(u)$	$N(u)\Theta_1(u)$	$iM(u)H(u)$	$N(u)\Theta_1(u)$
16.33.4 $\Theta(u)$	$\Theta(u)$	$\Theta_1(u)$	$\Theta(u)$	$iM(u)H(u)$	$-N(u)\Theta(u)$	$M(u)H_1(u)$	$-N(u)\Theta(u)$

where

$$M(u) = \left[\exp\left(-\frac{\pi i u}{2K}\right) \right] q^{-1/4},$$

$$N(u) = \left[\exp\left(-\frac{\pi i u}{K}\right) \right] q^{-1/4}$$

$H(u)$ and $H_1(u)$ have the period $4K$. $\Theta(u)$ and $\Theta_1(u)$ have the period $2K$.

$2iK'$ is a quasi-period for all four functions, that is to say, increase of the argument by $2iK'$ multiplies the function by a factor.

16.34. Relation of Jacobi's Zeta Function to the Theta Functions

$$Z(u) = \frac{\partial}{\partial u} \ln \Theta(u)$$

$$16.34.1 \quad Z(u) = \frac{\pi}{2K} \frac{\vartheta'_1\left(\frac{\pi u}{2K}\right)}{\vartheta_1\left(\frac{\pi u}{2K}\right)} - \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u}$$

$$16.34.2 \quad = \frac{\pi}{2K} \frac{\vartheta'_2\left(\frac{\pi u}{2K}\right)}{\vartheta_2\left(\frac{\pi u}{2K}\right)} + \frac{\operatorname{dn} u \operatorname{sn} u}{\operatorname{cn} u}$$

$$16.34.3 \quad = \frac{\pi}{2K} \frac{\vartheta'_3\left(\frac{\pi u}{2K}\right)}{\vartheta_3\left(\frac{\pi u}{2K}\right)} - m \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}$$

$$16.34.4 \quad = \frac{\pi}{2K} \frac{\vartheta'_4\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)}$$

16.35. Calculation of Jacobi's Zeta Function $Z(u|m)$ by Use of the Arithmetic-Geometric Mean

Form the A.G.M. scale 17.6 starting with

$$16.35.1 \quad a_0 = 1, b_0 = \sqrt{m_1}, c_0 = \sqrt{m}$$

terminating at the N th step when c_N is negligible to the accuracy required. Find φ_N in degrees where

$$16.35.2 \quad \varphi_N = 2^N a_N u \frac{180^\circ}{\pi}$$

and then compute successively $\varphi_{N-1}, \varphi_{N-2}, \dots, \varphi_1, \varphi_0$ from the recurrence relation

$$16.35.3 \quad \sin(2\varphi_{n-1} - \varphi_n) = \frac{c_n}{a_n} \sin \varphi_n.$$

Then

$$16.35.4$$

$$Z(u|m) = c_1 \sin \varphi_1 + c_2 \sin \varphi_2 + \dots + c_N \sin \varphi_N.$$

16.36. Neville's Notation for Theta Functions

These functions are defined in terms of Jacobi's theta functions of 16.31 by

$$16.36.1 \quad \vartheta_s(u) = \frac{H(u)}{H'(0)}, \vartheta_c(u) = \frac{H(u+K)}{H(K)}$$

$$16.36.2 \quad \vartheta_d(u) = \frac{\Theta(u+K)}{\Theta(K)}, \vartheta_n(u) = \frac{\Theta(u)}{\Theta(0)}.$$

If λ, μ are any integers positive, negative, or zero the points $u_0 + 2\lambda K + 2\mu iK'$ are said to be *congruent* to u_0 .

$\vartheta_s(u)$ has zeros at the points congruent to 0
 $\vartheta_c(u)$ has zeros at the points congruent to K
 $\vartheta_n(u)$ has zeros at the points congruent to iK'
 $\vartheta_d(u)$ has zeros at the points congruent to $K + iK'$

Thus the suffix secures that the function $\vartheta_p(u)$ has zeros at the points marked p in the introductory diagram in 16.1.2, and the constant by which Jacobi's function is divided secures that the leading coefficient of $\vartheta_p(u)$ at the origin is unity. Therefore the functions have the fundamentally important property that if p, q are any two of the letters s, c, n, d , the Jacobian elliptic function $pq u$ is given by

$$16.36.3 \quad pq u = \frac{\vartheta_p(u)}{\vartheta_q(u)}.$$

These functions also have the property

$$16.36.4 \quad m_1^{-1/4} \vartheta_c(K-u) = \vartheta_s(u)$$

$$16.36.5 \quad m_1^{-1/4} \vartheta_d(K-u) = \vartheta_n(u),$$

for complementary arguments u and $K-u$.

In terms of the theta functions defined in 16.27, let $v = \pi u/(2K)$, then

$$16.36.6 \quad \vartheta_s(u) = \frac{2K \vartheta_1(v)}{\vartheta_1'(0)}, \vartheta_c(u) = \frac{\vartheta_2(v)}{\vartheta_2(0)}$$

$$16.36.7 \quad \vartheta_d(u) = \frac{\vartheta_3(v)}{\vartheta_3(0)}, \vartheta_n(u) = \frac{\vartheta_4(v)}{\vartheta_4(0)}$$

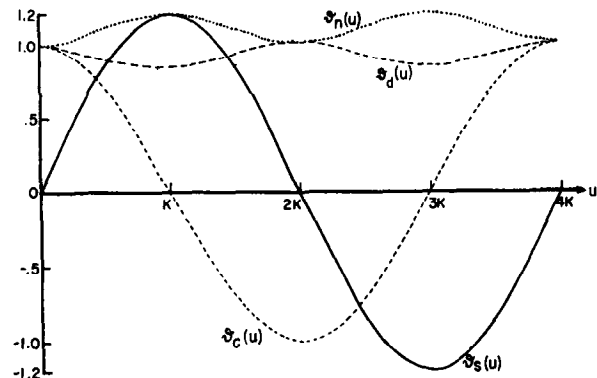


FIGURE 16.4. Neville's theta functions
 $\vartheta_s(u), \vartheta_c(u), \vartheta_d(u), \vartheta_n(u)$
 $m = \frac{1}{2}$

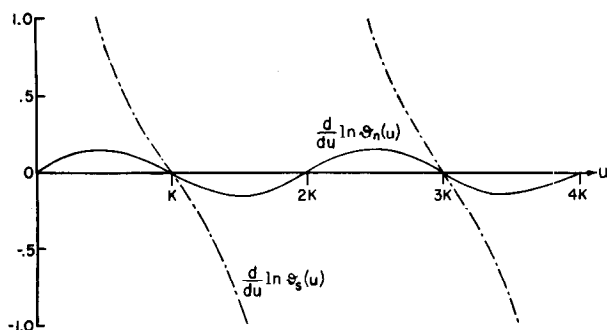


FIGURE 16.5. Logarithmic derivatives of theta functions

$$\frac{d}{du} \ln \vartheta_s(u), \frac{d}{du} \ln \vartheta_n(u) \\ m = \frac{1}{2}$$

16.37. Expression as Infinite Products

$$q = q(m), v = \pi u / (2K)$$

16.37.1

$$\vartheta_s(u) = \left(\frac{16q}{m m_1} \right)^{1/6} \sin v \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2v + q^{4n})$$

16.37.2

$$\vartheta_c(u) = \left(\frac{16q m_1^{1/2}}{m} \right)^{1/6} \cos v \prod_{n=1}^{\infty} (1 + 2q^{2n} \cos 2v + q^{4n})$$

16.37.3

$$\vartheta_d(u) = \left(\frac{m m_1}{16q} \right)^{1/12} \prod_{n=1}^{\infty} (1 + 2q^{2n-1} \cos 2v + q^{4n-2})$$

16.37.4

$$\vartheta_n(u) = \left(\frac{m}{16q m_1^2} \right)^{1/12} \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2v + q^{4n-2})$$

Numerical Methods

16.39. Use and Extension of the Tables

Example 1. Calculate $\text{nc}(1.99650|.64)$ to 4S. From Table 17.1, $1.99650 = K + .001$. From the table of principal terms

$$\begin{aligned} \text{nc } u &= -m_1^{-1/2}/(u-K) + \dots \\ \text{nc}(K+.001|.64) &= \frac{-(.36)^{-1/2}}{.001} + \dots \\ &= -\frac{10000}{6} + \dots \\ &= -1667 + \dots \end{aligned}$$

and since the next term is of order .001 this value -1667 is correct to at least 4S.

Example 2. Use the descending Landen transformation to calculate $\text{dn}(.20|.19)$ to 6D.

Here $m = .19$, $m_1^{1/2} = .9$ and so from 16.12.1

$$\mu = \left(\frac{1}{19} \right)^2, 1 + \mu^{1/2} = \frac{20}{19}, v = .19.$$

Also

16.38. Expression as Infinite Series

$$\text{Let } v = \pi u / (2K)$$

16.38.1

$$\vartheta_s(u) = \left[\frac{2\pi q^{1/2}}{m^{1/2} m_1^{1/2} K} \right]^{1/2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin(2n+1)v$$

$$16.38.2 \quad \vartheta_c(u) = \left[\frac{2\pi q^{1/2}}{m^{1/2} K} \right]^{1/2} \sum_{n=0}^{\infty} q^{n(n+1)} \cos(2n+1)v$$

$$16.38.3 \quad \vartheta_d(u) = \left[\frac{\pi}{2K} \right]^{1/2} \left\{ 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nv \right\}$$

16.38.4

$$\vartheta_n(u) = \left[\frac{\pi}{2m_1^{1/2} K} \right]^{1/2} \left\{ 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nv \right\}$$

$$16.38.5 \quad (2K/\pi)^{1/2} = 1 + 2q + 2q^4 + 2q^9 + \dots = \vartheta_3(0, q)$$

16.38.6

$$(2K'/\pi)^{1/2} = 1 + 2q_1 + 2q_1^4 + 2q_1^9 + \dots = \vartheta_3(0, q_1)$$

16.38.7

$$(2m^{1/2} K/\pi)^{1/2} = 2q^{1/4} (1 + q^2 + q^6 + q^{12} + q^{20} + \dots) \\ = \vartheta_2(0, q)$$

16.38.8

$$(2m_1^{1/2} K/\pi)^{1/2} = 1 - 2q + 2q^4 - 2q^9 + \dots = \vartheta_4(0, q).$$

$$\mu^2 = \left(\frac{1}{19} \right)^4 = 10^{-6} \times 7.67$$

which is negligible.

From 16.12.4

$$\text{dn}(.20|.19) = \frac{\text{dn}^2 \left[.19 \left| \left(\frac{1}{19} \right)^2 \right] - \left(1 - \frac{1}{19} \right) \right]}{\left(1 + \frac{1}{19} \right) - \text{dn}^2 \left[.19 \left| \left(\frac{1}{19} \right)^2 \right] \right]}$$

Now from 16.13.3

$$\text{dn} \left[.19 \left| \left(\frac{1}{19} \right)^2 \right] \right] = .999951$$

whence $\text{dn}(.20|.19) = .996253$.

Example 3. Use the ascending Landen transformation to calculate $\text{dn}(.20|.81)$ to 5D.

From 16.14.1

$$\mu = \frac{4(.9)}{(1.9)^2} = \frac{360}{361}, \mu_1 = \left(\frac{1}{19} \right)^2$$

$$1 + \mu_1^{1/2} = \frac{20}{19}, v = \frac{19}{20} \times .20 = .19,$$

μ_1^2 is negligible to 4D. Thus

$$\operatorname{dn}(.20|.81) = \frac{19}{20} \times \frac{\operatorname{dn}^2\left(.19 \left| \frac{360}{361} \right.\right) + \frac{1}{19}}{\operatorname{dn}\left(.19 \left| \frac{360}{361} \right.\right)}.$$

From 16.15.3

$$\begin{aligned} \operatorname{dn}\left(.19 \left| \frac{360}{361} \right.\right) &= \operatorname{sech}(.19) + \frac{1}{4} \times \frac{1}{361} \tanh .19 \operatorname{sech} .19 \\ &\quad [\sinh .19 \cosh .19 + .19] \\ &= .982218 + \frac{1}{4} \times \frac{1}{361} (.187746)(.982218) \\ &\quad [(.191145)(1.01810) + .19] \\ &= .982218 + \frac{1}{4} \times \frac{1}{361} (.184408)[.384605] \\ &= .982218 + .000049 = .982267. \end{aligned}$$

Thus $\operatorname{dn}(.20|.81) = .98406$.

Example 4. Use the ascending Landen transformation to calculate $\operatorname{cn}(.20|.81)$ to 6D.

Using 16.14.4, we calculate $\operatorname{dn}(.20|.81)$ and deduce $\operatorname{cn}(.20|.81)$ from 16.14.3 settling the sign from Figure 16.1.

As in the preceding example, we reduce the calculation of $\operatorname{dn}(.20|.81)$ to that of $\operatorname{dn}\left(.19 \left| \frac{360}{361} \right.\right)$, when

$$\operatorname{dn}\left(.19 \left| \frac{360}{361} \right.\right) = .982267$$

$$\operatorname{dn}(.20|.81) = .984056$$

$$\operatorname{cn}(.20|.81) = .980278.$$

Example 5. Use the A.G.M. scale to compute $\operatorname{dc}(.672|.36)$ to 4D.

From 16.9.6 we have $\operatorname{dc}^2(.672|.36) = .36 + \frac{.64}{1 - \operatorname{sn}^2(.672|.36)}$. We now calculate $\operatorname{sn}(.672|.36)$ by the method given in 16.4. Form the A.G.M. scale

n	a_n	b_n	c_n	$\frac{c_n}{a_n}$	φ_n	$\sin \varphi_n$	$\sin (2\varphi_{n-1} - \varphi_n)$	$2\varphi_{n-1} - \varphi_n$
0	1	.8	.6	.6	.65546	.60952		
1	.9	.89443	.1	.11111	1.2069	.93452	.10383	.10402
2	.89721	.89721	.00279	.00311	2.4117	.66679	.00207	.00207
3	.89721	.89721	0	0	4.8234	-.99384	0	0

$$\varphi_n = 2^n a_n u \quad \varphi_3 = 2^3 (.89721) (.672) = 4.8234$$

continuing until $c_n = 0$ to 5D.

Then complete as indicated in 16.4 to find φ_0 and so $\operatorname{sn} u$ and hence $\operatorname{dc} u$,

$$\varphi_0 = .65546 \quad \operatorname{sn} u = .60952 \quad \operatorname{dc} u = 1.1740.$$

Example 6. Use the A.G.M. scale to compute $\Theta(.6|.36)$ to 5D.

We use the method explained in 16.32 with $a_0 = 1$, $b_0 = .8$, $c_0 = .6$.

Computing the A.G.M. as explained in 17.6, we find

(For values of a_n , b_n , c_n , see Example 5.)

n	φ_n	$\sin \varphi_n$	$\sin (2\varphi_{n-1} - \varphi_n)$	$2\varphi_{n-1} - \varphi_n$	$\sec (2\varphi_{n-1} - \varphi_n)$	$\frac{1}{2^{n+1}} \ln \sec (2\varphi_{n-1} - \varphi_n)$
0	.58803	.55472				
1	1.0780	.88101	.09789	.09805	1.0048	.00120
2	2.1533	.83509	.00260	.00260	1.	0
3	4.3066	-.91879	0	0	1.	0

and then complete the calculation outlined in 16.32 to give

$$\begin{aligned} \ln \Theta(u|m) &= -.05734 + .02935 + .00120 \\ &= -.02679 \\ \Theta(u|m) &= .97357. \end{aligned}$$

The series expansion for Θ is preferable.

Example 7. Use the q -series to compute $\text{cs } (.53601\ 62|.09)$.

Here we use the series 16.23.12, $K=1.60804\ 862$, $q=.00589\ 414$, $v=\frac{\pi u}{2K}=\frac{\pi}{6}$ radians or 30° .

Since q^4 is negligible to 8D, we have to 7D $\text{cs } (.53601\ 62|.09)$

$$\begin{aligned} &= \frac{\pi}{2K} \cot 30^\circ - \frac{2\pi}{K} \left\{ \frac{q^2}{1+q^2} \sin 60^\circ \right\} \\ &= (.97683\ 3852)(1.73205\ 081) \\ &\quad - 3.90733\ 541[(.00003\ 4740)(.86602\ 5404)] \\ &= 1.69180\ 83. \end{aligned}$$

Example 8. Use theta functions to compute $\text{sn } (.61802|.5)$ to 5D.

Here $K(\frac{1}{2})=1.85407$

$$\epsilon^\circ = \frac{.61802}{1.85407} \times 90^\circ = 30^\circ$$

$$\sin^2 \alpha = 1/2, \alpha = 45^\circ.$$

Thus

$$\begin{aligned} \text{sn } (.61802|.5) &= \frac{\vartheta_s(30^\circ \backslash 45^\circ)}{\vartheta_n(30^\circ \backslash 45^\circ)} \\ &= \frac{.59128}{1.04729} = .56458 \end{aligned}$$

from Table 16.1.

Example 9. Use theta functions to compute $\text{sc } (.61802|.5)$ to 5D.

As in the preceding example

$$\epsilon^\circ = 30^\circ, \alpha^\circ = 45^\circ$$

so that

$$\text{sc } (.61802|.5) = \frac{\vartheta_s(30^\circ \backslash 45^\circ)}{\vartheta_c(30^\circ \backslash 45^\circ)}.$$

We use Table 16.1 to give

$$\vartheta_s(30^\circ \backslash 45^\circ) = .59128$$

$$(\sec 45^\circ)^{\frac{1}{2}} \vartheta_c(30^\circ \backslash 45^\circ) = 1.02796.$$

Therefore

$$\begin{aligned} \text{sc } (.61802|.5) &= \frac{.59128}{1.02796} (\sec 45^\circ)^{\frac{1}{2}} \\ &= .68402. \end{aligned}$$

Example 10. Find $\text{sn } (.75342|.7)$ by inverse interpolation in Table 17.5.

This method is explained in chapter 17, Example 7.

Example 11. Find u , given that $\text{cs } (u|.5) = .75$. From 16.9.4 we have

$$\text{sn}^2 u = \frac{1}{1 + \text{cs}^2 u}.$$

Thus

$$\text{sn}^2 (u|.5) = .64$$

and

$$\text{sn } (u|.5) = .8.$$

We have therefore replaced the problem by that of finding u given $\text{sn } (u|m)$, where m is known. If $\varphi = \text{am } u$

$$\sin \varphi = \text{sn } u \text{ and so}$$

$$\varphi = .9272952 \text{ radians or } 53.13010^\circ.$$

From Table 17.5,

$$u = F(53.13010^\circ \backslash 45^\circ) = .99391.$$

Alternatively, starting with the above value of φ we can use the A.G.M. scale to calculate $F(\varphi \backslash \alpha)$ as explained in 17.6. This method is to be preferred if more figures are required, or if α differs from a tabular value in Table 17.5.

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17. Elliptic Integrals

L. M. MILNE-THOMSON ¹

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17. Elliptic Integrals

Mathematical Properties

17.1. Definition of Elliptic Integrals

If $R(x, y)$ is a rational function of x and y , where y^2 is equal to a cubic or quartic polynomial in x , the integral

$$17.1.1 \quad \int R(x, y) dx$$

is called an *elliptic integral*.

The elliptic integral just defined can not, in general, be expressed in terms of elementary functions.

Exceptions to this are

- (i) when $R(x, y)$ contains no odd powers of y .
- (ii) when the polynomial y^2 has a repeated factor.

We therefore exclude these cases.

By substituting for y^2 and denoting by $p_s(x)$ a polynomial in x we get ²

$$\begin{aligned} R(x, y) &= \frac{p_1(x) + yp_2(x)}{p_3(x) + yp_4(x)} \\ &= \frac{[p_1(x) + yp_2(x)][p_3(x) - yp_4(x)]y}{\{[p_3(x)]^2 - y^2[p_4(x)]^2\}y} \\ &= \frac{p_5(x) + yp_6(x)}{yp_7(x)} = R_1(x) + \frac{R_2(x)}{y} \end{aligned}$$

where $R_1(x)$ and $R_2(x)$ are rational functions of x . Hence, by expressing $R_2(x)$ as the sum of a polynomial and partial fractions

$$\begin{aligned} \int R(x, y) dx &= \int R_1(x) dx + \sum_s A_s \int x^s y^{-1} dx \\ &\quad + \sum_s B_s \int [(x-c)^s y]^{-1} dx \end{aligned}$$

Reduction Formulae

Let

17.1.2

$$\begin{aligned} y^2 &= a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 \quad (|a_0| + |a_1| \neq 0) \\ &= b_0 (x-c)^4 + b_1 (x-c)^3 + b_2 (x-c)^2 + b_3 (x-c) + b_4 \\ &\quad (|b_0| + |b_1| \neq 0) \end{aligned}$$

$$17.1.3 \quad I_s = \int x^s y^{-1} dx, \quad J_s = \int [y(x-c)^s]^{-1} dx$$

By integrating the derivatives of yx^s and $y(x-c)^{-s}$ we get the reduction formulae

17.1.4

$$\begin{aligned} (s+2)a_0 I_{s+3} + \frac{1}{2}a_1(2s+3)I_{s+2} + a_2(s+1)I_{s+1} \\ + \frac{1}{2}a_3(2s+1)I_s + sa_4 I_{s-1} = x^s y \quad (s=0, 1, 2, \dots) \end{aligned}$$

² See [17.7] 22.72.

17.1.5

$$\begin{aligned} (2-s)b_0 J_{s-3} + \frac{1}{2}b_1(3-2s)J_{s-2} + b_2(1-s)J_{s-1} \\ + \frac{1}{2}b_3(1-2s)J_s - sb_4 J_{s+1} = y(x-c)^{-s} \\ (s=1, 2, 3, \dots) \end{aligned}$$

By means of these reduction formulae and certain transformations (see **Examples 1 and 2**) every elliptic integral can be brought to depend on the integral of a rational function and on three canonical forms for elliptic integrals.

17.2. Canonical Forms

Definitions

17.2.1

$m = \sin^2 \alpha$; m is the parameter,
 α is the modular angle

17.2.2

$$x = \sin \varphi = \operatorname{sn} u$$

17.2.3

$$\cos \varphi = \operatorname{cn} u$$

17.2.4

$(1-m \sin^2 \varphi)^{\frac{1}{2}} = \operatorname{dn} u = \Delta(\varphi)$, the delta amplitude

17.2.5 $\varphi = \arcsin(\operatorname{sn} u) = \operatorname{am} u$, the amplitude

Elliptic Integral of the First Kind

$$17.2.6 \quad F(\varphi \backslash \alpha) = F(\varphi | m) = \int_0^\varphi (1 - \sin^2 \alpha \sin^2 \theta)^{-\frac{1}{2}} d\theta$$

17.2.7

$$\begin{aligned} &= \int_0^x [(1-t^2)(1-mt^2)]^{-\frac{1}{2}} dt \\ &= \int_0^u dw = u \end{aligned}$$

Elliptic Integral of the Second Kind

$$17.2.8 \quad E(\varphi \backslash \alpha) = E(u | m) = \int_0^x (1-t^2)^{-\frac{1}{2}} (1-mt^2)^{\frac{1}{2}} dt$$

$$17.2.9 \quad = \int_0^\varphi (1 - \sin^2 \alpha \sin^2 \theta)^{\frac{1}{2}} d\theta$$

$$17.2.10 \quad = \int_0^u \operatorname{dn}^2 w dw$$

$$17.2.11 \quad = m_1 u + m \int_0^u \operatorname{cn}^2 w dw$$

$$17.2.12 \quad E(\varphi \backslash \alpha) = u - m \int_0^u \operatorname{sn}^2 w \, dw$$

$$17.2.13 \quad = \frac{\pi}{2K(m)} \frac{\partial'_4(\pi u/2K)}{\partial_4(\pi u/2K)} + \frac{E(m)u}{K(m)}$$

(For theta functions, see chapter 16.)

Elliptic Integral of the Third Kind

17.2.14

$$\Pi(n; \varphi \backslash \alpha) = \int_0^\varphi (1 - n \sin^2 \theta)^{-1} [1 - \sin^2 \alpha \sin^2 \theta]^{-1/2} d\theta$$

If $x = \operatorname{sn}(u|m)$,

17.2.15

$$\Pi(n; u|m) = \int_0^x (1 - nt^2)^{-1} [(1 - t^2)(1 - mt^2)]^{-1/2} dt$$

$$17.2.16 \quad = \int_0^u (1 - n \operatorname{sn}^2(w|m))^{-1} dw$$

The Amplitude φ

$$17.2.17 \quad \varphi = \operatorname{am} u = \arcsin(\operatorname{sn} u) = \arcsin x$$

can be calculated from Tables 17.5 and 4.14.

The Parameter m

Dependence on the *parameter* m is denoted by a vertical stroke preceding the parameter, e.g., $F(\varphi|m)$.

Together with the parameter we define the *complementary parameter* m_1 by

$$17.2.18 \quad m + m_1 = 1$$

When the parameter is real, it can always be arranged, see 17.4, that $0 \leq m \leq 1$.

The Modular Angle α

Dependence on the modular angle α , defined in terms of the parameter by 17.2.1, is denoted by a backward stroke $\backslash$ preceding the modular angle, thus $E(\varphi \backslash \alpha)$. The *complementary modular angle* is $\pi/2 - \alpha$ or $90^\circ - \alpha$ according to the unit and thus $m_1 = \sin^2(90^\circ - \alpha) = \cos^2 \alpha$.

The Modulus k

In terms of Jacobian elliptic functions (chapter 16), the modulus k and the complementary modulus are defined by

$$17.2.19 \quad k = \operatorname{ns}(K + iK'), \quad k' = \operatorname{dn} K.$$

They are related to the parameter by $k^2 = m$, $k'^2 = m_1$.

Dependence on the modulus is denoted by a comma preceding it, thus $\Pi(n; u, k)$.

In computation the modulus is of minimal importance, since it is the parameter and its complement which arise naturally. The parameter and the modular angle will be employed in this chapter to the exclusion of the modulus.

The Characteristic n

The elliptic integral of the third kind depends on three variables namely (i) the parameter, (ii) the amplitude, (iii) the characteristic n . When real, the characteristic may be any number in the interval $(-\infty, \infty)$. The properties of the integral depend upon the location of the characteristic in this interval, see 17.7.

17.3. Complete Elliptic Integrals of the First and Second Kinds

Referred to the canonical forms of 17.2, the elliptic integrals are said to be *complete* when the amplitude is $\frac{1}{2}\pi$ and so $x=1$. These complete integrals are designated as follows

17.3.1

$$[K(m)] = K = \int_0^1 [(1-t^2)(1-mt^2)]^{-1/2} dt \\ = \int_0^{\pi/2} (1 - m \sin^2 \theta)^{-1/2} d\theta$$

$$17.3.2 \quad K = F(\tfrac{1}{2}\pi|m) = F(\tfrac{1}{2}\pi \backslash \alpha)$$

17.3.3

$$E[K(m)] = E = \int_0^1 (1-t^2)^{-1/2} (1-mt^2)^{1/2} dt \\ = \int_0^{\pi/2} (1 - m \sin^2 \theta)^{1/2} d\theta$$

$$17.3.4 \quad E = E[K(m)] = E(m) = E(\tfrac{1}{2}\pi \backslash \alpha)$$

We also define

17.3.5

$$K' = K(m_1) = K(1-m) = \int_0^{\pi/2} (1 - m_1 \sin^2 \theta)^{-1/2} d\theta$$

$$17.3.6 \quad K' = F(\tfrac{1}{2}\pi|m_1) = F(\tfrac{1}{2}\pi \backslash \tfrac{1}{2}\pi - \alpha)$$

17.3.7

$$E' = E(m_1) = E(1-m) = \int_0^{\pi/2} (1 - m_1 \sin^2 \theta)^{1/2} d\theta$$

$$17.3.8 \quad E' = E[K(m_1)] = E(m_1) = E(\tfrac{1}{2}\pi \backslash \tfrac{1}{2}\pi - \alpha)$$

K and iK' are the "real" and "imaginary" *quarter-periods* of the corresponding Jacobian elliptic functions (see chapter 16).

Relation to the Hypergeometric Function

(see chapter 15)

$$17.3.9 \quad K = \frac{1}{2} \pi F\left(\frac{1}{2}, \frac{1}{2}; 1; m\right)$$

$$17.3.10 \quad E = \frac{1}{2} \pi F\left(-\frac{1}{2}, \frac{1}{2}; 1; m\right)$$

Infinite Series

17.3.11

$$K(m) = \frac{1}{2} \pi \left[1 + \left(\frac{1}{2}\right)^2 m + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 m^2 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 m^3 + \dots \right] \quad (|m| < 1)$$

17.3.12

$$E(m) = \frac{1}{2} \pi \left[1 - \left(\frac{1}{2}\right)^2 \frac{m}{1} - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{m^2}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{m^3}{5} - \dots \right] \quad (|m| < 1)$$

Legendre's Relation

$$17.3.13 \quad EK' + E'K - KK' = \frac{1}{2} \pi$$

Auxiliary Function

$$17.3.14 \quad L(m) = \frac{K'(m)}{\pi} \ln \frac{16}{m_1} - K(m)$$

$$17.3.15 \quad m = 1 - 16 \exp [-\pi(K(m) + L(m))/K'(m)]$$

$$17.3.16 \quad m = 16 \exp [-\pi(K'(m) + L(m_1))/K(m)]$$

 The function $L(m)$ is tabulated in Table 17.4.

 q -Series

 The Nome q and the Complementary Nome q_1

$$17.3.17 \quad q = q(m) = \exp [-\pi K'/K]$$

$$17.3.18 \quad q_1 = q(m_1) = \exp [-\pi K/K']$$

$$17.3.19 \quad \ln \frac{1}{q'} \ln \frac{1}{q_1} = \pi^2$$

17.3.20

$$\log_{10} \frac{1}{q} \log_{10} \frac{1}{q_1} = (\pi \log_{10} e)^2 = 1.86152 \ 28349 \text{ to } 10D$$

17.3.21

$$q = \exp [-\pi K'/K] = \frac{m}{16} + 8 \left(\frac{m}{16}\right)^2 + 84 \left(\frac{m}{16}\right)^3 + 992 \left(\frac{m}{16}\right)^4 + \dots \quad (|m| < 1)$$

$$17.3.22 \quad K = \frac{1}{2} \pi + 2\pi \sum_{s=1}^{\infty} \frac{q^s}{1+q^{2s}}$$

17.3.23

$$\frac{E}{K} = \frac{1}{3} (1 + m_1) + (\pi/K)^2 \left[1/12 - 2 \sum_{s=1}^{\infty} q^{2s} (1 - q^{2s})^{-2} \right]$$

$$17.3.24 \quad \text{am } u = v + \sum_{s=1}^{\infty} \frac{2q^s \sin 2sv}{s(1+q^{2s})} \text{ where } v = \pi u/(2K)$$

Limiting Values

$$17.3.25 \quad \lim_{m \rightarrow 0} K'(E - K) = 0$$

$$17.3.26 \quad \lim_{m \rightarrow 1} [K - \frac{1}{2} \ln (16/m_1)] = 0$$

$$17.3.27 \quad \lim_{m \rightarrow 0} m^{-1}(K - E) = \lim_{m \rightarrow 0} m^{-1}(E - m_1 K) = \pi/4$$

$$17.3.28 \quad \lim_{m \rightarrow 0} q/m = \lim_{m_1 \rightarrow 1} q_1/m_1 = 1/16$$

 Alternative Evaluations of K and E (see also 17.5)

17.3.29

$$K(m) = 2[1 + m_1^{1/2}]^{-1} K[(1 - m_1^{1/2})/(1 + m_1^{1/2})]^2$$

17.3.30

$$E(m) = (1 + m_1^{1/2}) E[(1 - m_1^{1/2})/(1 + m_1^{1/2})]^2 - 2m_1^{1/2} (1 + m_1^{1/2})^{-1} K[(1 - m_1^{1/2})/(1 + m_1^{1/2})]^2$$

$$17.3.31 \quad K(\alpha) = 2F(\arctan (\sec^{1/2} \alpha) \backslash \alpha)$$

$$17.3.32 \quad E(\alpha) = 2E(\arctan (\sec^{1/2} \alpha) \backslash \alpha) - 1 + \cos \alpha$$

 Polynomial Approximations ³ ($0 \leq m < 1$)

17.3.33

$$K(m) = [a_0 + a_1 m_1 + a_2 m_1^2] + [b_0 + b_1 m_1 + b_2 m_1^2] \ln (1/m_1) + \epsilon(m) \quad |\epsilon(m)| \leq 3 \times 10^{-6}$$

$a_0 = 1.38629 \ 44$	$b_0 = .5$
$a_1 = .11197 \ 23$	$b_1 = .12134 \ 78$
$a_2 = .07252 \ 96$	$b_2 = .02887 \ 29$

17.3.34

$$K(m) = [a_0 + a_1 m_1 + \dots + a_4 m_1^4] + [b_0 + b_1 m_1 + \dots + b_4 m_1^4] \ln (1/m_1) + \epsilon(m) \quad |\epsilon(m)| \leq 2 \times 10^{-8}$$

$a_0 = 1.38629 \ 436112$	$b_0 = .5$
$a_1 = .09666 \ 344259$	$b_1 = .12498 \ 593597$
$a_2 = .03590 \ 092383$	$b_2 = .06880 \ 248576$
$a_3 = .03742 \ 563713$	$b_3 = .03328 \ 355346$
$a_4 = .01451 \ 196212$	$b_4 = .00441 \ 787012$

³ The approximations 17.3.33-17.3.36 are from C. Hastings, Jr., Approximations for Digital Computers, Princeton Univ. Press, Princeton, N. J. (with permission).

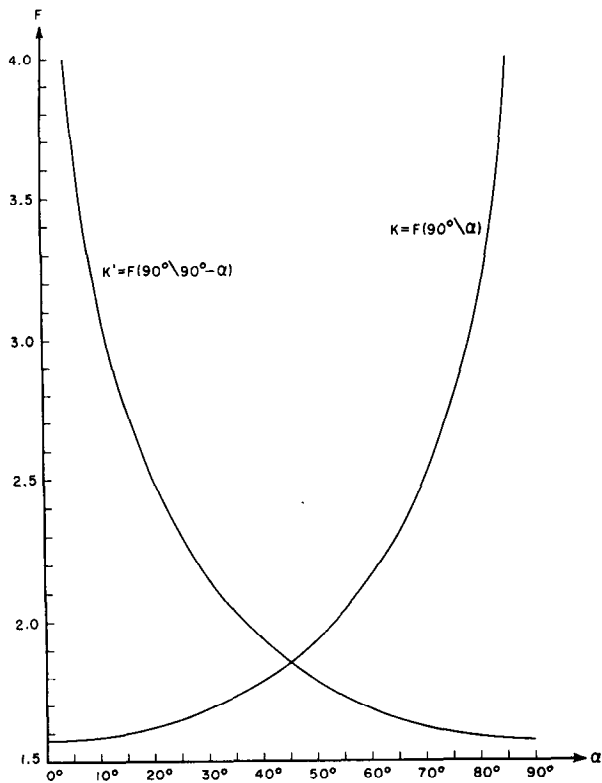


FIGURE 17.1. Complete elliptic integral of the first kind.

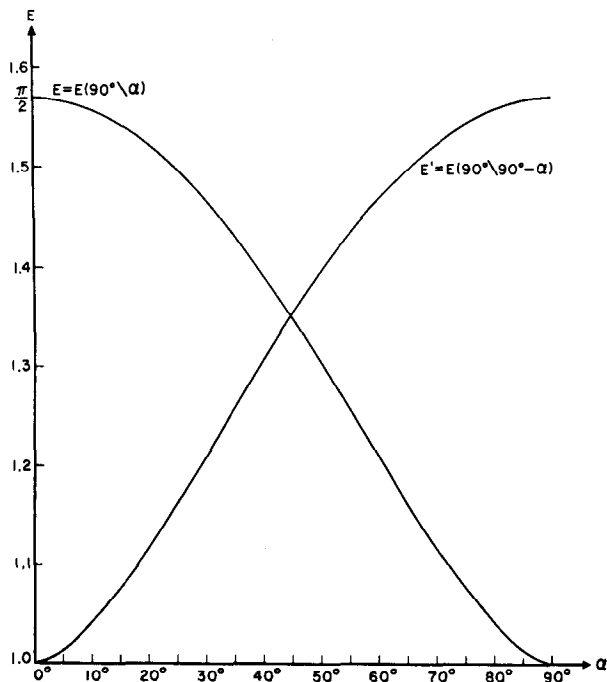


FIGURE 17.2. Complete elliptic integral of the second kind.

17.3.35

$$E(m) = [1 + a_1 m_1 + a_2 m_1^2] + [b_1 m_1 + b_2 m_1^2] \ln(1/m_1) + \epsilon(m)$$

$$|\epsilon(m)| < 4 \times 10^{-8}$$

$$a_1 = .46301 \quad 51 \quad b_1 = .24527 \quad 27$$

$$a_2 = .10778 \quad 12 \quad b_2 = .04124 \quad 96$$

17.3.36

$$E(m) = [1 + a_1 m_1 + \dots + a_4 m_1^4] + [b_1 m_1 + \dots + b_4 m_1^4] \ln(1/m_1) + \epsilon(m)$$

$$|\epsilon(m)| < 2 \times 10^{-8}$$

$$a_1 = .44325 \quad 141463 \quad b_1 = .24998 \quad 368310$$

$$a_2 = .06260 \quad 601220 \quad b_2 = .09200 \quad 180037$$

$$a_3 = .04757 \quad 383546 \quad b_3 = .04069 \quad 697526$$

$$a_4 = .01736 \quad 506451 \quad b_4 = .00526 \quad 449639$$

17.4. Incomplete Elliptic Integrals of the First and Second Kinds

Extension of the Tables

Negative Amplitude

$$17.4.1 \quad F(-\varphi|m) = -F(\varphi|m)$$

$$17.4.2 \quad E(-\varphi|m) = -E(\varphi|m)$$

Amplitude of Any Magnitude

$$17.4.3 \quad F(s\pi \pm \varphi|m) = 2sK \pm F(\varphi|m)$$

$$17.4.4 \quad E(u + 2K) = E(u) + 2E$$

$$17.4.5 \quad E(u + 2iK') = E(u) + 2i(K' - E')$$

17.4.6

$$E(u + 2mK + 2niK') = E(u) + 2mE + 2ni(K' - E')$$

$$17.4.7 \quad E(K - u) = E - E(u) + \operatorname{sn} u \operatorname{cd} u$$

Imaginary Amplitude

$$\text{If } \tan \theta = \sinh \varphi$$

$$17.4.8 \quad F(i\varphi|\alpha) = iF(\theta|\tfrac{1}{2}\pi - \alpha)$$

17.4.9

$$E(i\varphi|\alpha) = -iE(\theta|\tfrac{1}{2}\pi - \alpha) + iF(\theta|\tfrac{1}{2}\pi - \alpha) + i \tan \theta (1 - \cos^2 \alpha \sin^2 \theta)^{\frac{1}{2}}$$

Jacobi's Imaginary Transformation

17.4.10

$$E(iu|m) = i[u + \operatorname{dn}(u|m_1) \operatorname{sc}(u|m_1) - E(u|m_1)]$$

Complex Amplitude

$$17.4.11 \quad F(\varphi + i\psi|m) = F(\lambda|m) + iF(\mu|m_1)$$

where $\cot^2 \lambda$ is the positive root of the equation $x^2 - [\cot^2 \varphi + m \sinh^2 \psi \csc^2 \varphi - m_1]x - m_1 \cot^2 \varphi = 0$ and $m \tan^2 \mu = \tan^2 \varphi \cot^2 \lambda - 1$.

17.4.12

$$E(\varphi + i\psi \backslash \alpha) = E(\lambda \backslash \alpha) - iE(\mu \backslash 90^\circ - \alpha) + iF(\mu \backslash 90^\circ - \alpha) + \frac{b_1 + ib_2}{b_3}$$

where

$$b_1 = \sin^2 \alpha \sin \lambda \cos \lambda \sin^2 \mu (1 - \sin^2 \alpha \sin^2 \lambda)^{\frac{1}{2}}$$

$$b_2 = (1 - \sin^2 \alpha \sin^2 \lambda) (1 - \cos^2 \alpha \sin^2 \mu)^{\frac{1}{2}} \sin \mu \cos \mu$$

$$b_3 = \cos^2 \mu + \sin^2 \alpha \sin^2 \lambda \sin^2 \mu$$

Amplitude Near to $\pi/2$ (see also 17.5)

$$\text{If } \cos \alpha \tan \varphi \tan \psi = 1$$

$$17.4.13 \quad F(\varphi \backslash \alpha) + F(\psi \backslash \alpha) = F(\pi/2 \backslash \alpha) = K$$

17.4.14

$$E(\varphi \backslash \alpha) + E(\psi \backslash \alpha) = E(\pi/2 \backslash \alpha) + \sin^2 \alpha \sin \varphi \sin \psi$$

Values when φ is near to $\pi/2$ and m is near to unity can be calculated by these formulae.

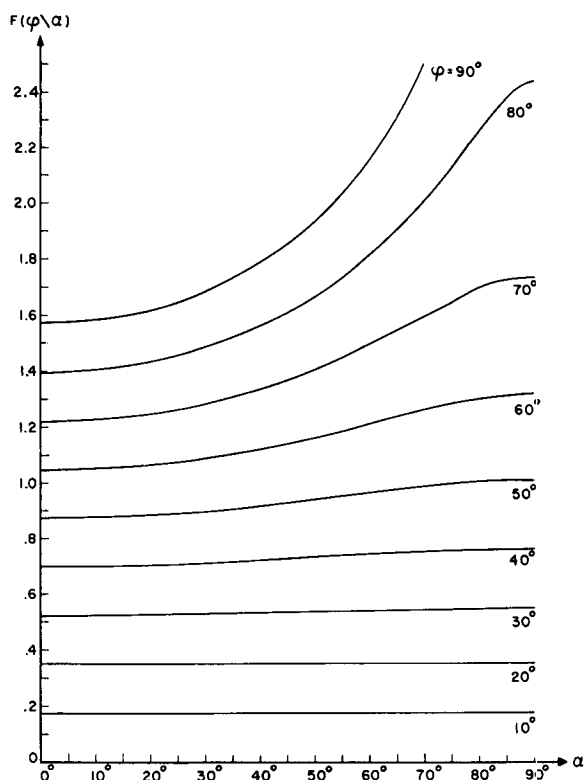


FIGURE 17.3. Incomplete elliptic integral of the first kind.

$F(\varphi \backslash \alpha)$, φ constant

Parameter Greater Than Unity

$$17.4.15 \quad F(\varphi | m) = m^{-\frac{1}{2}} F(\theta | m^{-1}), \quad \sin \theta = m^{\frac{1}{2}} \sin \varphi$$

$$17.4.16 \quad E(u | m) = m^{\frac{1}{2}} E(um^{\frac{1}{2}} | m^{-1}) - (m-1)u$$

by which a parameter greater than unity can be replaced by a parameter less than unity.

Negative Parameter

17.4.17

$$F(\varphi | -m) = (1+m)^{-\frac{1}{2}} K(m(1+m)^{-1})$$

$$- (1+m)^{-\frac{1}{2}} F\left(\frac{\pi}{2} - \varphi | m(1+m)^{-1}\right)$$

17.4.18

$$E(u | -m) = (1+m)^{\frac{1}{2}} \{ E(u(1+m)^{\frac{1}{2}} | m(1+m)^{-1})$$

$$- m(1+m)^{-\frac{1}{2}} \text{sn}(u(1+m)^{\frac{1}{2}} | m(1+m)^{-1})$$

$$\} \text{cd}(u(1+m)^{\frac{1}{2}} | m(1+m)^{-1}) \}$$

whereby computations can be made for negative parameters, and therefore for pure imaginary modulus.

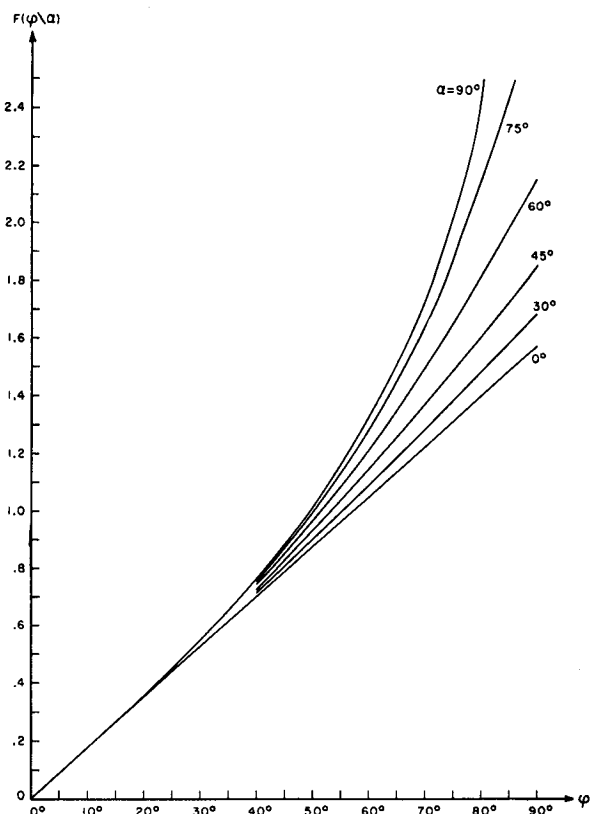


FIGURE 17.4. Incomplete elliptic integral of the first kind.

$F(\varphi \backslash \alpha)$, α constant

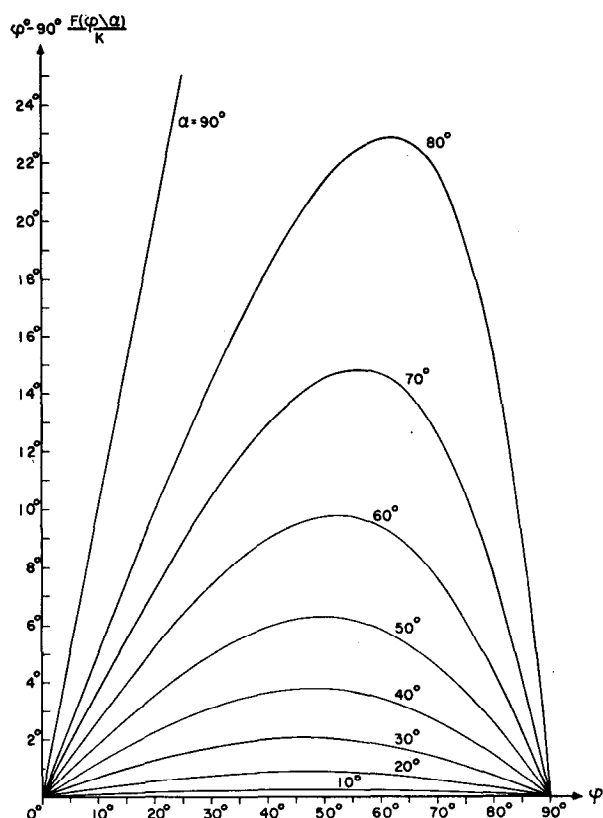


FIGURE 17.5. $\varphi - 90^\circ \frac{F(\varphi \setminus \alpha)}{K}$, α constant.

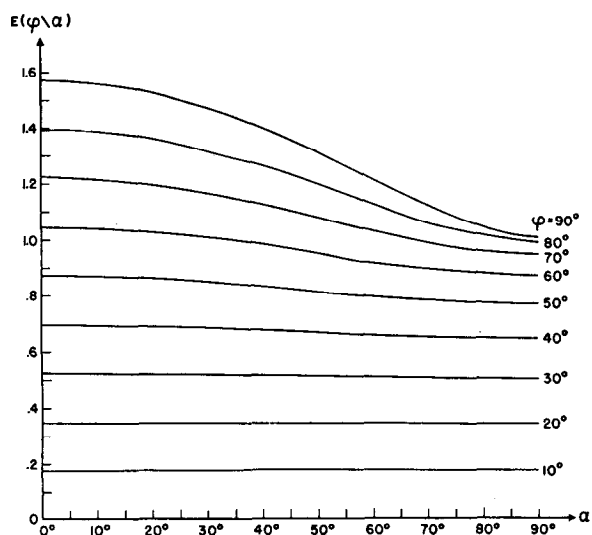


FIGURE 17.6. *Incomplete elliptic integral of the second kind.*

$E(\varphi \setminus \alpha)$, φ constant

Special Cases

17.4.19 $F(\varphi \setminus 0) = \varphi$

17.4.20 $F(i\varphi \setminus 0) = i\varphi$

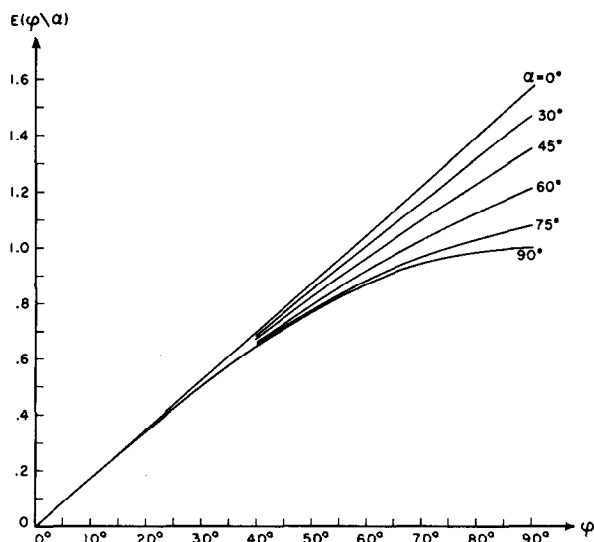


FIGURE 17.7. *Incomplete elliptic integral of the second kind.*

$E(\varphi \setminus \alpha)$, α constant

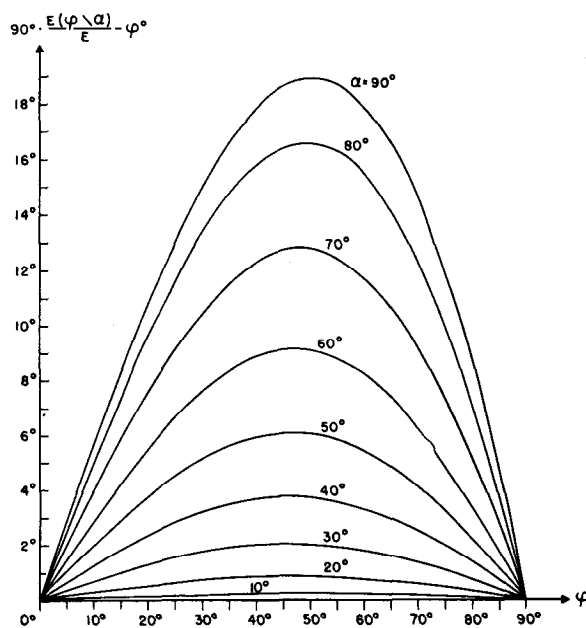


FIGURE 17.8. $90^\circ \frac{E(\varphi \setminus \alpha)}{E} - \varphi$, α constant.

17.4.21

$$F(\varphi \setminus 90^\circ) = \ln(\sec \varphi + \tan \varphi) = \ln \tan \left(\frac{\pi}{4} + \frac{\varphi}{2} \right)$$

17.4.22 $F(i\varphi \setminus 90^\circ) = i \arctan(\sinh \varphi)$

17.4.23 $E(\varphi \setminus 0) = \varphi$

17.4.24 $E(i\varphi \setminus 0) = i\varphi$

17.4.25 $E(\varphi \setminus 90^\circ) = \sin \varphi$

17.4.26 $E(i\varphi \setminus 90^\circ) = i \sinh \varphi$

Jacobi's Zeta Function

$$17.4.27 \quad Z(\varphi \backslash \alpha) = E(\varphi \backslash \alpha) - E(\alpha)F(\varphi \backslash \alpha)/K(\alpha)$$

$$17.4.28 \quad Z(u|m) = Z(u) = E(u) - uE(m)/K(m)$$

$$17.4.29 \quad Z(-u) = -Z(u)$$

$$17.4.30 \quad Z(u+2K) = Z(u)$$

$$17.4.31 \quad Z(K-u) = -Z(K+u)$$

$$17.4.32 \quad Z(u) = Z(u-K) - m \operatorname{sn}(u-K) \operatorname{cd}(u-K)$$

Special Values

$$17.4.33 \quad Z(u|0) = 0$$

$$17.4.34 \quad Z(u|1) = \tanh u$$

Addition Theorem

$$17.4.35$$

$$Z(u+v) = Z(u) + Z(v) - m \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u+v)$$

Jacobi's Imaginary Transformation

$$17.4.36$$

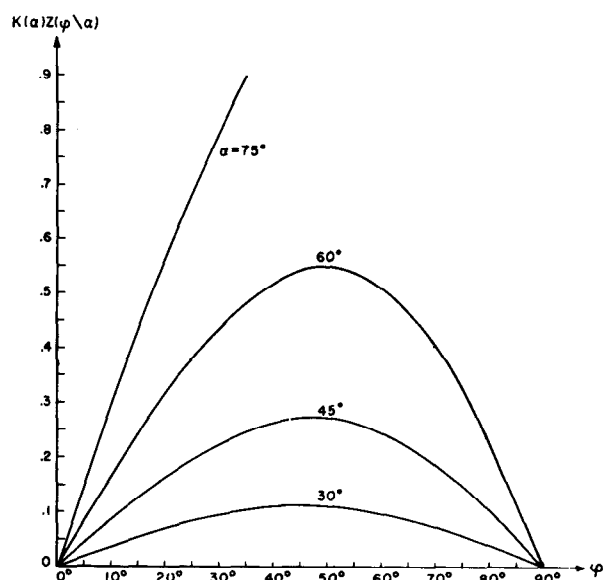
$$iZ(iu|m) = Z(u|m_1) + \frac{\pi u}{2KK'} - \operatorname{dn}(u|m_1) \operatorname{sc}(u|m_1)$$

Relation to Jacobi's Theta Function

$$17.4.37 \quad Z(u) = \Theta'(u)/\Theta(u) = \frac{d}{du} \ln \Theta(u)$$

q-Series

$$17.4.38 \quad Z(u) = \frac{2\pi}{K} \sum_{s=1}^{\infty} q^s (1-q^{2s})^{-1} \sin(\pi s u/K)$$


 FIGURE 17.9. Jacobian zeta function $K(\alpha)Z(\varphi \backslash \alpha)$.

*See page II.

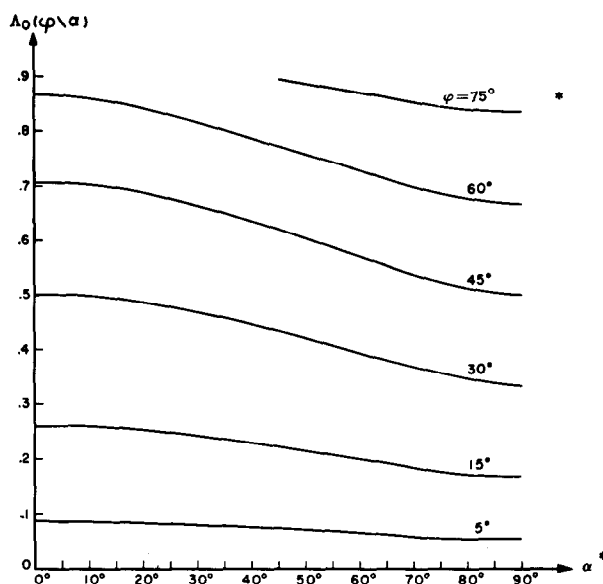
Heuman's Lambda Function

$$17.4.39$$

$$\Lambda_0(\varphi \backslash \alpha) = \frac{F(\varphi \backslash 90^\circ - \alpha)}{K'(\alpha)} + \frac{2}{\pi} K(\alpha) Z(\varphi \backslash 90^\circ - \alpha)$$

$$17.4.40$$

$$= \frac{2}{\pi} \{ K(\alpha) E(\varphi \backslash 90^\circ - \alpha) - [K(\alpha) - E(\alpha)] F(\varphi \backslash 90^\circ - \alpha) \}$$


 FIGURE 17.10. Heuman's lambda function $\Lambda_0(\varphi \backslash \alpha)$.

Numerical Evaluation of Incomplete Integrals of the First and Second Kinds

For the numerical evaluation of an elliptic integral the quartic (or cubic⁴) under the radical should first be expressed in terms of t^2 , see **Examples 1 and 2**. In the resulting quartic there are only six possible sign patterns or combinations of the factors namely

$$(t^2 + a^2)(t^2 + b^2), (a^2 - t^2)(t^2 - b^2), (a^2 - t^2)(b^2 - t^2), (t^2 - a^2)(t^2 - b^2), (t^2 + a^2)(t^2 - b^2), (t^2 + a^2)(b^2 - t^2).$$

The list which follows is then exhaustive for integrals which reduce to $F(\varphi \backslash \alpha)$ or $E(\varphi \backslash \alpha)$.

The value of the elliptic integral of the first kind is also expressed as an *inverse* Jacobian elliptic function. Here, for example, the notation $u = \operatorname{sn}^{-1} x$ means that $x = \operatorname{sn} u$.

The column headed "t substitution" gives the Jacobian elliptic function substitution which is appropriate to reduce every elliptic integral which contains the given quartic.

⁴ For an alternate treatment of cubics see 17.4.61 and 17.4.70.

	$F(\varphi \backslash \alpha)$	Equivalent Inverse Jacobian Elliptic Function	φ	t Substitution	$E(\varphi \backslash \alpha)$
$\cos \alpha = b/a$ $a > b$ $m = (a^2 - b^2)/a^2$	17.4.41 $a \int_0^x \frac{dt}{[(t^2 + a^2)(t^2 + b^2)]^{1/2}}$	$\operatorname{sc}^{-1} \left(\frac{x}{b} \middle \frac{a^2 - b^2}{a^2} \right)$	$\tan \varphi = \frac{x}{b}$	$t = b \operatorname{sc} v$	$\frac{b^2}{a} \int_0^x \frac{(t^2 + a^2)}{(t^2 + b^2)} \frac{dt}{[(t^2 + a^2)(t^2 + b^2)]^{1/2}}$
	17.4.42 $a \int_x^\infty \frac{dt}{[(t^2 + a^2)(t^2 + b^2)]^{1/2}}$	$\operatorname{cs}^{-1} \left(\frac{x}{a} \middle \frac{a^2 - b^2}{a^2} \right)$	$\tan \varphi = \frac{a}{x}$	$t = a \operatorname{cs} v$	$a \int_x^\infty \frac{(t^2 + b^2)}{(t^2 + a^2)} \frac{dt}{[(t^2 + a^2)(t^2 + b^2)]^{1/2}}$
	17.4.43 $a \int_b^x \frac{dt}{[(a^2 - t^2)(t^2 - b^2)]^{1/2}}$	$\operatorname{nd}^{-1} \left(\frac{x}{b} \middle \frac{a^2 - b^2}{a^2} \right)$	$\sin^2 \varphi = \frac{a^2(x^2 - b^2)}{x^2(a^2 - b^2)}$	$t = b \operatorname{nd} v$	$ab^2 \int_b^x \frac{1}{t^2} \frac{dt}{[(a^2 - t^2)(t^2 - b^2)]^{1/2}}$
	17.4.44 $a \int_x^a \frac{dt}{[(a^2 - t^2)(t^2 - b^2)]^{1/2}}$	$\operatorname{dn}^{-1} \left(\frac{x}{a} \middle \frac{a^2 - b^2}{a^2} \right)$	$\sin^2 \varphi = \frac{a^2 - x^2}{a^2 - b^2}$	$t = a \operatorname{dn} v$	$\frac{1}{a} \int_x^a \frac{t^2 dt}{[(a^2 - t^2)(t^2 - b^2)]^{1/2}}$
	17.4.45 $a \int_0^x \frac{dt}{[(a^2 - t^2)(b^2 - t^2)]^{1/2}}$	$\operatorname{sn}^{-1} \left(\frac{x}{b} \middle \frac{b^2}{a^2} \right)$	$\sin \varphi = \frac{x}{b}$	$t = b \operatorname{sn} v$	$\frac{1}{a} \int_0^x \frac{(a^2 - t^2) dt}{[(a^2 - t^2)(b^2 - t^2)]^{1/2}}$
$\sin \alpha = b/a$ $a > b$ $m = b^2/a^2$	17.4.46 $a \int_x^b \frac{dt}{[(a^2 - t^2)(b^2 - t^2)]^{1/2}}$	$\operatorname{cd}^{-1} \left(\frac{x}{b} \middle \frac{b^2}{a^2} \right)$	$\sin^2 \varphi = \frac{a^2(b^2 - x^2)}{b^2(a^2 - x^2)}$	$t = b \operatorname{cd} v$	$a(a^2 - b^2) \int_x^b \left(\frac{1}{a^2 - t^2} \right) \frac{dt}{[(a^2 - t^2)(b^2 - t^2)]^{1/2}}$
	17.4.47 $a \int_a^x \frac{dt}{[(t^2 - a^2)(t^2 - b^2)]^{1/2}}$	$\operatorname{dc}^{-1} \left(\frac{x}{a} \middle \frac{b^2}{a^2} \right)$	$\sin^2 \varphi = \frac{x^2 - a^2}{x^2 - b^2}$	$t = a \operatorname{dc} v$	$\frac{a^2 - b^2}{a} \int_a^x \left(\frac{t^2}{t^2 - b^2} \right) \frac{dt}{[(t^2 - a^2)(t^2 - b^2)]^{1/2}}$
	17.4.48 $a \int_x^\infty \frac{dt}{[(t^2 - a^2)(t^2 - b^2)]^{1/2}}$	$\operatorname{ns}^{-1} \left(\frac{x}{a} \middle \frac{b^2}{a^2} \right)$	$\sin \varphi = \frac{a}{x}$	$t = a \operatorname{ns} v$	$a \int_x^\infty \left(\frac{t^2 - b^2}{t^2} \right) \frac{dt}{[(t^2 - a^2)(t^2 - b^2)]^{1/2}}$
$\cot \alpha = \frac{b}{a}$ $m = a^2/(a^2 + b^2)$	17.4.49 $(a^2 + b^2)^{1/2} \int_b^x \frac{dt}{[(t^2 + a^2)(t^2 - b^2)]^{1/2}}$	$\operatorname{nc}^{-1} \left(\frac{x}{b} \middle \frac{a^2}{a^2 + b^2} \right)$	$\cos \varphi = \frac{b}{x}$	$t = b \operatorname{nc} v$	$\frac{b^2}{(a^2 + b^2)^{1/2}} \int_b^x \frac{t^2 + a^2}{t^2} \frac{dt}{[(t^2 + a^2)(t^2 - b^2)]^{1/2}}$
	17.4.50 $(a^2 + b^2)^{1/2} \int_x^\infty \frac{dt}{[(t^2 + a^2)(t^2 - b^2)]^{1/2}}$	$\operatorname{ds}^{-1} \left(\frac{x}{(a^2 + b^2)^{1/2}} \middle \frac{a^2}{a^2 + b^2} \right)$	$\sin^2 \varphi = \frac{a^2 + b^2}{a^2 + x^2}$	$t = (a^2 + b^2)^{1/2} \operatorname{ds} v$	$(a^2 + b^2)^{1/2} \int_x^\infty \frac{t^2}{(t^2 + a^2)} \frac{dt}{[(t^2 + a^2)(t^2 - b^2)]^{1/2}}$
$\tan \alpha = \frac{b}{a}$ $m = b^2/(a^2 + b^2)$	17.4.51 $(a^2 + b^2)^{1/2} \int_0^x \frac{dt}{[(t^2 + a^2)(b^2 - t^2)]^{1/2}}$	$\operatorname{sd}^{-1} \left(\frac{x(a^2 + b^2)^{1/2}}{ab} \middle \frac{b^2}{a^2 + b^2} \right)$	$\sin^2 \varphi = \frac{x^2(a^2 + b^2)}{b^2(a^2 + x^2)}$	$t = \frac{ab}{(a^2 + b^2)^{1/2}} \operatorname{sd} v$	$a^2(a^2 + b^2)^{1/2} \int_0^x \frac{1}{(t^2 + a^2)} \frac{dt}{[(t^2 + a^2)(b^2 - t^2)]^{1/2}}$
	17.4.52 $(a^2 + b^2)^{1/2} \int_x^b \frac{dt}{[(t^2 + a^2)(b^2 - t^2)]^{1/2}}$	$\operatorname{cn}^{-1} \left(\frac{x}{b} \middle \frac{b^2}{a^2 + b^2} \right)$	$\cos \varphi = \frac{x}{b}$	$t = b \operatorname{cn} v$	$\frac{1}{(a^2 + b^2)^{1/2}} \int_x^b \frac{(t^2 + a^2) dt}{[(t^2 + a^2)(b^2 - t^2)]^{1/2}}$

Some Important Special Cases

$\frac{1}{2}F(\varphi \backslash \alpha)$	$\cos \varphi$	α	$\frac{1}{3^{1/4}}F(\varphi \backslash \alpha)$	$\cos \varphi$	α
17.4.53 $\int_x^\infty \frac{dt}{(1+t^4)^{1/2}}$	$\frac{x^2-1}{x^2+1}$	45°	17.4.57 $\int_x^\infty \frac{dt}{(t^3-1)^{1/2}}$	$\frac{x-1-\sqrt{3}}{x-1+\sqrt{3}}$	15°
17.4.54 $\int_0^x \frac{dt}{(1+t^4)^{1/2}}$	$\frac{1-x^2}{1+x^2}$	45°	17.4.58 $\int_1^x \frac{dt}{(t^3-1)^{1/2}}$	$\frac{\sqrt{3}+1-x}{\sqrt{3}-1+x}$	15°
17.4.55 $\frac{1}{2}^{1/2} \int_1^x \frac{dt}{(t^4-1)^{1/2}}$	$\frac{1}{x}$	45°	17.4.59 $\int_x^1 \frac{dt}{(1-t^3)^{1/2}}$	$\frac{\sqrt{3}-1+x}{\sqrt{3}+1-x}$	75°
17.4.56 $\frac{1}{2}^{1/2} \int_x^1 \frac{dt}{(1-t^4)^{1/2}}$	x	45°	17.4.60 $\int_{-\infty}^x \frac{dt}{(1-t^3)^{1/2}}$	$\frac{1-\sqrt{3}-x}{1+\sqrt{3}-x}$	75°

Reduction of $\int dt/\sqrt{P}$ where $P=P(t)$ is a cubic polynomial with three real factors $P=(t-\beta_1)(t-\beta_2)(t-\beta_3)$ where $\beta_1 > \beta_2 > \beta_3$. Write

17.4.61

$$\lambda = \frac{1}{2}(\beta_1 - \beta_3)^{1/2}, \quad m = \sin^2 \alpha = \frac{\beta_2 - \beta_3}{\beta_1 - \beta_3},$$

$$m_1 = \cos^2 \alpha = \frac{\beta_1 - \beta_2}{\beta_1 - \beta_3}$$

17.4.62 $\lambda \int_{\beta_3}^x \frac{dt}{\sqrt{P}}$	$F(\varphi \backslash \alpha)$	$\sin^2 \varphi = \frac{x - \beta_3}{\beta_2 - \beta_3}$
17.4.63 $\lambda \int_x^{\beta_2} \frac{dt}{\sqrt{P}}$	$F(\varphi \backslash \alpha)$	$\cos^2 \varphi = \frac{(\beta_1 - \beta_2)(x - \beta_3)}{(\beta_2 - \beta_3)(\beta_1 - x)}$
17.4.64 $\lambda \int_{\beta_1}^x \frac{dt}{\sqrt{P}}$	$F(\varphi \backslash \alpha)$	$\sin^2 \varphi = \frac{x - \beta_1}{x - \beta_2}$
17.4.65 $\lambda \int_x^\infty \frac{dt}{\sqrt{P}}$	$F(\varphi \backslash \alpha)$	$\cos^2 \varphi = \frac{x - \beta_1}{x - \beta_3}$
17.4.66 $\lambda \int_{-\infty}^x \frac{dt}{\sqrt{-P}}$	$F(\varphi \backslash (90^\circ - \alpha^\circ))$	$\sin^2 \varphi = \frac{\beta_1 - \beta_3}{\beta_1 - x}$
17.4.67 $\lambda \int_x^{\beta_3} \frac{dt}{\sqrt{-P}}$	$F(\varphi \backslash (90^\circ - \alpha^\circ))$	$\cos^2 \varphi = \frac{\beta_2 - \beta_3}{\beta_2 - x}$
17.4.68 $\lambda \int_{\beta_2}^x \frac{dt}{\sqrt{-P}}$	$F(\varphi \backslash (90^\circ - \alpha^\circ))$	$\sin^2 \varphi = \frac{(\beta_1 - \beta_3)(x - \beta_2)}{(\beta_1 - \beta_2)(x - \beta_3)}$
17.4.69 $\lambda \int_x^{\beta_1} \frac{dt}{\sqrt{-P}}$	$F(\varphi \backslash (90^\circ - \alpha^\circ))$	$\cos^2 \varphi = \frac{x - \beta_2}{\beta_1 - \beta_2}$

Reduction of $\int dt/\sqrt{P}$ when $P=P(t)=t^3+a_2t+a_3$ is a cubic polynomial with only one real root $t=\beta$. We form the first and second derivatives $P'(t)$, $P''(t)$ with respect to t and then write

$$17.4.70 \quad \lambda^2 = [P'(\beta)]^{1/2}, \quad m = \sin^2 \alpha = \frac{1}{2} - \frac{1}{8} \frac{P''(\beta)}{[P'(\beta)]^{1/2}}$$

17.4.71 $\lambda \int_{\beta}^x \frac{dt}{\sqrt{P}}$	$F(\varphi \backslash \alpha)$	$\cos \varphi = \frac{\lambda^2 - (x - \beta)}{\lambda^2 + (x - \beta)}$
17.4.72 $\lambda \int_x^\infty \frac{dt}{\sqrt{P}}$	$F(\varphi \backslash \alpha)$	$\cos \varphi = \frac{(x - \beta) - \lambda^2}{(x - \beta) + \lambda^2}$
17.4.73 $\lambda \int_{-\infty}^x \frac{dt}{\sqrt{(-P)}}$	$F(\varphi \backslash (90^\circ - \alpha^\circ))$	$\cos \varphi = \frac{(\beta - x) - \lambda^2}{(\beta - x) + \lambda^2}$
17.4.74 $\lambda \int_x^{\beta} \frac{dt}{\sqrt{(-P)}}$	$F(\varphi \backslash (90^\circ - \alpha^\circ))$	$\cos \varphi = \frac{\lambda^2 - (\beta - x)}{\lambda^2 + (\beta - x)}$

17.5. Landen's Transformation

 Descending Landen Transformation ⁵

Let α_n, α_{n+1} be two modular angles such that

$$17.5.1 \quad (1 + \sin \alpha_{n+1})(1 + \cos \alpha_n) = 2 \quad (\alpha_{n+1} < \alpha_n)$$

and let φ_n, φ_{n+1} be two corresponding amplitudes such that

$$17.5.2 \quad \tan(\varphi_{n+1} - \varphi_n) = \cos \alpha_n \tan \varphi_n \quad (\varphi_{n+1} > \varphi_n)$$

⁵ The emphasis here is on the modular angle since this is an argument of the Tables. All formulae concerning Landen's transformation may also be expressed in terms of the modulus $k = m^{\frac{1}{2}} = \sin \alpha$ and its complement $k' = m^{\frac{1}{2}} = \cos \alpha$.

Thus the step from n to $n+1$ decreases the modular angle but increases the amplitude. By iterating the process we can descend from a given modular angle to one whose magnitude is negligible, when 17.4.19 becomes applicable.

With $\alpha_0 = \alpha$ we have

17.5.3

$$F(\varphi \setminus \alpha) = (1 + \cos \alpha)^{-1} F(\varphi_1 \setminus \alpha_1) \\ = \frac{1}{2} (1 + \sin \alpha_1) F(\varphi_1 \setminus \alpha_1)$$

$$17.5.4 \quad F(\varphi \setminus \alpha) = 2^{-n} \prod_{s=1}^n (1 + \sin \alpha_s) F(\varphi_n \setminus \alpha_n)$$

$$17.5.5 \quad F(\varphi \setminus \alpha) = \Phi \prod_{s=1}^{\infty} (1 + \sin \alpha_s)$$

$$17.5.6 \quad \Phi = \lim_{n \rightarrow \infty} \frac{1}{2^n} F(\varphi_n \setminus \alpha_n) = \lim_{n \rightarrow \infty} \frac{\varphi_n}{2^n}$$

$$17.5.7 \quad K = F(\tfrac{1}{2}\pi \setminus \alpha) = \tfrac{1}{2}\pi \prod_{s=1}^{\infty} (1 + \sin \alpha_s)$$

$$17.5.8 \quad F(\varphi \setminus \alpha) = 2\pi^{-1} K \Phi$$

17.5.9

$$E(\varphi \setminus \alpha) = F(\varphi \setminus \alpha) \left[1 - \frac{1}{2} \sin^2 \alpha \left(1 + \frac{1}{2} \sin \alpha_1 \right. \right. \\ \left. \left. + \frac{1}{2^2} \sin \alpha_1 \sin \alpha_2 + \dots \right) \right] + \sin \alpha \left[\frac{1}{2} (\sin \alpha_1)^{1/2} \sin \varphi_1 \right. \\ \left. + \frac{1}{2^2} (\sin \alpha_1 \sin \alpha_2)^{1/2} \sin \varphi_2 + \dots \right]$$

17.5.10

$$E = K \left[1 - \frac{1}{2} \sin^2 \alpha \left(1 + \frac{1}{2} \sin \alpha_1 + \frac{1}{2^2} \sin \alpha_1 \sin \alpha_2 \right. \right. \\ \left. \left. + \frac{1}{2^3} \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 + \dots \right) \right]$$

Ascending Landen Transformation

Let α_n, α_{n+1} be two modular angles such that

$$17.5.11 \quad (1 + \sin \alpha_n)(1 + \cos \alpha_{n+1}) = 2 \quad (\alpha_{n+1} > \alpha_n)$$

and let φ_n, φ_{n+1} be two corresponding amplitudes such that

$$17.5.12 \quad \sin(2\varphi_{n+1} - \varphi_n) = \sin \alpha_n \sin \varphi_n \quad (\varphi_{n+1} < \varphi_n)$$

Thus the step from n to $n+1$ increases the modular angle but decreases the amplitude. By iterating the process we can ascend from a given modular angle to one whose difference from a right angle is so small that 17.4.21 becomes applicable.

With $\alpha_0 = \alpha$ we have

$$17.5.13 \quad F(\varphi \setminus \alpha) = 2(1 + \sin \alpha)^{-1} F(\varphi_1 \setminus \alpha_1)$$

$$17.5.14 \quad F(\varphi \setminus \alpha) = 2^n \prod_{s=1}^n (1 + \sin \alpha_s)^{-1} F(\varphi_n \setminus \alpha_n)$$

$$17.5.15 \quad F(\varphi \setminus \alpha) = \prod_{s=1}^n (1 + \cos \alpha_s) F(\varphi_n \setminus \alpha_n)$$

$$17.5.16 \quad F(\varphi \setminus \alpha) = [\csc \alpha \prod_{s=1}^{\infty} \sin \alpha_s]^{\frac{1}{2}} \ln \tan \left(\frac{1}{4} \pi + \frac{1}{2} \Phi \right)$$

$$17.5.17 \quad \Phi = \lim_{n \rightarrow \infty} \varphi_n$$

Neighborhood of a Right Angle (see also 17.4.13)

When both φ and α are near to a right angle, interpolation in the table $F(\varphi \setminus \alpha)$ is difficult. Either Landen's transformation can then be used with advantage to increase the modular angle and decrease the amplitude or vice-versa.

17.6. The Process of the Arithmetic-Geometric Mean

Starting with a given number triple (a_0, b_0, c_0) we proceed to determine number triples $(a_1, b_1, c_1), (a_2, b_2, c_2), \dots, (a_N, b_N, c_N)$ according to the following scheme of arithmetic and geometric means

17.6.1

a_0	b_0	
$a_1 = \frac{1}{2}(a_0 + b_0)$	$b_1 = (a_0 b_0)^{\frac{1}{2}}$	
$a_2 = \frac{1}{2}(a_1 + b_1)$	$b_2 = (a_1 b_1)^{\frac{1}{2}}$	
$\vdots$	$\vdots$	
$a_N = \frac{1}{2}(a_{N-1} + b_{N-1})$	$b_N = (a_{N-1} b_{N-1})^{\frac{1}{2}}$	
		c_0
		$c_1 = \frac{1}{2}(a_0 - b_0)$
		$c_2 = \frac{1}{2}(a_1 - b_1)$
		$\vdots$
		$c_N = \frac{1}{2}(a_{N-1} - b_{N-1})$

We stop at the N th step when $a_N = b_N$, i.e., when $c_N = 0$ to the degree of accuracy to which the numbers are required.

To determine the complete elliptic integrals $K(\alpha), E(\alpha)$ we start with

$$17.6.2 \quad a_0 = 1, b_0 = \cos \alpha, c_0 = \sin \alpha$$

whence

$$17.6.3 \quad K(\alpha) = \frac{\pi}{2a_N}$$

$$17.6.4 \quad \frac{K(\alpha) - E(\alpha)}{K(\alpha)} = \frac{1}{2} [c_0^2 + 2c_1^2 + 2^2c_2^2 + \dots + 2^N c_N^2]$$

To determine $K'(\alpha)$, $E'(\alpha)$ we start with

$$17.6.5 \quad a'_0 = 1, b'_0 = \sin \alpha, c'_0 = \cos \alpha$$

whence

$$17.6.6 \quad K'(\alpha) = \frac{\pi}{2a'_N}$$

17.6.7

$$\frac{K'(\alpha) - E'(\alpha)}{K'(\alpha)} = \frac{1}{2} [c_0'^2 + 2c_1'^2 + 2^2c_2'^2 + \dots + 2^N c_N'^2]$$

To calculate $F(\varphi \setminus \alpha)$, $E(\varphi \setminus \alpha)$ start from 17.5.2 which corresponds to the descending Landen transformation and determine $\varphi_1, \varphi_2, \dots, \varphi_N$ successively from the relation

$$17.6.8 \quad \tan(\varphi_{n+1} - \varphi_n) = (b_n/a_n) \tan \varphi_n, \varphi_0 = \varphi$$

Then to the prescribed accuracy

$$17.6.9 \quad F(\varphi \setminus \alpha) = \varphi_N / (2^N a_N) \quad *$$

17.6.10

$$Z(\varphi \setminus \alpha) = E(\varphi \setminus \alpha) - (E/K) F(\varphi \setminus \alpha)$$

$$* \quad = c_1 \sin \varphi_1 + c_2 \sin \varphi_2 + \dots + c_N \sin \varphi_N$$

17.7. Elliptic Integrals of the Third Kind

17.7.1

$$\Pi(n; \varphi \setminus \alpha) = \int_0^\varphi (1 - n \sin^2 \theta)^{-1} (1 - \sin^2 \alpha \sin^2 \theta)^{-1/2} d\theta$$

$$17.7.2 \quad \Pi(n; \frac{1}{2}\pi \setminus \alpha) = \Pi(n \setminus \alpha)$$

Case (i) Hyperbolic Case $0 < n < \sin^2 \alpha$

$$\epsilon = \arcsin(n/\sin^2 \alpha)^{1/2}, \quad 0 \leq \epsilon \leq \frac{1}{2}\pi$$

$$\beta = \frac{1}{2}\pi F(\epsilon \setminus \alpha)/K(\alpha)$$

$$q = q(\alpha)$$

$$v = \frac{1}{2}\pi F(\varphi \setminus \alpha)/K(\alpha),$$

$$\delta_1 = [n(1-n)^{-1}(\sin^2 \alpha - n)^{-1}]^{1/2}$$

17.7.3

$$\Pi(n; \varphi \setminus \alpha) = \delta_1 \left[-\frac{1}{2} \ln \left[\vartheta_4(v+\beta)/\vartheta_4(v-\beta) \right] \right.$$

$$\left. + v \vartheta_1'(\beta)/\vartheta_1(\beta) \right]$$

17.7.4

$$\frac{1}{2} \ln \frac{\vartheta_4(v+\beta)}{\vartheta_4(v-\beta)} = 2 \sum_{s=1}^{\infty} s^{-1} q^s (1 - q^{2s})^{-1} \sin 2sv \sin 2s\beta$$

17.7.5

$$\frac{\vartheta_1'(\beta)}{\vartheta_1(\beta)} = \cot \beta + 4 \sum_{s=1}^{\infty} q^{2s} (1 - 2q^{2s} \cos 2\beta + q^{4s})^{-1} \sin 2\beta$$

In the above we can also use Neville's theta functions 16.36.

$$17.7.6 \quad \Pi(n \setminus \alpha) = K(\alpha) + \delta_1 K(\alpha) Z(\epsilon \setminus \alpha)$$

Case (ii) Hyperbolic Case $n > 1$

The case $n > 1$ can be reduced to the case $0 < N < \sin^2 \alpha$ by writing

$$17.7.7 \quad N = n^{-1} \sin^2 \alpha, p_1 = [(n-1)(1-n^{-1} \sin^2 \alpha)]^{1/2}$$

17.7.8

$$\begin{aligned} \Pi(n; \varphi \setminus \alpha) &= -\Pi(N; \varphi \setminus \alpha) + F(\varphi \setminus \alpha) \\ &\quad + \frac{1}{2p_1} \ln [(\Delta(\varphi) + p_1 \tan \varphi)(\Delta(\varphi) - p_1 \tan \varphi)^{-1}] \end{aligned}$$

where $\Delta(\varphi)$ is the delta amplitude, 17.2.4.

$$17.7.9 \quad \Pi(n \setminus \alpha) = K(\alpha) - \Pi(N \setminus \alpha)$$

Case (iii) Circular Case $\sin^2 \alpha < n < 1$

$$\epsilon = \arcsin[(1-n)/\cos^2 \alpha]^{1/2}, \quad 0 \leq \epsilon \leq \frac{1}{2}\pi$$

$$\beta = \frac{1}{2}\pi F(\epsilon \setminus 90^\circ - \alpha)/K(\alpha)$$

$$q = q(\alpha)$$

17.7.10

$$v = \frac{1}{2}\pi F(\varphi \setminus \alpha)/K(\alpha), \delta_2 = [n(1-n)^{-1}(n - \sin^2 \alpha)^{-1}]^{1/2}$$

$$17.7.11 \quad \Pi(n; \varphi \setminus \alpha) = \delta_2 (\lambda - 4\mu v)$$

17.7.12

$$\begin{aligned} \lambda &= \arctan(\tanh \beta \tan v) \\ &\quad + 2 \sum_{s=1}^{\infty} (-1)^{s-1} s^{-1} q^{2s} (1 - q^{2s})^{-1} \sin 2sv \sinh 2s\beta \end{aligned}$$

17.7.13

$$\mu = \left[\sum_{s=1}^{\infty} s q^{2s} \sinh 2s\beta \right] \left[1 + 2 \sum_{s=1}^{\infty} q^{2s} \cosh 2s\beta \right]^{-1}$$

$$17.7.14 \quad \Pi(n \setminus \alpha) = K(\alpha) + \frac{1}{2}\pi \delta_2 [1 - \Lambda_0(\epsilon \setminus \alpha)]$$

where Λ_0 is Heuman's Lambda function, 17.4.39.

*See page II.

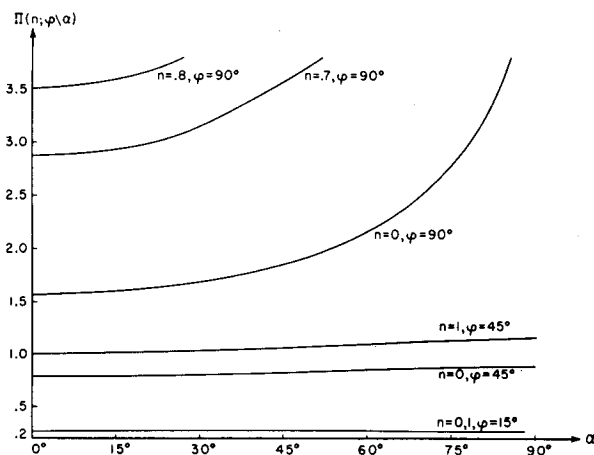


FIGURE 17.11. *Elliptic integral of the third kind*
 $\Pi(n; \varphi|\alpha)$.

Case (iv) Circular Case $n < 0$

The case $n < 0$ can be reduced to the case $\sin^2 \alpha < N < 1$ by writing

17.7.15

$$N = (\sin^2 \alpha - n)(1 - n)^{-1}$$

$$p_2 = [-n(1 - n)^{-1}(\sin^2 \alpha - n)]^{\frac{1}{2}}$$

17.7.16

$$\begin{aligned} & [(1 - n)(1 - n^{-1} \sin^2 \alpha)]^{\frac{1}{2}} \Pi(n; \varphi|\alpha) \\ &= [(1 - N)(1 - N^{-1} \sin^2 \alpha)]^{\frac{1}{2}} \Pi(N; \varphi|\alpha) \\ &+ p_2^{-1} \sin^2 \alpha F(\varphi|\alpha) + \arctan [\tfrac{1}{2} p_2 \sin 2\varphi / \Delta(\varphi)] \end{aligned}$$

17.7.17

$$\begin{aligned} \Pi(n|\alpha) &= (-n \cos^2 \alpha)(1 - n)^{-1}(\sin^2 \alpha - n)^{-1} \Pi(N|\alpha) \\ &+ \sin^2 \alpha (\sin^2 \alpha - n)^{-1} K(\alpha) \end{aligned}$$

Special Cases

17.7.18

$$n = 0$$

$$\Pi(0; \varphi|\alpha) = F(\varphi|\alpha)$$

17.7.19

$$n = 0, \alpha = 0$$

$$\Pi(0; \varphi|0) = \varphi$$

17.7.20

$$\alpha = 0$$

$$\begin{aligned} \Pi(n; \varphi|0) &= (1 - n)^{-\frac{1}{2}} \arctan [(1 - n)^{\frac{1}{2}} \tan \varphi], & n < 1 \\ &= (n - 1)^{-\frac{1}{2}} \operatorname{arctanh} [(n - 1)^{\frac{1}{2}} \tan \varphi], & n > 1 \\ &= \tan \varphi & n = 1 \end{aligned}$$

17.7.21

$$\alpha = \pi/2$$

$$\begin{aligned} \Pi(n; \varphi|\pi/2) &= (1 - n)^{-1} [\ln (\tan \varphi + \sec \varphi) \\ &- \tfrac{1}{2} n^{\frac{1}{2}} \ln (1 + n^{\frac{1}{2}} \sin \varphi)(1 - n^{\frac{1}{2}} \sin \varphi)^{-1}] & n \neq 1 \end{aligned}$$

17.7.22

$$n = \pm \sin \alpha$$

$$\begin{aligned} (1 \mp \sin \alpha) \{ 2\Pi(\pm \sin \alpha; \varphi|\alpha) - F(\varphi|\alpha) \} \\ = \arctan [(1 \mp \sin \alpha) \tan \varphi / \Delta(\varphi)] \end{aligned}$$

17.7.23

$$n = 1 \pm \cos \alpha$$

$$\begin{aligned} 2 \cos \alpha \Pi(1 \pm \cos \alpha; \varphi|\alpha) &= \pm \tfrac{1}{2} \ln [(1 + \tan \varphi \\ &\cdot \Delta(\varphi))(1 - \tan \varphi \cdot \Delta(\varphi))^{-1}] + \tfrac{1}{2} \ln [(\Delta(\varphi) \\ &+ \cos \alpha \cdot \tan \varphi)(\Delta(\varphi) - \cos \alpha \tan \varphi)^{-1}] \\ &\mp (1 \mp \cos \alpha) F(\varphi|\alpha) \end{aligned}$$

17.7.24

$$n = \sin^2 \alpha$$

$$\Pi(\sin^2 \alpha; \varphi|\alpha) = \sec^2 \alpha E(\varphi|\alpha) - (\tan^2 \alpha \sin 2\varphi) / (2\Delta(\varphi))$$

17.7.25

$$n = 1$$

$$\Pi(1; \varphi|\alpha) = F(\varphi|\alpha) - \sec^2 \alpha E(\varphi|\alpha) + \sec^2 \alpha \tan \varphi \Delta(\varphi)$$

Numerical Methods

17.8. Use and Extension of the Tables

Example 1. Reduce to canonical form $\int y^{-1} dx$, where

$$y^2 = -3x^4 + 34x^3 - 119x^2 + 172x - 90$$

By inspection or by solving an equation of the fourth degree we find that

$$y^2 = Q_1 Q_2 \text{ where } Q_1 = 3x^2 - 10x + 9, Q_2 = -x^2 + 8x - 10$$

First Method

$Q_1 - \lambda Q_2 = (3 + \lambda)x^2 - (10 + 8\lambda)x + 9 + 10\lambda$ is a perfect square if the discriminant

$$(10 + 8\lambda)^2 - 4(3 + \lambda)(9 + 10\lambda) = 0; \text{ i.e., if } \lambda = -\frac{2}{3} \text{ or } \frac{1}{2}$$

and then

$$Q_1 + \frac{2}{3} Q_2 = \frac{7}{3} (x - 1)^2, Q_1 - \frac{1}{2} Q_2 = \frac{7}{2} (x - 2)^2$$

Solving for Q_1 and Q_2 we get

$$Q_1 = (x - 1)^2 + 2(x - 2)^2, Q_2 = 2(x - 1)^2 - 3(x - 2)^2$$

The substitution $t = (x - 1)/(x - 2)$ then gives

$$\int y^{-1} dx = \pm \int [(t^2 + 2)(2t^2 - 3)]^{-\frac{1}{2}} dt$$

*See page II.

If the quartic $y^2=0$ has four real roots in x (or in the case of a cubic all three roots are real), we must so combine the factors that no root of $Q_1=0$ lies between the roots of $Q_2=0$ and no root of $Q_2=0$ lies between the roots of $Q_1=0$. Provided this condition is observed the method just described will always lead to real values of λ . These values may, however, be irrational.

Second Method

Write

$$t^2 = \frac{Q_1}{Q_2} = \frac{3x^2 - 10x + 9}{-x^2 + 8x - 10}$$

and let the discriminant of $Q_2 t^2 - Q_1$ be

$$4T^2 = (8t^2 + 10)^2 - 4(t^2 + 3)(10t^2 + 9) \\ = 4(3t^2 + 2)(2t^2 - 1)$$

Then

$$\int y^{-1} dx = \pm \int T^{-1} dt = \pm \int [(3t^2 + 2)(2t^2 - 1)]^{-1/2} dt$$

This method will succeed if, as here, T^2 as a function of t^2 has real factors. If the coefficients of the given quartic are rational numbers, the factors of T^2 will likewise be rational.

Third Method

Write

$$w = \frac{Q_1}{Q_2} = \frac{3x^2 - 10x + 9}{-x^2 + 8x - 10}$$

and let the discriminant of $Q_2 w - Q_1$ be

$$4W = 4(3w + 2)(2w - 1) = 4(Aw^2 + Bw + C)$$

Then if

$$z^2 = W/w \text{ and } Z^2 = (B - z^2)^2 - 4AC = (z^2 - 1)^2 + 48$$

$$\int y^{-1} dx = \pm \int Z^{-1} dz$$

However, in this case the factors of Z are complex and the method fails.

Of the second and third methods one will always succeed where the other fails, and if the coefficients of the given quartic are rational numbers, the factors of T^2 or Z^2 , as the case may be, will be rational.

Example 2. Reduce to canonical form $\int y^{-1} dx$ where $y^2 = x(x-1)(x-2)$.

We use the third method of **Example 1** taking $Q_1 = (x-1)$, $Q_2 = x(x-2)$ and writing

$$w = \frac{Q_1}{Q_2} = \frac{x-1}{x^2-2x}$$

The discriminant of $Q_2 w - Q_1 = x^2 w - (2w+1)x + 1$ is

$$4W = (2w+1)^2 - 4w = 4w^2 + 1$$

so that

$$W = Aw^2 + Bw + C \text{ where } A=1, B=0, C=\frac{1}{4}$$

and if we write $z^2 = W/w$ and

$$Z^2 = (B - z^2)^2 - 4AC = (z^2)^2 - 1 = (z^2 - 1)(z^2 + 1),$$

$$\int y^{-1} dx = \pm \int [(z^2 - 1)(z^2 + 1)]^{-1/2} dz$$

The first method of **Example 1** fails with the above values of Q_1 and Q_2 since the root of $Q_1=0$ lies between the roots of $Q_2=0$, and we get imaginary values of λ . The method succeeds, however, if we take $Q_1 = x$, $Q_2 = (x-1)(x-2)$, for then the roots of $Q_1=0$ do not lie between those of $Q_2=0$.

Example 3. Find $K(80/81)$.

First Method

Use 17.3.29 with $m=80/81$, $m_1=1/81$, $m_1^{1/2}=1/9$. Since $[(1 - m_1^{1/2})(1 + m_1^{1/2})^{-1}]^2 = .64$, $K(80/81) = 1.8 K(.64) = 3.59154 \text{ 500 to 8D}$, taking $K(.64)$ from **Table 17.1**.

Second Method

Table 17.4 giving $L(m)$ is useful for computing $K(m)$ when m is near unity or $K'(m)$ when m is near zero.

$$K(80/81) = \frac{1}{\pi} K'(80/81) \ln(16 \times 81) - L(80/81).$$

By interpolation in **Tables 17.1** and **17.4**, since $80/81 = .98765 \text{ 43210}$,

$$K'(80/81) = 1.57567 \text{ 8423}$$

$$L(80/81) = .00311 \text{ 16543}$$

$$K(80/81) = \pi^{-1}(1.57567 \text{ 8423})(7.16703 \text{ 7877})$$

$$= .00311 \text{ 16543}$$

$$= 3.59154 \text{ 5000 to 9D.}$$

Third Method

The polynomial approximation 17.3.34 gives to 8D

$$K(80/81) = 3.59154 \text{ 501}$$

Fourth Method, Arithmetic-Geometric Mean

Here $\sin^2 \alpha = 80/81$ and we start with

$$a_0 = 1, b_0 = \frac{1}{9}, c_0 = \sqrt{80/81} = .99380 \text{ 79900}$$

giving

n	a_n	b_n	c_n
0	1.00000 00000	.11111 11111	.99380 79900
1	.55555 55555	.33333 33333	.44444 44444
2	.44444 44444	.43033 14829	.11111 11111
3	.43738 79636	.43733 10380	.00705 64808
4	.43735 95008	.43735 94999	.00002 84628
5	.43735 95003	.43735 95003	0

Thus $K(80/81) = \frac{1}{2} \pi a_5^{-1} = 3.59154 \ 5001$.

Example 4. Find $E(80/81)$.

First Method

Use 17.3.30 which gives, with $m=80/81$

$$\begin{aligned} E(80/81) &= \frac{10}{9} E(.64) - \frac{1}{5} K(.64) \\ &= 1.01910 \ 6047 \end{aligned}$$

taking $E(.64)$ and $K(.64)$ from Table 17.1.

Second Method

Polynomial approximation, 17.3.36 gives $E(80/81) = 1.01910 \ 6060$. The last two figures must be dropped to keep within the limit of accuracy of the method.

Third Method

Arithmetic-geometric mean, 17.6. The numbers were calculated in Example 3, fourth method, and we have

$$\begin{aligned} \frac{K(80/81) - E(80/81)}{K(80/81)} &= \frac{1}{2} [c_0^2 + 2c_1^2 + 2^2c_2^2 + \dots + 2^5c_5^2] \\ &= \frac{1}{2} [1.43249 \ 71298] \\ &= .71624 \ 85649. \end{aligned}$$

Using the value of $K(80/81)$ found in Example 3, fourth method, we have

$$E(80/81) = 1.01910 \ 6048 \text{ to } 9D.$$

Example 5. Find q when $m=.9995$.

Here $m_1=.0005$ and so from Table 17.4

$$\begin{aligned} Q(m) &= .06251 \ 563013 \\ q_1 = m_1 Q(m) &= .00003 \ 12578 \ 15. \end{aligned}$$

From 17.3.19

$$\begin{aligned} \ln\left(\frac{1}{q}\right) &= \pi^2 / \ln\left(\frac{1}{q_1}\right) = \pi^2 / 10.37324 \ 1132 \\ &= .95144 \ 84701 \\ q &= .38618 \ 125. \end{aligned}$$

The computation could also be made using common logarithms with the aid of 17.3.20. The point of this procedure is that it enables us to calculate q_1 without the loss of significant figures which would result from direct interpolation in Table 17.1. By this means $\ln(1/q_1)$ can be found without loss of accuracy.

Example 6. Find m to 10D when $K'/K=.25$ and when $K'/K=3.5$.

From 17.3.15 with $K'/K=.25$ we can write the iteration formula

$$m^{(n+1)} = 1 - 16e^{-4\pi} \exp[-\pi L(m^{(n)})/K'(m^{(n)})].$$

Then by iteration using Tables 17.1 and 17.4

n	$m^{(n)}$
0	1.
1	.99994 42025
2	.99994 42041
3	.99994 42041

Thus $m=.99994 \ 42041$.

From 17.3.16 with $K'/K=3.5$ we can write the iteration formula,

$$m^{(n+1)} = 16e^{-3.5\pi} \exp[-\pi L(m_1^{(n)})/K(m^{(n)})]$$

n	$m^{(n)}$
0	0
1	.(3)26841 25043
2	.(3)26837 65
3	.(3)26837 65

Thus $m=.00026 \ 83765$.

The above methods in conjunction with the auxiliary Table 17.4 of $L(m)$ enable us to extend Table 17.3 for $K'/K > 3$, and for $K'/K < .3$.

Example 7. Calculate to 5D the Jacobian elliptic function $\operatorname{sn}(.75342|.7)$ using Table 17.5.

Here

$$m = \sin^2 \alpha = .7, \quad \alpha = 56.789089^\circ.$$

Thus, $\operatorname{sn}(.75342|.7) = \sin \varphi$ where φ is determined from

$$F(\varphi \backslash 56.789089^\circ) = .75342.$$

Inspection of Table 17.5 shows that φ lies between 40° and 45° . We have from the table of $F(\varphi \backslash \alpha)$

$\varphi \backslash \alpha$	56°	58°	60°
35°	.63803	.63945	.64085
40°	.73914	.74138	.74358
45°	.84450	.84788	.85122
50°	.95479	.95974	.96465

From this we form the table of $F(\varphi \backslash 56.789089^\circ)$

φ	F	Δ	Δ_2	Δ_3
35°	.63859	10144	437	72
40°	.74003			
45°	.84584			
50°	.95674			

A rough estimate now shows that φ lies between 40° and 41° . We therefore form the following table of $F(\varphi \backslash 56.789089^\circ)$ by direct interpolation in the foregoing table

φ	F
40.0°	.74003
40.5°	.75040
41.0°	.76082

whence by linear inverse interpolation

$$\varphi = 40.5^\circ + .5^\circ \left[\frac{.75342 - .75040}{.76082 - .75040} \right] = 40.6449^\circ$$

and so $\sin \varphi = .65137 = \text{sn}(.75342 | .7)$.

This method of bivariate interpolation is given merely as an illustration. Other more direct methods such as that of the arithmetic-geometric mean described in 17.6 and illustrated for the Jacobian functions in chapter 16 are less laborious.

Example 8. Evaluate

$$\int_2^3 [(2t^2+1)(t^2-2)]^{-1/2} dt.$$

First Method, Bivariate Interpolation

From 17.4.50 we have

$$\sqrt{5} \int_2^3 [(2t^2+1)(t^2-2)]^{-1/2} dt = F(\varphi_1 \backslash \alpha) - F(\varphi_2 \backslash \alpha),$$

where

$$\sin^2 \alpha = \frac{1}{5}, \cos \varphi_1 = \frac{\sqrt{2}}{3}, \cos \varphi_2 = \frac{\sqrt{2}}{2}$$

Thus $\alpha = 26.5650512^\circ$, $\varphi_1 = 61.8744943^\circ$, $\varphi_2 = 45^\circ$, $F(\varphi_1 \backslash \alpha) = 1.115921$ and $F(\varphi_2 \backslash \alpha) = .800380$ and therefore the integral is equal to .141114.

Second Method, Numerical Quadrature

Simpson's formula with 11 ordinates and interval .1 gives .141117.

Example 9. Evaluate

$$\int_2^4 [(t^2-2)(t^2-4)]^{-1/2} dt.$$

First Method, Reduction to Standard Form and Bivariate Interpolation

Here we can use 17.4.48 noting that $a^2=4$, $b^2=2$, and that

$$\begin{aligned} \int_2^4 [(t^2-2)(t^2-4)]^{-1/2} dt &= \int_2^\infty - \int_4^\infty \\ &= \frac{1}{2} [F(\varphi_1 \backslash 45^\circ) - F(\varphi_2 \backslash 45^\circ)] \\ &= \frac{1}{2} [1.854075 - .535623] = .659226 \end{aligned}$$

where

$$\sin \varphi_1 = \frac{2}{2}, \sin \varphi_2 = \frac{2}{4}, \sin^2 \alpha = \frac{2}{4}$$

Thus

$$\alpha = 45^\circ, \varphi_1 = 90^\circ, \varphi_2 = 30^\circ.$$

Second Method, Numerical Integration

If we wish to use numerical integration we must observe that the integrand has a singularity at $t=2$ where it behaves like $[8(t-2)]^{-1/2}$.

We remove the singularity at $t=2$, by writing

$$\int_2^4 [(t^2-2)(t^2-4)]^{-1/2} dt = \int_2^4 f(t) dt + \int_2^4 [8(t-2)]^{-1/2} dt$$

where

$$f(t) = [(t^2-2)(t^2-4)]^{-1/2} - [8(t-2)]^{-1/2}.$$

If we define $f(2)=0$,

$$\int_2^4 f(t) dt$$

can be calculated by numerical quadrature. Also

$$\int_2^4 [8(t-2)]^{-1/2} dt = \left[\frac{1}{\sqrt{2}} (t-2)^{1/2} \right]_2^4 = 1$$

and thus we calculate the integral as

$$1 + \int_2^4 f(t) dt = 1 - .340773 = .659227.$$

Example 10. Evaluate

$$u = \int_{17}^\infty (x^3 - 7x + 6)^{-1/2} dx.$$

$x^3 - 7x + 6 = (x-1)(x-2)(x+3)$ and we use 17.4.65 with $\beta_1=2$, $\beta_2=1$, $\beta_3=-3$,

$$m = \sin^2 \alpha = 4/5, \lambda = \sqrt{5}/2, \cos^2 \varphi = 3/4.$$

Thus $\alpha = 63.434949^\circ$, $\varphi = 30^\circ$ and

$$\begin{aligned} u &= 2(5)^{-1/2} F(30^\circ \setminus 63.434949^\circ) \\ &= 2(5)^{-1/2} (.543604) = .486214 \text{ from Table 17.5.} \end{aligned}$$

The above integral is of the Weierstrass type and in fact $17 = \mathcal{P}(\frac{1}{2}u; 28, -24)$ (see chapter 18).

Example 11. Evaluate

$$\int_0^{2/3} (24 - 12t + 2t^2 - t^3)^{-1/2} dt.$$

We have

$$24 - 12t + 2t^2 - t^3 = -(t-2)(t^2 + 12) = -P(t).$$

There is only one real zero and we therefore use 17.4.74 with $P(t) = t^3 - 2t^2 + 12t - 24$, $\beta = 2$ so that $P'(2) = 16$, $P''(2) = 8$, $\lambda = 2$ and therefore

$$m = \sin^2 \alpha = \frac{1}{4}, \quad \alpha = 30^\circ$$

Therefore the given integral is

$$\int_0^2 - \int_{2/3}^2 = \frac{1}{2} [F(\varphi_1 \setminus 60^\circ) - F(\varphi_2 \setminus 60^\circ)]$$

where

$$\cos \varphi_1 = \frac{1}{3}, \quad \varphi_1 = 70.52877 \ 93^\circ$$

$$\cos \varphi_2 = \frac{1}{2}, \quad \varphi_2 = 60^\circ$$

and the integral $= \frac{1}{2} [1.510344 - 1.212597] = .148874$.

Example 12. Use Landen's transformation to evaluate

$$\int_0^{\pi/2} \left(1 - \frac{1}{4} \sin^2 \theta\right)^{-1/2} d\theta \text{ to 5D.}$$

First Method, Descending Transformation

We use 17.5.1 to give

$$1 + \sin \alpha_1 = \frac{2}{1 + \cos 30^\circ} = 1.071797$$

$$\cos \alpha_1 = [(1 - \sin \alpha_1)(1 + \sin \alpha_1)]^{1/2} = .997419$$

$$1 + \sin \alpha_2 = \frac{2}{1 + \cos \alpha_1} = 1.001292; \cos \alpha_2 = .999999$$

$$1 + \sin \alpha_3 = \frac{2}{1 + \cos \alpha_2} = 1.000000$$

Thus from 17.5.7,

$$\begin{aligned} \text{the integral} &= F(90^\circ \setminus 30^\circ) = \frac{\pi}{2} (1.071797)(1.001292) \\ &= 1.68575 \text{ to 5D.} \end{aligned}$$

Second Method, Ascending Transformation

We use 17.5.11 to give

$$1 + \cos \alpha_{n+1} = 2/(1 + \sin \alpha_n)$$

n	$\cos \alpha_n$	$\sin \alpha_n$
1	.33333 333	.94280 904
2	.02943 725	.99956 663
3	.00021 673	.99999 998

$$\begin{aligned} \sin (2\varphi_1 - 90^\circ) &= \sin 30^\circ, & \varphi_1 &= 60^\circ \\ \sin (2\varphi_2 - \varphi_1) &= \sin \alpha_1 \sin \varphi_1, & \varphi_2 &= 57.367805^\circ \\ \sin (2\varphi_3 - \varphi_2) &= \sin \alpha_2 \sin \varphi_2, & \varphi_3 &= 57.348426^\circ \\ \sin (2\varphi_4 - \varphi_3) &= \sin \alpha_3 \sin \varphi_3, & \varphi_4 &= 57.348425^\circ = \Phi. \end{aligned}$$

From 17.5.16

$$\begin{aligned} F(90^\circ \setminus 30^\circ) &= \frac{2}{1.5} \frac{2}{1.94280 \ 904} \frac{2}{1.99956 \ 663} \\ &\quad \frac{2}{1.99999 \ 998} \ln \tan \left(45^\circ + \frac{1}{2} \Phi\right) \\ &= 1.37288 \ 050 \ln \tan 73.674213^\circ \\ &= 1.37288 \ 050 (1.22789 \ 30) \end{aligned}$$

$$F(90^\circ \setminus 30^\circ) = 1.68575 \text{ to 5D.}$$

Example 13. Find the value of $F(89.5^\circ \setminus 89.5^\circ)$.

First Method

This is a case where interpolation in Table 17.5 is not possible. We use 17.4.13 which gives

$$F(89.5^\circ \setminus 89.5^\circ) = F(90^\circ \setminus 89.5^\circ) - F(\psi \setminus 89.5^\circ)$$

where

$$\begin{aligned} \cot \psi &= \sin (.5^\circ) \cot (.5^\circ) = \cos (.5^\circ) \\ \psi &= 45.00109 \ 084^\circ \end{aligned}$$

and $F(\psi \setminus 89.5^\circ) = .881390$ from Table 17.5.

$$\begin{aligned} F(90^\circ \setminus 89.5^\circ) &= K(\sin^2 89.5^\circ) = K(.99992 \ 38476) \\ &= 6.12777 \ 88 \end{aligned}$$

$$\text{Thus } F(89.5^\circ \setminus 89.5^\circ) = 5.246389.$$

Second Method

Landen's ascending transformation, 17.5.11, gives

$$\cos \alpha_1 = (1 - \sin 89.5^\circ) / (1 + \sin 89.5^\circ)$$

$$\sin \alpha_1 = [(1 - \cos \alpha_1)(1 + \cos \alpha_1)]^{\frac{1}{2}} = .99999 \ 99997$$

$$\cos \alpha_2 = 0$$

$$\sin \alpha_2 = 1.$$

17.5.12 then gives

$$\begin{aligned} \sin (2\varphi_1 - 89.5^\circ) &= \sin 89.5^\circ \sin 89.5^\circ \\ &= .99992 \ 38476 \end{aligned}$$

$$2\varphi_1 - 89.5^\circ = 89.2929049^\circ, \varphi_1 = 89.39645 \ 245^\circ$$

$$\sin (2\varphi_2 - \varphi_1) = \sin \alpha_1 \sin \varphi_1, \quad \varphi_2 = 89.39645 \ 602^\circ$$

$$\sin (2\varphi_3 - \varphi_2) = \sin \varphi_2, \quad \varphi_3 = \varphi_2 = \Phi.$$

Thus 17.5.16 gives

$$F(89.5^\circ \setminus 89.5^\circ) =$$

$$\left(\frac{1}{.99996 \ 19231} \right)^{\frac{1}{2}} \ln (\tan 89.69822 \ 801^\circ) = 5.24640.$$

Example 14. Evaluate

$$\int_1^2 [(9-t^2)(16+t^2)^3]^{-\frac{1}{2}} dt \text{ to 5D.}$$

From 17.4.51 the given integral

$$= \int_0^2 - \int_0^1 = \frac{1}{80} [E(\varphi_1 \setminus \alpha) - E(\varphi_2 \setminus \alpha)]$$

where

$$\sin \alpha = \frac{3}{5}, \quad \alpha = 36.86990^\circ$$

$$\sin \varphi_1 = \frac{1}{3} \sqrt{5}, \quad \varphi_1 = 48.18968^\circ$$

$$\sin \varphi_2 = \frac{5}{3\sqrt{17}}, \quad \varphi_2 = 23.84264^\circ.$$

By bivariate interpolation in Table 17.6 we find that the given integral

$$= \frac{1}{80} [.80904 - .41192] = .00496.$$

Simpson's rule with 3 ordinates gives

$$\frac{1}{8} [.00504 + .01975 + .005] = .00496.$$

Example 15. Evaluate

$$\Pi\left(\frac{1}{16}; 45^\circ \setminus 30^\circ\right) =$$

$$\int_0^{\pi/4} (1 - \frac{1}{16} \sin^2 \theta)^{-1} (1 - \frac{1}{4} \sin^2 \theta)^{-\frac{1}{2}} d\theta \text{ to 6D.}$$

This is case (i) of integrals of the third kind, $0 < n < \sin^2 \alpha$, 17.7.3

$$n = \frac{1}{16}, \varphi = 45^\circ, \alpha = 30^\circ,$$

$$\epsilon = \arcsin (n/\sin^2 \alpha)^{\frac{1}{2}} = 30^\circ,$$

$$\beta = \frac{1}{2} \pi F(30^\circ \setminus 30^\circ) / K(30^\circ) = .49332 \ 60$$

$$v = \frac{1}{2} \pi F(45^\circ \setminus 30^\circ) / K(30^\circ) = .74951 \ 51,$$

$$\delta_1 = (16/45)^{\frac{1}{2}}$$

and so from 17.7.3

$$\Pi\left(\frac{1}{16}; 45^\circ \setminus 30^\circ\right) =$$

$$(16/45)^{\frac{1}{2}} \left\{ -\frac{1}{2} \ln \frac{\vartheta_4(v+\beta)}{\vartheta_4(v-\beta)} + \frac{\vartheta_1'(\beta)}{\vartheta_1(\beta)} v \right\}$$

$$q = .01797 \ 24.$$

Using the q -series, 16.27, for the ϑ functions we get

$$\begin{aligned} \Pi\left(\frac{1}{16}; 45^\circ \setminus 30^\circ\right) &= (16/45)^{\frac{1}{2}} \{ -.02995 \ 89 \\ &\quad + (1.86096 \ 21)(.74951 \ 51) \} = .813845. \end{aligned}$$

Table 17.9 gives .81385 with 4 point Lagrangian interpolation.

Example 16. Evaluate the complete elliptic integral

$$\Pi\left(\frac{1}{8} \setminus 30^\circ\right) \text{ to 6D.}$$

From 17.7.6 we have

$$\Pi\left(\frac{1}{8} \setminus 30^\circ\right) = K(30^\circ) + (16/45)^{1/2} K(\alpha) Z(\epsilon \setminus 30^\circ)$$

where $\epsilon = \arcsin (n/\sin^2 \alpha)^{\frac{1}{2}} = 30^\circ$. Thus using Table 17.7

$$\Pi\left(\frac{1}{8} \setminus 30^\circ\right) = 1.743055.$$

Table 17.9 gives 1.74302 with 5 point Lagrangian interpolation.

Example 17. Evaluate

$$\Pi\left(\frac{5}{8}; 45^\circ \setminus 30^\circ\right)$$

$$= \int_0^{\pi/4} (1 - \frac{5}{8} \sin^2 \theta)^{-1} (1 - \frac{1}{4} \sin^2 \theta)^{-1/2} d\theta$$

to 6D.

This is case (iii) of integrals of the third kind, $\sin^2 \alpha < n < 1$,

$$n = \frac{5}{8}, \varphi = 45^\circ, \alpha = 30^\circ$$

$$\epsilon = \arcsin [(1-n)/\cos^2 \alpha]^{\frac{1}{2}} = 45^\circ$$

$$\beta = \frac{1}{2}\pi F(45^\circ \backslash 60^\circ)/K(30^\circ) = .79317\ 74$$

$$v = \frac{1}{2}\pi F(45^\circ \backslash 30^\circ)/K(30^\circ) = .74951\ 51$$

$$\delta_2 = (40/9)^{\frac{1}{2}}$$

$$q = .01797\ 24$$

and so from 17.7.11

$$\begin{aligned} \Pi\left(\frac{5}{8}; 45^\circ \backslash 30^\circ\right) &= (40/9)^{1/2}(\lambda - 4\mu v) \\ &= 2.10818\ 51\{.55248\ 32 - 4(.03854\ 26) \\ &\quad (.74951\ 51)\} = .921129. \end{aligned}$$

Table 17.9 gives .92113 with 4 point Lagrangian interpolation.

Example 18. Evaluate the complete elliptic integral

$$\Pi\left(\frac{5}{8} \backslash 30^\circ\right) \text{ to 5D.}$$

From 17.7.14 we have

$$\Pi\left(\frac{5}{8} \backslash 30^\circ\right) = K(30^\circ) + \frac{\pi}{2} \sqrt{\frac{40}{9}} [1 - \Lambda_0(\epsilon \backslash 30^\circ)]$$

where $\epsilon = \arcsin [(1-n)/\cos^2 \alpha]^{\frac{1}{2}} = 45^\circ$. Thus using **Table 17.8**

$$\Pi\left(\frac{5}{8} \backslash 30^\circ\right) = 2.80099.$$

Table 17.9 gives 2.80126 by 6 point Lagrangian interpolation. The discrepancy results from interpolation with respect to n for $\varphi = 90^\circ$ in **Table 17.9**.

Example 19. Evaluate

$$\begin{aligned} \Pi\left(\frac{5}{8}; 45^\circ \backslash 30^\circ\right) \\ = \int_0^{\pi/4} (1 - \frac{5}{4} \sin^2 \theta)^{-1} (1 - \frac{1}{4} \sin^2 \theta)^{-1/2} d\theta \end{aligned}$$

to 5D.

Here $n = \frac{5}{4}$, $\varphi = 45^\circ$, $\alpha = 30^\circ$ and since the characteristic is greater than unity we use 17.7.7

$$\begin{aligned} N &= n^{-1} \sin^2 \alpha = .2, \quad p_1 = (1/5)^{\frac{1}{2}} \\ \Pi\left(\frac{5}{8}; 45^\circ \backslash 30^\circ\right) &= -\Pi(2; 45^\circ \backslash 30^\circ) + F(45^\circ \backslash 30^\circ) \\ &\quad + (\frac{1}{2}\sqrt{5}) \ln \frac{(7/8)^{\frac{1}{2}} + (1/5)^{\frac{1}{2}}}{(7/8)^{\frac{1}{2}} - (1/5)^{\frac{1}{2}}} \\ &= -.83612 + .80437 \\ &\quad + \frac{1}{2}\sqrt{5} \ln \frac{\sqrt{35} + \sqrt{8}}{\sqrt{35} - \sqrt{8}} \\ &= 1.13214. \end{aligned}$$

Numerical quadrature gives the same result.

Example 20. Evaluate

$$\begin{aligned} \Pi\left(-\frac{1}{4}; 45^\circ \backslash 30^\circ\right) \\ = \int_0^{\pi/4} (1 + \frac{1}{4} \sin^2 \theta)^{-1} (1 - \frac{1}{4} \sin^2 \theta)^{-1} d\theta \end{aligned}$$

to 5D.

Here the characteristic is negative and we therefore use 17.7.15 with $n = -\frac{1}{4}$, $\sin^2 \alpha = \frac{1}{4}$

$$N = (1-n)^{-1}(\sin^2 \alpha - n) = .4, \quad p_2 = \sqrt{.1}$$

and therefore

$$\begin{aligned} (5/2)^{\frac{1}{2}} \Pi\left(-\frac{1}{4}; 45^\circ \backslash 30^\circ\right) &= (9/40)^{\frac{1}{2}} \Pi\left(\frac{5}{8}; 45^\circ \backslash 30^\circ\right) \\ &\quad + \frac{1}{2}(5/2)^{\frac{1}{2}} F(45^\circ \backslash 30^\circ) + \arctan (35)^{-1} \end{aligned}$$

Using **Tables 4.14, 17.5, and 17.9** we get

$$\Pi\left(-\frac{1}{4}; 45^\circ \backslash 30^\circ\right) = .76987$$

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18. Weierstrass Elliptic and Related Functions

THOMAS H. SOUTHARD¹

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Non-Negative Discriminant ($3 \leq \bar{g}_2 \leq \infty$)

$$\omega g_3^{1/6}, \frac{\omega' g_3^{1/6}}{i} + \frac{\sqrt{6}}{12} \ln(\bar{g}_2 - 3); \bar{g}_2 = 3(.05)3.4, 7D$$

$$\omega g_3^{1/6}, \omega' g_3^{1/6}/i; \bar{g}_2 = 3.4(.1)5(.2)10, 7D$$

$$\omega g_3^{1/6} \bar{g}_2^{1/4}, \omega' g_3^{1/6} \bar{g}_2^{1/4}/i; \bar{g}_2^{-1} = .1(-.01)0, 7D$$

Non-Positive Discriminant ($-\infty \leq \bar{g}_2 \leq 3$)

$$\omega_2 g_3^{1/6} |\bar{g}_2|^{1/4}, \omega'_2 g_3^{1/6} |\bar{g}_2|^{1/4}/i; \bar{g}_2^{-1} = 0(-.01) -.2, 7D$$

$$\omega_2 g_3^{1/6}, \omega'_2 g_3^{1/6}/i; \bar{g}_2^{-1} = -.2(-.05) -1, 7D$$

$$\omega_2 g_3^{1/6}, \frac{\omega'_2}{i} g_3^{1/6} + \frac{\sqrt{6}}{6} \ln(3 - \bar{g}_2); \bar{g}_2 = -1(.2)3, 7D$$

Table 18.2. Table for Obtaining $\mathcal{P}$, $\mathcal{P}'$ and ζ on $0x$ and $0y$ (Unit
 Real Half-Period—Period Ratio a). 674

Positive Discriminant ($0 \leq x \leq 1, 0 \leq y \leq a$)

$$z^2 \mathcal{P}(z), z^3 \mathcal{P}'(z), z\zeta(z), a = 1, 1.05, 1.1, 1.2, 1.4, 2, 4 \\ x = 0(.05)1, y = 0(.05) 1.1, 1.2 (.2) a, 6-8D$$

Negative Discriminant ($0 \leq x \leq 1, 0 \leq y \leq a/2$)

$$z^2 \mathcal{P}(z), z^3 \mathcal{P}'(z), z\zeta(z), a = 1, 1.05, 1.15, 1.3, 1.5, 2, 4 \\ x = 0(.05)1, y = 0(.05)1 (.1)b(b \geq a/2), 7D$$

Table 18.3. Invariants and Values at Half-Periods ($1 \leq a \leq \infty$) (Unit
 Real Half-Period). 680
 $a = 1(.02)1.6(.05)2.3(.1) 4, \infty, 6-8D$

Non-Negative Discriminant

$$g_2, g_3, e_1 = \mathcal{P}(1), e_3 = \mathcal{P}(\omega'), \eta = \zeta(1), \eta'/i = \zeta(\omega')/i, \sigma(1), \sigma(\omega')/i, \sigma(\omega_2)$$

Non-Positive Discriminant

$$g_2, g_3, e_1, \eta_2 = \zeta(1), \eta'_2/i = \zeta(\omega'_2)/i, \sigma(1), \sigma(\omega'_2)/i, \sigma(\omega')$$

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18. Weierstrass Elliptic and Related Functions

Mathematical Properties

18.1. Definitions, Symbolism, Restrictions and Conventions

An elliptic function is a single-valued doubly periodic function of a single complex variable which is analytic except at poles and whose only singularities in the finite plane are poles. If ω and ω' are a pair of (primitive) half-periods of such a function $f(z)$, then $f(z+2M\omega+2N\omega')=f(z)$, M and N being integers. Thus the study of any such function can be reduced to consideration of its behavior in a *fundamental period parallelogram* (FPP). An elliptic function has a finite number of poles (and the same number of zeros) in a FPP; the number of such poles (zeros) (an irreducible set) is the *order* of the function (poles and zeros are counted according to their multiplicity). All other poles (zeros) are called *congruent* to the irreducible set. The simplest (non-trivial) elliptic functions are of order two. One may choose as the standard function of order two either a function with two simple poles (Jacobi's choice) or one double pole (Weierstrass' choice) in a FPP.

Weierstrass' $\mathcal{P}$ -Function. Let ω, ω' denote a pair of complex numbers with $\mathcal{I}(\omega'/\omega) > 0$. Then $\mathcal{P}(z) = \mathcal{P}(z|\omega, \omega')$ is an elliptic function of order two with periods $2\omega, 2\omega'$ and having a double pole at $z=0$, whose principal part is z^{-2} ; $\mathcal{P}(z) - z^{-2}$ is analytic in a neighborhood of the origin and vanishes at $z=0$.

Weierstrass' ζ -Function $\zeta(z) = \zeta(z|\omega, \omega')$ satisfies the condition $\zeta'(z) = -\mathcal{P}(z)$; further, $\zeta(z)$ has a simple pole at $z=0$ whose principal part is z^{-1} ; $\zeta(z) - z^{-1}$ vanishes at $z=0$ and is analytic in a neighborhood of the origin. $\zeta(z)$ is *NOT* an elliptic function, since it is not periodic. However, it is quasi-periodic (see "period" relations), so reduction to FPP is possible.

Weierstrass' σ -Function $\sigma(z) = \sigma(z|\omega, \omega')$ satisfies the condition $\sigma'(z)/\sigma(z) = \zeta(z)$; further, $\sigma(z)$ is an entire function which vanishes at the origin. Like ζ , it is *NOT* an elliptic function, since it is not periodic. However, it is quasi-periodic (see "period" relations), so reduction to FPP is possible.

Invariants g_2 and g_3

Let $W = 2M\omega + 2N\omega'$, M and N being integers. Then

$$18.1.1 \quad g_2 = 60\Sigma' W^{-4} \text{ and } g_3 = 140\Sigma' W^{-6}$$

are the INVARIANTS, summation being over all pairs M, N except $M=N=0$.

Alternate Symbolism Emphasizing Invariants

$$18.1.2 \quad \mathcal{P}(z) = \mathcal{P}(z; g_2, g_3)$$

$$18.1.3 \quad \mathcal{P}'(z) = \mathcal{P}'(z; g_2, g_3)$$

$$18.1.4 \quad \zeta(z) = \zeta(z; g_2, g_3)$$

$$18.1.5 \quad \sigma(z) = \sigma(z; g_2, g_3)$$

Fundamental Differential Equation, Discriminant and Related Quantities

$$18.1.6 \quad \mathcal{P}''(z) = 4\mathcal{P}^3(z) - g_2\mathcal{P}(z) - g_3$$

$$18.1.7$$

$$= 4(\mathcal{P}(z) - e_1)(\mathcal{P}(z) - e_2)(\mathcal{P}(z) - e_3)$$

$$18.1.8$$

$$\Delta = g_2^3 - 27g_3^2 = 16(e_2 - e_3)^2(e_3 - e_1)^2(e_1 - e_2)^2$$

$$18.1.9$$

$$g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3) = 2(e_1^2 + e_2^2 + e_3^2)$$

$$18.1.10 \quad g_3 = 4e_1e_2e_3 = \frac{4}{3}(e_1^3 + e_2^3 + e_3^3)$$

$$18.1.11 \quad e_1 + e_2 + e_3 = 0$$

$$18.1.12 \quad e_1^4 + e_2^4 + e_3^4 = g_2^2/8$$

$$18.1.13 \quad 4e_i^3 - g_2e_i - g_3 = 0 (i=1, 2, 3)$$

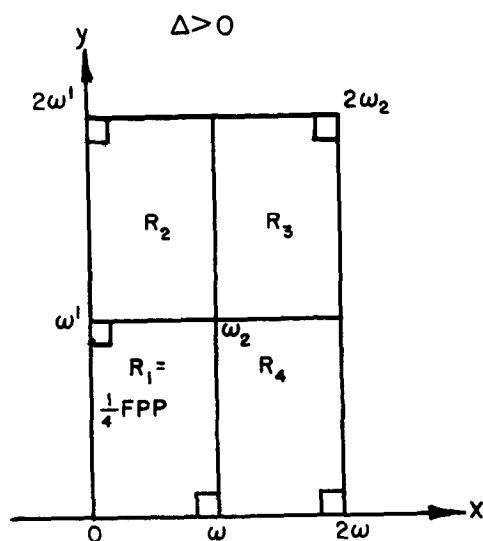
Agreement about Values of Invariants (and Discriminant)

We shall consider, in this chapter, only *real* g_2 and g_3 (this seems to cover most applications)—hence Δ is real. We shall dichotomize most of what follows (either $\Delta > 0$ or $\Delta < 0$). Homogeneity relations 18.2.1–18.2.15 enable a further restriction to non-negative g_3 (except for one case when $\Delta = 0$).

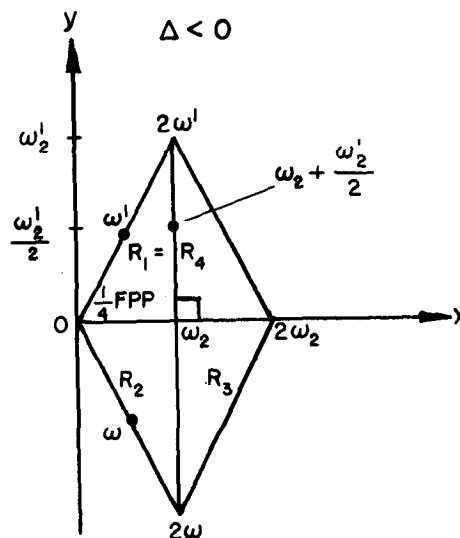
Note on Symbolism for Roots of Complex Numbers and for Conjugate Complex Numbers

In this chapter, $z^{1/n}$ (n a positive integer) is used to denote the principal n th root of z , as in chapter 3; $\bar{z}$ is used to denote the complex conjugate of z .

FPP's, Symbols for Periods, etc.



RECTANGLE

 ω REAL ω' PURE IMAG. $|\omega'| \geq \omega$, since $g_3 \geq 0$ 

RHOMBUS

 ω_2 REAL ω'_2 PURE IMAG. $|\omega'_2| \geq \omega_2$, since $g_3 \geq 0$

FIGURE 18.1

$$\omega_1 = \omega$$

$$\omega_3 = \omega + \omega'$$

$$\omega_5 = \omega'$$

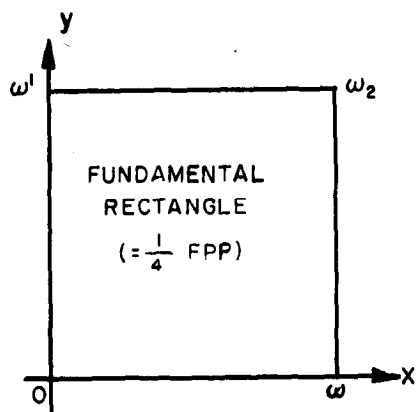
$$\omega'_2 = \omega' - \omega$$

Fundamental Rectangles

Study of all four functions ($\wp, \wp', \zeta, \sigma$) can be reduced to consideration of their values in a Fundamental Rectangle including the origin (see 18.2 on homogeneity relations, reduction formulas and processes).

 $\Delta > 0$

Fundamental Rectangle is $\frac{1}{4}$ FPP, which has vertices $0, \omega, \omega_2$ and ω'

 $\Delta < 0$

Fundamental Rectangle has vertices $0, \omega_2, \omega_2 + \frac{\omega'_2}{2}, \frac{\omega'_2}{2}$

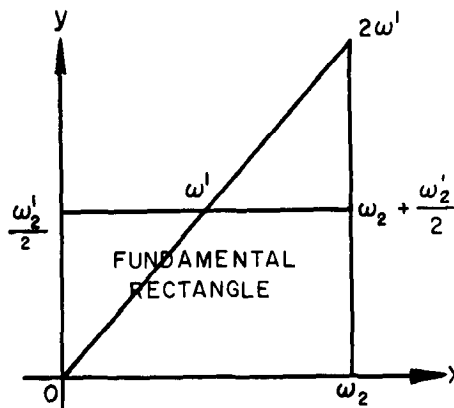


FIGURE 18.2

There is a point on the right boundary of Fundamental Rectangle where $\wp = 0$. Denote it by z_0 .

The Case $g_3 < 0$

Put $t=i$ and obtain, e.g.,

18.2.16 $\mathcal{P}(z; g_2, g_3) = -\mathcal{P}(iz; g_2, -g_3)$

Thus the case $g_3 < 0$ can be reduced to one where $g_3 > 0$.

"Period" Relations and Reduction to the FPP (M, N integers)

18.2.17 $\mathcal{P}'(z+2M\omega+2N\omega')=\mathcal{P}'(z)$

18.2.18 $\mathcal{P}(z+2M\omega+2N\omega')=\mathcal{P}(z)$

18.2.19

$$\zeta(z+2M\omega+2N\omega')=\zeta(z)+2M\eta+2N\eta'$$

18.2.20

$$\sigma(z+2M\omega+2N\omega')$$

$$=(-1)^{M+N+MN}\sigma(z)\exp[(z+M\omega+N\omega')(2M\eta+2N\eta')]$$

18.2.21 where $\eta = \zeta(\omega)$, $\eta' = \zeta(\omega')$

“Conjugate” Values

$f(\bar{z}) = \bar{f}(z)$, where f is any one of the functions $\mathcal{P}, \mathcal{P}', \zeta, \sigma$.

P (See Figure 18.1)

$\Delta > 0$

($\bar{s}$ denotes conjugate of s)

$\Delta < 0$

Point z_4 in R_4

18.2.22 $\mathcal{P}'(z_4) = -\overline{\mathcal{P}'(2\omega - z_4)}$

$$\varphi'(\bar{z}_4) = -\overline{\varphi'(2\omega_3 - z_4)}$$

18.2.23 $\mathcal{P}(z_1) = \overline{\mathcal{P}}(2\omega - z_1)$

$$\mathcal{P}(z_4) = \overline{\mathcal{P}(2\omega_2 - z_4)}$$

18.2.24 $\zeta(z_4) = -\zeta(\overline{2\omega - z_4}) + 2\pi$

$$\zeta(z_4) = -\overline{\zeta(2\omega_2 - z_4)} + 2(\eta + \eta')$$

18.2.25 $\sigma(z_4) = \overline{\sigma(2\omega - z_4)} \exp [2\eta(z_4 - \omega)]$

$$\sigma(z_1) = \overline{\sigma(2\omega_2 - z_1)} \exp [2(\eta + \eta')(z_1 - \omega_2)]$$

18.2.26 $\varphi'(z_3) = -\varphi'(2\omega_2 - z_3)$

Point x_3 in R_3

$$\varphi'(z_3) = -\varphi'(2\omega_2 - z_3)$$

18.2.27 $\mathcal{P}(z_3) = \mathcal{P}(2\omega_2 - z_3)$

$$\mathcal{P}(z_3) = \mathcal{P}(2\omega_2 - z_3)$$

18.2.28 $\zeta(z_3) = -\zeta(2\omega_2 - z_3) + 2(\eta + \eta')$

$$\zeta(z_3) = -\zeta(2\omega_2 - z_3) + 2(\eta + \eta')$$

$$18.2.29 \quad \sigma(z_3) = \sigma(2\omega_2 - z_3) \exp [2(\eta + \eta')(z_3 - \omega_3)]$$

$$\sigma(z_3) = \sigma(2\omega_2 - z_3) \exp [2(\eta + \eta')(z_3 - \omega_2)]$$

18.2.30 $\varphi'(z) = \overline{\varphi'(z - 2\omega')}$

Point $\mathbf{z}_2$ in R_2

$$\mathcal{P}'(z) = \overline{\mathcal{P}'}(\bar{z})$$

18.2.31 $\varphi(z) = \overline{\varphi(z - 2\omega')}$

$$P(z_0) = \overline{P}(\overline{z_0})$$

18.2.32 $\zeta(z_2) = \overline{\zeta(\overline{z_2 - 2\omega'})} + 2\eta'$

$$\zeta(z_2) = \overline{\zeta(\overline{z_2})}$$

18.2.33 $\sigma(z_2) = -\overline{\sigma(z_2 - 2\omega') \exp [2\eta'(z_2 - \omega')]}$

$$\sigma(z_2) = \overline{\sigma(\overline{z_2})}$$

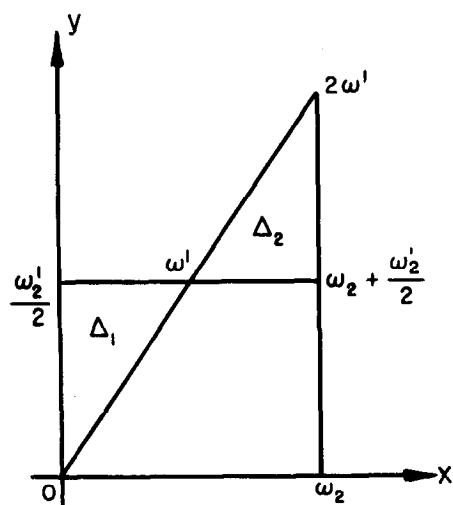


FIGURE 18.3

Reduction from $\frac{1}{4}$ FPP to Fundamental Rectangle in Case $\Delta < 0$

We need only be concerned with the case when z is in triangle Δ_2 (therefore $2\omega' - z$ is in triangle Δ_1).

$$18.2.34 \quad \mathcal{P}(z) = \mathcal{P}(2\omega' - z)$$

$$18.2.35 \quad \mathcal{P}'(z) = -\mathcal{P}'(2\omega' - z)$$

$$18.2.36 \quad \zeta(z) = 2\eta' - \zeta(2\omega' - z)$$

$$18.2.37 \quad \sigma(z) = \sigma(2\omega' - z) \exp [2\eta'(z - \omega')]$$

Reduction to Case where Real Half-Period is Unity

(preserving period ratio)

$$\Delta > 0$$

$$\Delta < 0$$

$$(\omega_2 = \omega + \omega')$$

$$18.2.38 \quad \mathcal{P}'(z|\omega, \omega') = \omega^{-3} \mathcal{P}'\left(z\omega^{-1} \middle| 1, \frac{\omega'}{\omega}\right) \quad \mathcal{P}'(z|\omega, \omega') = \omega_2^{-3} \mathcal{P}'\left(z\omega_2^{-1} \middle| \frac{\omega}{\omega_2}, \frac{\omega'}{\omega_2}\right)$$

$$18.2.39 \quad \mathcal{P}(z|\omega, \omega') = \omega^{-2} \mathcal{P}\left(z\omega^{-1} \middle| 1, \frac{\omega'}{\omega}\right) \quad \mathcal{P}(z|\omega, \omega') = \omega_2^{-2} \mathcal{P}\left(z\omega_2^{-1} \middle| \frac{\omega}{\omega_2}, \frac{\omega'}{\omega_2}\right)$$

$$18.2.40 \quad \zeta(z|\omega, \omega') = \omega^{-1} \zeta\left(z\omega^{-1} \middle| 1, \frac{\omega'}{\omega}\right) \quad \zeta(z|\omega, \omega') = \omega_2^{-1} \zeta\left(z\omega_2^{-1} \middle| \frac{\omega}{\omega_2}, \frac{\omega'}{\omega_2}\right)$$

$$18.2.41 \quad \sigma(z|\omega, \omega') = \omega \sigma\left(z\omega^{-1} \middle| 1, \frac{\omega'}{\omega}\right) \quad \sigma(z|\omega, \omega') = \omega_2 \sigma\left(z\omega_2^{-1} \middle| \frac{\omega}{\omega_2}, \frac{\omega'}{\omega_2}\right)$$

$$18.2.42 \quad g_2(\omega, \omega') = \omega^{-4} g_2\left(1, \frac{\omega'}{\omega}\right) \quad g_2(\omega, \omega') = \omega_2^{-4} g_2\left(\frac{\omega}{\omega_2}, \frac{\omega'}{\omega_2}\right)$$

$$18.2.43 \quad g_3(\omega, \omega') = \omega^{-6} g_3\left(1, \frac{\omega'}{\omega}\right) \quad g_3(\omega, \omega') = \omega_2^{-6} g_3\left(\frac{\omega}{\omega_2}, \frac{\omega'}{\omega_2}\right)$$

$$18.2.44 \quad e_i(\omega, \omega') = \omega^{-2} e_i\left(1, \frac{\omega'}{\omega}\right) \quad e_i(\omega, \omega') = \omega_2^{-2} e_i\left(\frac{\omega}{\omega_2}, \frac{\omega'}{\omega_2}\right)$$

$$(i=1, 2, 3)$$

$$(i=1, 2, 3)$$

$$18.2.45 \quad \Delta(\omega, \omega') = \omega^{-12} \Delta\left(1, \frac{\omega'}{\omega}\right) \quad \Delta(\omega, \omega') = \omega_2^{-12} \Delta\left(\frac{\omega}{\omega_2}, \frac{\omega'}{\omega_2}\right)$$

NOTE: New real half-period is

$$\frac{\omega}{\omega_2} + \frac{\omega'}{\omega_2} = \frac{\omega + \omega'}{\omega_2} = 1$$

18.3. Special Values and Relations

Values at Periods

$\mathcal{P}$, $\mathcal{P}'$, and ζ are infinite, σ is zero at $z=2\omega_i$, $i=1, 2, 3$ and at $2\omega'_i$ ($\Delta < 0$).

$$\Delta > 0$$

$$\Delta < 0$$

Half-Periods

$$18.3.1 \quad \mathcal{P}(\omega_i) = e_i \quad (i=1, 2, 3)$$

$$18.3.2 \quad \mathcal{P}'(\omega_i) = 0 \quad (i=1, 2, 3)$$

$$18.3.3 \quad \eta_i = \zeta(\omega_i) \quad (i=1, 2, 3)$$

$$18.3.4 \quad \eta_1 = \eta, \eta_2 = \eta + \eta', \eta_3 = \eta'$$

$$18.3.5 \quad H_i^2 = 2e_i^2 + e_j e_k \quad (i, j, k=1, 2, 3; i \neq j, i \neq k, j \neq k)$$

$$18.3.6 \quad = (e_i - e_j)(e_i - e_k) = 2e_i^2 + \frac{g_3}{4e_i} = 3e_i^2 - \frac{g_2}{4}$$

$$18.3.7 \quad e_i \text{ real} \quad e_2 \text{ real and non-negative}$$

$$18.3.8 \quad e_1 > 0 \geq e_2 > e_3 \quad (e_2 = 0 \text{ when } g_3 = 0)$$

$$(\text{equality when } g_3 = 0) \quad e_1 = -\alpha + i\beta, e_3 = \bar{e}_1$$

$$\text{where } \alpha \geq 0, \beta > 0$$

$$(\text{equality when } g_3 = 0)$$

$$18.3.9 \quad \eta > 0 \quad \eta'_2 = \zeta(\omega'_2) = \eta' - \eta$$

$$18.3.10 \quad \eta'/i \leq 0 \text{ if } \eta_2 > 0$$

$$18.3.11 \quad |\omega'|/\omega \leq 1.91014 \ 050 \text{ (approx.)} \quad \eta'_2/i \leq 0 \text{ if } |\omega'_2|/\omega_2 \leq 3.81915 \ 447 \text{ (approx.)}$$

$$18.3.12 \quad H_1 > 0, H_3 > 0 \quad H_2 > 0$$

$$18.3.13 \quad H_2 = i\sqrt{-H_3^2} \quad \pi/4 < \arg(H_3) \leq \pi/2 \text{ (equality if } g_3 = 0); H_1 = \bar{H}_3$$

$$18.3.14 \quad \sigma(\omega) = e^{\eta\omega/2}/H_1^{1/2} \quad \sigma(\omega_2) = e^{\eta_2\omega_2/2}/H_2^{1/2}$$

$$18.3.15 \quad \sigma(\omega') = ie^{\eta'\omega'/2}/H_3^{1/2} \quad \sigma(\omega'_2) = ie^{\eta'_2\omega'_2/2}/H_3^{1/2}$$

$$18.3.16 \quad \sigma^2(\omega_2) = e^{\eta_2\omega_2}/(-H_2) \quad \sigma^2(\omega') = e^{\eta'\omega'}/(-H_3)$$

$$18.3.17 \quad \arg[\sigma(\omega_2)] = \frac{\eta'\omega}{i} + \frac{\pi}{2} \quad \arg[\sigma(\omega')] = \frac{\eta'_2\omega_2}{4i} + \frac{\pi}{2} - \frac{1}{2} \arg(e_2 + H_2 - e_i)$$

Quarter Periods

$$18.3.18 \quad \mathcal{P}(\omega/2) = e_1 + H_1 > e_1 \quad \mathcal{P}(\omega_2/2) = e_2 + H_2 > e_2$$

$$18.3.19 \quad \mathcal{P}'(\omega/2) = -2H_1\sqrt{2H_1+3e_1} \quad \mathcal{P}'(\omega_2/2) = -2H_2\sqrt{2H_2+3e_2}$$

$$18.3.20 \quad \zeta(\omega/2) = \frac{1}{2}[\eta + \sqrt{2H_1+3e_1}] \quad \zeta(\omega_2/2) = \frac{1}{2}[\eta_2 + \sqrt{2H_2+3e_2}]$$

$$\Delta > 0$$

$$\Delta < 0$$

$$18.3.21 \quad \sigma(\omega/2) = \frac{e^{\eta\omega/8}}{2^{1/4}H_1^{3/8}(2H_1+3e_1)^{1/8}}$$

$$\sigma(\omega_2/2) = \frac{e^{\eta_2\omega_2/8}}{2^{1/4}H_2^{3/8}(2H_2+3e_2)^{1/8}}$$

$$18.3.22 \quad \mathcal{P}(\omega'/2) = e_3 - H_3 < e_3 < 0$$

$$\mathcal{P}(\omega'_2/2) = e_2 - H_2 = \mathcal{P}(\omega_2 + \omega'_2/2) < e_2 < 0$$

$$18.3.23 \quad \mathcal{P}'(\omega'/2) = -2H_3i\sqrt{2H_3-3e_3}$$

$$\mathcal{P}'(\omega'_2/2) = -2H_2i\sqrt{2H_2-3e_2} = \overline{\mathcal{P}'}(\omega_2 + \omega'_2/2)$$

$$18.3.24 \quad \zeta(\omega'/2) = \frac{1}{2}[\eta' - i\sqrt{2H_3-3e_3}]$$

$$\zeta(\omega'_2/2) = \frac{1}{2}[\eta'_2 - i\sqrt{2H_2-3e_2}] = -\zeta(\omega_2 + \omega'_2/2) + 2\eta'$$

$$18.3.25 \quad \sigma(\omega'/2) = \frac{ie^{\eta'\omega'/8}}{2^{1/4}H_3^{3/8}(2H_3-3e_3)^{1/8}}$$

$$\sigma(\omega'_2/2) = \frac{ie^{\eta'_2\omega'_2/8}}{2^{1/4}H_2^{3/8}(2H_2-3e_2)^{1/8}}$$

$$= \sigma(\omega_2 + \omega'_2/2) \exp[-\eta'\omega_2]$$

$$18.3.26 \quad \mathcal{P}(\omega_2/2) = e_2 - H_2$$

$$\mathcal{P}(\omega'/2) = e_3 - H_3$$

$$18.3.27 \quad \mathcal{P}'(\omega_2/2) = -2H_2i(2H_2-3e_2)^{1/2}$$

$$\mathcal{P}'(\omega'/2) = -2iH_3(2H_3-3e_3)^{1/2}$$

$$18.3.28 \quad \zeta(\omega_2/2) = \frac{1}{2}[\eta_2 - i(2H_2-3e_2)^{1/2}]$$

$$\zeta(\omega'/2) = \frac{1}{2}[\eta' - i(2H_3-3e_3)^{1/2}]$$

$$18.3.29 \quad \sigma(\omega_2/2) = \frac{e^{\eta_2\omega_2/8}e^{i\pi/4}}{[4H_2^3(2H_2-3e_2)]^{1/8}}$$

$$\sigma(\omega'/2) = \frac{e^{\eta'\omega'/8}e^{i\pi/4}}{[4H_3^3(2H_3-3e_3)]^{1/8}}$$

One-Third Period Relations

$$\text{At } z=2\omega_i/3 (i=1, 2, 3) \text{ or } 2\omega'_i/3, \mathcal{P}''^2 = 12\mathcal{P}\mathcal{P}';$$

equivalently:

$$18.3.30 \quad 48\mathcal{P}^4 - 24g_2\mathcal{P}^2 - 48g_3\mathcal{P} - g_2^2 = 0$$

$$\Delta > 0$$

$$\Delta < 0$$

$$18.3.31 \quad \zeta(2\omega/3) = \frac{2\eta}{3} + \left[\frac{\mathcal{P}(2\omega/3)}{3} \right]^{1/2}$$

$$\zeta(2\omega_2/3) = \frac{2\eta_2}{3} + \left[\frac{\mathcal{P}(2\omega_2/3)}{3} \right]^{1/2}$$

$$18.3.32 \quad \zeta(2\omega'/3) = \frac{2\eta'}{3} - \left[\frac{\mathcal{P}(2\omega'/3)}{3} \right]^{1/2}$$

$$\zeta(2\omega'_2/3) = \frac{2\eta'_2}{3} - \left[\frac{\mathcal{P}(2\omega'_2/3)}{3} \right]^{1/2}$$

$$18.3.33 \quad \zeta(2\omega_2/3) = \frac{2\eta_2}{3} + \left[\frac{\mathcal{P}(2\omega_2/3)}{3} \right]^{1/2}$$

$$\zeta(2\omega'/3) = \frac{2\eta'}{3} + \left[\frac{\mathcal{P}(2\omega'/3)}{3} \right]^{1/2}$$

$$18.3.34 \quad \sigma(2\omega/3) = \frac{-\exp[2\eta\omega/9]}{\sqrt[3]{\mathcal{P}'(2\omega/3)}}$$

$$\sigma(2\omega_2/3) = \frac{-\exp[2\eta_2\omega_2/9]}{\sqrt[3]{\mathcal{P}'(2\omega_2/3)}}$$

$$18.3.35 \quad \sigma(2\omega'/3) = \frac{-\exp[2\eta'\omega'/9]}{[\mathcal{P}'(2\omega'/3)]^{1/3}e^{2\pi i/3}}$$

$$\sigma(2\omega'_2/3) = \frac{-\exp[2\eta'_2\omega'_2/9]}{[\mathcal{P}'(2\omega'_2/3)]^{1/3}e^{2\pi i/3}}$$

$$18.3.36 \quad \sigma(2\omega_2/3) = \frac{-\exp[2\eta_2\omega_2/9]}{[\mathcal{P}'(2\omega_2/3)]^{1/3}e^{2\pi i/3}}$$

$$\sigma(2\omega'/3) = \frac{-\exp[2\eta'\omega'/9]}{[\mathcal{P}'(2\omega'/3)]^{1/3}e^{2\pi i/3}}$$

Legendre's Relation

$$18.3.37 \quad \eta\omega' - \eta'\omega = \pi i/2$$

$$\eta_2\omega'_2 - \eta'_2\omega_2 = \pi i$$

(also valid for $\Delta < 0$)

Relations Among the H_i

$$18.3.38 \quad H_1^2 + H_2^2 + H_3^2 = 3g_2/4$$

$$18.3.39 \quad H_1^2H_2^2 + H_2^2H_3^2 + H_3^2H_1^2 = 0$$

$$18.3.40 \quad H_1^2 H_2^2 H_3^2 = -\Delta/16$$

$$18.3.41 \quad 16H_i^3 - 12g_2H_i^4 + \Delta = 0 (i=1, 2, 3)$$

18.4. Addition and Multiplication Formulas

Addition Formulas² ($z_1 \neq z_2$)

$$18.4.1 \quad \mathcal{P}(z_1 + z_2) = \frac{1}{4} \left[\frac{\mathcal{P}'(z_1) - \mathcal{P}'(z_2)}{\mathcal{P}(z_1) - \mathcal{P}(z_2)} \right]^2 - \mathcal{P}(z_1) - \mathcal{P}(z_2)$$

$$18.4.2 \quad \mathcal{P}'(z_1 + z_2) = \frac{\mathcal{P}(z_1 + z_2)[\mathcal{P}'(z_1) - \mathcal{P}'(z_2)] + \mathcal{P}(z_1)\mathcal{P}'(z_2) - \mathcal{P}'(z_1)\mathcal{P}(z_2)}{\mathcal{P}(z_2) - \mathcal{P}(z_1)}$$

$$18.4.3 \quad \zeta(z_1 + z_2) = \zeta(z_1) + \zeta(z_2) + \frac{1}{2} \frac{\mathcal{P}'(z_1) - \mathcal{P}'(z_2)}{\mathcal{P}(z_1) - \mathcal{P}(z_2)}$$

$$18.4.4 \quad \sigma(z_1 + z_2)\sigma(z_1 - z_2) = -\sigma^2(z_1)\sigma^2(z_2)[\mathcal{P}(z_1) - \mathcal{P}(z_2)]$$

Duplication and Triplication Formulas

$$\left[\text{Note that } \mathcal{P}'' = 6\mathcal{P}^2(z) - \frac{g_2}{2}, \mathcal{P}'^2(z) = 4\mathcal{P}^3(z) - g_2\mathcal{P}(z) - g_3 \text{ and } \mathcal{P}'''(z) = 12\mathcal{P}(z)\mathcal{P}'(z) \right]$$

$$18.4.5 \quad \mathcal{P}(2z) = -2\mathcal{P}(z) + \left[\frac{\mathcal{P}''(z)}{2\mathcal{P}'(z)} \right]^2$$

$$18.4.6 \quad \mathcal{P}'(2z) = \frac{-4\mathcal{P}'^4(z) + 12\mathcal{P}(z)\mathcal{P}'^2(z)\mathcal{P}''(z) - \mathcal{P}''^3(z)}{4\mathcal{P}'^3(z)}$$

$$18.4.7 \quad \zeta(2z) = 2\zeta(z) + \mathcal{P}''(z)/2\mathcal{P}'(z)$$

$$18.4.8 \quad \sigma(2z) = -\mathcal{P}'(z)\sigma^4(z)$$

$$18.4.9 \quad \zeta(3z) = 3\zeta(z) + \frac{4\mathcal{P}'^3(z)}{\mathcal{P}'(z)\mathcal{P}'''(z) - \mathcal{P}''^2(z)}$$

$$18.4.10 \quad \sigma(3z) = -\mathcal{P}'^2(z)\sigma^9(z)[\mathcal{P}(2z) - \mathcal{P}(z)]$$

18.5. Series Expansions

Laurent Series

$$18.5.1 \quad \mathcal{P}(z) = z^{-2} + \sum_{k=2}^{\infty} c_k z^{2k-2}$$

$$18.5.2 \quad \text{where} \quad c_2 = g_2/20, \quad c_3 = g_3/28$$

and

$$18.5.3 \quad c_k = \frac{3}{(2k+1)(k-3)} \sum_{m=2}^{k-2} c_m c_{k-m}, \quad k \geq 4$$

$$18.5.4 \quad \mathcal{P}'(z) = -2z^{-3} + \sum_{k=2}^{\infty} (2k-2)c_k z^{2k-3}$$

$$18.5.5 \quad \zeta(z) = z^{-1} - \sum_{k=2}^{\infty} c_k z^{2k-1}/(2k-1)$$

$$18.5.6 \quad \sigma(z) = \sum_{m,n=0}^{\infty} a_{m,n} \left(\frac{1}{2}g_2\right)^m (2g_3)^n \cdot \frac{z^{4m+6n+1}}{(4m+6n+1)!}$$

² Formulas for ζ and σ are not true algebraic addition formulas.

18.5.7

where $a_{0,0}=1$ and

$$18.5.8 \quad a_{m,n} = 3(m+1)a_{m+1,n-1} + \frac{16}{3}(n+1)a_{m-2,n+1} - \frac{1}{3}(2m+3n-1)(4m+6n-1)a_{m-1,n},$$

it being understood that $a_{m,n}=0$ if either subscript is negative.

(The radius of convergence of the above series for $\mathcal{P}-z^{-2}$, $\mathcal{P}'+2z^{-3}$ and $\zeta-z^{-1}$ is equal to the smallest of $|2\omega|$, $|2\omega'|$ and $|2\omega \pm 2\omega'|$; series for σ converges for all z .)

Values of Coefficients³ c_k in Terms of c_2 and c_3

18.5.9

$$c_4 = c_2^2/3$$

18.5.10

$$c_5 = 3c_2c_3/11$$

18.5.11

$$c_6 = [2c_2^3 + 3c_3^2]/39$$

18.5.12

$$c_7 = 2c_2^2c_3/33$$

18.5.13

$$c_8 = 5c_2(11c_2^3 + 36c_3^2)/7293$$

18.5.14

$$c_9 = c_3(29c_2^3 + 11c_3^2)/2717$$

18.5.15

$$c_{10} = (242c_2^5 + 1455c_2^2c_3^2)/240669$$

18.5.16

$$c_{11} = 14c_2c_3(389c_2^3 + 369c_3^2)/3187041$$

18.5.17

$$c_{12} = (114950c_2^6 + 1080000c_2^3c_3^2 + 166617c_3^4)/891678645$$

18.5.18

$$c_{13} = 10c_2^2c_3(297c_2^3 + 530c_3^2)/11685817$$

18.5.19

$$c_{14} = \frac{2c_2(528770c_2^5 + 7164675c_2^2c_3^2 + 2989602c_3^4)}{(306735)(215441)}$$

18.5.20

$$c_{15} = \frac{4c_3(62921815c_2^6 + 179865450c_2^3c_3^2 + 14051367c_3^4)}{(179685)(38920531)}$$

18.5.21

$$c_{16} = \frac{c_2^2(58957855c_2^6 + 1086511320c_2^3c_3^2 + 875341836c_3^4)}{(5909761)(5132565)}$$

18.5.22

$$c_{17} = \frac{c_2c_3(30171955c_2^6 + 126138075c_2^3c_3^2 + 28151739c_3^4)}{(920205)(6678671)}$$

18.5.23

$$c_{18} = \frac{1541470 \cdot 949003c_2^9 + 30458088737 \cdot 1155c_2^6c_3^2 + 122378650673 \cdot 378c_2^3c_3^4 + 2348703 \cdot 887777c_3^6}{(1342211013)(4695105713)}$$

18.5.24

$$c_{19} = \frac{2c_2^2c_3(3365544215c_2^6 + 429852433 \cdot 45c_2^3c_3^2 + 8527743477c_3^4)}{(91100295)(113537407)}$$

³ NOTES:

1. c_4 - c_{16} were computed and checked independently by D. H. Lehmer; these were double-checked by substituting $g_2=20$, $g_3=28$ c_3 in values given in [18.10].

2. c_{17} - c_{18} were derived from values in [18.10] by the same substitution. These were checked (numerically) for particular values of g_2 , g_3 .

3. c_{19} is given incorrectly in [18.12] (factor 13 is missing in denominator of third term of bracket); this value was computed independently.

4. No factors of any of the above integers with more than ten digits are known to the author. This is not necessarily true of smaller integers, which have, in many instances, been arranged for convenient use with a desk calculator.

Value⁴ of Coefficients $a_{m,n}$

$n \uparrow$	0	1	2	3	4	5	6	7	8	9	10	11	12
8	$^{-7,12}$ -2-3-5-59 107895773												
7	$^{-7,10}$ -2-3-5-23 257-18049	$^{-7,11}$ -2-3-5-59 107895773											
6	$^{-6,9}$ -2-3-5 229-2683	$^{-7,9}$ -2-3-5-23 257 18049	$^{-6,11}$ -2-3-5-7 181-1699 2803	$^{-8,10}$ -2-3-5-7 41-6047 4922497									
5	$^{-5,8}$ 2-3-5 9163	$^{-6,8}$ -2-3-5-229 2683	$^{-5,10}$ -2-3-5 40570423	$^{-6,9}$ -2-3-5-7 59-179 142231	$^{-4,10}$ -2-3-5-7 1321 1415535763								
4	$^{-4,8}$ 2-3-5 31	$^{-5,7}$ 2-3-5-9103 2-3-5-31	$^{-5,8}$ -2-3-5-7 13-37-41	$^{-5,9}$ -2-3-5-691 83609	$^{-4,10,2}$ -2-3-5-11-31 313-190387	$^{-5,9}$ -2-3-5-7-19 263-4848953	$^{-5,9}$ -2-3-5-7-19 1752686144977						
3	$^{-3,4}$ 2-3-23 2-3-23	$^{-3,4,3}$ 2-3-5-31 2-3-5-31	$^{-3,7}$ 2-3-5-17 109	$^{-3,8}$ -2-3-5-83 3911	$^{-4,8}$ -2-3-5-503 156217	$^{-4,8}$ -2-3-31 315989669	$^{-4,7}$ -2-3-5-7-11-29 83-1129-9551	$^{-4,8}$ -2-3-5-7-613 17605225081					
2	$^{-2,3}$ -2-3 -3	$^{-2,3}$ 2-3-23 -2-3	$^{-2,5}$ 2-3-5-53 3-19	$^{-3,4}$ 2-3-5-37 167 3037	$^{-2,3,5}$ -2-3-5-17 3037	$^{-4,7}$ -2-3-61-151 653	$^{-4,7}$ -2-3-5-17-53 2387260103	$^{-4,7}$ -2-3-5-17-53 2957-41189	$^{-2,3,5,7}$ -2-3-5-7-17 67-195651059	$^{-3,7,2}$ 2-3-5-7 35866647631901			
1	$^{-1}$ -1 -1	$^{-2,3}$ -2-3 -1	$^{-3}$ 3-19 -3	$^{-3,3}$ 2-3-311 3-5-20807	$^{-3,5}$ 3-5-20807 3-107	$^{-2,3,11}$ -2-3-11 2609	$^{-3,17}$ -3-17 1578257	$^{-3,5}$ -2-3-7-13 2742587	$^{-7}$ -3 248882935409	$^{-2,3,5,7,193}$ -2-3-5-7-193 13679-274973	$^{-6}$ 3-7 89555603641079		
0	$^{-0}$ 1 0	$^{-1}$ -1 1	$^{-3}$ 3-23 9	$^{-3,23}$ 3-23 68	$^{-3,107}$ 3-107 321	$^{-3,7,23,37}$ 3-7-23-37 160759	$^{-3,313,503}$ 3-313-503 1416951	$^{-3,7}$ -3-7 685973	$^{-3,11,37}$ 3-11-37 257981	$^{-3,7,23,14387}$ -3-7-23-14387 40763	$^{-3,7}$ -3-7 176302760639	$^{-3,7,11,23}$ -3-7-11-23 383-739-18539	$^{-3,7,24733}$ 3-7-24733 198922785511

→ m 38994681

⁴ Values of $a_{m,n}$ in unfactored form for $4m+6n+1 \leq 35$ are given in [18.25], p. 7; of $(a_{m,n})^{3-n}$ in factored form in [18.15], Vol. 4, p. 89 for $4m+6n+1 \leq 25$. Additional values were computed and checked on desk calculators; primality of large factors was established with the aid of SWAC (National Bureau of Standards Western Automatic Computer).

Reversed Series⁵ for Large $|\mathcal{P}|$

18.5.25

$$\begin{aligned}
z = & \frac{1}{2} \left[2u + c_2 u^5 + c_3 u^7 + \frac{\alpha_2^2}{3} u^9 + \frac{6\alpha_2 \alpha_3}{11} u^{11} \right. \\
& + \frac{1}{13} (3\alpha_3^2 + 5\alpha_2^3) u^{13} + \frac{\alpha_2^2 \alpha_3}{68} (12\alpha_3^2 + 7\alpha_2^3) u^{17} \\
& + \frac{5\alpha_3}{19} (\alpha_3^2 + 7\alpha_2^3) u^{19} + \frac{\alpha_2^2}{4} (3\alpha_3^3 + 10\alpha_2^3) u^{21} \\
& + \frac{35\alpha_2 \alpha_3}{92} (9\alpha_2^3 + 4\alpha_3^2) u^{23} \\
& + \frac{7}{200} (33\alpha_2^6 + 180\alpha_2^3 \alpha_3^2 + 10\alpha_3^4) u^{25} \\
& + \frac{7\alpha_2^2 \alpha_3}{12} (11\alpha_2^3 + 10\alpha_3^2) u^{27} \\
& + \frac{3\alpha_2}{2^3 \cdot 29} (143\alpha_2^6 + 1155\alpha_2^3 \alpha_3^2 + 210\alpha_3^4) u^{29} \\
& + \frac{21\alpha_3}{2^3 \cdot 31} (143\alpha_2^6 + 220\alpha_2^3 \alpha_3^2 + 6\alpha_3^4) u^{31} \\
& + \frac{3\alpha_2^2}{2^6} (65\alpha_2^6 + 728\alpha_2^3 \alpha_3^2 + 280\alpha_3^4) u^{33} \\
& + \frac{33\alpha_2 \alpha_3}{2^3 \cdot 5 \cdot 7} (195\alpha_2^6 + 455\alpha_2^3 \alpha_3^2 + 42\alpha_3^4) u^{35} \\
& + \frac{11}{2^6 \cdot 37} (1105\alpha_2^9 + 16380\alpha_2^6 \alpha_3^2 + 10920\alpha_2^3 \alpha_3^4 \\
& + 168\alpha_3^6) u^{37} + \frac{33\alpha_2^2 \alpha_3}{2^6} (85\alpha_2^6 + 280\alpha_2^3 \alpha_3^2 + 56\alpha_3^4) u^{39} \\
& + \frac{143\alpha_2}{2^7 \cdot 41} (323\alpha_2^9 + 6120\alpha_2^6 \alpha_3^2 + 6300\alpha_2^3 \alpha_3^4 + 336\alpha_3^6) u^{41} \\
& + \frac{143\alpha_3}{2^6 \cdot 43} (1615\alpha_2^9 + 7140\alpha_2^6 \alpha_3^2 + 2520\alpha_2^3 \alpha_3^4 + 24\alpha_3^6) u^{43} \\
& \left. + O(u^{45}) \right],
\end{aligned}$$

18.5.26 where $\alpha_2 = g_2/8$

18.5.27 $\alpha_3 = g_3/8$

18.5.28 $u = (\mathcal{P}^{-1})^{\frac{1}{2}}$

Reversed Series for Large $|\mathcal{P}'|$

18.5.29 $z = A_1 u + A_5 u^5 + A_7 u^7 + A_9 u^9 + \dots$

18.5.30 where $u = (\mathcal{P}'^{1/3})^{-1} e^{i\tau/3}$

18.5.31 $A_1 = 2^{1/3}$

18.5.32 $A_5 = -\frac{a_2}{5} A_1^2$

18.5.33 $A_7 = \frac{-4a_3 A_1}{7}$

18.5.34 $A_9 = 0$

18.5.35 $A_{11} = 8a_2 a_3 A_1^2/11$

18.5.36 $A_{13} = \frac{10A_1}{39} (a_2^3 + 6a_3^2)$

18.5.37 $A_{15} = -96a_2^2 a_3/175$

18.5.38 $A_{17} = -\frac{14a_2 A_1^2}{51} (a_2^3 + 12a_3^2)$

18.5.39 where $a_2 = g_2/6$, $a_3 = g_3/6$

Reversed Series for Large $|\zeta|$

18.5.40 $z = u + A_5 u^5 + A_7 u^7 + A_9 u^9 + \dots$

18.5.41 where $u = \zeta^{-1}$

18.5.42 $A_5 = -\delta_2/5$

18.5.43 $A_7 = -\delta_3/7$

18.5.44 $A_9 = \delta_2^2/7$

18.5.45 $A_{11} = 3\delta_2 \delta_3/11$

18.5.46 $A_{13} = \frac{17}{1001} (-8\delta_2^3 + 7\delta_3^2)$

18.5.47 $A_{15} = -41\delta_2^2 \delta_3/91$

18.5.48 $A_{17} = \frac{\delta_2}{9163} (1349\delta_2^2 - 4116\delta_3^2)$

18.5.49 $A_{19} = \frac{2\delta_3}{323323} (115431\delta_2^2 - 22568\delta_3^2)$

18.5.50 where $\delta_2 = g_2/12$

18.5.51 $\delta_3 = g_3/20$

⁵ In this and other series a choice of the value of the root has been made so that z will be in the Fundamental Rectangle (Figure 18.2), whenever the value of the given function is appropriate.

Other Series Involving $\mathcal{P}$ Series near z_0 [$\mathcal{P}(z_0)=0$]

18.5.52

$$\begin{aligned}\mathcal{P} = \mathcal{P}'_0 u \left[1 - 3c_2 u^4 - 4c_3 u^6 + \frac{10c_2^2}{3} u^8 + \frac{114c_2 c_3}{11} u^{10} \right. \\ \left. + \frac{7(12c_3^2 - 5c_2^3)}{13} u^{12} - \frac{488c_2^2 c_3}{33} u^{14} \right] + u^2 \left[-5c_2 - 14c_3 u^2 \right. \\ \left. + 5c_2^2 u^4 + 33c_2 c_3 u^6 + \frac{84c_2^3 - 10c_2^3}{3} u^8 - \frac{1363c_2^2 c_3}{33} u^{10} \right. \\ \left. + \frac{5c_2(55c_2^3 - 2316c_3^2)u^{12}}{143} \right] + \dots\end{aligned}$$

18.5.53

where $u = (z - z_0)$, $\mathcal{P}'_0 = \mathcal{P}'(z_0) = i\sqrt{g_3}$

18.5.54

$$\begin{aligned}u = \mathcal{P}'_0 \left[v + av^2 + 2a^2 v^3 + \left(\frac{g_3 \mathcal{P}'_0{}^2}{2} + 5a^3 \right) v^4 + \frac{a}{5} (3 \mathcal{P}'_0{}^4 \right. \\ \left. + 15g_3 \mathcal{P}'_0{}^2 + 70a^3) v^5 + 2a^2 (2 \mathcal{P}'_0{}^4 + 7g_3 \mathcal{P}'_0{}^2 + 21a^3) v^6 \right. \\ \left. + \left(\frac{g_3 \mathcal{P}'_0{}^6}{7} + \{g_3^2 + 20a^3\} \mathcal{P}'_0{}^4 + 15a^2 g_3 \mathcal{P}'_0{}^2 + 132a^6 \right) v^7 \right. \\ \left. + 15a \left(\frac{g_3 \mathcal{P}'_0{}^6}{4} + \left\{ \frac{3g_3^2}{4} + 6a^3 \right\} \mathcal{P}'_0{}^4 + \frac{33ag_3}{2} \mathcal{P}'_0{}^2 \right. \right. \\ \left. \left. + \frac{143a^6}{5} \right) v^8 + \frac{5a^2}{2} \left(\frac{2}{3} \mathcal{P}'_0{}^8 + 15g_3 \mathcal{P}'_0{}^6 \right. \right. \\ \left. \left. + \{154a^3 + 33g_3^2\} \mathcal{P}'_0{}^4 + \frac{2002a^3 g_3 \mathcal{P}'_0{}^2}{5} + 572a^6 \right) v^9 \right. \\ \left. + \frac{1}{4} \left(3 \{28a^3 + g_3^2\} \mathcal{P}'_0{}^8 + 11g_3 \{98a^3 + g_3^2\} \mathcal{P}'_0{}^6 \right. \right. \\ \left. \left. + 2002a^3 \left\{ \frac{16}{5} a^3 + g_3^2 \right\} \mathcal{P}'_0{}^4 \right. \right. \\ \left. \left. + 16016 a^6 g_3 \mathcal{P}'_0{}^2 + 19448 a^9 \right) v^{10} \right] + \dots\end{aligned}$$

18.5.55 where $v = \mathcal{P} / (\mathcal{P}'_0)^2$ and $a = g_2/4$ Series near ω_i

18.5.56

$$\begin{aligned}(\mathcal{P} - e_i) = (3e_i^2 - 5c_2)u + (10c_2 e_i + 21c_3)u^2 + (7c_2 e_i^2 \\ + 21c_3 e_i + 5c_2^2)u^3 + (18c_3 e_i^2 + 30c_2^2 e_i \\ + 33c_2 c_3)u^4 + (22c_2^2 e_i^2 + 92c_2 c_3 e_i + 105c_3^2 \\ - \frac{10c_2^3}{3})u^5 + \left(\frac{728}{11} c_2 c_3 e_i^2 + \frac{220}{3} c_2^2 e_i + 84c_3^2 e_i \right. \\ \left. + \frac{1214}{11} c_2^2 c_3 \right)u^6 + \left(\frac{635}{13} c_2^2 e_i^2 + \frac{855}{13} c_2^2 e_i^2 \right. \\ \left. + \frac{3405}{11} c_2^2 c_3 e_i + \frac{45750}{143} c_2 c_3^2 + \frac{25}{13} c_2^3 \right)u^7 + \dots,\end{aligned}$$

18.5.57

where $u = (z - \omega_i)^2$ Other Series Involving $\mathcal{P}'$ Series near z_0

18.5.58

$$\begin{aligned}(\mathcal{P}' - \mathcal{P}'_0) = \left[-10c_2 u - 56c_3 u^3 + 30c_2^2 u^5 + 264c_2 c_3 u^7 \right. \\ \left. + \frac{(840c_3^2 - 100c_2^3)}{3} u^9 - \frac{5452c_2^2 c_3}{11} u^{11} \right. \\ \left. + \frac{70c_2(55c_2^3 - 2316c_3^2)}{143} u^{13} \right] \\ + \mathcal{P}'_0 \left[-15c_2 u^4 - 28c_3 u^6 + 30c_2^2 u^8 + 114c_2 c_3 u^{10} \right. \\ \left. + 7(12c_3^2 - 5c_2^3)u^{12} - \frac{2440c_2^2 c_3}{11} u^{14} \right] + \dots\end{aligned}$$

18.5.59

where $u = (z - z_0)$

18.5.60

$$\begin{aligned}(z - z_0) = A - bA^3 - \frac{3\mathcal{P}'_0}{2} A^4 + 3(c_2 + b^2)A^5 \\ + 10b \mathcal{P}'_0 A^6 - 3[36c_3 - 3\mathcal{P}'_0 + 4b^3]A^7 \\ - 3\mathcal{P}'_0 \left(\frac{25}{2} c_2 + 21b^2 \right) A^8 + \frac{5}{12} (285b^2 c_2 \\ + 100c_2^2 - 279\mathcal{P}'_0{}^2 b + 132b^4) A^9 + \dots\end{aligned}$$

18.5.61

where $A = (\mathcal{P}' - \mathcal{P}'_0)/(-10c_2)$

18.5.62

and $b = 4g_3/g_2$ Series near ω_i

18.5.63

$$\begin{aligned}\mathcal{P}' = 2(3e_i^2 - 5c_2)\alpha + 4(10c_2 e_i + 21c_3)\alpha^3 + 6(7c_2 e_i^2 \\ + 21c_3 e_i + 5c_2^2)\alpha^5 + 24(6c_3 e_i^2 + 10c_2^2 e_i \\ + 11c_2 c_3)\alpha^7 + 10 \left(22c_2^2 e_i^2 + 92c_2 c_3 e_i + 105c_3^2 \right. \\ \left. - \frac{10c_2^3}{3} \right) \alpha^9 + 24 \left(\frac{364}{11} c_2 c_3 e_i^2 + \frac{110}{3} c_2^2 e_i \right. \\ \left. + 42c_3^2 e_i + \frac{607}{11} c_2^2 c_3 \right) \alpha^{11} + 70 \left(\frac{127}{13} c_2^2 e_i^2 \right. \\ \left. + \frac{171}{13} c_2^2 e_i^2 + \frac{681}{11} c_2^2 c_3 e_i + \frac{9150}{143} c_2 c_3^2 + \frac{5}{13} c_2^3 \right) \alpha^{13} \\ + \dots,\end{aligned}$$

18.5.64

where $\alpha = (z - \omega_i)$.

Other Series Involving ζ Series near z_0 [$\mathcal{P}(z_0)=0$]

18.5.65

$$\zeta - \zeta_0 = \mathcal{P}'_0 \left[-\frac{u^2}{2} + \frac{c_2 u^6}{2} + \frac{c_3 u^8}{2} - \frac{c_2^2 u^{10}}{3} - \frac{19c_2 c_3 u^{12}}{22} \right. \\ \left. + \frac{(5c_2^3 - 12c_3^2)}{26} u^{14} + \frac{61c_2^2 c_3 u^{16}}{66} \right] + \left[\frac{5c_2 u^3}{3} \right. \\ \left. + \frac{7c_3 u^5}{2} - \frac{5c_2^2 u^7}{7} - \frac{11c_2 c_3 u^9}{3} + \frac{(10c_2^3 - 84c_3^2)}{33} u^{11} \right. \\ \left. + \frac{1363c_2^2 c_3}{429} u^{13} + \frac{c_2(2316c_3^2 - 55c_2^3)}{429} u^{15} \right] + \dots,$$

18.5.66 where $u = (z - z_0)$,

18.5.67 $\zeta_0 \equiv \zeta(z_0)$

Series near ω_i

18.5.68

$$(\zeta - \eta_i) = -e_i \alpha - \frac{(3e_i^2 - 5c_2)}{3} \alpha^3 - \frac{(10c_2 e_i + 21c_3) \alpha^5}{5} \\ - \frac{(7c_2 e_i^2 + 21c_3 e_i + 5c_2^2) \alpha^7}{7} \\ - \frac{(6c_3 e_i^2 + 10c_2^2 e_i + 11c_2 c_3) \alpha^9}{3} \\ - \frac{\left(22c_2^2 e_i^2 + 92c_2 c_3 e_i + 105c_3^2 - \frac{10}{3} c_2^3 \right) \alpha^{11}}{11} \\ - \frac{2}{13} \left(\frac{364}{11} c_2 c_3 e_i^2 + \frac{110}{3} c_2^2 e_i + 42c_3^2 e_i \right. \\ \left. + \frac{607}{11} c_2^2 c_3 \right) \alpha^{13} - \frac{1}{3} \left(\frac{127}{13} c_2^3 e_i^2 + \frac{171}{13} c_2^2 e_i^2 \right. \\ \left. + \frac{681}{11} c_2^2 c_3 e_i + \frac{9150}{143} c_2 c_3^2 + \frac{5}{13} c_2^4 \right) \alpha^{15} - \dots,$$

18.5.69

where $\alpha = (z - \omega_i)$ Reversed Series for Small $|\sigma|$

18.5.70

$$z = \sigma + \frac{\gamma_2}{5} \sigma^5 + \frac{\gamma_3}{7} \sigma^7 + \frac{3\gamma_2^2}{14} \sigma^9 \\ + \frac{19\gamma_2 \gamma_3}{55} \sigma^{11} + \frac{3842\gamma_2^2 + 861\gamma_3^2}{6006} \sigma^{13} + \dots,$$

18.5.71 where $\gamma_2 = g_2/48$

18.5.72 $\gamma_3 = g_3/120$

For reversion of Maclaurin series, see 3.6.25 and [18.18].

18.6. Derivatives and Differential Equations

Ordinary ($c_2 = g_2/20$, $c_3 = g_3/28$)

18.6.1 $\zeta'(z) = -\mathcal{P}(z)$

18.6.2 $\sigma'(z)/\sigma(z) = \zeta(z)$

18.6.3

$\mathcal{P}''(z) = 4\mathcal{P}^3(z) - g_2\mathcal{P}(z) - g_3 = 4(\mathcal{P}^3 - 5c_2\mathcal{P} - 7c_3)$

18.6.4 $\mathcal{P}''(z) = 6\mathcal{P}^2(z) - \frac{1}{2}g_2 = 6\mathcal{P}^2 - 10c_2$

18.6.5 $\mathcal{P}'''(z) = 12\mathcal{P}\mathcal{P}'$

18.6.6

$$\mathcal{P}^{(4)}(z) = 12(\mathcal{P}\mathcal{P}'' + \mathcal{P}'\mathcal{P}') \\ = 5! \left[\mathcal{P}^3 - 3c_2\mathcal{P} - \frac{14c_3}{5} \right]$$

18.6.7

$$\mathcal{P}^{(5)}(z) = 12(\mathcal{P}\mathcal{P}''' + 2\mathcal{P}'\mathcal{P}'' + \mathcal{P}''\mathcal{P}') \\ = 3 \cdot 5! \mathcal{P}'[\mathcal{P}^2 - c_2]$$

18.6.8

$$\mathcal{P}^{(6)}(z) = 12(\mathcal{P}\mathcal{P}^{(4)} + 3\mathcal{P}'\mathcal{P}''' + 3\mathcal{P}''\mathcal{P}'' \\ + \mathcal{P}'''\mathcal{P}')$$

18.6.9 $= 7! [\mathcal{P}^4 - 4c_2\mathcal{P}^2 - 4c_3\mathcal{P} + 5c_2^2/7]$

18.6.10 $\mathcal{P}^{(7)}(z) = 4 \cdot 7! \mathcal{P}'[\mathcal{P}^3 - 2c_2\mathcal{P} - c_3]$

18.6.11

$$\mathcal{P}^{(8)}(z) = 9! [\mathcal{P}^5 - 5c_2\mathcal{P}^3 - 5c_3\mathcal{P}^2 \\ + (10c_2^2\mathcal{P} + 11c_2c_3)/3]$$

18.6.12

$\mathcal{P}^{(9)}(z) = 5 \cdot 9! \mathcal{P}'[\mathcal{P}^4 - 3c_2\mathcal{P}^2 - 2c_3\mathcal{P} + 2c_2^2/3]$

18.6.13

$$\mathcal{P}^{(10)}(z) = 11! [\mathcal{P}^6 - 6c_2\mathcal{P}^4 - 6c_3\mathcal{P}^3 + 7c_2^2\mathcal{P}^2 \\ + (342c_2c_3\mathcal{P} + 84c_2^3 - 10c_3^2)/33]$$

18.6.14

$$\mathcal{P}^{(11)}(z) = 6 \cdot 11! \mathcal{P}'[\mathcal{P}^5 - 4c_2\mathcal{P}^3 - 3c_3\mathcal{P}^2 \\ + (77c_2^2\mathcal{P} + 57c_2c_3)/33]$$

18.6.15

$$\mathcal{P}^{(12)}(z) = 13! [\mathcal{P}^7 - 7c_2\mathcal{P}^5 - 7c_3\mathcal{P}^4 + 35c_2^2\mathcal{P}^3/3 \\ + 210c_2c_3\mathcal{P}^2/11 + (84c_2^3 - 35c_3^2)\mathcal{P}/13 - 1363c_2^2c_3/429]$$

18.6.16

$$\mathcal{P}^{(13)}(z) = 7 \cdot 13! \mathcal{P}'[\mathcal{P}^6 - 5c_2\mathcal{P}^4 - 4c_3\mathcal{P}^3 \\ + 5c_2^2\mathcal{P}^2 + 60c_2c_3\mathcal{P}/11 + (12c_2^3 - 5c_3^2)/13]$$

18.6.17

$$\mathcal{P}^{(14)}(z) = 15! [\mathcal{P}^8 - 8c_2\mathcal{P}^6 - 8c_3\mathcal{P}^5 + 52c_2^2\mathcal{P}^4/3 \\ + 328c_2c_3\mathcal{P}^3/11 + (444c_2^3 - 328c_3^2)\mathcal{P}^2/39 \\ - 488c_2^2c_3\mathcal{P}/33 + c_2(55c_2^3 - 2316c_3^2)/429]$$

18.6.18

$$\mathcal{P}^{(15)}(z) = 8 \cdot 15! \mathcal{P}'[\mathcal{P}^7 - 6c_2\mathcal{P}^5 - 5c_3\mathcal{P}^4 + 26c_2^2\mathcal{P}^3/3 \\ + 123c_2c_3\mathcal{P}^2/11 + (111c_2^3 - 82c_3^2)\mathcal{P}/39 - 61c_2^2c_3/33]$$

Partial Derivatives with Respect to Invariants

18.6.19

$$\Delta \frac{\partial \mathcal{P}}{\partial g_3} = \mathcal{P}' \left(3g_2 \zeta - \frac{9}{2} g_3 z \right) + 6g_2 \mathcal{P}^2 - 9g_3 \mathcal{P} - g_2^2$$

18.6.20

$$\Delta \frac{\partial \mathcal{P}}{\partial g_2} = \mathcal{P}' \left(-\frac{9}{2} g_3 \zeta + \frac{g_2^2 z}{4} \right) - 9g_3 \mathcal{P}^2 + \frac{g_2^2}{2} \mathcal{P} + \frac{3}{2} g_2 g_3$$

18.6.21

$$\Delta \frac{\partial \zeta}{\partial g_3} = -3\zeta \left(g_2 \mathcal{P} + \frac{3}{2} g_3 \right) + \frac{1}{2} z \left(9g_3 \mathcal{P} + \frac{1}{2} g_2^2 \right) - \frac{3}{2} g_2 \mathcal{P}'$$

18.6.22

$$\Delta \frac{\partial \zeta}{\partial g_2} = \frac{1}{2} \zeta \left(9g_3 \mathcal{P} + \frac{1}{2} g_2^2 \right) - \frac{1}{2} g_2 z \left(\frac{1}{2} g_2 \mathcal{P} + \frac{3}{4} g_3 \right) + \frac{9}{4} g_3 \mathcal{P}'$$

$$18.6.23 \quad \Delta \frac{\partial \sigma}{\partial g_3} = \frac{3}{2} g_2 \sigma'' + \frac{9}{2} g_3 \sigma + \frac{1}{8} g_2^2 z^2 \sigma - \frac{9}{2} g_2 z \sigma'$$

18.6.24

$$\Delta \frac{\partial \sigma}{\partial g_2} = -\frac{9}{4} g_3 \sigma'' - \frac{1}{4} g_2^2 \sigma - \frac{3}{16} g_2 g_3 z^2 \sigma + \frac{1}{4} g_2^2 z \sigma'$$

(here ' denotes $\frac{\partial}{\partial z}$)

Differential Equations

18.6.25

Equation

Solution

$$y'^3 = y^2(y-a)^2 \quad y = \frac{a}{2} + \frac{27}{16} \mathcal{P}' \left(\frac{z}{2}; 0, -\frac{64a^2}{729} \right)$$

18.6.26

$$y'^3 = (y^3 - 3ay^2 + 3y)^2 \quad y = \frac{2}{a-3} \mathcal{P}'(z; 0, g_3),$$

$$g_3 = \frac{4-3a^2}{27}$$

18.6.27

$$y'^4 = \frac{128}{3} (y+a)^2 (y+b)^3 \quad y = 6 \mathcal{P}^2(z; g_2, 0) - b,$$

$$g_2 = -\frac{2}{3} (a-b)$$

$$y'' = [a \mathcal{P}(z) + b]y \text{ (Lamé's equation)—see [18.8], 2.26}$$

For other (more specialized) equations (of orders 1-3) involving $\mathcal{P}(z)$, see [18.8], nos. 1.49, 2.28, 2.72-3, 2.439-440, 3.9-12.

For the use of $\mathcal{P}(z)$ in solving differential equations of the form $y'^m + A(z, y) = 0$, where $A(z, y)$ is a polynomial in y of degree $2m$, with coefficients which are analytic functions of z , see [18.7], p. 312ff.

18.7. Integrals

Indefinite

$$18.7.1 \quad \int \mathcal{P}^2(z) dz = \frac{1}{6} \mathcal{P}'(z) + \frac{1}{12} g_2 z$$

$$18.7.2 \quad \int \mathcal{P}^3(z) dz = \frac{1}{120} \mathcal{P}'''(z) - \frac{3}{20} g_2 \zeta(z) + \frac{1}{10} g_3 z$$

(formulas for higher powers may be derived by integration of formulas for $\mathcal{P}^{(2k)}(z)$)

For $\int \mathcal{P}^n(z) dz$, n any positive integer, see [18.15] vol. 4, pp. 108-9.

If $\mathcal{P}'(a) \neq 0$

18.7.3

$$\mathcal{P}'(a) \int \frac{dz}{\mathcal{P}(z) - \mathcal{P}(a)} = 2z\zeta(a) + \ln \sigma(z-a) - \ln \sigma(z+a)$$

For $\int dz / [\mathcal{P}(z) - \mathcal{P}(a)]^n$, ($\mathcal{P}'(a) \neq 0$) n any positive integer, see [18.15], vol. 4, pp. 109-110.

Definite

$$\Delta > 0$$

$$\Delta < 0$$

18.7.4

$$\omega = \int_{a_1}^{\infty} \frac{dt}{\sqrt{s(t)}} \quad \omega_2 = \int_{a_2}^{\infty} \frac{dt}{\sqrt{s(t)}}$$

18.7.5

$$\omega' = i \int_{-\infty}^{a_3} \frac{dt}{\sqrt{|s(t)|}} \quad \omega'_2 = i \int_{-\infty}^{a_2} \frac{dt}{\sqrt{|s(t)|}}$$

18.7.6

where t is real and

18.7.7

$$s(t) = 4t^3 - g_2 t - g_3$$

18.8 Conformal Mapping

$$w = u + iv$$

$$\Delta > 0$$

$w = \wp(z)$ maps the Fundamental Rectangle onto the half-plane $v \leq 0$; if $|\omega'| = \omega(g_3 = 0)$, the isosceles triangle $0\omega\omega_2$ is mapped onto $u \geq 0, v \leq 0$.

$w = \wp'(z)$ maps the Fundamental Rectangle onto the w -plane less quadrant III; if $|\omega'| = \omega$, the triangle $0\omega\omega_2$ is mapped onto $v \geq 0, v \geq u$.

(a = period ratio)

$w = \zeta(z)$ maps the Fundamental Rectangle onto the half-plane $u \geq 0$. If $a \leq 1.9$ (approx.), $v \leq 0$; otherwise the image extends into quadrant I. For very large a , the image has a large area in quadrant I.

$w = \sigma(z)$ maps the Fundamental Rectangle onto quadrant I if $a < 1.9$ (approx.), onto quadrants I and II if $1.9 \leq a < 3.8$ (approx.). For large a , $\arg[\sigma(\omega_2)] \approx \frac{\pi^2 a}{12}$; consequently the image winds around the origin for large a .

Other maps are described in [18.23] arts. 13.7 (square on circle), 13.11 (ring on plane with 2 slits in line) and in [18.24], p. 35 (double half equilateral triangle on half-plane).

$$\Delta < 0$$

$w = \wp(z)$ maps the Fundamental Rectangle onto the half-plane $v \leq 0$; if $|\omega'_2| = \omega_2(g_3 = 0)$, the isosceles triangle $0\omega_2\omega'$ is mapped onto $u \geq 0, v \leq 0$.

$w = \wp'(z)$ maps the Fundamental Rectangle onto most of the w -plane less quadrant III; if $|\omega'_2| = \omega_2$, the triangle $0\omega_2\omega'$ is mapped onto $v \geq 0, v \geq u$.

$w = \zeta(z)$ maps the Fundamental Rectangle onto the half-plane $u \geq 0$. The image is mostly in quadrant IV for small a , entirely so for (approx.) $1.3 \leq a \leq 3.8$. For very large a , the image has a large area in quadrant I.

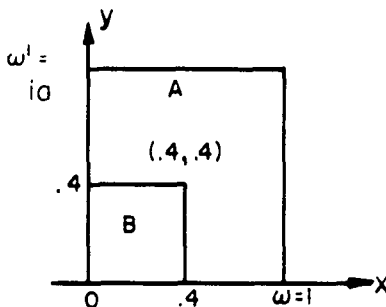
$w = \sigma(z)$ maps the Fundamental Rectangle onto quadrant I if $a < 3.8$ (approx.), onto quadrants I and II if $3.8 \leq a < 7.6$ (approx.). For large a , $\arg\left[\sigma\left(\omega_2 + \frac{\omega'_2}{2}\right)\right] \approx \frac{\pi^2 a}{24}$; consequently the image winds around the origin for large a .

Other maps are described in [18.23] arts. 13.8 (equilateral triangle on half-plane) and 13.9 (isosceles triangle on half-plane).

Obtaining $\wp'$ from $\wp'^2$

Fundamental Rectangle
 $\Delta > 0$

FUNDAMENTAL RECTANGLE

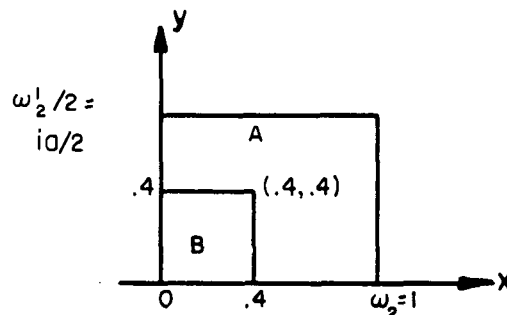


In region A

$\Re(\wp') \geq 0$ if $y \geq .4$ and $x \leq .5$; $\Im(\wp') \geq 0$ elsewhere

Fundamental Rectangle
 $\Delta < 0$

FUNDAMENTAL RECTANGLE



In region A

(1) If $a \geq 1.05$, use criterion for region A for $\Delta > 0$.

(2) If $1 \leq a < 1.05$: $\Re(\wp') \geq 0$ if $y \geq .4$ and $x \leq .4$, $-\pi/4 < \arg(\wp') < 3\pi/4$ if $.4 < y \leq .5$ and $.4 < x \leq .5$. $\Im(\wp') \geq 0$ elsewhere

FIGURE 18.4

In region B

The sign (indeed, perhaps one or more significant digits) of $\mathcal{P}'$ is obtainable from the first term, $-2/z^3$, of the Laurent series for $\mathcal{P}'$.

(Precisely similar criteria apply when the real half-period $\neq 1$)

In region B

Use the criterion for region B for $\Delta > 0$.

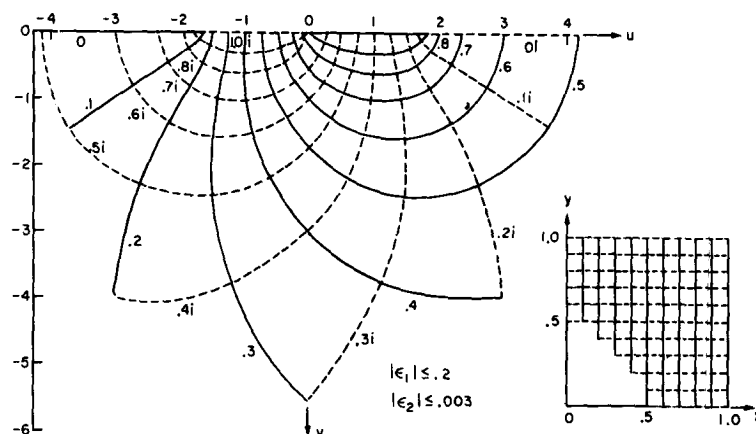
$$\Delta > 0 \quad \omega = 1$$

$$\text{Map: } \mathcal{P}(z) = u + iv$$

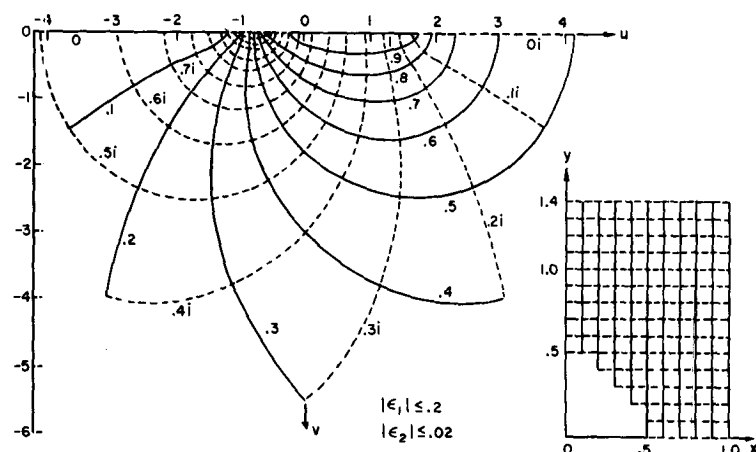
$$\text{Near zero: } \mathcal{P}(z) = \frac{1}{z^2} + \epsilon_1$$

$$\mathcal{P}(z) = \frac{1}{z^2} + c_2 z^2 + \epsilon_2$$

$$\omega' = i$$



$$\omega' = 1.4i$$



$$\omega' = 2.0i$$

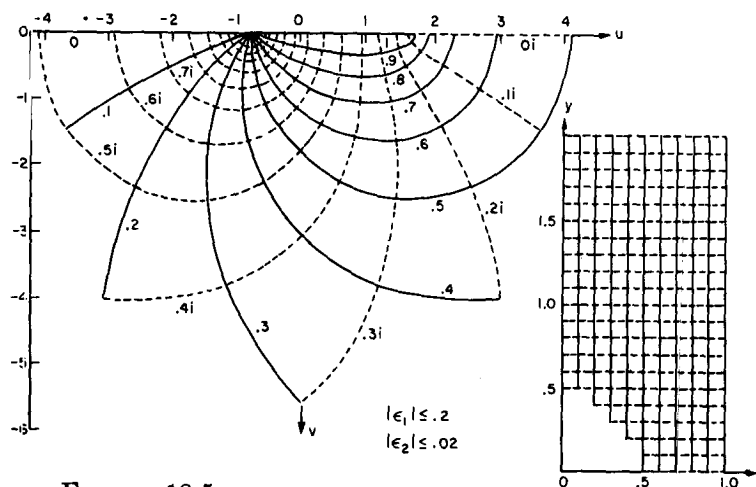


FIGURE 18.5

$$\Delta < 0 \quad \omega_2 = 1$$

$$\text{Map: } \mathcal{P}(z) = u + iv$$

$$\text{Near zero: } \mathcal{P}(z) = \frac{1}{z^2} + \epsilon_1$$

$$\mathcal{P}(z) = \frac{1}{z^2} + c_2 z^2 + \epsilon_2$$

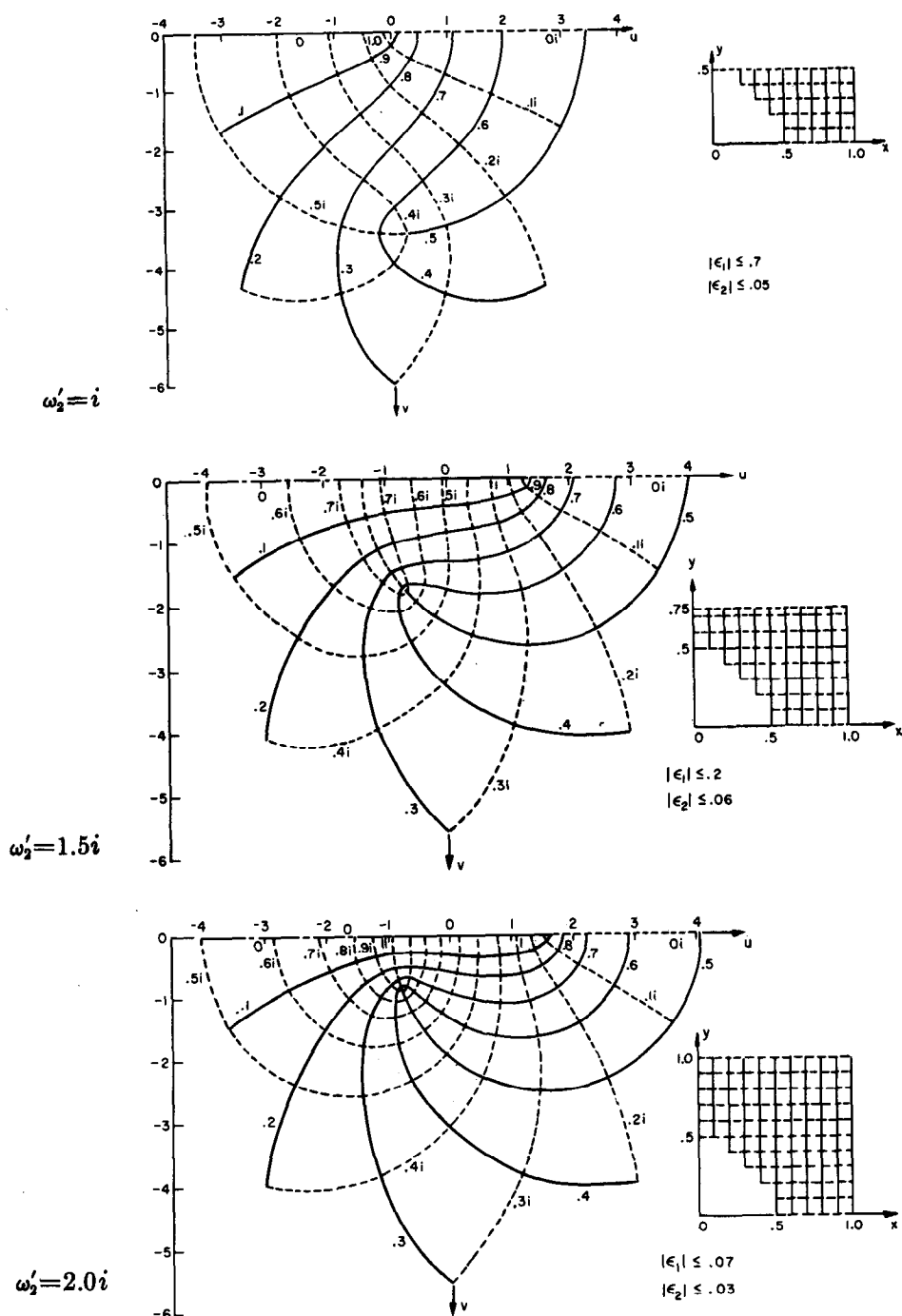


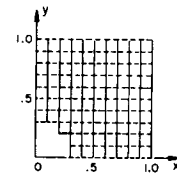
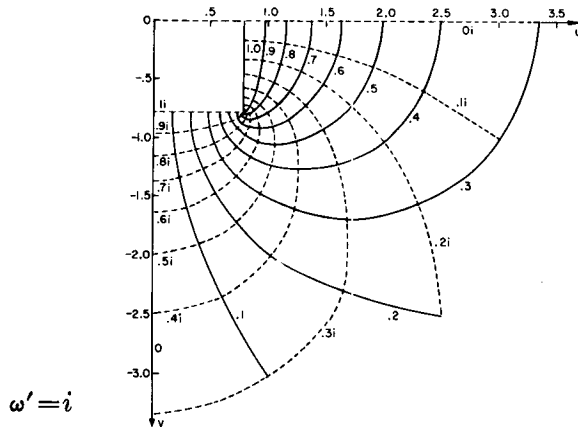
FIGURE 18.6

$$\Delta > 0 \quad \omega = 1$$

$$\text{Map: } \zeta(z) = u + iv$$

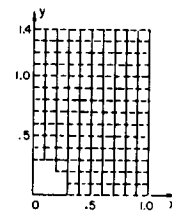
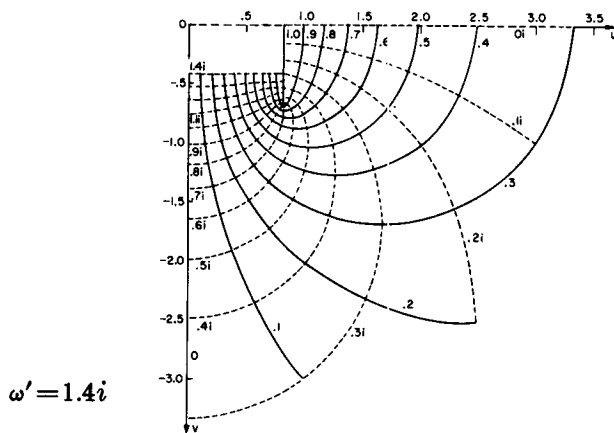
$$\text{Near zero: } \zeta(z) = \frac{1}{z} + \epsilon_1$$

$$\zeta(z) = \frac{1}{z} - \frac{c_2 z^3}{3} + \epsilon_2$$



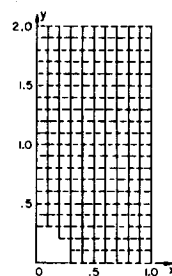
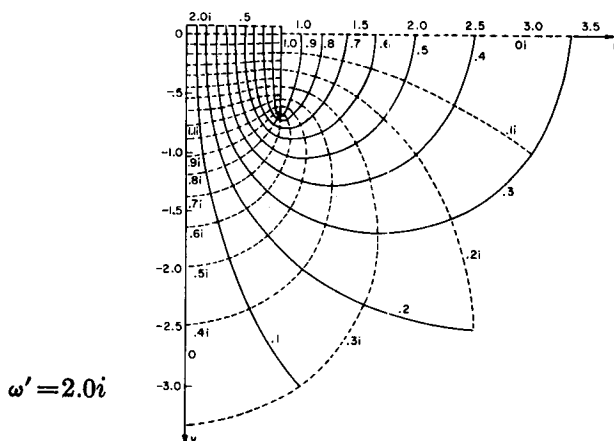
$$|\epsilon_1| \leq .01$$

$$|\epsilon_2| \leq 2 \times 10^{-5}$$



$$|\epsilon_1| \leq .007$$

$$|\epsilon_2| \leq .0002$$



$$|\epsilon_1| \leq .007$$

$$|\epsilon_2| \leq .0002$$

FIGURE 18.7

$\Delta < 0 \quad \omega_2 = 1$

Map: $\zeta(z) = u + iv$

Near zero: $\zeta(z) = \frac{1}{z} + \epsilon_1$

$\zeta(z) = \frac{1}{z} - \frac{c_2 z^3}{3} + \epsilon_2$

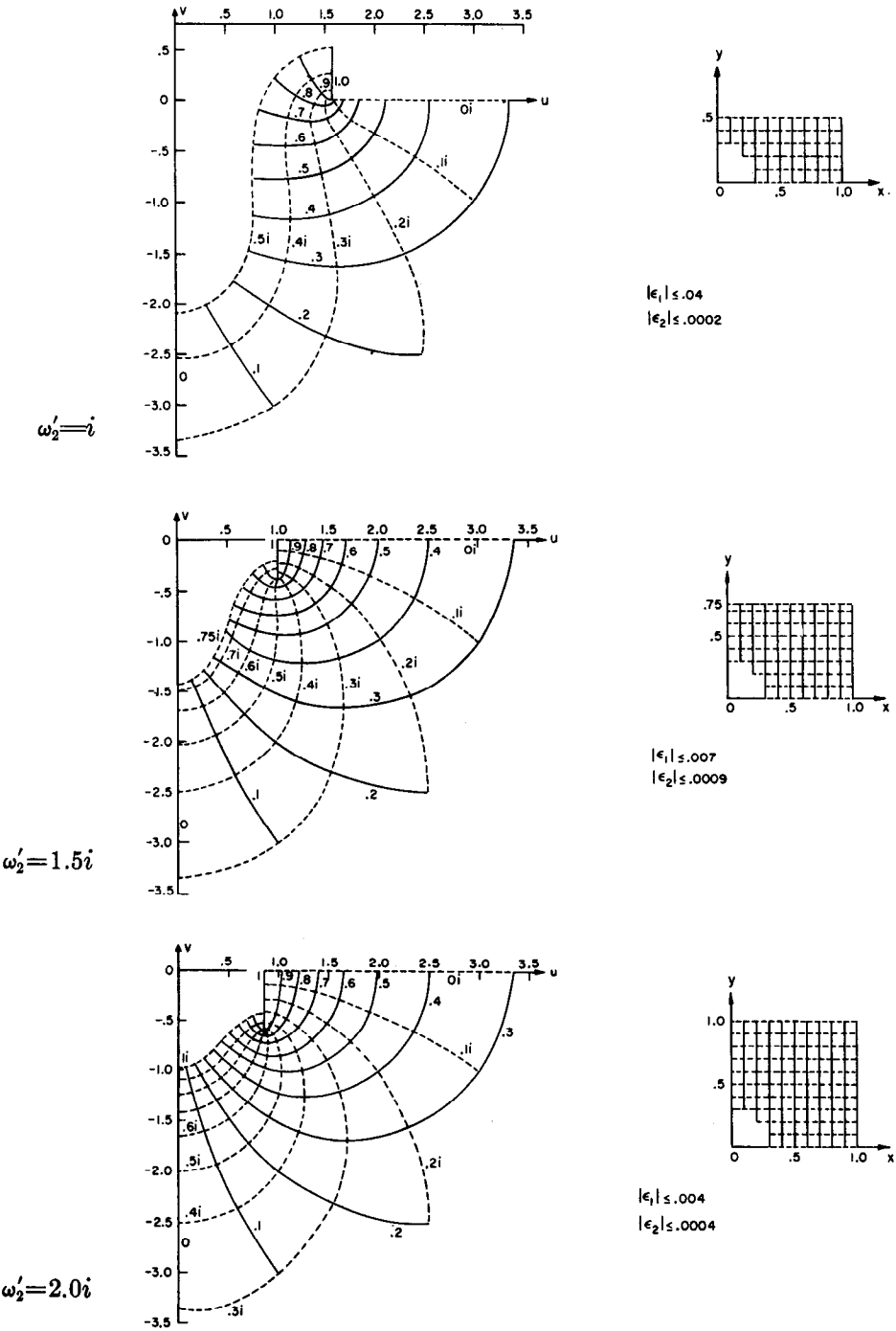


FIGURE 18.8

$$\Delta > 0 \quad \omega = 1$$

Map: $\sigma(z) = u + iv$

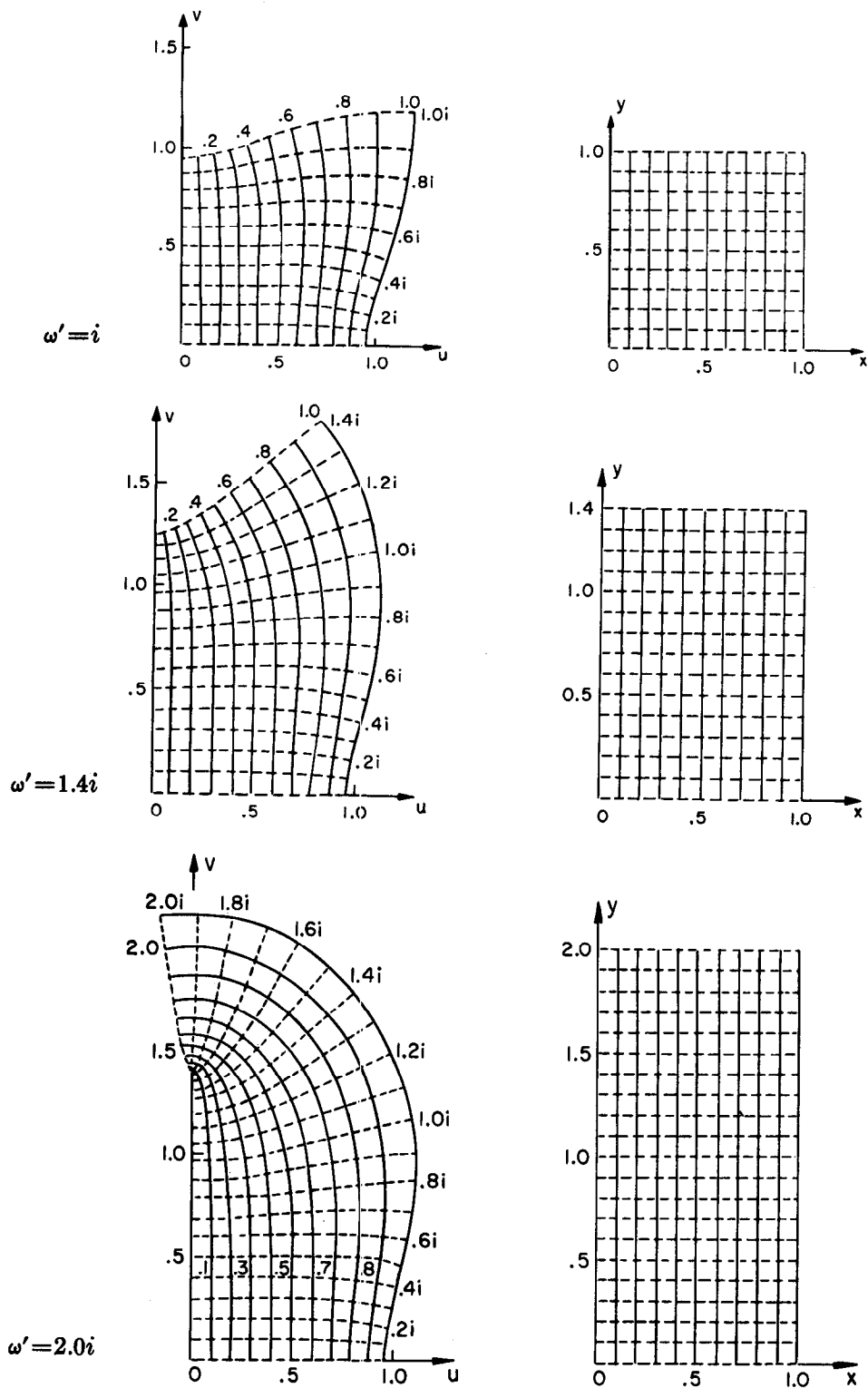


FIGURE 18.9

$$\Delta < 0 \quad \omega_2 = 1$$

$$\text{Map: } \sigma(z) = u + iv$$

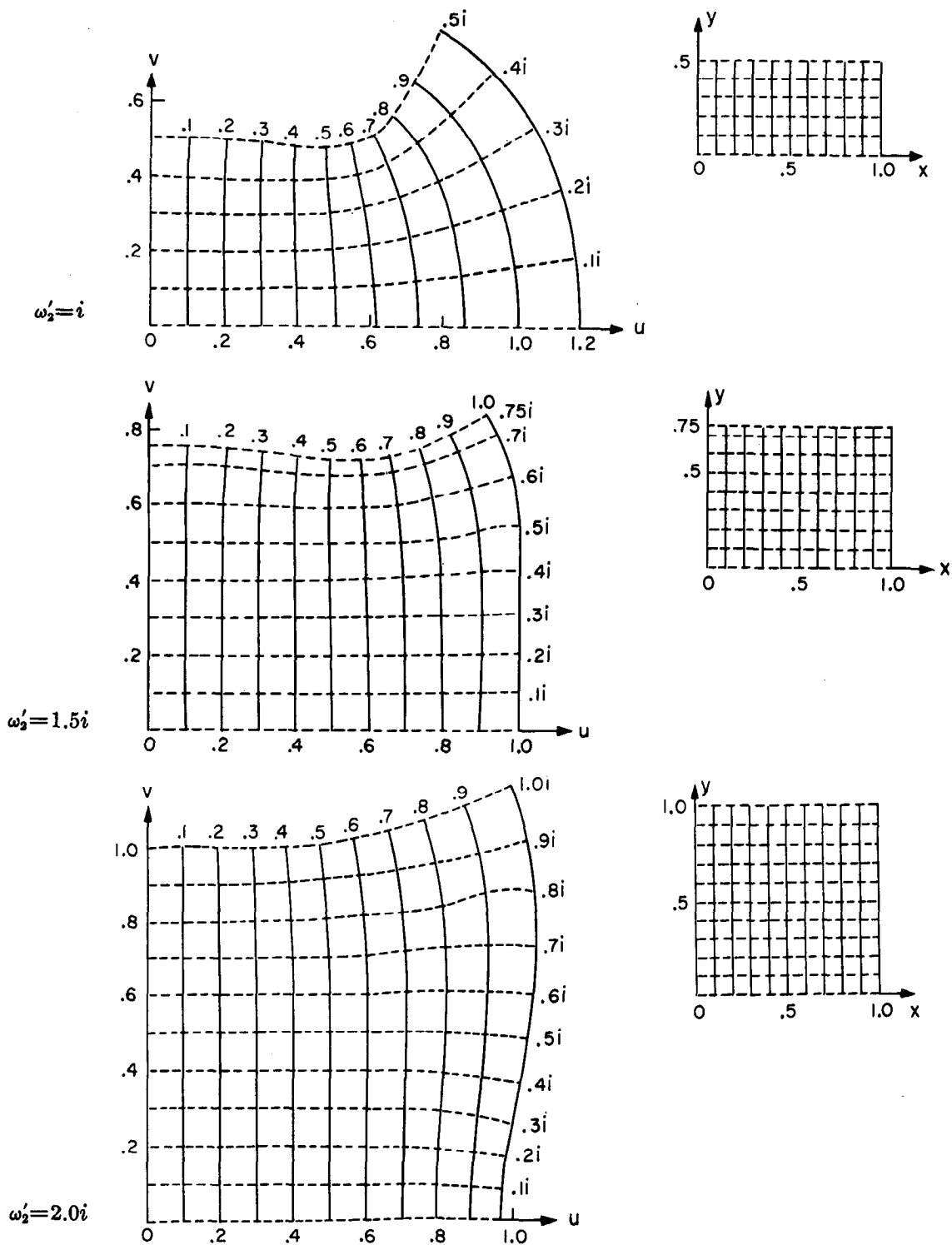


FIGURE 18.10

18.9. Relations with Complete Elliptic Integrals K and K' and Their Parameter m

and with Jacobi's Elliptic Functions (see chapter 16)

(Here $K(m)$ and $K'(m)=K(1-m)$ are complete elliptic integrals of the 1st kind; see chapter 17.)

$$\Delta > 0$$

$$\Delta < 0$$

18.9.1
$$e_1 = \frac{(2-m)K^2(m)}{3\omega^2}$$

$$e_1 = \frac{(2m-1)+6i\sqrt{m-m^2}}{3\omega_2^2} K^2(m)$$

18.9.2
$$e_2 = \frac{(2m-1)K^2(m)}{3\omega^2}$$

$$e_2 = \frac{2(1-2m)K^2(m)}{3\omega_2^2}$$

18.9.3
$$e_3 = \frac{-(m+1)K^2(m)}{3\omega^2}$$

$$e_3 = \frac{(2m-1)-6i\sqrt{m-m^2}}{3\omega_2^2} K^2(m)$$

18.9.4
$$g_2 = \frac{4(m^2-m+1)K^4(m)}{3\omega^4}$$

$$g_2 = \frac{4(16m^2-16m+1)K^4(m)}{3\omega_2^4}$$

18.9.5
$$g_3 = \frac{4(m-2)(2m-1)(m+1)K^6(m)}{27\omega^6}$$

$$g_3 = \frac{8(2m-1)(32m^2-32m-1)K^6(m)}{27\omega_2^6}$$

18.9.6
$$\Delta = \frac{16m^2(m-1)^2K^{12}(m)}{\omega^{12}}$$

$$\Delta = \frac{-256(m-m^2)K^{12}(m)}{\omega_2^{12}}$$

18.9.7
$$\omega' = \frac{iK'(m)\omega}{K(m)}$$

$$\omega'_2 = \frac{iK'(m)\omega_2}{K(m)}$$

18.9.8
$$\omega = K(m)/(e_1 - e_3)^{1/2}$$

$$\omega_2 = K(m)/H_2^{1/2}$$

18.9.9
$$m = (e_2 - e_3)/(e_1 - e_3)$$

$$m = \frac{1}{2} \frac{3e_2}{4H_2}$$

18.9.10
$$[0 < m \leq \frac{1}{2}, \text{ since } g_3 \geq 0]$$

18.9.11
$$\mathcal{P}(z) = e_3 + (e_1 - e_3)/\text{sn}^2(z^*|m)$$

$$\mathcal{P}(z) = e_2 + H_2 \frac{1 + \text{cn}(z'|m)}{1 - \text{cn}(z'|m)}$$

18.9.12

$$\mathcal{P}'(z) = -2(e_1 - e_3)^{3/2} \cdot \text{cn}(z^*|m) \text{dn}(z^*|m) / \text{sn}^3(z^*|m)$$

$$\mathcal{P}'(z) = \frac{-4H_2^{3/2} \text{sn}(z'|m) \text{dn}(z'|m)}{[1 - \text{cn}(z'|m)]^2}$$

where

$$z^* = (e_1 - e_3)^{1/2} z$$

where

$$z' = 2zH_2^{1/2}$$

18.9.13
$$\eta = \zeta(\omega) = \frac{K(m)}{3\omega} [3E(m) + (m-2)K(m)]$$

$$\eta_2 = \zeta(\omega_2) = \frac{K(m)}{3\omega_2} [6E(m) + (4m-5)K(m)]$$

18.9.14
$$\eta' = \zeta(\omega') = \frac{\eta\omega' - \frac{1}{2}\pi i}{\omega}$$

$$\eta'_2 = \zeta(\omega'_2) = \frac{\eta_2\omega'_2 - \pi i}{\omega_2}$$

[$E(m)$ is a complete elliptic integral of the 2d kind (see chapter 17).]

18.10. Relations with Theta Functions (chapter 16)

The formal definitions of the four ϑ functions are given by the series 16.27.1–16.27.4 which converge for all complex z and all q defined below. (Some authors use πz , instead of z , as the independent variable.)

These functions depend on z and on a parameter q , which is usually suppressed. Note that

$$\vartheta_1'(0) = \vartheta_2(0)\vartheta_3(0)\vartheta_4(0), \text{ where } \vartheta_i(0) = \vartheta_i(0, q).$$

$$\Delta > 0$$

$$\Delta < 0$$

18.10.1

$$\tau = \omega'/\omega$$

$$\tau_2 = \omega_2'/2\omega_2$$

18.10.2

$$q = e^{i\pi\tau} = e^{-\pi K'/K}$$

$$q = iq_2 = ie^{i\pi\tau_2} = ie^{-\pi|\omega_2'|/2\omega_2}$$

18.10.3

q is real and since $g_3 \geq 0$ ($|\omega'| \geq \omega$), $0 < q \leq e^{-\pi}$

q is pure imaginary and since $g_3 \geq 0$ ($|\omega_2'| \geq \omega_2$), $0 < |q| \leq e^{-\pi/2}$

18.10.4

$$(v = \pi z/2\omega)$$

$$(v = \pi z/2\omega_2)$$

18.10.5

$$\mathcal{P}(z) = e_j + \frac{\pi^2}{4\omega^2} \left[\frac{\vartheta_1'(0)\vartheta_{j+1}(v)}{\vartheta_{j+1}(0)\vartheta_1(v)} \right]^2$$

$$\mathcal{P}(z) = e_2 + \frac{\pi^2}{4\omega_2^2} \left[\frac{\vartheta_1'(0)\vartheta_2(v)}{\vartheta_2(0)\vartheta_1(v)} \right]^2$$

$$j = 1, 2, 3$$

18.10.6

$$\mathcal{P}'(z) = -\frac{\pi^3}{4\omega^3} \frac{\vartheta_2(v)\vartheta_3(v)\vartheta_4(v)\vartheta_1'^3(0)}{\vartheta_2(0)\vartheta_3(0)\vartheta_4(0)\vartheta_1^3(v)}$$

$$\mathcal{P}'(z) = -\frac{\pi^3}{4\omega_2^3} \frac{\vartheta_2(v)\vartheta_3(v)\vartheta_4(v)\vartheta_1'^3(0)}{\vartheta_2(0)\vartheta_3(0)\vartheta_4(0)\vartheta_1^3(v)}$$

18.10.7

$$\zeta(z) = \frac{\eta z}{\omega} + \frac{\pi\vartheta_1'(v)}{2\omega\vartheta_1(v)}$$

$$\zeta(z) = \frac{\eta_2 z}{\omega_2} + \frac{\pi\vartheta_1'(v)}{2\omega_2\vartheta_1(v)}$$

18.10.8

$$\sigma(z) = \frac{2\omega}{\pi} \exp\left(\frac{\eta z^2}{2\omega}\right) \frac{\vartheta_1(v)}{\vartheta_1'(0)}$$

$$\sigma(z) = \frac{2\omega_2}{\pi} \exp\left(\frac{\eta_2 z^2}{2\omega_2}\right) \frac{\vartheta_1(v)}{\vartheta_1'(0)}$$

18.10.9

$$12\omega^2 e_1 = \pi^2 [\vartheta_2^4(0) + \vartheta_4^4(0)]$$

$$12\omega_2^2 e_1 = \pi^2 [\vartheta_2^4(0) - \vartheta_4^4(0)]$$

18.10.10

$$12\omega^2 e_2 = \pi^2 [\vartheta_2^4(0) - \vartheta_4^4(0)]$$

$$12\omega_2^2 e_2 = \pi^2 [\vartheta_2^4(0) + \vartheta_4^4(0)]$$

18.10.11

$$12\omega^2 e_3 = -\pi^2 [\vartheta_2^4(0) + \vartheta_4^4(0)]$$

$$12\omega_2^2 e_3 = -\pi^2 [\vartheta_2^4(0) - \vartheta_4^4(0)]$$

18.10.12

$$(e_2 - e_3)^{\frac{1}{2}} = -i(e_3 - e_2)^{\frac{1}{2}} = \frac{\pi}{2\omega} \vartheta_2^2(0)$$

$$(e_2 - e_3)^{\frac{1}{2}} = i(e_3 - e_2)^{\frac{1}{2}} = \frac{\pi}{2\omega_2} \vartheta_2^2(0)$$

18.10.13

$$(e_1 - e_3)^{\frac{1}{2}} = -i(e_3 - e_1)^{\frac{1}{2}} = \frac{\pi}{2\omega} \vartheta_3^2(0)$$

$$(e_1 - e_3)^{\frac{1}{2}} = i(e_3 - e_1)^{\frac{1}{2}} = \frac{\pi}{2\omega_2} \vartheta_3^2(0)$$

18.10.14

$$(e_1 - e_2)^{\frac{1}{2}} = -i(e_2 - e_1)^{\frac{1}{2}} = \frac{\pi}{2\omega} \vartheta_4^2(0)$$

$$(e_2 - e_1)^{\frac{1}{2}} = -i(e_1 - e_2)^{\frac{1}{2}} = \frac{\pi}{2\omega_2} \vartheta_4^2(0)$$

18.10.15

$$g_2 = \frac{2}{3} \left(\frac{\pi}{2\omega} \right)^4 [\vartheta_2^8(0) + \vartheta_3^8(0) + \vartheta_4^8(0)]$$

$$g_2 = \frac{2}{3} \left(\frac{\pi}{2\omega_2} \right)^4 [\vartheta_2^8(0) + \vartheta_3^8(0) + \vartheta_4^8(0)]$$

18.10.16

$$g_3 = 4e_1 e_2 e_3$$

$$g_3 = 4e_1 e_2 e_3$$

18.10.17

$$\Delta^{\frac{1}{2}} = \frac{\pi^3}{4\omega^3} \vartheta_1'^2(0)$$

$$(-\Delta)^{\frac{1}{2}} = \frac{\pi^3}{4\omega_2^3} \vartheta_1'^2(0) e^{-i\pi/4}$$

18.10.18

$$\eta \equiv \zeta(\omega) = -\frac{\pi^2 \vartheta_1'''(0)}{12\omega \vartheta_1'(0)}$$

$$\eta_2 \equiv \zeta(\omega_2) = -\frac{\pi^2 \vartheta_1'''(0)}{12\omega_2 \vartheta_1'(0)}$$

18.10.19

$$\eta' \equiv \zeta(\omega') = \frac{\eta\omega' - \frac{1}{2}\pi i}{\omega}$$

$$\eta_2' \equiv \zeta(\omega_2') = \frac{\eta_2\omega_2' - \pi i}{\omega_2}$$

Series

18.10.20

$$\vartheta_1(0)=0$$

18.10.21

$$\vartheta_2(0)=2q^{\frac{1}{2}}[1+q^{1\cdot 2}+q^{2\cdot 3}+q^{3\cdot 4}+\dots+q^{n(n+1)}+\dots]$$

18.10.22

$$\vartheta_3(0)=1+2[q+q^4+q^9+\dots+q^{n^2}+\dots]$$

18.10.23

$$\vartheta_4(0)=1+2[-q+q^4-q^9+\dots+(-1)^n q^{n^2}+\dots]$$

Attainable Accuracy

$$\Delta > 0$$

$$\Delta < 0$$

Note: $\vartheta_j(0) > 0$, $j=2, 3, 4$ Note: $\vartheta_2(0) = Ae^{i\pi/8}$, $A > 0$;

$$\Re \vartheta_3(0) > 0; \vartheta_4(0) = \overline{\vartheta_3(0)}$$

 $\vartheta_j(0)$: 2 terms give at least 5S

2 terms give at least 3S

 $j=2, 3, 4$ 3 terms give at least 11S

3 terms give at least 5S

4 terms give at least 21S

4 terms give at least 10S

18.11 Expressing any Elliptic Function in Terms of $\mathcal{P}$ and $\mathcal{P}'$ If $f(z)$ is any elliptic function and $\mathcal{P}(z)$ has same periods, write

18.11.1

$$f(z) = \frac{1}{2}[f(z) + f(-z)] + \frac{1}{2}[\{f(z) - f(-z)\} \{\mathcal{P}'(z)\}^{-1}] \mathcal{P}'(z).$$

Since both brackets represent even elliptic functions, we ask how to express an even elliptic function $g(z)$ (of order $2k$) in terms of $\mathcal{P}(z)$. Because of the evenness, an irreducible set of zeros can be denoted by a_i ($i=1, 2, \dots, k$) and the set of points congruent to $-a_i$ ($i=1, 2, \dots, k$); correspondingly in connection with the poles we consider the points $\pm b_i$, $i=1, 2, \dots, k$. Then

18.11.2

$$g(z) = A \prod_{i=1}^k \left\{ \frac{\mathcal{P}(z) - \mathcal{P}(a_i)}{\mathcal{P}(z) - \mathcal{P}(b_i)} \right\}, \text{ where } A \text{ is}$$

a constant. If any a_i or b_i is congruent to the origin, the corresponding factor is omitted from the product. Factors corresponding to multiple poles (zeros) are repeated according to the multiplicity.

18.12. Case $\Delta=0$ ($c>0$)

Subcase I

18.12.1 $g_2 > 0$, $g_3 < 0$: ($e_1=e_2=c$, $e_3=-2c$)18.12.2 $H_1=H_2=0$, $H_3=3c$

18.12.3

$$\mathcal{P}(z; 12c^2, -8c^3) = c + 3c \{ \sinh [(3c)^{\frac{1}{2}} z] \}^{-2}$$

18.12.4

$$\zeta(z; 12c^2, -8c^3) = -cz + (3c)^{\frac{1}{2}} \coth [(3c)^{\frac{1}{2}} z]$$

18.12.5

$$\sigma(z; 12c^2, -8c^3) = (3c)^{-\frac{1}{2}} \sinh [(3c)^{\frac{1}{2}} z] e^{-cz^2/2}$$

18.12.6

$$\omega = \infty, \omega' = (12c)^{-\frac{1}{2}} \pi i$$

18.12.7

$$\eta = \zeta(\omega) = -\infty$$

18.12.8

$$\eta' = \zeta(\omega') = -c\omega'$$

18.12.9

$$q=1, \quad m=1$$

18.12.10

$$\sigma(\omega)=0$$

18.12.11

$$\sigma(\omega') = \frac{2\omega' e^{\pi^2/24}}{\pi}$$

18.12.12

$$\sigma(\omega_2)=0$$

18.12.13

$$\mathcal{P}(\omega/2)=c$$

18.12.14

$$\mathcal{P}'(\omega/2)=0$$

18.12.15

$$\zeta(\omega/2)=-\infty$$

18.12.16

$$\sigma(\omega/2)=0$$

18.12.17

$$\mathcal{P}(\omega'/2)=-5c$$

18.12.18

$$\mathcal{P}'(\omega'/2) = \frac{-\pi^3}{2\omega'^3}$$

18.12.19

$$\zeta(\omega'/2) = \frac{1}{2}(-c\omega' + \pi/\omega')$$

Reduction for z_2 in Δ_2 : $z_1 = \epsilon \bar{z}_2$ is in Δ_1 .

$$18.13.1 \quad \mathcal{P}(z_2) = \epsilon^{-2} \overline{\mathcal{P}}(z_1)$$

$$18.13.2 \quad \mathcal{P}'(z_2) = -\overline{\mathcal{P}'}(z_1)$$

$$18.13.3 \quad \zeta(z_2) = \epsilon^{-1} \bar{\zeta}(z_1)$$

$$18.13.4 \quad \sigma(z_2) = \epsilon \bar{\sigma}(z_1)$$

Reduction for z_3 in Δ_3 : $z_1 = \epsilon^{-1}(2\omega' - z_3)$ is in Δ_1

$$18.13.5 \quad \mathcal{P}(z_3) = \epsilon^{-2} \mathcal{P}(z_1)$$

$$18.13.6 \quad \mathcal{P}'(z_3) = \mathcal{P}'(z_1)$$

$$18.13.7 \quad \zeta(z_3) = -\epsilon^{-1} \zeta(z_1) + 2\eta', \quad \eta' = \zeta(\omega')$$

$$18.13.8 \quad \sigma(z_3) = \epsilon \sigma(z_1) \exp[(z_3 - \omega')(2\eta')]$$

Special Values and Formulas

18.13.9

$$\Delta = -27, \quad H_1 = \sqrt{3}(4^{-1/3})\bar{\epsilon},$$

$$H_2 = \sqrt{3}(4^{-1/3}), \quad H_3 = \sqrt{3}(4^{-1/3})\epsilon$$

$$18.13.10 \quad m = \sin^2 15^\circ = \frac{2 - \sqrt{3}}{4}, \quad q = ie^{-\pi\sqrt{3}/2}$$

$$18.13.11 \quad \vartheta_2(0) = Ae^{i\pi/8}$$

$$18.13.12 \quad \vartheta_3(0) = Ae^{i\pi/24}$$

$$18.13.13 \quad \vartheta_4(0) = Ae^{-i\pi/24}$$

18.13.14

$$\text{where } A = (\omega_2/\pi)^{1/2} 2^{1/3} 3^{1/8} \approx 1.0086 \ 67$$

$$18.13.15 \quad \omega_2 = \frac{K(m)2^{1/3}}{3^{1/4}} = \frac{\Gamma^3(1/3)}{4\pi}$$

Values at Half-periods

	$\mathcal{P}$	$\mathcal{P}'$	ζ	σ
18.13.16 $\omega \equiv \omega_1$	$e_1 = 4^{-1/3}\epsilon^2$	0	$\eta = \epsilon\pi/2\omega_2\sqrt{3}$	$\epsilon^{-1}\sigma(\omega_2)$
18.13.17 ω_2	$e_2 = 4^{-1/3}$	0	$\eta_2 = \eta + \eta' = \pi/2\omega_2\sqrt{3}$	$\frac{e^{\pi/4\sqrt{3}}(2^{1/3})}{3^{1/2}}$
18.13.18 $\omega' \equiv \omega_3$	$e_3 = 4^{-1/3}\epsilon^{-2}$	0	$\eta' = \epsilon^{-1}\pi/2\omega_2\sqrt{3}$	$\epsilon\sigma(\omega_2)$
18.13.19 ω_2'	$e_2 = 4^{-1/3}$	0	$\eta_2' = -\pi i/2\omega_2 = \eta' - \eta$	$\frac{ie^{3\pi/4\sqrt{3}}(2^{1/3})}{3^{1/2}}$

Values ⁷ along $(0, \omega_2)$

	$\mathcal{P}$	$\mathcal{P}'$	ζ	σ
18.13.20 $2\omega_2/9$	$\frac{\sqrt[3]{\cos 80^\circ}}{\sqrt[3]{\cos 20^\circ} - \sqrt[3]{\cos 40^\circ}}$	$-\frac{\sqrt{3}[\sqrt[3]{\cos 20^\circ} + \sqrt[3]{\cos 40^\circ}]}{\sqrt[3]{\cos 20^\circ} - \sqrt[3]{\cos 40^\circ}}$		
18.13.21 $\omega_2/3$	$1/(2^{1/3} - 1)$	$-\sqrt{3}(2^{1/3} + 1)/(2^{1/3} - 1)$	$\frac{\eta_2 + \sqrt{3}(2^{2/3} + 2 + 2^{4/3})}{6}$	$\frac{e^{\pi/36\sqrt{3}}}{3^{1/6}} \sqrt[4]{\frac{2^{1/3} - 1}{2^{1/3} + 1}}$
18.13.22 $4\omega_2/9$	$\frac{\sqrt[3]{\cos 40^\circ}}{\sqrt[3]{\cos 20^\circ} - \sqrt[3]{\cos 80^\circ}}$	$-\frac{\sqrt{3}[\sqrt[3]{\cos 20^\circ} + \sqrt[3]{\cos 80^\circ}]}{\sqrt[3]{\cos 20^\circ} - \sqrt[3]{\cos 80^\circ}}$		
18.13.23 $\omega_2/2$	$e_2 + H_2$	$-3^{3/4}\sqrt{2 + \sqrt{3}}$	$(\pi/4\omega_2\sqrt{3}) + (3^{1/4}\sqrt{2 + \sqrt{3}}/2^{4/3})$	$\frac{e^{\pi/18\sqrt{3}}(2^{1/12})}{3^{1/4}\sqrt{2 + \sqrt{3}}}$
18.13.24 $2\omega_2/3$	1	$-\sqrt{3}$	$\frac{2}{3}(\eta_2) + 3^{-1/2}$	$e^{\pi/9\sqrt{3}}/3^{1/6}$
18.13.25 $8\omega_2/9$	$\frac{\sqrt[3]{\cos 20^\circ}}{\sqrt[3]{\cos 40^\circ} + \sqrt[3]{\cos 80^\circ}}$	$-\frac{\sqrt{3}[\sqrt[3]{\cos 40^\circ} - \sqrt[3]{\cos 80^\circ}]}{\sqrt[3]{\cos 40^\circ} + \sqrt[3]{\cos 80^\circ}}$		

⁷ Values at $2\omega_2/9$, $4\omega_2/9$ and $8\omega_2/9$ from [18.14].

Values along $(0, z_0)$

	$\mathcal{P}$	$\mathcal{P}'$	ζ	σ
18.13.26 $z_0/2$	$-2^{1/3}\epsilon^2$	$3i$	$\left[\frac{\eta_2}{\sqrt{3}} + 2^{-1/3}\right]e^{-i\pi/6}$	$\frac{e^{\pi/12}\sqrt{3}e^{i\pi/6}}{3^{1/4}}$
18.13.27 $3z_0/4$	$\epsilon^2(e_2 - H_2)$	$i(3^{3/4})\sqrt{2-\sqrt{3}}$	$\left[\frac{\pi}{4\omega_2} + \frac{3^{1/4}\sqrt{2-\sqrt{3}}}{2^{1/3}}\right]e^{-i\pi/6}$	$\frac{e^{3\pi/16}\sqrt{3}(2^{1/12})e^{i\pi/6}}{3^{1/4}\sqrt{2-\sqrt{3}}}$
18.13.28 z_0	0	i	$\frac{2\eta_2}{\sqrt{3}}e^{-i\pi/6}$	$\frac{e^{\pi/3}\sqrt{3}e^{i\pi/6}}{3^{1/4}}$

Duplication Formulas

$$18.13.29 \quad \mathcal{P}(2z) = \frac{\mathcal{P}(z)[\mathcal{P}^3(z) + 2]}{4\mathcal{P}^3(z) - 1}$$

$$18.13.30 \quad \mathcal{P}'(2z) = \frac{2\mathcal{P}^6(z) - 10\mathcal{P}^3(z) - 1}{[\mathcal{P}'(z)]^3}$$

$$18.13.31 \quad \zeta(2z) = 2\zeta(z) + \frac{3\mathcal{P}^2(z)}{\mathcal{P}'(z)}$$

$$18.13.32 \quad \sigma(2z) = -\mathcal{P}'(z)\sigma^4(z)$$

Trisection Formulas (x real)

$$18.13.33 \quad \mathcal{P}\left(\frac{x}{3}\right) = \frac{\sqrt[3]{\cos \frac{\phi - \pi}{3}}}{\sqrt[3]{\cos \frac{\phi}{3}} - \sqrt[3]{\cos \frac{\phi + \pi}{3}}}$$

$$18.13.34 \quad \mathcal{P}'\left(\frac{x}{3}\right) = -\sqrt{3} \frac{\sqrt[3]{\cos \frac{\phi}{3}} + \sqrt[3]{\cos \frac{\phi + \pi}{3}}}{\sqrt[3]{\cos \frac{\phi}{3}} - \sqrt[3]{\cos \frac{\phi + \pi}{3}}}$$

where $\tan \phi = \mathcal{P}'(x)$, $0 < x < 2\omega_2$ and we must choose ϕ in intervals

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \text{ to get}$$

$$\mathcal{P}\left(\frac{x}{3}\right), \mathcal{P}\left(\frac{x}{3} + \frac{2\omega_2}{3}\right), \mathcal{P}\left(\frac{x}{3} + \frac{4\omega_2}{3}\right), \text{ respectively.}$$

Complex Multiplication

$$18.13.35 \quad \mathcal{P}(\epsilon z) = \epsilon^{-2}\mathcal{P}(z)$$

$$18.13.36 \quad \mathcal{P}'(\epsilon z) = -\mathcal{P}'(z)$$

$$18.13.37 \quad \zeta(\epsilon z) = \epsilon^{-1}\zeta(z)$$

$$18.13.38 \quad \sigma(\epsilon z) = \epsilon\sigma(z)$$

In the above, ϵ denotes (as it does throughout section 18.13), $e^{i\pi/3}$. The above equations are useful as follows, e.g.:

If z is real, ϵz is on $0\omega'$ (Figure 18.11); if ϵz were purely imaginary, z would be on $0z_0$ (Figure 18.11).

Conformal Maps

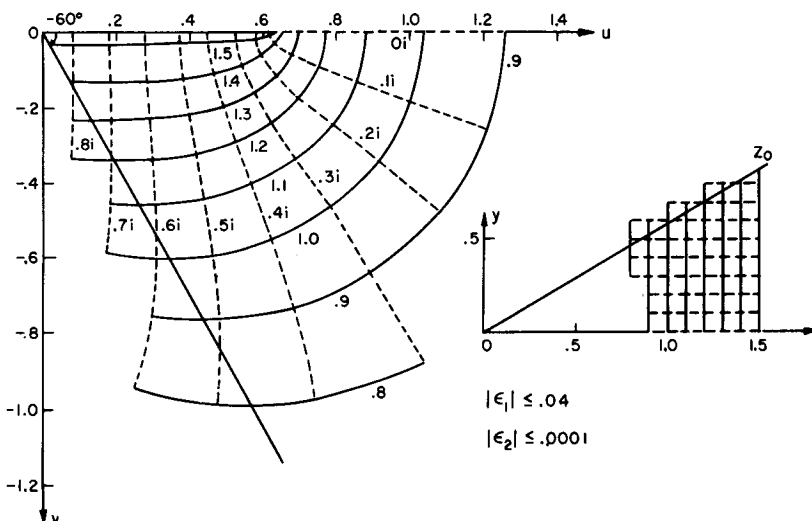
Equianharmonic Case

$$\text{Map: } f(z) = u + iv$$

$$\mathcal{P}(z)$$

$$\text{Near zero: } \mathcal{P}(z) = \frac{1}{z^2} + \epsilon_1$$

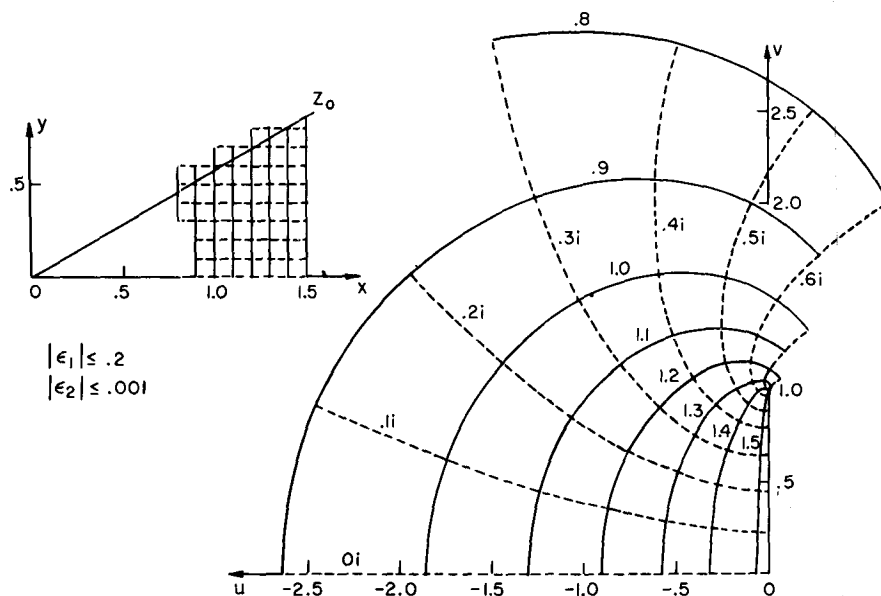
$$\mathcal{P}(z) = \frac{1}{z^2} + \frac{z^4}{28} + \epsilon_2$$



$$\mathcal{P}'(z)$$

Near zero: $\mathcal{P}'(z) = \frac{-2}{z^3} + \epsilon_1$

$$\mathcal{P}'(z) = \frac{-2}{z^3} + \frac{z^3}{7} + \epsilon_2$$



$$\zeta(z)$$

Near zero: $\zeta(z) = \frac{1}{z} + \epsilon_1$

$$\zeta(z) = \frac{1}{z} - \frac{z^5}{140} + \epsilon_2$$

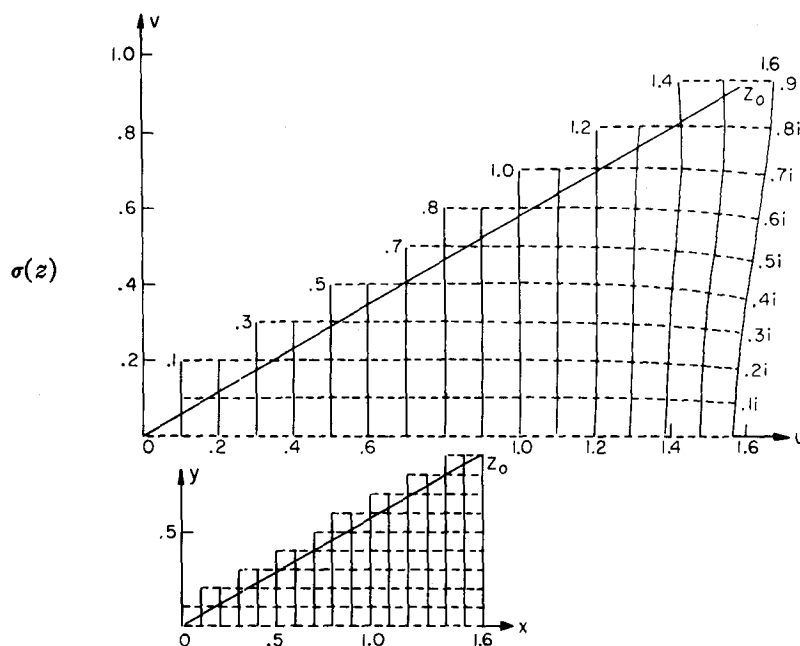
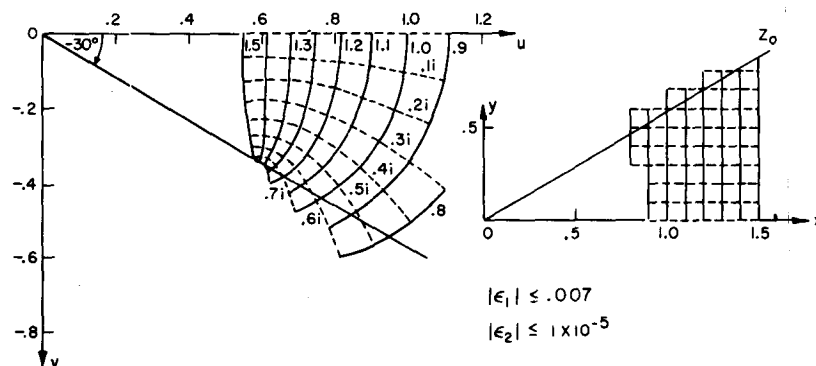


FIGURE 18.12

Coefficients for Laurent Series for $\wp$, $\wp'$ and ζ $(c_m = 0 \text{ for } m \neq 3k)$

k	EXACT c_{3k}	APPROXIMATE c_{3k}
1	1/28	3.5714 28571 42857 $\dots \times 10^{-2}$
2	$1/(13 \cdot 28^2) = 1/10192$	9.8116 16954 47409 73312 40188×10^{-5}
3	$1/(13 \cdot 19 \cdot 28^3) = 1/5422144$	1.8442 88901 21693 55885 78983×10^{-7}
4	$3/(5 \cdot 13^2 \cdot 19 \cdot 28^4) = 234375/(7709611 \times 10^8)$	3.0400 36650 35758 61350 20301×10^{-10}
5	$4/(5 \cdot 13^2 \cdot 19 \cdot 31 \cdot 28^5) = 78125/(16729 \cdot 85587 \times 10^9)$	4.6697 95161 83961 00384 33643×10^{-13}
6	$(7 \cdot 43)/(13^3 \cdot 19^2 \cdot 31 \cdot 37 \cdot 28^6)$	6.8662 18676 79393 36788 98×10^{-16}
7	$(6 \cdot 431)/(5 \cdot 13^3 \cdot 19^2 \cdot 31 \cdot 37 \cdot 43 \cdot 28^7)$	9.7990 31742 57961 41839 66×10^{-19}
8	$(3 \cdot 7 \cdot 313)/(5^2 \cdot 13^4 \cdot 19^3 \cdot 31 \cdot 37 \cdot 43 \cdot 28^8)$	1.3685 06574 79360 13026 87×10^{-21}
9	$(4 \cdot 1201)/(5^2 \cdot 13^4 \cdot 19^3 \cdot 31 \cdot 37 \cdot 43 \cdot 28^9)$	1.8800 72610 01329 79236 40×10^{-24}
10	$(2^2 \cdot 3 \cdot 41 \cdot 1823)/(5 \cdot 13^5 \cdot 19^4 \cdot 31^2 \cdot 37 \cdot 43 \cdot 61 \cdot 28^{10})$	2.5497 66946 68202 63683 $\times 10^{-27}$
11	$(3 \cdot 79 \cdot 733)/(5 \cdot 13^5 \cdot 19^4 \cdot 31^2 \cdot 37 \cdot 43 \cdot 61 \cdot 67 \cdot 28^{11})$	3.4222 48599 51463 05316 $\times 10^{-30}$
12	$3 \cdot 1153 \cdot 13963 \cdot 29059$	4.5541 38864 99184 30391 $\times 10^{-33}$
13	$\frac{5^3 \cdot 13^5 \cdot 19^4 \cdot 31^2 \cdot 37^2 \cdot 43 \cdot 61 \cdot 67 \cdot 73 \cdot 28^{13}}{2^2 \cdot 3^2 \cdot 7 \cdot 11 \cdot 2647111}$	6.0171 15776 98241 99591 $\times 10^{-36}$

First 5 approximate values determined from exact values of c_{3k} ; subsequent values determined by using exact ratios c_{3k}/c_{3k-3} , using at least double precision arithmetic with a desk calculator. All approximate c 's were checked with the use of the recursion relation; $c_3 - c_{27}$ are believed correct to at least 21S; $c_{30} - c_{39}$ are believed correct to 20S.

$$c_{3k} \leq \frac{c_3}{13^{k-1} \cdot 28^{k-1}}, \quad k=2, 3, 4, \dots$$

Other Series Involving $\wp$ Reversed Series for Large $|\wp|$

18.13.39

$$z = (\wp^{-1})^{1/2} \left[1 + \frac{u}{7} + \frac{3u^2}{26} + \frac{5u^3}{38} + \frac{7u^4}{40} + \frac{63u^5}{248} + \frac{231u^6}{592} + \frac{429u^7}{688} + O(u^8) \right],$$

18.13.40 where $u = \wp^{-3}/8$ and z is in the Fundamental Triangle (Figure 18.11) if $\wp$ has an appropriate value.

Series near z_0

18.13.41

$$\wp = iu \left[1 - \frac{u^6}{7} + \frac{3u^{12}}{364} \right] + u^4 \left[-\frac{1}{2} + \frac{u^6}{28} \right] + O(u^{16})$$

18.13.42

$$u = -i\wp \left[1 + \frac{\wp^3}{2} + \frac{6\wp^6}{7} + 2\wp^9 + \frac{70\wp^{12}}{13} + O(\wp^{15}) \right],$$

18.13.43 where $u = (z - z_0)$

Series near ω_2

18.13.44

$$(\wp - e_2) = 3e_2^2 u \left[1 + x + x^2 + \frac{6}{7}x^3 + \frac{5}{7}x^4 + \frac{4}{7}x^5 + \frac{285}{637}x^6 + O(x^7) \right],$$

18.13.45 where $u = (z - \omega_2)^2$, $x = e_2 u$

18.13.46

$$u = e_2^{-1} \left[w - w^2 + w^3 - \frac{6}{7}w^4 + \frac{3}{7}w^5 + \frac{3}{7}w^6 - \frac{1143}{637}w^7 + O(w^8) \right],$$

18.13.47 where $w = (\wp - e_2)/3e_2$

Other Series Involving $\wp'$ Reversed Series for Large $|\wp'|$

18.13.48

$$z = 2^{1/3} (\wp'^{1/3})^{-1} e^{i\pi/3} \left[1 - \frac{2}{21} (\wp')^{-2} + \frac{5}{117} (\wp')^{-4} + O(\wp'^{-6}) \right],$$

z being in the Fundamental Triangle (Figure 18.11) if $\wp'$ has an appropriate value.

Series near z_0

18.13.49

$$(\wp' - i) = x \left[-2 - ix + \frac{5}{14}x^2 + \frac{3i}{28}x^3 + O(x^4) \right]$$

18.13.50 where $x = (z - z_0)^3$

$$18.13.51 \quad x = 2\alpha \left[1 - i\alpha - \frac{9}{7}\alpha^2 + \frac{13i\alpha^3}{7} + O(\alpha^4) \right],$$

18.13.52 where $\alpha = (\wp' - i)/(-4)$

Series near ω_2

18.13.53

$$\mathcal{P}' = 6e_2^2(z - \omega_2) \left[1 + 2v + 3v^2 + \frac{24}{7}v^3 + \frac{25}{7}v^4 + \frac{24}{7}v^5 + \frac{285}{91}v^6 + O(v^7) \right],$$

18.13.54 where $v = e_2(z - \omega_2)^2$

18.13.55

$$(z - \omega_2) = (\mathcal{P}'/6e_2^2) \left[1 - 2w + 9w^2 - \frac{360}{7}w^3 + 330w^4 - 2268w^5 + \frac{212058}{13}w^6 + O(w^7) \right],$$

18.13.56 where $w = \mathcal{P}'^{1/2}/9$ Other Series Involving ζ Reversed Series for Large $|\zeta|$

18.13.57

$$z = \zeta^{-1} \left[1 - \frac{\gamma}{7} + \frac{17\gamma^2}{143} - \frac{496\gamma^3}{3553} + O(\gamma^4) \right],$$

18.13.58

$$\gamma = \zeta^{-6}/20$$

Series near z_0

18.13.59

$$(\zeta - \zeta_0) = i \left[-\frac{u^2}{2} + \frac{u^8}{56} - \frac{3u^{14}}{5096} \right] + \left[\frac{u^5}{8} - \frac{u^{11}}{308} \right] + O(u^{17}),$$

18.13.60 where $u = (z - z_0)$ Series near ω_2

18.13.61

$$(\zeta - \eta_2) = -e_2(z - \omega_2) \left[1 + v + \frac{3}{5}v^2 + \frac{3}{7}v^3 + \frac{2}{7}v^4 + \frac{15}{77}v^5 + \frac{12}{91}v^6 + \frac{57}{637}v^7 + O(v^8) \right],$$

18.13.62

$$v = e_2(z - \omega_2)^2$$

18.13.63

$$(z - \omega_2) = \frac{(\zeta - \eta_2)}{-e_2} \left[1 - w + \frac{12w^2}{5} - \frac{267w^3}{35} + \frac{139w^4}{5} - \frac{30192w^5}{275} + \frac{1634208}{3575}w^6 + O(w^7) \right],$$

18.13.64

$$w = (\zeta - \eta_2)^2/e_2$$

Series Involving σ

18.13.65

$$\sigma = z - \frac{2 \cdot 3}{7!} z^7 - \frac{2^3 \cdot 3^3}{13!} z^{13} + \frac{2^6 \cdot 3^4 \cdot 23}{19!} z^{19}$$

$$+ \frac{2^7 \cdot 3^5 \cdot 5^2 \cdot 31}{25!} z^{25} + \frac{2^8 \cdot 3^8 \cdot 5 \cdot 9103}{31!} z^{31} - \frac{2^{12} \cdot 3^9 \cdot 5 \cdot 229 \cdot 2683}{37!} z^{37} - \frac{2^{14} \cdot 3^{10} \cdot 5 \cdot 23 \cdot 257 \cdot 18049}{43!} z^{43} - \frac{2^{15} \cdot 3^{12} \cdot 5 \cdot 59 \cdot 107895773}{49!} z^{49} + O(z^{55})$$

18.13.66

$$z = \sigma + \frac{\sigma^7}{2^3 \cdot 3 \cdot 5 \cdot 7} + \frac{41\sigma^{13}}{2^7 \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 13} + \frac{13 \cdot 337\sigma^{19}}{2^{10} \cdot 3^4 \cdot 5^3 \cdot 11 \cdot 17 \cdot 19} + \frac{31 \cdot 101\sigma^{25}}{2^{15} \cdot 3^5 \cdot 5 \cdot 11^2 \cdot 17 \cdot 23} + O(\sigma^{31})$$

Economized Polynomials ($0 \leq x \leq 1.53$)

$$18.13.67 \quad x^2 \mathcal{P}(x) = \sum_0^6 a_n x^{6n} + \epsilon(x) \\ |\epsilon(x)| < 2 \times 10^{-7}$$

$$\begin{array}{ll} a_0 = (-1)9.99999 \ 96 & a_4 = -(-9)2.20892 \ 47 \\ a_1 = (-2)3.57143 \ 20 & a_5 = (-10)1.74915 \ 35 \\ a_2 = (-5)9.80689 \ 93 & a_6 = -(-12)4.46863 \ 93 \\ a_3 = (-7)2.00835 \ 02 \end{array}$$

$$18.13.68 \quad x^3 \mathcal{P}'(x) = \sum_0^6 a_n x^{6n} + \epsilon(x) \\ |\epsilon(x)| < 4 \times 10^{-7}$$

$$\begin{array}{ll} a_0 = -2.00000 \ 00 & a_4 = -(-9)2.12719 \ 66 \\ a_1 = (-1)1.42857 \ 22 & a_5 = (-10)6.53654 \ 67 \\ a_2 = (-4)9.81018 \ 03 & a_6 = -(-11)1.70510 \ 78 \\ a_3 = (-6)3.00511 \ 93 \end{array}$$

$$18.13.69 \quad x\zeta(x) = \sum_0^6 a_n x^{6n} + \epsilon(x) \\ |\epsilon(x)| < 3 \times 10^{-8}$$

$$\begin{array}{ll} a_0 = (-1)9.99999 \ 98 & a_4 = (-10)6.12486 \ 14 \\ a_1 = -(-3)7.14285 \ 86 & a_5 = (-11)4.66919 \ 85 \\ a_2 = -(-6)8.91165 \ 65 & a_6 = (-12)1.25014 \ 65 \\ a_3 = -(-8)1.44381 \ 84 \end{array}$$

18.14. Lemniscatic Case

$$(g_2=1, g_3=0)$$

If $g_2 > 0$ and $g_3 = 0$, homogeneity relations allow us to reduce our consideration of $\mathcal{P}$ to $\mathcal{P}(z; 1, 0)$ ($\mathcal{P}'$, ζ and σ are handled similarly). Thus $\mathcal{P}(z; g_2, 0) = g_2^{\frac{1}{2}} \mathcal{P}(zg_2^{\frac{1}{2}}; 1, 0)$. The case $g_2=1, g_3=0$ is called the LEMNISCATIC case.

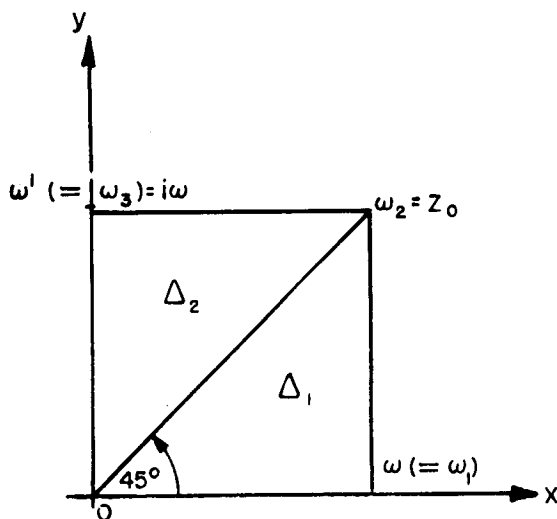


FIGURE 18.13

$\frac{1}{2}$ FPP; Reduction to Fundamental Triangle

$\Delta_1 \equiv \Delta 0\omega\omega_2$ is the Fundamental Triangle

$$\omega \approx 1.8540\ 74677\ 30137\ 192^8$$

Reduction for z_2 in Δ_2 : $z_1 = i\bar{z}_2$ is in Δ_1

$$18.14.1 \quad \mathcal{P}(z_2) = -\overline{\mathcal{P}}(z_1)$$

$$18.14.2 \quad \mathcal{P}'(z_2) = i\overline{\mathcal{P}'}(z_1)$$

$$18.14.3 \quad \zeta(z_2) = -i\overline{\zeta}(z_1)$$

$$18.14.4 \quad \sigma(z_2) = i\overline{\sigma}(z_1)$$

Special Values and Formulas

18.14.5

$$\Delta=1, H_1=H_3=2^{-\frac{1}{2}}, H_2=i/2,$$

$$m = \sin^2 45^\circ = \frac{1}{2}, q = e^{-\pi}$$

$$18.14.6 \quad \vartheta_2(0) = \vartheta_4(0) = (\omega\sqrt{2}/\pi)^{\frac{1}{2}}; \vartheta_3(0) = (2\omega/\pi)^{\frac{1}{2}}$$

$$18.14.7 \quad \omega = K(\sin^2 45^\circ) = \frac{\Gamma^2(\frac{1}{2})}{4\sqrt{\pi}} = \frac{\tilde{\omega}}{\sqrt{2}} \text{ where}$$

$\tilde{\omega} \approx 2.62205\ 75542\ 92119\ 81046\ 48395\ 89891\ 11941\ 36827\ 54951\ 43162$ is the Lemniscate constant [18.9]

⁸ This value was computed and checked by double precision methods on a desk calculator and is believed correct to 18S.

Values at Half-periods

	$\mathcal{P}$	$\mathcal{P}'$	ζ	σ
18.14.8 $\omega = \omega_1$	$e_1 = \frac{1}{2}$	0	$\eta = \pi/4\omega$	$e^{\pi/8}(2^{1/4})$
18.14.9 $\omega_2 = z_0$	$e_2 = 0$	0	$\eta + \eta'$	$e^{\pi/4}(\sqrt{2})e^{i\pi/4}$
18.14.10 $\omega' = \omega_3$	$e_3 = -\frac{1}{2}$	0	$\eta' = -\pi i/4\omega$	$ie^{\pi/8}(2^{1/4})$

Values along $(0, \omega)$

	$\mathcal{P}$	$\mathcal{P}'$	ζ	σ
18.14.11 $\omega/4$	$\frac{\sqrt{\alpha}}{2}(\sqrt{\alpha+2^{1/4}})(1+2^{1/4})$			
18.14.12 $\omega/2$	$\alpha/2$	$-\alpha$	$\frac{\pi}{8\omega} + \frac{\alpha}{2\sqrt{2}}$	$\frac{e^{\pi/32}(2^{1/16})}{\alpha^{\frac{1}{2}}}$
18.14.13 $2\omega/3$	$\frac{1}{2}\sqrt{1+\sec 30^\circ}$	$-\frac{\sqrt{2\sqrt{3}+3}}{\sqrt{3}}$	$\frac{2\eta}{3} + \sqrt{\frac{\mathcal{P}(2\omega/3)}{3}}$	$\frac{e^{\pi/16}(3^{1/8})}{(2+\sqrt{3})^{1/16}}$
18.14.14 $3\omega/4$	$\frac{\sqrt{\alpha}}{2}(\sqrt{\alpha-2^{1/4}})(1+2^{1/4})$			

$$\alpha = 1 + \sqrt{2}$$

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Values along $(0, z_0)$

	$\mathcal{P}$	$\mathcal{P}'$	ζ	σ
18.14.15 $z_0/4$	$-\frac{i}{2}(\alpha + \sqrt{2\alpha})$	$\alpha(\sqrt{\alpha} + \sqrt{2})e^{i\pi/4}$		$\frac{e^{\pi/64}(21/32)}{\alpha^{1/4}(\sqrt{\alpha} + \sqrt{2})^{1/4}} e^{i\pi/4}$
18.14.16 $z_0/2$	$-i/2$	$e^{i\pi/4}$	$\left[\frac{\pi}{4\omega\sqrt{2}} + \frac{1}{2}\right] e^{-i\pi/4}$	$e^{\pi/16}(21/8)e^{i\pi/4}$
18.14.17 $2z_0/3$	$-\frac{i}{2}\sqrt{\sec 30^\circ - 1}$	$\frac{e^{i\pi/4}\sqrt[4]{2\sqrt{3}-3}}{\sqrt{3}}$	$\frac{2\eta_2}{3} + \left[\frac{\mathcal{P}(2z_0/3)}{3}\right]^{1/2}$	$\frac{e^{\pi/9}e^{i\pi/4}(31/6)}{\sqrt[12]{2\sqrt{3}-3}}$
18.14.18 $3z_0/4$	$-\frac{i}{2}(\alpha - \sqrt{2\alpha})$	$\alpha(\sqrt{\alpha} - \sqrt{2})e^{i\pi/4}$		$\frac{e^{3\pi/64}(21/32)}{\alpha^{1/4}(\sqrt{\alpha} - \sqrt{2})^{1/4}} e^{i\pi/4}$

$$\alpha = 1 + \sqrt{2}$$

Duplication Formulas

$$\begin{aligned} \mathbf{18.14.19} \quad \mathcal{P}(2z) &= [\mathcal{P}^2(z) + \tfrac{1}{4}]^2 / \{\mathcal{P}(z)[4\mathcal{P}^2(z) - 1]\} \\ &= [\mathcal{P}^2(z) + \tfrac{1}{4}]^2 / \{\mathcal{P}(z)[4\mathcal{P}^2(z) - 1]\} \end{aligned}$$

18.14.20

$$\mathcal{P}'(2z) = (\beta + 1)(\beta^2 - 6\beta + 1) / [32\mathcal{P}'^3(z)], \quad \beta = 4\mathcal{P}^2(z)$$

$$\mathbf{18.14.21} \quad \zeta(2z) = 2\zeta(z) + \frac{6\mathcal{P}^2(z) - \frac{1}{2}}{2\mathcal{P}'(z)}$$

$$\mathbf{18.14.22} \quad \sigma(2z) = -\mathcal{P}'(z)\sigma^4(z)$$

Bisection Formulas ($0 < x < 2\omega$)**18.14.23**

$$\begin{aligned} \mathcal{P}\left(\frac{x}{2}\right) &= [\mathcal{P}^{\frac{1}{2}}(x) + \{\mathcal{P}(x) + \tfrac{1}{2}\}^{\frac{1}{2}}][\mathcal{P}^{\frac{1}{2}}(x) \pm \{\mathcal{P}(x) - \tfrac{1}{2}\}^{\frac{1}{2}}] \\ & \quad [\text{Use } + \text{ on } 0 < x \leq \omega, - \text{ on } \omega \leq x < 2\omega] \end{aligned}$$

18.14.24

$$\begin{aligned} \tfrac{1}{2}\mathcal{P}'\left(\frac{x}{2}\right) &= \mathcal{P}'(x) \mp [2\mathcal{P}(x) + \tfrac{1}{2}]\sqrt{\mathcal{P}(x) - \tfrac{1}{2}} \\ & \quad - [2\mathcal{P}(x) - \tfrac{1}{2}]\sqrt{\mathcal{P}(x) + \tfrac{1}{2}} \\ & \quad - 2\mathcal{P}^{3/2}(x) \quad (\text{See [18.13].}) \end{aligned}$$

[Use $-$ on $0 < x \leq \omega$, $+$ on $\omega \leq x < 2\omega$]

Complex Multiplication

$$\mathbf{18.14.25} \quad \mathcal{P}(iz) = -\mathcal{P}(z)$$

$$\mathbf{18.14.26} \quad \mathcal{P}'(iz) = i\mathcal{P}'(z)$$

$$\mathbf{18.14.27} \quad \zeta(iz) = -i\zeta(z)$$

$$\mathbf{18.14.28} \quad \sigma(iz) = i\sigma(z)$$

The above equations could be used as follows, e.g.: If z were real, iz would be purely imaginary.

Conformal Maps

Lemniscatic Case

Map: $f(z) = u + iv$

$$\mathcal{P}(z)$$

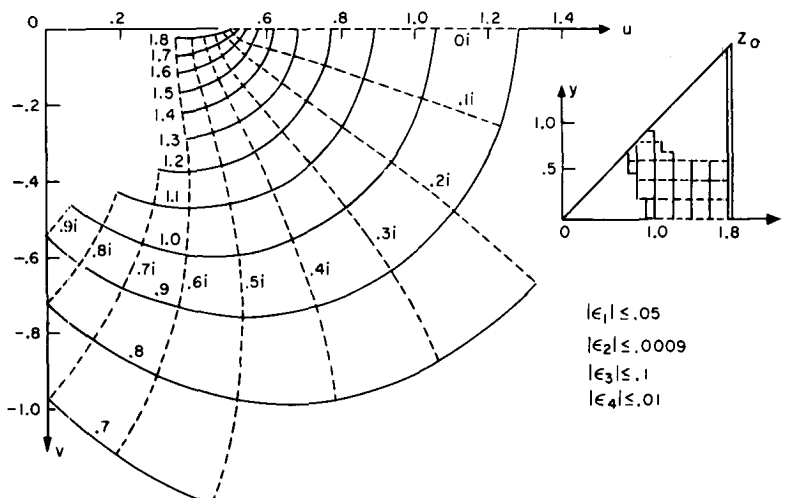
$$\text{Near zero: } \mathcal{P}(z) = \frac{1}{z^2} + \epsilon_1$$

$$\mathcal{P}(z) = \frac{1}{z^2} + \frac{z^2}{20} + \epsilon_2, \quad |z| < 1$$

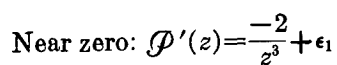
$$\text{Near } z_0: \mathcal{P}(z) = \frac{-(z - z_0)^2}{4} + \epsilon_3,$$

$$|z - z_0| < \sqrt{2}$$

$$\mathcal{P}(z) = \frac{-(z - z_0)^2}{4} + \frac{(z - z_0)^6}{80} + \epsilon_4$$



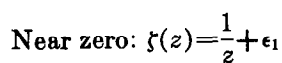
$$\begin{aligned} |\epsilon_1| &\leq .05 \\ |\epsilon_2| &\leq .0009 \\ |\epsilon_3| &\leq .1 \\ |\epsilon_4| &\leq .01 \end{aligned}$$



Near z_0 : $\mathcal{P}'(z) = \frac{-(z-z_0)}{2} + \epsilon_3$

$$\mathcal{P}'(z) = \frac{-2}{z^3} + \frac{z}{10} + \epsilon_2$$

$$\mathcal{P}'(z) = \frac{-(z-z_0)}{2} + \frac{3(z-z_0)^3}{40} + \epsilon_1$$



$$f(z) = \frac{1}{z} - \frac{z^3}{60} + \epsilon_2, \quad |z| < 1$$

Near z_0 : $\zeta(z) = \zeta_0 + \frac{(z-z_0)^3}{12} + \epsilon_3$,

$$|z - z_0| < \sqrt{2}$$

$$\zeta(z) = \zeta_0 + \frac{(z-z_0)^3}{12} - \frac{(z-z_0)^7}{560} + \epsilon_4$$

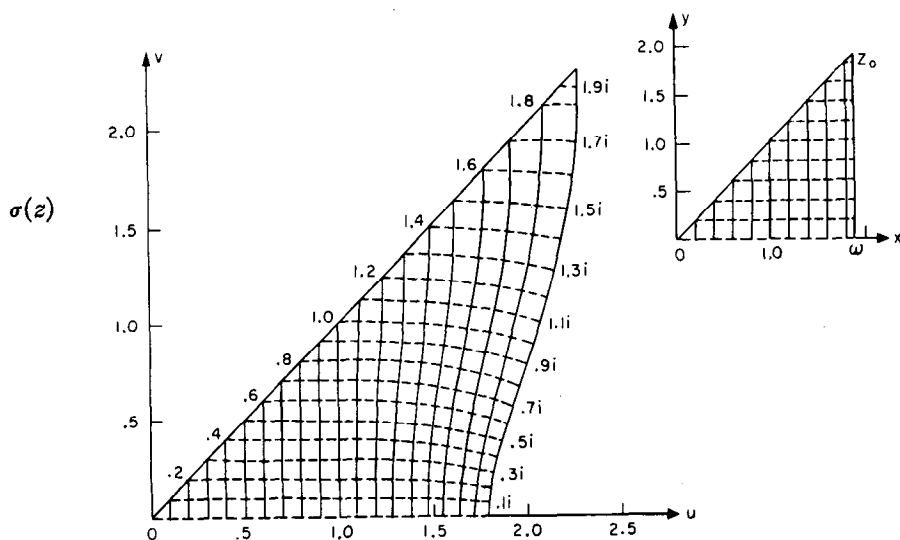
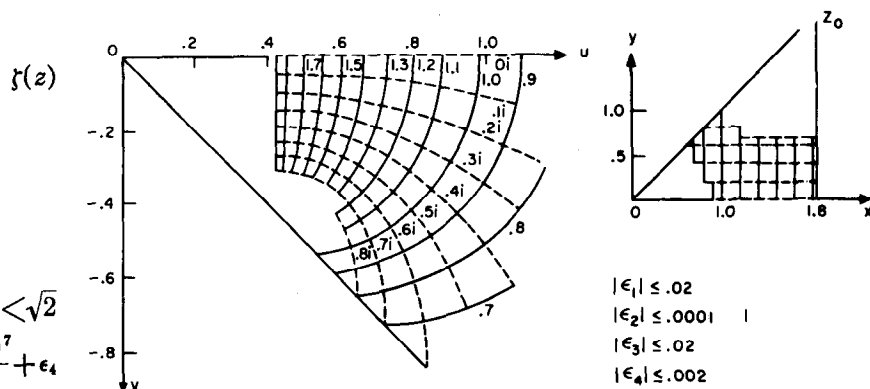


FIGURE 18.14

Coefficients for Laurent Series for $\mathcal{P}$, $\mathcal{P}'$, and ζ $(c_n = 0 \text{ for } m \text{ odd})$

k	EXACT c_{2k}	APPROXIMATE c_{2k}
1	1/20	.05
2	$1/(3 \cdot 20^2) = 1/1200$	.8333 $\dots \times 10^{-3}$
3	$2/(3 \cdot 13 \cdot 20^3) = 1/156000$	.641025 $\dots \times 10^{-5}$
4	$5/(3 \cdot 13 \cdot 17 \cdot 20^4) = 1/21216000$	.47134 23831 07088 98944×10^{-7}
5	$2/(3^2 \cdot 13 \cdot 17 \cdot 20^5) = 1/(31824 \times 10^5)$	.31422 82554 04725 99296×10^{-9}
6	$10/(3^3 \cdot 13^2 \cdot 17 \cdot 20^6) = 1/(4964544 \times 10^5)$	.20142 83688 49183 32882×10^{-11}
7	$4/(3 \cdot 13^2 \cdot 17 \cdot 29 \cdot 20^7) = 1/(7998432 \times 10^7)$	.12502 45048 02941 37651×10^{-13}
8	$2453/(3^4 \cdot 11 \cdot 13^2 \cdot 17^2 \cdot 29 \cdot 20^8) = 958203125/(1262002599 \times 10^{18})$	.75927 19109 76468 59917×10^{-16}
9	$2 \cdot 5 \cdot 7 \cdot 61/(3^3 \cdot 13^3 \cdot 17^2 \cdot 29 \cdot 37 \cdot 20^9) = 833984375/(18394643943 \times 10^{17})$	.45338 43533 93461 06092×10^{-18}

$$c_{2k} \leq \frac{c_2^k}{3^{k-1}}, \quad k=1, 2, \dots$$

Other Series Involving $\mathcal{P}$ Reversed Series for Large $|\mathcal{P}|$

18.14.29

$$z = (\mathcal{P}^{-1})^{1/2} \left[1 + \frac{w}{5} + \frac{w^2}{6} + \frac{5w^3}{26} + \frac{35w^4}{136} + \frac{3w^5}{8} + \frac{231w^6}{400} + \frac{429w^7}{464} + \frac{195w^8}{128} + \frac{12155w^9}{4736} + \frac{46189w^{10}}{10496} + O(w^{11}) \right],$$

18.14.30 $w = \mathcal{P}^{-2}/8$, and z is in the Fundamental Triangle (Figure 18.13) if $\mathcal{P}$ has an appropriate value.

Series near z_0

$$18.14.31 \quad 2\mathcal{P} = -x + \frac{x^3}{5} - \frac{2x^5}{75} + \frac{x^7}{325} + O(x^9),$$

$$18.14.32 \quad x = (z - z_0)^2/2$$

$$18.14.33 \quad x = -\left[w + \frac{w^3}{5} + \frac{7w^5}{75} + \frac{11w^7}{195} + O(w^9) \right]$$

$$w = 2\mathcal{P}$$

Series near ω

18.14.34

$$(\mathcal{P} - e_1) = v + v^2 + \frac{4v^3}{5} + \frac{3v^4}{5} + \frac{32v^5}{75} + \frac{22v^6}{75} + \frac{64v^7}{325} + O(v^8),$$

$$18.14.35 \quad v = (z - \omega)^2/2$$

18.14.36

$$v = y \left[1 - y + \frac{6y^2}{5} - \frac{8y^3}{5} + \frac{172y^4}{75} - \frac{52y^5}{15} + \frac{1064y^6}{195} + O(y^7) \right],$$

$$18.14.37 \quad y = (\mathcal{P} - e_1)$$

Other Series Involving $\mathcal{P}'$ Reversed Series for Large $|\mathcal{P}'|$

18.14.38

$$z = Au \left[1 - \frac{v}{5} + \frac{5v^3}{39} - \frac{7v^4}{51} + O(v^5) \right], \quad u = (\mathcal{P}'^{1/3})^{-1} e^{i\pi/3},$$

18.14.39 $A = 2^{1/3}$, $v = Au^4/6$, and z is in the Fundamental Triangle (Figure 18.13) if $\mathcal{P}'$ has an appropriate value.

Series near z_0

18.14.40

$$\mathcal{P}' = \frac{1}{2}(z - z_0) \left[-1 + 3w - \frac{10w^2}{3} + \frac{35w^3}{13} + O(w^4) \right],$$

18.14.41

$$w = (z - z_0)^4/20$$

18.14.42

$$(z - z_0) = 2\mathcal{P}' \left[1 + \frac{3u}{5} + \frac{5u^2}{3} + \frac{84u^3}{13} + O(u^4) \right],$$

18.14.43

$$u = 4\mathcal{P}'^4$$

Series near ω

18.14.44

$$\mathcal{P}' = x \left[1 + x^2 + \frac{3}{5}x^4 + \frac{3}{10}x^6 + \frac{2}{15}x^8 + \frac{11}{200}x^{10} + O(x^{12}) \right],$$

18.14.45

$$x = (z - \omega)$$

18.14.46

$$x = \mathcal{P}' - \mathcal{P}'^3 + \frac{12\mathcal{P}'^5}{5} - \frac{15\mathcal{P}'^7}{2} + \frac{80\mathcal{P}'^9}{3} - \frac{819\mathcal{P}'^{11}}{8} + O(\mathcal{P}'^{13})$$

Other Series Involving ζ Reversed Series for Large $|\zeta|$

$$18.14.47 \quad z = \zeta^{-1} \left[1 - \frac{v}{5} + \frac{v^2}{7} - \frac{136v^3}{1001} + \frac{1349v^4}{9163} + O(v^5) \right],$$

18.14.48

$$v = \zeta^{-4}/12$$

Series near z_0

18.14.49

$$(\zeta - \zeta_0) = \frac{1}{4}(z - z_0)^3 \left[\frac{1}{3} - \frac{v}{7} + \frac{2v^2}{33} - \frac{v^3}{39} + O(v^4) \right],$$

18.14.50

$$v = (z - z_0)^4/20$$

Series near ω

18.14.51

$$(\zeta - \eta) = -\frac{x}{2} - \frac{x^3}{6} - \frac{x^5}{20} - \frac{x^7}{70} - \frac{x^9}{240} - \frac{x^{11}}{825} - \frac{11x^{13}}{31200} - \frac{x^{15}}{9750} + O(x^{17}),$$

18.14.52

$$x = (z - \omega)$$

18.14.53

$$x = w - \frac{w^3}{3} + \frac{7w^5}{30} - \frac{13w^7}{63} + \frac{929w^9}{4536} - \frac{194w^{11}}{891} + \frac{942883w^{13}}{3891888} + O(w^{15})$$

18.14.54 $w = -2(\zeta - \eta)$ Series Involving σ

18.14.55

$$\sigma = z - \frac{z^5}{2 \cdot 5!} + \frac{3^2 z^9}{2^2 \cdot 9!} + \frac{3 \cdot 23 z^{13}}{2^3 \cdot 13!} + \frac{3 \cdot 107 z^{17}}{2^4 \cdot 17!} + \frac{3^3 \cdot 7 \cdot 23 \cdot 37 z^{21}}{2^5 \cdot 21!} + \frac{3^2 \cdot 313 \cdot 503 z^{25}}{2^6 \cdot 25!} - \frac{3^4 \cdot 7 \cdot 685973 z^{29}}{2^7 \cdot 29!} + O(z^{33})$$

18.14.56

$$z = \sigma + \frac{\sigma^5}{2^4 \cdot 3 \cdot 5} + \frac{\sigma^9}{2^9 \cdot 3 \cdot 7} + \frac{17 \cdot 113 \sigma^{13}}{2^{13} \cdot 3^4 \cdot 7 \cdot 11 \cdot 13} + \frac{122051 \sigma^{17}}{2^{19} \cdot 3^5 \cdot 7^2 \cdot 11 \cdot 17} + \frac{5 \cdot 13 \sigma^{21}}{2^{23} \cdot 3^2 \cdot 11 \cdot 19} + O(\sigma^{25})$$

Economized Polynomials ($0 \leq x \leq 1.86$)

$$18.14.57 \quad x^2 \mathcal{P}(x) = \sum_0^6 a_n x^{4n} + \epsilon(x)$$

$$|\epsilon(x)| < 2 \times 10^{-7}$$

$$\begin{aligned} a_0 &= (-1)9.99999 \ 98 & a_4 &= (-8)4.81438 \ 20 \\ a_1 &= (-2)4.99999 \ 62 & a_5 &= (-10)2.29729 \ 21 \\ a_2 &= (-4)8.33352 \ 77 & a_6 &= (-12)4.94511 \ 45 \\ a_3 &= (-6)6.40412 \ 86 \end{aligned}$$

$$18.14.58 \quad x^3 \mathcal{P}'(x) = \sum_0^6 a_n x^{4n} + \epsilon(x)$$

$$|\epsilon(x)| < 4 \times 10^{-7}$$

$$\begin{aligned} a_0 &= -2.00000 \ 00 & a_4 &= (-7)6.58947 \ 52 \\ a_1 &= (-1)1.00000 \ 02 & a_5 &= (-9)5.59262 \ 49 \\ a_2 &= (-3)4.99995 \ 38 & a_6 &= (-11)5.54177 \ 69 \\ a_3 &= (-5)6.41145 \ 59 \end{aligned}$$

$$18.14.59 \quad x\zeta(x) = \sum_0^6 a_n x^{4n} + \epsilon(x)$$

$$|\epsilon(x)| < 3 \times 10^{-8}$$

$$\begin{aligned} a_0 &= (-1)9.99999 \ 99 & a_4 &= -(-9)2.57492 \ 62 \\ a_1 &= -(-2)1.66666 \ 74 & a_5 &= -(-11)5.67008 \ 00 \\ a_2 &= -(-4)1.19036 \ 70 & a_6 &= (-13)9.70015 \ 80 \\ a_3 &= -(-7)5.86451 \ 63 \end{aligned}$$

18.15. Pseudo-Lemniscatic Case

$$(g_2 = -1, g_3 = 0)$$

If $g_2 < 0$ and $g_3 = 0$, homogeneity relations allow us to reduce our consideration of $\mathcal{P}$ to $\mathcal{P}(z; -1, 0)$. Thus

$$18.15.1 \quad \mathcal{P}(z; g_2, 0) = |g_2|^{1/2} \mathcal{P}(z|g_2|^{1/4}; -1, 0)$$

[$\mathcal{P}'$, ζ and σ are handled similarly]. Because of its similarity to the lemniscatic case, we refer to the case $g_2 = -1$, $g_3 = 0$ as the pseudo-lemniscatic case. It plays the same role (period ratio unity) for $\Delta < 0$ as does the lemniscatic case for $\Delta > 0$.

$$\begin{aligned} \omega_2 &= \sqrt{2} \times (\text{real half-period for lemniscatic case}) \\ &= \tilde{\omega} \text{ (the Lemniscate Constant—see 18.14.7)} \end{aligned}$$

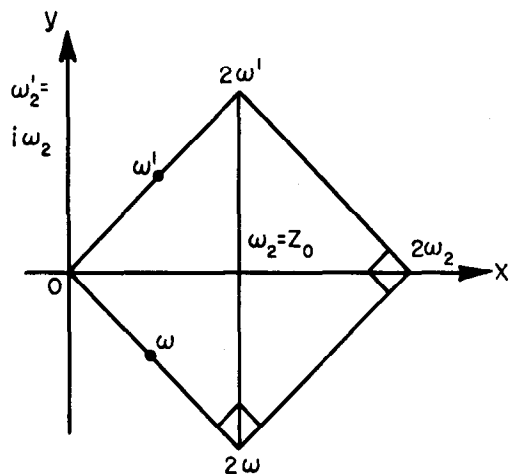


FIGURE 18.15

Special Values and Relations

$$18.15.2 \quad \Delta = -1, g_2 = -1, g_3 = 0$$

18.15.3

$$H_1 = -i/\sqrt{2}, H_2 = \frac{1}{2}, H_3 = i/\sqrt{2}, \quad m = \frac{1}{2}, q = ie^{-\pi/2}$$

18.15.4

$$\vartheta_2(0) = R2^{1/4}e^{i\pi/8}, \vartheta_3(0) = Re^{i\pi/8}, \vartheta_4(0) = Re^{-i\pi/8},$$

18.15.5

$$\text{where } R = \sqrt{\omega_2 \sqrt{2}/\pi}$$

Values at Half-Periods

	$\mathcal{P}$	$\mathcal{P}'$	ζ	σ
18.15.6 $\omega \equiv \omega_1$	$i/2$	0	$\frac{1}{2}(\eta_2 - \eta_2')$	$e^{-i\pi/4}e^{\pi/8}(2^{1/4})$
18.15.7 ω_2	0	0	$\eta_2 = \pi/2\omega_2$	$e^{\pi/4}\sqrt{2}$
18.15.8 $\omega' \equiv \omega_3$	$-i/2$	0	$\frac{1}{2}(\eta_2 + \eta_2')$	$e^{i\pi/4}e^{\pi/8}(2^{1/4})$
18.15.9 ω_2'	0	0	$\eta_2' = -i\eta_2$	$i\sigma(\omega_2)$

Relations with Lemniscatic Values

$$18.15.10 \quad \mathcal{P}(z; -1, 0) = i\mathcal{P}(ze^{i\pi/4}; 1, 0)$$

$$18.15.12 \quad \zeta(z; -1, 0) = e^{i\pi/4}\zeta(ze^{i\pi/4}; 1, 0)$$

$$18.15.11 \quad \mathcal{P}'(z; -1, 0) = e^{3\pi i/4}\mathcal{P}'(ze^{i\pi/4}; 1, 0)$$

$$18.15.13 \quad \sigma(z; -1, 0) = e^{-i\pi/4}\sigma(ze^{i\pi/4}; 1, 0)$$

Numerical Methods

18.16. Use and Extension of the Tables

Example 1. Lemniscatic Case

(a) Given $z = x + iy$ in the Fundamental Triangle, find $\mathcal{P}(\mathcal{P}', \zeta, \sigma)$ more accurately than can be done with the maps.

σ —Use Maclaurin series throughout the Fundamental Triangle. Five terms give at least six significant figures, six terms at least ten. $\mathcal{P}, \zeta$ —Use Laurent's series directly "near" 0, (if $|z| < 1$, four terms give at least eight significant figures for $\mathcal{P}$, nine for ζ ; five terms at least ten significant figures for $\mathcal{P}$, eleven for ζ). Use Taylor's series directly "near" z_0 . Elsewhere (unless approximately seven or eight significant figures are insufficient) use economized polynomials to obtain $\mathcal{P}(x)$, $\mathcal{P}'(x)$ and/or $\zeta(x)$ as appropriate. To get $\mathcal{P}(iy)$, $\mathcal{P}'(iy)$ and/or $\zeta(iy)$, use Laurent's series for "small" y , otherwise use economized polynomials to compute $\mathcal{P}(y)$, $\mathcal{P}'(y)$ and/or $\zeta(y)$, then use complex multiplication to obtain $\mathcal{P}(iy)$, $\mathcal{P}'(iy)$ and/or $\zeta(iy)$. Finally, use appropriate addition formula to get $\mathcal{P}(z)$ and/or $\zeta(z)$.

$\mathcal{P}'$ —Use Laurent's series directly "near" 0 (if $|z| < 1$, four terms give at least six significant figures, five terms at least eight significant figures). Elsewhere, either use economized polynomials and addition formula as for $\mathcal{P}$ and ζ , or get $\mathcal{P}'^2 = 4\mathcal{P}^3 - \mathcal{P}$ and extract appropriate square root ($\mathcal{P}' \geq 0$).

(b) Given $\mathcal{P}(\mathcal{P}', \zeta, \sigma)$ corresponding to a point in the Fundamental Triangle, compute z more accurately than can be done with the maps. Only a few significant figures are obtainable from the use of any of the given (truncated) reversed series, except in a small neighborhood of the center of the series. For greater accuracy, use inverse interpolation procedures.

Example 2. Equianharmonic Case

(a) Given $z = x + iy$ in the Fundamental Triangle, find $\mathcal{P}(\mathcal{P}', \zeta, \sigma)$ more accurately than can be done with the maps.

σ —Use Maclaurin series throughout the Fundamental Triangle. Four terms give at least eleven significant figures, five terms at least twenty one.

$\mathcal{P}, \zeta$ —Use Laurent's series directly "near" 0 (if $|z| < 1$, four terms give at least 10S for $\mathcal{P}$, 11S for ζ ; five terms at least 13S for $\mathcal{P}$, 14S for ζ). Elsewhere (unless approximately seven or eight significant figures are insufficient) use economized polynomials to obtain $\mathcal{P}(x)$, $\mathcal{P}'(x)$ and/or $\zeta(x)$, as appropriate. To get $\mathcal{P}(iy)$, $\mathcal{P}'(iy)$ and/or $\zeta(iy)$, use Laurent's series. Then use appropriate addition formula to get $\mathcal{P}(z)$ and/or $\zeta(z)$.

$\mathcal{P}'$ —Use Laurent's series directly "near" 0 (if $|z| < 1$, four terms give at least 8S, five terms at least 11S). Elsewhere, either proceed as for $\mathcal{P}$ and ζ , or get $\mathcal{P}'^2 = 4\mathcal{P}^3 - 1$ and extract appropriate square root ($\mathcal{G}\mathcal{P}' \geq 0$).

(b) Given $\mathcal{P}(\mathcal{P}', \zeta, \sigma)$ corresponding to a point in the Fundamental Triangle, compute z more accurately than can be done with the maps. Only a few significant figures are obtainable from the use of any of the given (truncated) reversed series, except in a small neighborhood of the center of the series. For greater accuracy, use inverse interpolation procedures.

Example 3. Given period ratio a , find parameters m (of elliptic integrals and Jacobi's functions of chapter 16) and q (of $\wp$ functions).

m —In both the cases $\Delta > 0$ and $\Delta < 0$, the period ratio is equal to $K'(m)/K(m)$ (see 18.9). Knowing K'/K , if $1 < K'/K \leq 3$, use **Table 17.3** to find m ; if $K'/K > 3$, use the method of **Example 6** in chapter 17. An alternative method is to use **Table 18.3** to obtain the necessary entries, thence use

$$m = (e_2 - e_3)/(e_1 - e_3) \text{ in case } \Delta > 0,$$

$$m = \frac{1}{2} - 3e_2/4H_2 \text{ in case } \Delta < 0.$$

q —In both the cases $\Delta > 0$ and $\Delta < 0$, the period ratio determines the exponent for q [$q = e^{-\pi a}$ if $\Delta > 0$, $q = ie^{-\pi a/2}$ if $\Delta < 0$]. Hence enter **Table 4.16** [$e^{-\pi x}$, $x = 0(.01)1$] and multiply the results as appropriate [e.g., $e^{-4.72\pi} = (e^{-\pi})^4(e^{-.72\pi})$].

Determination of Values at Half-Periods, Invariants and Related Quantities from Given Periods (Table 18.3)

$\Delta > 0$

Given ω and ω' , form $\omega'/i\omega$ and enter **Table 18.3**. Multiply the results obtained by the appropriate power of ω (see footnotes of **Table 18.3**) to obtain value desired.

Example 4.

Given $\omega = 10$, $\omega' = 11i$, find e_i , g_i , and Δ .

Here $\omega'/i\omega = 1.1$, so that direct reading of **Table 18.3** gives

$$\begin{aligned} e_1(1) &= 1.6843 \ 041 \\ e_2(1) &= -.2166 \ 258 \ (= -e_1 - e_3) \\ e_3(1) &= -1.4676 \ 783 \\ g_2(1) &= 10.0757 \ 7364 \\ g_3(1) &= 2.1420 \ 1000. \end{aligned}$$

Multiplying by appropriate powers of $\omega = 10$ we obtain

$$\begin{aligned} e_1 &= .01684 \ 3041 \\ e_2 &= -.00216 \ 6258 \\ e_3 &= -.01467 \ 6783 \\ g_2 &= 1.0075 \ 77364 \times 10^{-3} \\ g_3 &= 2.1420 \ 1000 \times 10^{-6} \\ \Delta &= 8.9902 \ 3191 \times 10^{-10} \end{aligned}$$

whence

$\Delta < 0$

Given ω_2 and ω_2' , form $\omega_2'/i\omega_2$ and enter **Table 18.3**. Multiply the results obtained by the appropriate power of ω_2 (see footnotes of **Table 18.3**) to obtain value desired.

Example 4.

Given $\omega_2 = 10$, $\omega_2' = 11i$, find e_i , g_i , and Δ .

Here $\omega_2'/i\omega_2 = 1.1$, so that direct reading of **Table 18.3** gives

$$\begin{aligned} e_1(1) &= -.2166 \ 2576 + 3.0842 \ 589i \\ e_2(1) &= .4332 \ 5152 = -2\mathcal{R}(e_1) \\ e_3(1) &= \bar{e}_1(1) \\ g_2(1) &= -37.4874 \ 912 \\ g_3(1) &= 16.5668 \ 099. \end{aligned}$$

Multiplying by appropriate powers of $\omega_2 = 10$ we obtain

$$\begin{aligned} e_1 &= -.00216 \ 62576 + .03084 \ 2589i \\ e_2 &= .00433 \ 25152 \\ e_3 &= \bar{e}_1 \\ g_2 &= -3.7487 \ 4912 \times 10^{-3} \\ g_3 &= 1.6566 \ 8099 \times 10^{-6} \\ \Delta &= -6.0092 \ 019 \times 10^{-8} \end{aligned}$$

whence

Example 5. ($\Delta > 0$)

Given $\omega=10$, $\omega'=55i$, find η , η' , $\sigma(\omega)$, $\sigma(\omega')$ and $\sigma(\omega_2)$.

Forming $\omega'/i\omega=5.5$ and entering **Table 18.3** we obtain $\eta=.82246704$, $\sigma(\omega)=.96045\ 40$. Using Legendre's relation we find $\eta'=\eta\omega'-\pi i/2=2.9527\ 723i$. Since interpolation for $\sigma(\omega')$ and $\sigma(\omega+\omega')$ is difficult, use is made of **18.3.15–18.3.17** together with **18.3.4** and **18.3.6**. Values of g_2 , g_3 and e_1 can be read directly to eight significant figures and e_3 to about five significant figures giving $g_2=8.1174\ 243$, $g_3=4.4508\ 759$, $e_1=1.6449\ 341$, and $e_3=-.82247$. Use of **18.3.6** yields $H_3=.00174\ 69$ and $H_2=.00174\ 69i$. Application of **18.3.15–18.3.17** yields $\sigma(\omega')/i=.0071177$ and $\sigma(\omega_2)=-.002016-.01055i$. Multiplying the results obtained by the appropriate powers of ω we obtain $\eta=.08224\ 6704$, $\eta'=.29527\ 723i$, $\sigma(\omega)=9.6045\ 40$, $\sigma(\omega')=.071177i$ and $\sigma(\omega_2)=-.02016-.1055i$.

Example 5. ($\Delta < 0$)

Given $\omega_2=1000$, $\omega_2'=1004i$, find η_2 , η_2' , $\sigma(\omega_2)$, $\sigma(\omega_2')$ and $\sigma(\omega')$.

With $\omega_2'/i\omega_2=1.004$, four point interpolation in **Table 18.3** gives $\eta_2=1.5626\ 756$, $\eta_2'=-1.5726\ 664i$, $\sigma(\omega_2)=1.1805\ 028$, $\sigma(\omega_2')=1.1901\ 52i$ and $\sigma(\omega')=.475084+.476717i$.

Multiplying the results obtained by the appropriate powers of ω_2 gives $\eta_2=.00156\ 26756$, $\eta_2'=-.00157\ 26664i$, $\sigma(\omega_2)=1180.5028$, $\sigma(\omega_2')=1190.152i$ and $\sigma(\omega')=475.084+476.717i$.

Determination of Periods from Given Invariants (Table 18.1.) **$\Delta > 0$**

Given $g_2 > 0$ and $g_3 > 0$ such that $\Delta = g_2^3 - 27g_3^2 > 0$ (if $g_3 = 0$, $|\omega'| = \omega$; see lemniscatic case), compute $\bar{g}_2 = g_2g_3^{-2/3}$. From **Table 18.1**, determine $\omega g_3^{1/6}$ and $\omega' g_3^{1/6}$, thence ω and ω' .

Example 6.

Given $g_2=10$, $g_3=2$, find ω and ω' . With $\bar{g}_2 = g_2g_3^{-2/3}=6.2996\ 05249$, from **Table 18.1** $\omega g_3^{1/6}=1.1267\ 806$ and $\omega' g_3^{1/6}=1.2324\ 295i$ whence $\omega=1.003847$ and $\omega'=1.097970i$.

Example 7.

Given $g_2=8$, $g_3=4$, find ω and ω' . With $\bar{g}_2 = g_2g_3^{-2/3}=3.1748\ 02104$, from **Table 18.1** $\omega g_3^{1/6}=1.2718\ 310$ and $\omega' g_3^{1/6}=1.8702\ 425i$ whence $\omega=1.009453$ and $\omega'=1.484413i$.

 $\Delta < 0$

Given g_2 and $g_3 > 0$ such that $\Delta = g_2^3 - 27g_3^2 < 0$ (if $g_3 = 0$, $|\omega_2'| = \omega_2$; see pseudo-lemniscatic case), compute $\bar{g}_2 = g_2g_3^{-2/3}$. From **Table 18.1**, determine $\omega_2 g_3^{1/6}$ and $\omega_2' g_3^{1/6}$, thence ω_2 and ω_2' .

Example 6.

Given $g_2=-10$, $g_3=2$, find ω_2 and ω_2' . With $\bar{g}_2 = g_2g_3^{-2/3}=-10/1.5874\ 0105=-6.2996\ 053$, from **Table 18.1** $\omega_2 g_3^{1/6}=1.5741\ 349$ and $\omega_2' g_3^{1/6}=1.7124\ 396i$ whence $\omega_2=1.40239\ 48$ and $\omega_2'=1.52561\ 02i$.

Example 7.

Given $g_2=7$, $g_3=6$, find ω_2 and ω_2' . With $\bar{g}_2 = g_2g_3^{-2/3}=7/3.3019\ 2725=2.119974$, from **Table 18.1** $\omega_2 g_3^{1/6}=1.3423\ 442$ and $\omega_2' g_3^{1/6}=3.1441\ 141i$ whence $\omega_2=.99579\ 976$ and $\omega_2'=2.33241\ 83i$.

Computation of $\mathcal{P}$, $\mathcal{P}'$, or ζ for Given z and Arbitrary g_2, g_3

(or arbitrary periods from which g_2 and g_3 can be computed—
in any case, periods must be known, at least approximately)

First reduce the problem (if necessary) to computation for a point z in the Fundamental Rectangle by use of appropriate results from **18.2**.

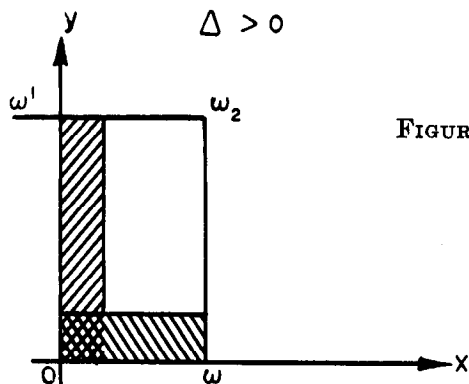
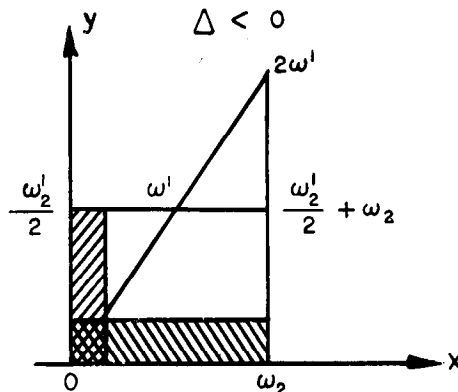


FIGURE 18.16



Method 1 (as accurate as desired)

If both x and y are "small," (point in double-cross hatched region) use Laurent's series in z directly. If either x or y is "large," use Laurent's series on $0x$, then on $0y$ and finally use an addition formula. (For $\mathcal{P}'$ an alternative is to get $\mathcal{P}$, then compute the appropriate root of $\mathcal{P}'^2 = 4\mathcal{P}^3 - g_2\mathcal{P} - g_3$; see 18.8.)

$$\Delta > 0$$

Method 2 (for $\mathcal{P}$ or $\mathcal{P}'$ only)

Compute e_i ($i=1,2,3$) (if only g_2, g_3 are given use Table 18.1 to get the periods, then get e_i in Table 18.3; if periods are also given, use Table 18.3 directly). In any case, obtain $m(=[e_2-e_3]/[e_1-e_3])$, thence Jacobi's functions $\operatorname{sn}(z^*|m)$, $\operatorname{cn}(z^*|m)$, $\operatorname{dn}(z^*|m)$, from 16.4 and 16.21 and $\mathcal{P}$ or $\mathcal{P}'$ from 18.9.11–18.9.12.

Method 3 (accuracy limited by Table 4.16 of $e^{-\pi x}$ and by the method of getting periods).

Obtain periods, their ratio a , then $q=e^{-\pi a}$ from Table 4.16. Hence get $\vartheta_i(0)$, $i=2,3,4$ from truncated series 18.10.21–23. Compute appropriate ϑ functions for $z=x$ and for $z=iy$, whence get $\mathcal{P}(x)$, $\mathcal{P}'(x)$ and/or $\zeta(x)$, $\mathcal{P}(iy)$, $\mathcal{P}'(iy)$ and/or $\zeta(iy)$, then use an addition formula (if either x or y is "small", it is probably easier to use Laurent's series).

Example 8. Given $z=.07+.1i$, $g_2=10$, $g_3=2$, find $\mathcal{P}$.

Using Laurent's series directly with

$$\begin{aligned} c_2 &= .5 \\ c_3 &= .07142 \ 85714 \\ c_4 &= .08333 \ 33333 \\ c_5 &= .00974 \ 02597 \\ z^{-2} &= -22.97193 \ 820 - 63.06022 \ 25i \\ +c_2 z^2 &= -\ .00255 \ 000 + \ .00700 \ 00i \\ +c_3 z^4 &= -\ .00001 \ 214 - \ .00001 \ 02i \\ +c_4 z^6 &= + \ .00000 \ 024 - \ .00000 \ 01i \end{aligned}$$

$$\mathcal{P}(z) = -22.97450 \ 010 - 63.05323 \ 28i.$$

Example 9. Given $z=15+73i$, $g_2=8$, $g_3=4$, find $\mathcal{P}$. From Example 7, $\omega=1.009453$, $\omega'=1.484413i$. From Table 18.3, $e_1=1.61803 \ 37$, $e_3=-.99999 \ 96$, whence $m=.14589 \ 79$. From 18.2.18 with $M=7$ and $N=24$, $\mathcal{P}(.867658+1.748176i)=\mathcal{P}(15+73i)$. Since z lies in R_2 , by 18.2.31 $\mathcal{P}(15+73i)=\overline{\mathcal{P}}(.867658+1.22065i)$. From 16.4 with $z^*=1.40390+1.97505i$, $\operatorname{sn}(z^*|m)=2.46550+1.96527i$. Using 18.9.11, $\mathcal{P}(15+73i)=-.57743+.067797i$.

$$\Delta < 0$$

Method 2 (for $\mathcal{P}$ or $\mathcal{P}'$ only)

Compute e_2 and H_2 (if only g_2, g_3 are given, use Table 18.1 to get the periods, then get e_i in Table 18.3; if periods are also given use Table 18.3 directly). In any case, obtain $m(=\frac{1}{2}-3e_2/4H_2)$ thence Jacobi's functions $\operatorname{sn}(z'|m)$, $\operatorname{cn}(z'|m)$, $\operatorname{dn}(z'|m)$, from 16.4 and 16.21 and $\mathcal{P}$ or $\mathcal{P}'$ from 18.9.11–18.9.12.

Method 3 (accuracy limited as in the case $\Delta > 0$).

Obtain periods, their ratio a , thence $q_2=e^{-\pi a/2}$ from Table 4.16. Then proceed as in the case $\Delta > 0$, using corresponding formulas.

Example 8. Given $z=.1+.03i$, $g_2=-10$, $g_3=2$, find $\mathcal{P}$.

Using Laurent's series directly with

$$\begin{aligned} c_2 &= -.5 \\ c_3 &= .07142 \ 85714 \\ c_4 &= .80333 \ 33333 \\ z^{-2} &= 76.59287 \ 938 - 50.50079 \ 960i \\ c_2 z^2 &= -.00455 \ 000 - \ .00300 \ 000i \\ c_3 z^4 &= +.00000 \ 334 + \ .00000 \ 780i \\ c_4 z^6 &= -.00000 \ 002 + \ .00000 \ 011i \end{aligned}$$

$$\mathcal{P}(z) = 76.58833 \ 270 - 50.50379 \ 169i.$$

Example 9. Given $z=1.75+3.6i$, $g_2=7$, $g_3=6$, find $\mathcal{P}$. From Example 7, $\omega_2=.99579 \ 98$, $\omega'_2=2.33241 \ 83i$. Using 18.2.18 with $M=1$, $N=1$, $\mathcal{P}(1.75+3.6i)=\mathcal{P}(-.24159 \ 96-1.064836i)=\mathcal{P}(.24159 \ 96+1.0648 \ 36i)$. With $\Delta < 0$ from Table 18.3, $e_1=-.81674 \ 362 + .50120 \ 90i$, $e_2=1.63348 \ 724$, $e_3=-.81674 \ 362 - .50120 \ 90i$ whence $m=.01014 \ 3566$, $H_1^2=1.58144 \ 50$, so that $z'=2zH_1^2=.76415 \ 29+3.367959i$. From 16.4, $\operatorname{cn}(z'|m)=4.00543 \ 66 - 12.32465 \ 69i$. Applying 18.9.11, $\mathcal{P}(1.75+3.6i)=-.960894-.383068i$.

$$\Delta > 0$$

Example 10. Given $\omega=10$, $\omega'=20i$, find $\zeta(9+19i)$ by use of theta functions, 18.10 and addition formulas.

For the period ratio $a=\omega'/\omega i=2$ with the aid of **Table 4.16**, $q=e^{-2\pi}=0.00186\ 74427$.

Using the truncated approximations 18.10.21–18.10.23 we compute the theta functions for argument zero. Using 16.27.1–16.27.4 we compute the theta functions for arguments v where $z=x$ and $z=iy$. Then, with 18.10.5–18.10.7 together with 18.10.9 and 18.10.18 we obtain $\zeta(9)=.09889\ 5484$, $\zeta(19i)=-.00120\ 0155i$, $\mathcal{P}(9)=.01706\ 9647$, $\mathcal{P}'(9)=-.00125\ 8460$, $\mathcal{P}(19i)=-.00861\ 2615$, $\mathcal{P}'(19i)=-.00003\ 757i$. Using the addition formula 18.4.3, we obtain $\zeta(9+19i)=.07439\ 49-.00046\ 88i$.

Use of Table 18.2 in Computing $\mathcal{P}$, $\mathcal{P}'$, ζ for Special Period Ratios

If the problem is reduced to computing $\mathcal{P}$, $\mathcal{P}'$, ζ in the Fundamental Rectangle for the case when the real half-period is unity and pure imaginary half-period is ia , for certain values of a **Table 18.2** may be used. Consider $\mathcal{P}$ as an example. If $|z|$ is "small", then use Laurent's series directly for $\mathcal{P}(z)$ [invariants for use in the series are given in **Table 18.3**].

If x is "large" and y "small" use **Table 18.2** to obtain $x^2\mathcal{P}(x)$ and $x^3\mathcal{P}'(x)$, thence $\mathcal{P}(x)$ and $\mathcal{P}'(x)$; use Laurent's series to obtain $\mathcal{P}(iy)$ and $\mathcal{P}'(iy)$; finally, use addition formula 18.4.1.

For x "small" and y "large", reverse the procedure. For both x and y "large," use **Table 18.2** to obtain $\mathcal{P}(x)$, $\mathcal{P}'(x)$, $\mathcal{P}(iy)$ and $\mathcal{P}'(iy)$, thence use addition formula 18.4.1.

Similar procedures apply to $\mathcal{P}'$ or ζ . For $\mathcal{P}'$, one can also first obtain $\mathcal{P}$, then compute $\mathcal{P}'^2=4\mathcal{P}^3-g_2\mathcal{P}-g_3$ and extract the appropriate square root (see 18.8 re choice of sign for $\mathcal{P}'$).

$$\Delta > 0$$

Example 11. Compute $\mathcal{P}(.8+i)$ when $a=1.2$. Using **Table 18.2** or Laurent's series 18.5.1–4 with $g_2=9.15782\ 851$ and $g_3=3.23761\ 717$ from **Table 18.3**, $\mathcal{P}(.8)=1.92442\ 11$, $\mathcal{P}'(.8)=-2.76522\ 05$, $\mathcal{P}(i)=-1.40258\ 06$ and $\mathcal{P}'(i)=-1.19575\ 58i$. Using the addition formula 18.4.1 $\mathcal{P}(.8+i)=-.381433-.149361i$.

Example 12. Compute $\zeta(.02+3i)$ for $a=4$. Using **Table 18.2** or Laurent's series 18.5.1–5 with $g_2=8.11742\ 426$ and $g_3=4.45087\ 587$ from **Table 18.3**,

$$\begin{aligned}\zeta(.02) &= 49.99999\ 89, \\ \mathcal{P}(.02) &= 2500.00016, \\ \mathcal{P}'(.02) &= -249999.98376, \\ \zeta(3i) &= .89635\ 173i, \\ \mathcal{P}(3i) &= -.82326\ 511, \\ \mathcal{P}'(3i) &= -.00249\ 829i.\end{aligned}$$

Applying the addition formula 18.4.3, $\zeta(.02+3i)=.016465+.89635i$.

$$\Delta < 0$$

Example 10. Given $\omega_2=5$, $\omega'_2=7i$ find $\mathcal{P}'(3+2i)$ by use of theta functions, 18.10 and addition formulas.

With the use of **Table 4.16** and 18.10.2, $q=ie^{-.7\pi}=0.11090\ 12784i$.

The theta functions are computed for argument zero using 18.10.21–18.10.23 and the theta functions for arguments v_1 and v_2 corresponding to $z=z_1+z_2$ using 16.27.1–16.27.4. Using 18.10.5–18.10.6 together with 18.10.10, we find $\mathcal{P}(3)=.10576\ 946$, $\mathcal{P}(2i)=-.24497\ 773$, $\mathcal{P}'(3)=-.07474140$, $\mathcal{P}'(2i)=-.25576\ 007i$. The addition formula 18.4.1 yields $\mathcal{P}(3+2i)=.01763\ 210-.07769\ 187i$, and 18.4.2 yields $\mathcal{P}'(3+2i)=-.00069\ 182+.04771\ 305i$.

$$\Delta < 0$$

Example 11. Compute $\mathcal{P}(.9+.1i)$ for $a=1.05$. Using **Table 18.2** or Laurent's series 18.5.1–4 with $g_2=-42.41653\ 54$ and $g_3=9.92766\ 62$ from **Table 18.3**, $\mathcal{P}(.9)=.34080\ 33$, $\mathcal{P}'(.9)=-2.164801$, $\mathcal{P}(.1i)=-99.97876$, $\mathcal{P}'(.1i)=-2000.4255i$. With the addition formula 18.4.1 $\mathcal{P}(.9+.1i)=.231859-.215149i$.

Example 12. Compute $\mathcal{P}'(.4+.9i)$ for $a=2$. Using **Table 18.2** or Laurent's series 18.5.1–4, with

$$\begin{aligned}g_2 &= 4.54009\ 85, \\ g_3 &= 8.38537\ 94\end{aligned}$$

from **Table 18.3**,

$$\begin{aligned}\mathcal{P}(.4) &= 6.29407\ 07, \\ \mathcal{P}'(.4) &= -30.99041, \\ \mathcal{P}(.9i) &= -1.225548, \\ \mathcal{P}'(.9i) &= -3.19127\ 03i.\end{aligned}$$

Using the addition formulas 18.4.1–2,

$$\mathcal{P}'(.4+.9i)=1.10519\ 76-.56489\ 00i.$$

Computation of σ for Given z and Arbitrary g_2 and g_3

(or periods from which g_2 and g_3 can be computed—in any case, periods must be known, at least approximately)

First reduce the problem (if necessary) to computation for a point z in the Fundamental Rectangle (see 18.2). After final reduction let z denote the point obtained.

$$\Delta > 0$$

If $\Re z > \omega/2$ or,

$\Im z > \omega'/2$, use duplication formula

$$\sigma(z) = -\mathcal{P}'(z/2)\sigma^4(z/2),$$

obtaining $\sigma(z/2)$ by use of Maclaurin series for σ and $\mathcal{P}'(z/2)$ by method explained above. Otherwise, simply use Maclaurin series for σ directly.

An alternate method is to use theta functions 18.10 first computing q and $\vartheta_i(0)$, $i=2, 3, 4$.

$$\Delta > 0$$

Example 13. Compute $\sigma(.4+1.3i)$ for $g_2=8$, $g_3=4$. From **Example 7**, $\omega=1.009453$ and $\omega'=1.484413i$. Since $\Im z > \omega'/2$, the Maclaurin series 18.5.6 is used to obtain $\sigma(z/2)=\sigma(.2+.65i)=.19543\ 86+.64947\ 28i$, the Laurent series 18.5.4 to obtain $\mathcal{P}'(.2+.65i)=5.02253\ 80-3.56066\ 93i$. The duplication formula 18.4.8 gives $\sigma(.4+1.3i)=.278080+1.272785i$.

$$\Delta < 0$$

If $\Re z > \omega_2/2$ or

$\Im z > \omega'_2/4$, use duplication formula as in case $\Delta > 0$. Otherwise, use Maclaurin series for σ directly.

$$\Delta < 0$$

Example 13. Compute $\sigma(.8+.4i)$ for $g_2=7$, $g_3=6$. From **Example 7**, $\omega_2=.99579\ 976$, $\omega'_2=2.33241\ 83i$. Since $\Re z > \omega_2/2$, the Maclaurin series 18.5.6 is used to obtain $\sigma(z/2)=\sigma(.4+.2i)=.40038\ 019+.19962\ 017i$, the Laurent series 18.5.4 to obtain $\mathcal{P}'(.4+.2i)=-3.70986\ 70+22.218544i$. The duplication formula 18.4.8 gives $\sigma(.8+.4i)=.81465\ 765+.38819\ 473i$.

Given $\sigma(\mathcal{P}, \mathcal{P}', \zeta)$ corresponding to a point in the Fundamental Rectangle, as well as g_2 and g_3 or the equivalent, find z .

Only a few significant figures are obtainable from the use of any of the given (truncated) reversed series, except in a small neighborhood of the center of the series. For greater accuracy, use inverse interpolation procedures.

If the given function does not correspond to a value of z in the Fundamental Rectangle (see Conformal Maps) the problem can always be reduced to this case by the use of appropriate reduction formulas in 18.2. This process is relatively simple for $\mathcal{P}(z)$, more difficult for the other functions (e.g. if $\Delta > 0$ and $\mathcal{P}=a+ib$, where $b > 0$, simply consider $\overline{\mathcal{P}}=a-ib$ and find z_1 in R_1 [Figure 18.1]; then compute $z_2=\overline{z}_1+2\omega'$, the point in R_2 corresponding to the given $\mathcal{P}$).

$$\Delta > 0$$

Example 14. Given $\mathcal{P}=1-i$, $g_2=10$, $g_3=2$, find z . Using the first three terms of the reversed series 18.5.25 $z_1 \approx .727+.423i$. The Laurent series 18.5.1 gives

$$\mathcal{P}(z_1) = \mathcal{P}(.727+.423i) = .825-.895i$$

and

$$\mathcal{P}(z_2) = \mathcal{P}(.697+.393i) = .938-1.038i.$$

Inverse interpolation gives $z_1^{(1)} = .707+.380i$. Repeated applications of the above procedure yield $z = .706231+.379893i$.

$$\Delta < 0$$

Example 14. Given $\mathcal{P}=1+i$, $g_2=-10$, $g_3=2$, find z . From **Example 6**, $\omega_2=1.40239\ 48$ and $\omega'_2=1.52561\ 02i$. Since $b > 0$, z exists in R_2 and z is computed with $\overline{\mathcal{P}}$. Using 18.5.25 with $\alpha_2=-1.25$, $\alpha_3=.25$, $u=[(\overline{\mathcal{P}})^{-1}]^{1/2}$ and the coefficients c_n from **Example 8**

$$2u = 1.55377\ 3973+.64359\ 42493i$$

$$c_2 u^5 = .08044\ 9281-.19422\ 17466i$$

$$c_3 u^7 = -.01961\ 9359+.00812\ 66047i$$

$$\frac{\alpha_2^2 u^9}{3} = -.10115\ 7160-.04190\ 06673i$$

$\Delta > 0$

Example 15. Given $\zeta = 10 - 15i$, $g_2 = 8$, $g_3 = 4$, find z . Using the reversed series 18.5.40 with

$$A_5 = -.13333\ 333,$$

$$A_7 = -.02857\ 14286,$$

$$u = .03076\ 923076 + .04615\ 384615i$$

$$A_8 u^5 = -.00000\ 001402 + .00000\ 006860i$$

$$A_7 u^7 = -.00000\ 000004 - .00000\ 000003i$$

$$z = .03076\ 921670 + .04615\ 391472i.$$

 $\Delta < 0$

Stopping with the term in u^7 , $z_1 \approx .81 + .23i$. Assuming $\Delta z = -.03 - .01i$, using 18.5.1, $\mathcal{P}(.81 + .23i) = .91410\ 95 - .86824\ 37i$, $\mathcal{P}(.78 + .22i) = 1.03191\ 60 - .91795\ 22i$; with inverse interpolation $z_1^{(1)} = .7725 + .2404i$. Repeated applications of inverse interpolation yield $z = .772247 - .239258i$.

Example 15. Given $\sigma = .4 + .1i$, $g_2 = 7$, $g_3 = 6$, find z . Using the reversed series 18.5.70 with $\gamma_2 = .14583$, $\gamma_3 = .05$

$$\sigma = +.40000\ 000 + .10000\ 000i$$

$$\frac{\gamma_2 \sigma^5}{5} = +.00011\ 783 + .00032\ 696i$$

$$\frac{\gamma_3 \sigma^7}{7} = -.00000\ 208 + .00001\ 432i$$

$$\frac{3\gamma_2^2 \sigma^9}{14} = -.00000\ 093 + .00000\ 126i$$

$$\frac{19\gamma_2\gamma_3\sigma^{11}}{55} = -.00000\ 013 + .00000\ 006i$$

$$z = .40011\ 469 + .10034\ 260i$$

Methods of Computation of $\mathcal{P}(\mathcal{P}', \zeta$ or $\sigma)$ for Given z and Given g_2, g_3 (or the equivalent), with the Use of Automatic Digital Computing Machinery

(a) Integration of Differential Equation

$\mathcal{P}$ and $\mathcal{P}'$ may be generated for any z close enough to a "known point" $z^*(\mathcal{P}(z^*)$ and $\mathcal{P}'(z^*)$ being given) by integrating $\mathcal{P}'' = 6\mathcal{P}^2 - g_2/2$. A program to do this on SWAC, via a modification of the Hammer-Hollingsworth method (MTAC, July 1955, pp. 92-96) due to Dr. P. Henrici, exists at Numerical Analysis Research, UCLA (code number 00600, written by W. L. Wilson, Jr.). The program has been tested numerically in the equianharmonic case, using integration steps of various sizes. For example, if one starts with $z^* = \omega_2$, using an "integration step" (h, k) , where h and k are respectively the horizontal and vertical components of a step, with (h, k) having one of the six values $(\pm 2h_0, 0)$, $(\pm h_0, \pm k_0)$, $h_0 = \omega_2/2000$, $k_0 = |\omega_2'|/2000$, one can expect almost 8S in $\mathcal{P}$ and 7S in $\mathcal{P}'$ after 1000 steps, unless z is too near a pole.

(b) Use of Series

The process of reducing the computation problem to one in which z is in the Fundamental Rectangle can obviously be mechanized. Inside the Fundamental Rectangle the direct use of Laurent's series is appropriate when the period

ratio a is not too large. However, if $a \geq \sqrt{3}(\Delta > 0)$ or $a \geq 2\sqrt{3}(\Delta < 0)$, the series will diverge at the far corner of the Fundamental Rectangle, so that use may be made of an appropriate duplication formula. Alternatively, one may compute the functions on $0x$ and $0y$, then use an addition formula. Even so, the series will diverge at $z = ia$ if $a \geq 2(\Delta > 0)$ and at $z = ia/2$ if $a \geq 4(\Delta < 0)$.

For great accuracy, multiple precision operations might be necessary. Double precision floating point mode has been used in a program, written for SWAC, to compute $\mathcal{P}$, $\mathcal{P}'$ and ζ .

For computation of σ , use of the Maclaurin series throughout the Fundamental Rectangle is probably simplest (series converges for all z).

Mention should be made of the possible use of the series defining the ϑ functions. These series converge for all complex v , and the computation of $\mathcal{P}$, $\mathcal{P}'$, ζ and σ by 18.10.5-18.10.8 could easily be mechanized. The series involved have the advantage of converging very fast, even in case $\Delta < 0$, where $|q| \leq e^{-\pi/2}$ ($q \leq e^{-\pi}$ if $\Delta > 0$).

Use of Maps

If the problem (of computing $\mathcal{P}$, $\mathcal{P}'$, ζ or σ for given z) is reduced to the case where the real half-period is unity and imaginary half-period is one of those used in the maps in 18.8 inspection of the

appropriate figure will give the value of $\mathcal{P}(z)$ [$\xi(z)$ or $\sigma(z)$] to 2–3S. If $\mathcal{P}'$ is wanted instead, get $\mathcal{P}$, use 18.6.3 to obtain $\mathcal{P}'^2$ and select sign (s) of $\mathcal{P}'$ appropriately. (See Conformal Mapping (18.8) for choice of sign of square root of $\mathcal{P}'^2$).

Computation of z_0

Given g_2, g_3 (or equivalent)

Since $z_0^2 \mathcal{P}(z_0) = 0$, the Laurent's series gives

$$0 = 1 + c_2 u^2 + c_3 u^3 + c_4 u^4 + \dots$$

where $u = z_0^2$. We may solve this equation [by Graeffe's (root-squaring) process or otherwise] for its absolutely smallest root [having found an

approximation to $|z_0|$ by Graeffe's process, we may use the fact that $z_0 = \omega + iy_0 (\Delta > 0)$, $z_0 = \omega_2 + iy_0 (\Delta < 0)$ to obtain an approximation to z_0].

It is noted that y_0/ω is a monotonic decreasing function of (period ratio) $a \geq 1$ for $\Delta > 0$ and

$$[1 \geq y_0/\omega > \frac{2}{\pi} \operatorname{arccosh} \sqrt{3} (\approx .7297)].$$

y_0/ω_2 is a monotonic increasing function of a for $\Delta < 0$ and

$$[0 \leq y_0/\omega_2 < \frac{2}{\pi} \operatorname{arccosh} \sqrt{3}]$$

Further data is available from Table 18.2 or from Conformal Maps defined by $\mathcal{P}(z)$.

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19. Parabolic Cylinder Functions

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The author acknowledges permission from H.M. Stationery Office to draw freely from [19.11] the material in the introduction, and the tabular values of $W(a, x)$ for $a = -5(1)5$, $\pm x = 0(.1)5$. Other tables of $W(a, x)$ and the tables of $U(a, x)$ and $V(a, x)$ were prepared on EDSAC 2 at the University Mathematical Laboratory, Cambridge, England, using a program prepared by Miss Joan Walsh for solution of general second order linear homogeneous differential equations with quadratic polynomial coefficients. The auxiliary tables were prepared at the Computation Laboratory of the National Bureau of Standards.

¹ The University Mathematical Laboratory, Cambridge, England. (Prepared under contract with the National Bureau of Standards.)

19. Parabolic Cylinder Functions

Mathematical Properties

19.1. The Parabolic Cylinder Functions

Introductory

These are solutions of the differential equation

$$19.1.1 \quad \frac{d^2 y}{dx^2} + (ax^2 + bx + c)y = 0$$

with two real and distinct standard forms

$$19.1.2 \quad \frac{d^2 y}{dx^2} - (\tfrac{1}{4}x^2 + a)y = 0$$

$$19.1.3 \quad \frac{d^2 y}{dx^2} + (\tfrac{1}{4}x^2 - a)y = 0$$

The functions

19.1.4

$$y(a, x) \quad y(a, -x) \quad y(-a, ix) \quad y(-a, -ix)$$

are all solutions either of 19.1.2 or of 19.1.3 if any one is such a solution.

Replacement of a by $-ia$ and x by $xe^{i\pi}$ converts 19.1.2 into 19.1.3. If $y(a, x)$ is a solution of 19.1.2, then 19.1.3 has solutions:

19.1.5

$$y(-ia, xe^{i\pi}) \quad y(-ia, -xe^{i\pi}) \\ y(ia, -xe^{-i\pi}) \quad y(ia, xe^{-i\pi})$$

Both variable x and the parameter a may take on general complex values in this section and in many subsequent sections. Practical applications appear to be confined to real solutions of real equations; therefore attention is confined to such solutions, and, in general, formulas are given for the two equations 19.1.2 and 19.1.3 independently. The principal computational consequence of the remarks above is that reflection in the y -axis produces an independent solution in almost all cases (Hermite functions provide an exception), so that tables may be confined either to positive x or to a single solution of 19.1.2 or 19.1.3.

$$\text{The Equation } \frac{d^2 y}{dx^2} - \left(\frac{1}{4}x^2 + a\right)y = 0$$

19.2. Power Series in x

Even and odd solutions of 19.1.2 are given by

19.2.1

$$y_1 = e^{-\frac{1}{2}x^2} M\left(\tfrac{1}{2}a + \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}x^2\right) \\ = e^{-\frac{1}{2}x^2} \left\{ 1 + (a + \tfrac{1}{2}) \frac{x^2}{2!} + (a + \tfrac{1}{2})(a + \tfrac{5}{2}) \frac{x^4}{4!} + \dots \right\} \\ = e^{-\frac{1}{2}x^2} {}_1F_1\left(\tfrac{1}{2}a + \tfrac{1}{4}; \tfrac{1}{2}; \tfrac{1}{2}x^2\right)$$

19.2.2

$$= e^{\frac{1}{2}x^2} M\left(-\tfrac{1}{2}a + \tfrac{1}{4}, \tfrac{1}{2}, -\tfrac{1}{2}x^2\right) \\ = e^{\frac{1}{2}x^2} \left\{ 1 + (a - \tfrac{1}{2}) \frac{x^2}{2!} + (a - \tfrac{1}{2})(a - \tfrac{5}{2}) \frac{x^4}{4!} + \dots \right\}$$

19.2.3

$$y_2 = xe^{-\frac{1}{2}x^2} M\left(\tfrac{1}{2}a + \tfrac{3}{4}, \tfrac{3}{2}, \tfrac{1}{2}x^2\right) \\ = e^{-\frac{1}{2}x^2} \left\{ x + (a + \tfrac{3}{2}) \frac{x^3}{3!} + (a + \tfrac{3}{2})(a + \tfrac{7}{2}) \frac{x^5}{5!} + \dots \right\}$$

19.2.4

$$= xe^{\frac{1}{2}x^2} M\left(-\tfrac{1}{2}a + \tfrac{3}{4}, \tfrac{3}{2}, -\tfrac{1}{2}x^2\right) \\ = e^{\frac{1}{2}x^2} \left\{ x + (a - \tfrac{3}{2}) \frac{x^3}{3!} + (a - \tfrac{3}{2})(a - \tfrac{7}{2}) \frac{x^5}{5!} + \dots \right\}$$

these series being convergent for all values of x (see chapter 13 for $M(a, c, z)$).

Alternatively,

19.2.5

$$y_1 = 1 + a \frac{x^2}{2!} + \left(a^2 + \tfrac{1}{2}\right) \frac{x^4}{4!} + \left(a^3 + \tfrac{7}{2}a\right) \frac{x^6}{6!} \\ + \left(a^4 + 11a^2 + \tfrac{15}{4}\right) \frac{x^8}{8!} + \left(a^5 + 25a^3 + \tfrac{211}{4}a\right) \frac{x^{10}}{10!} + \dots$$

19.2.6

$$y_2 = x + a \frac{x^3}{3!} + \left(a^2 + \tfrac{3}{2}\right) \frac{x^5}{5!} + \left(a^3 + \tfrac{13}{2}a\right) \frac{x^7}{7!} \\ + \left(a^4 + 17a^2 + \tfrac{63}{4}\right) \frac{x^9}{9!} + \left(a^5 + 35a^3 + \tfrac{531}{4}a\right) \frac{x^{11}}{11!} + \dots$$

in which non-zero coefficients a_n of $x^n/n!$ are connected by

$$19.2.7 \quad a_{n+2} = a \cdot a_n + \frac{1}{4}n(n-1)a_{n-2}$$

19.3. Standard Solutions

These have been chosen to have the asymptotic behavior exhibited in 19.8. The first is Whitaker's function [19.8, 19.9] in a more symmetrical notation.

19.3.1

$$U(a, x) = D_{-a-\frac{1}{2}}(x) = \cos \pi(\frac{1}{4} + \frac{1}{2}a) \cdot Y_1 - \sin \pi(\frac{1}{4} + \frac{1}{2}a) \cdot Y_2$$

19.3.2

$$V(a, x) = \frac{1}{\Gamma(\frac{1}{2}-a)} \{ \sin \pi(\frac{1}{4} + \frac{1}{2}a) \cdot Y_1 + \cos \pi(\frac{1}{4} + \frac{1}{2}a) \cdot Y_2 \}$$

in which

$$19.3.3 \quad Y_1 = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{1}{4}-\frac{1}{2}a)}{2^{\frac{1}{2}a+\frac{1}{2}}} y_1 = \sqrt{\pi} \frac{\sec \pi(\frac{1}{4} + \frac{1}{2}a)}{2^{\frac{1}{2}a+\frac{1}{2}} \Gamma(\frac{3}{4} + \frac{1}{2}a)} y_1$$

$$19.3.4 \quad Y_2 = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{3}{4}-\frac{1}{2}a)}{2^{\frac{1}{2}a-\frac{1}{2}}} y_2 = \sqrt{\pi} \frac{\csc \pi(\frac{1}{4} + \frac{1}{2}a)}{2^{\frac{1}{2}a-\frac{1}{2}} \Gamma(\frac{1}{4} + \frac{1}{2}a)} y_2$$

19.3.5

$$U(a, 0) = \frac{\sqrt{\pi}}{2^{\frac{1}{2}a+\frac{1}{2}} \Gamma(\frac{3}{4} + \frac{1}{2}a)} \\ U'(a, 0) = -\frac{\sqrt{\pi}}{2^{\frac{1}{2}a-\frac{1}{2}} \Gamma(\frac{1}{4} + \frac{1}{2}a)}$$

19.3.6

$$V(a, 0) = \frac{2^{\frac{1}{2}a+\frac{1}{2}} \sin \pi(\frac{3}{4} - \frac{1}{2}a)}{\Gamma(\frac{3}{4} - \frac{1}{2}a)} \\ V'(a, 0) = \frac{2^{\frac{1}{2}a+\frac{1}{2}} \sin \pi(\frac{1}{4} - \frac{1}{2}a)}{\Gamma(\frac{1}{4} - \frac{1}{2}a)}$$

In terms of the more familiar $D_n(x)$ of Whitaker,

$$19.3.7 \quad U(a, x) = D_{-a-\frac{1}{2}}(x)$$

19.3.8

$$V(a, x) = \frac{1}{\pi} \Gamma(\frac{1}{2}+a) \{ \sin \pi a \cdot D_{-a-\frac{1}{2}}(x) + D_{-a-\frac{1}{2}}(-x) \}$$

19.4. Wronskian and Other Relations

$$19.4.1 \quad W\{U, V\} = \sqrt{2/\pi}$$

19.4.2

$$\pi V(a, x) = \Gamma(\frac{1}{2}+a) \{ \sin \pi a \cdot U(a, x) + U(a, -x) \}$$

19.4.3

$$\Gamma(\frac{1}{2}+a)U(a, x) = \pi \sec^2 \pi a \{ V(a, -x) - \sin \pi a \cdot V(a, x) \}$$

19.4.4

$$\frac{\Gamma(\frac{1}{4}-\frac{1}{2}a) \cos \pi(\frac{1}{4} + \frac{1}{2}a)}{\sqrt{\pi} 2^{\frac{1}{2}a-\frac{1}{2}}} y_1 = 2 \sin \pi(\frac{3}{4} + \frac{1}{2}a) \cdot Y_1 \\ = U(a, x) + U(a, -x)$$

19.4.5

$$-\frac{\Gamma(\frac{3}{4}-\frac{1}{2}a) \sin \pi(\frac{1}{4} + \frac{1}{2}a)}{\sqrt{\pi} 2^{\frac{1}{2}a-\frac{1}{2}}} y_2 = 2 \cos \pi(\frac{3}{4} + \frac{1}{2}a) \cdot Y_2 \\ = U(a, x) - U(a, -x)$$

19.4.6

$$\sqrt{2\pi}U(-a, \pm ix) = \\ \Gamma(\frac{1}{2}+a) \{ e^{-i\pi(\frac{1}{2}a-\frac{1}{2})}U(a, \pm x) + e^{i\pi(\frac{1}{2}a-\frac{1}{2})}U(a, \mp x) \}$$

19.4.7

$$\sqrt{2\pi}U(a, \pm x) = \\ \Gamma(\frac{1}{2}-a) \{ e^{-i\pi(\frac{1}{2}a+\frac{1}{2})}U(-a, \pm ix) + e^{i\pi(\frac{1}{2}a+\frac{1}{2})}U(-a, \mp ix) \}$$

19.5. Integral Representations

A full treatment is given in [19.11] section 4. Representations are given here for $U(a, z)$ only; others may be derived by use of the relations given in 19.4.

$$19.5.1 \quad U(a, z) = \frac{\Gamma(\frac{1}{2}-a)}{2\pi i} e^{-\frac{1}{2}z^2} \int_{\alpha} e^{zs - \frac{1}{2}s^2} s^{a-\frac{1}{2}} ds$$

$$19.5.2 \quad = \frac{\Gamma(\frac{1}{2}-a)}{2\pi i} e^{\frac{1}{2}z^2} \int_{\beta} e^{-\frac{1}{2}t^2} (z+t)^{a-\frac{1}{2}} dt$$

where α and β are the contours shown in Figures 19.1 and 19.2.

When $a + \frac{1}{2}$ is a positive integer these integrals become indeterminate; in this case

$$19.5.3 \quad U(a, z) = \frac{1}{\Gamma(\frac{1}{2}+a)} e^{-\frac{1}{2}z^2} \int_0^{\infty} e^{-zs - \frac{1}{2}s^2} s^{a-\frac{1}{2}} ds$$

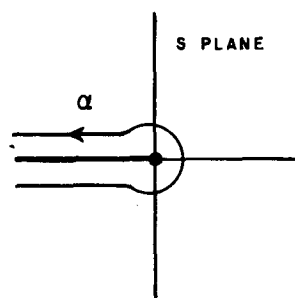


FIGURE 19.1

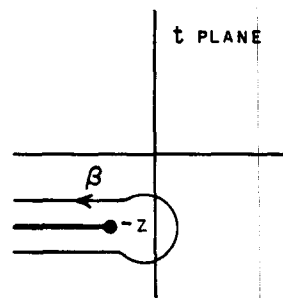
 $-\pi < \arg s < \pi$


FIGURE 19.2

 $-\pi < \arg(z+t) < \pi$

$$19.5.4 \quad U(a, z) = \frac{1}{\sqrt{2\pi i}} e^{iz^2} \int_{\epsilon} e^{-zs + \frac{1}{2}s^2} s^{-a-\frac{1}{2}} ds$$

$$19.5.5 \quad = \frac{e^{(a-\frac{1}{2})\pi i}}{\sqrt{2\pi i}} e^{iz^2} \int_{\epsilon_3} e^{zs + \frac{1}{2}s^2} s^{-a-\frac{1}{2}} ds$$

$$19.5.6 \quad = \frac{e^{-(a-\frac{1}{2})\pi i}}{\sqrt{2\pi i}} e^{iz^2} \int_{\epsilon_4} e^{zs + \frac{1}{2}s^2} s^{-a-\frac{1}{2}} ds$$

where ϵ , ϵ_3 and ϵ_4 are shown in Figures 19.3 and 19.4.

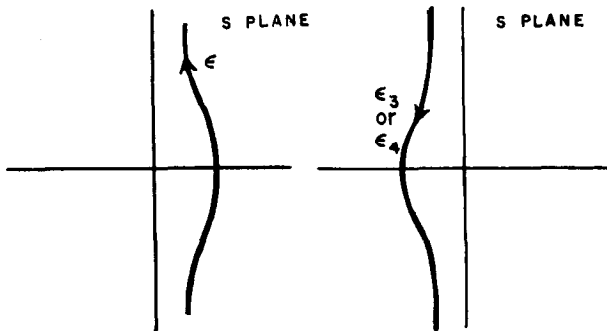


FIGURE 19.3

 $-\frac{1}{2}\pi < \arg s < \frac{1}{2}\pi$

FIGURE 19.4

 On ϵ_3 $\frac{1}{2}\pi < \arg s < \frac{3}{2}\pi$

 On ϵ_4 $-\frac{3}{2}\pi < \arg s < -\frac{1}{2}\pi$

19.5.7

$$U(a, z) = \frac{\Gamma(\frac{3}{4} - \frac{1}{2}a)}{2^{1/2 + \frac{1}{2}\pi i}} \int_{\zeta_1} e^{iz^2 t} (1+t)^{1/2 - \frac{1}{2}a} (1-t)^{-1/2 - \frac{1}{2}a} dt$$

19.5.8

$$= \frac{\Gamma(\frac{3}{4} - \frac{1}{2}a)}{2^{1/2 + \frac{1}{2}\pi i}} \int_{\eta_1} \frac{1}{2} z e^{v(\frac{1}{4}z^2 + v)} v^{1/2 - \frac{1}{2}a} (\frac{1}{4}z^2 - v)^{-1/2 - \frac{1}{2}a} dv$$

19.5.9

$$U(a, z) = \frac{i\Gamma(\frac{1}{4} - \frac{1}{2}a)}{2^{1/2 + \frac{1}{2}\pi i}} \int_{\eta_1} \frac{1}{2} z e^{-iz^2 t} (1+t)^{-1/2 - \frac{1}{2}a} (1-t)^{1/2 - \frac{1}{2}a} dt$$

19.5.10

$$= \frac{i\Gamma(\frac{1}{4} - \frac{1}{2}a)}{2^{1/2 + \frac{1}{2}\pi i}} \int_{\eta_1} e^{-v(\frac{1}{4}z^2 + v)} v^{-1/2 - \frac{1}{2}a} (\frac{1}{4}z^2 - v)^{1/2 - \frac{1}{2}a} dv$$

The contour ζ_1 is such that $(\frac{1}{4}z^2 + v)$ goes from $\infty e^{-i\pi}$ to $\infty e^{i\pi}$ while $v = \frac{1}{4}z^2$ is not encircled; $(\frac{1}{4}z^2 - v)^{-1/2 - \frac{1}{2}a}$ has its principal value except possibly in the immediate neighborhood of the branch-point when encirclement is being avoided. Likewise η_1 is such that $(\frac{1}{4}z^2 - v)$ goes from $\infty e^{i\pi}$ to $\infty e^{-i\pi}$ while encirclement of $v = -\frac{1}{4}z^2$ is similarly avoided. The contours (ζ_1) and (η_1) may be obtained from ζ_1 and η_1 by use of the substitution $v = \frac{1}{4}z^2 t$.

The expressions 19.5.7 and 19.5.8 become indeterminate when $a = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots$; for these values

19.5.11

$$U(a, z) = \frac{1}{\Gamma(\frac{1}{4} + \frac{1}{2}a)} z e^{-iz^2} \int_0^\infty e^{-s} s^{1/2 - \frac{1}{2}a} (z^2 + 2s)^{-1/2 - \frac{1}{2}a} ds$$

Again 19.5.9 and 19.5.10 become indeterminate when $a = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$; for these values

19.5.12

$$U(a, z) = \frac{1}{\Gamma(\frac{3}{4} + \frac{1}{2}a)} e^{-iz^2} \int_0^\infty e^{-s} s^{1/2 - \frac{1}{2}a} (z^2 + 2s)^{-1/2 - \frac{1}{2}a} ds$$

Barnes-Type Integrals

$$19.5.13 \quad U(a, z) = \frac{e^{-iz^2}}{2\pi i} z^{-a-\frac{1}{2}} \int_{-\infty i}^{+\infty i} \frac{\Gamma(s) \Gamma(\frac{1}{2} + a - 2s)}{\Gamma(\frac{1}{2} + a)} (\sqrt{2}z)^{2s} ds \quad (|\arg z| < \frac{3}{4}\pi)$$

where the contour separates the zeros of $\Gamma(s)$ from those of $\Gamma(a + \frac{1}{2} - 2s)$. Similarly

$$19.5.14 \quad V(a, z) = \sqrt{\frac{2}{\pi}} \frac{e^{iz^2}}{2\pi i} z^{a-\frac{1}{2}} \int_{-\infty i}^{+\infty i} \frac{\Gamma(s) \Gamma(\frac{1}{2} - a - 2s)}{\Gamma(\frac{1}{2} - a)} (\sqrt{2}z)^{2s} \cos s\pi ds \quad (|\arg z| < \frac{1}{4}\pi)$$

19.6. Recurrence Relations

$$19.6.1 \quad U'(a, x) + \frac{1}{2}xU(a, x) + (a + \frac{1}{2})U(a+1, x) = 0$$

$$19.6.2 \quad U'(a, x) - \frac{1}{2}xU(a, x) + U(a-1, x) = 0$$

$$19.6.3 \quad 2U'(a, x) + U(a-1, x) + (a + \frac{1}{2})U(a+1, x) = 0$$

$$19.6.4 \quad xU(a, x) - U(a-1, x) + (a + \frac{1}{2})U(a+1, x) = 0$$

These are also satisfied by $\Gamma(\frac{1}{2} - a)V(a, x)$.

$$19.6.5 \quad V'(a, x) - \frac{1}{2}xV(a, x) - (a - \frac{1}{2})V(a-1, x) = 0$$

$$19.6.6 \quad V'(a, x) + \frac{1}{2}xV(a, x) - V(a+1, x) = 0$$

19.6.7

$$2V'(a, x) - V(a+1, x) - (a - \frac{1}{2})V(a-1, x) = 0$$

19.6.8

$$xV(a, x) - V(a+1, x) + (a - \frac{1}{2})V(a-1, x) = 0$$

These are also satisfied by $U(a, x)/\Gamma(\frac{1}{2} - a)$

$$19.6.9 \quad y_1'(a, x) + \frac{1}{2}xy_1(a, x) = (a + \frac{1}{2})y_2(a+1, x)$$

$$19.6.10 \quad y_1'(a, x) - \frac{1}{2}xy_1(a, x) = (a - \frac{1}{2})y_2(a-1, x)$$

$$19.6.11 \quad y'_2(a, x) + \frac{1}{2}xy_2(a, x) = y_1(a+1, x)$$

$$19.6.12 \quad y'_2(a, x) - \frac{1}{2}xy_2(a, x) = y_1(a-1, x)$$

Asymptotic Expansions

19.7. Expressions in Terms of Airy Functions

When a is large and negative, write, for $0 \leq x < \infty$

$$x = 2\sqrt{|a|}\xi \quad t = (4|a|)^{1/3}\tau$$

19.7.1

$$\tau = -(\frac{2}{3}\vartheta_3)^{1/3}$$

$$\vartheta_3 = \frac{1}{2} \int_{\xi}^1 \sqrt{1-s^2} ds = \frac{1}{4} \arccos \xi - \frac{1}{4}\xi \sqrt{1-\xi^2} \quad (\xi \leq 1)$$

19.7.2

$$\tau = +(\frac{2}{3}\vartheta_2)^{1/3}$$

$$\vartheta_2 = \frac{1}{2} \int_1^{\xi} \sqrt{s^2-1} ds = \frac{1}{4}\xi \sqrt{\xi^2-1} - \frac{1}{4} \operatorname{arccosh} \xi \quad (\xi \geq 1)$$

Then for $x \geq 0$, $a \rightarrow -\infty$

19.7.3

$$U(a, x) \sim 2^{-1/3} \Gamma(\frac{1}{4} - \frac{1}{2}a) \left(\frac{t}{\xi^2-1}\right)^{1/4} \operatorname{Ai}(t)$$

19.7.4

$$\Gamma(\frac{1}{2}-a) V(a, x) \sim 2^{-1/3} \Gamma(\frac{1}{4} - \frac{1}{2}a) \left(\frac{t}{\xi^2-1}\right)^{1/4} \operatorname{Bi}(t)$$

Table 19.3 gives τ as a function of ξ .

See [19.5] for further developments.

19.8. Expansions for x Large and a Moderate

When $x \gg |a|$

19.8.1

$$U(a, x) \sim e^{-1/2 x^2} x^{-a-1/2} \left\{ 1 - \frac{(a+\frac{1}{2})(a+\frac{3}{2})}{2x^2} + \frac{(a+\frac{1}{2})(a+\frac{3}{2})(a+\frac{5}{2})(a+\frac{7}{2})}{2 \cdot 4x^4} - \dots \right\} \\ (x \rightarrow +\infty)$$

19.8.2

$$V(a, x) \sim \sqrt{\frac{2}{\pi}} e^{1/2 x^2} x^{a-1/2} \left\{ 1 + \frac{(a-\frac{1}{2})(a-\frac{3}{2})}{2x^2} + \frac{(a-\frac{1}{2})(a-\frac{3}{2})(a-\frac{5}{2})(a-\frac{7}{2})}{2 \cdot 4x^4} + \dots \right\} \\ (x \rightarrow +\infty)$$

These expansions form the basis for the choice of standard solutions in 19.3. The former is valid for complex x , with $|\arg x| < \frac{1}{2}\pi$, in the complete

sense of Watson [19.6], although valid for a wider range of $|\arg x|$ in Poincaré's sense; the second series is completely valid *only for x real and positive*.

19.9. Expansions for a Large With x Moderate

(i) a positive

When $a \gg x^2$, with $p = \sqrt{a}$, then

$$19.9.1 \quad U(a, x) = \frac{\sqrt{\pi}}{2^{1/2} \Gamma(\frac{3}{4} + \frac{1}{2}a)} \exp(-px + v_1)$$

$$19.9.2 \quad U(a, -x) = \frac{\sqrt{\pi}}{2^{1/2} \Gamma(\frac{3}{4} + \frac{1}{2}a)} \exp(px + v_2)$$

where

$$19.9.3 \quad v_1, v_2 \sim \mp \frac{\frac{2}{3}(\frac{1}{2}x)^3}{2p} - \frac{(\frac{1}{2}x)^2}{(2p)^2} \mp \frac{\frac{1}{2}x - \frac{2}{3}(\frac{1}{2}x)^5}{(2p)^3} \\ + \frac{2(\frac{1}{2}x)^4}{(2p)^4} \pm \frac{(\frac{1}{8}\frac{1}{2}x)^3 - \frac{4}{3}(\frac{1}{2}x)^7}{(2p)^5} + \dots \\ (a \rightarrow +\infty)$$

The upper sign gives the first function, and the lower sign the second function.

(ii) a negative

When $-a \gg x^2$, with $p = \sqrt{-a}$, then

19.9.4

$$U(a, x) + i\Gamma(\frac{1}{2}-a) \cdot V(a, x) \\ = \frac{e^{i\pi(\frac{1}{4}+\frac{1}{2}a)} \Gamma(\frac{1}{4}-\frac{1}{2}a)}{2^{1/2} \Gamma(\frac{3}{4}+\frac{1}{2}a)} e^{ipx} \exp(v_r + iv_i)$$

where

19.9.5

$$v_r \sim + \frac{(\frac{1}{2}x)^2}{(2p)^2} + \frac{2(\frac{1}{2}x)^4}{(2p)^4} - \frac{9(\frac{1}{2}x)^2 - \frac{1}{3}(\frac{1}{2}x)^6}{(2p)^6} - \dots \\ v_i \sim - \frac{\frac{2}{3}(\frac{1}{2}x)^3}{2p} + \frac{\frac{1}{2}x + \frac{2}{3}(\frac{1}{2}x)^5}{(2p)^3} + \frac{\frac{1}{8}(\frac{1}{2}x)^3 - \frac{4}{3}(\frac{1}{2}x)^7}{(2p)^5} - \dots \\ (a \rightarrow -\infty)$$

Further expansions of a similar type will be found in [19.11].

19.10. Darwin's Expansions

(i) a positive, $x^2 + 4a$ large. Write

$$19.10.1 \quad X = \sqrt{x^2 + 4a}$$

$$\theta = 4a\vartheta_1(x/2\sqrt{a}) = \frac{1}{2} \int_0^x X dx = \frac{1}{4}xX + a \ln \frac{x+X}{2\sqrt{a}}$$

$$= \frac{x}{4} \sqrt{x^2 + 4a} + a \operatorname{arcsinh} \frac{x}{2\sqrt{a}}$$

(see Table 19.3 for ϑ_1), then

$$19.10.2 \quad U(a, x) = \frac{(2\pi)^{1/4}}{\sqrt{\Gamma(\frac{1}{2}+a)}} \exp \{-\theta + v(a, x)\}$$

$$19.10.3 \quad U(a, -x) = \frac{(2\pi)^{1/4}}{\sqrt{\Gamma(\frac{1}{2}+a)}} \exp \{\theta + v(a, -x)\}$$

where

19.10.4

$$v(a, x) \sim -\frac{1}{2} \ln X + \sum_{s=1}^{\infty} (-1)^s d_{3s} / X^{3s} \quad (a > 0, x^2 + 4a \rightarrow +\infty)$$

and d_{3s} is given by 19.10.13.

(ii) a negative, $x^2 + 4a$ large and positive. Write

$$19.10.5 \quad X = \sqrt{x^2 - 4|a|}$$

$$\theta = 4|a|\vartheta_2(x/2\sqrt{|a|}) = \frac{1}{2} \int_{2\sqrt{|a|}}^x X dx = \frac{1}{4} xX + a \ln \frac{x+X}{2\sqrt{|a|}}$$

$$= \frac{1}{4} x\sqrt{x^2 - 4|a|} + a \operatorname{arccosh} \frac{x}{2\sqrt{|a|}}$$

(see Table 19.3 for ϑ_2), then

$$19.10.6 \quad U(a, x) = \frac{\sqrt{\Gamma(\frac{1}{2}-a)}}{(2\pi)^{1/4}} \exp \{-\theta + v(a, x)\}$$

19.10.7

$$V(a, x) = \frac{2}{(2\pi)^{1/4} \sqrt{\Gamma(\frac{1}{2}-a)}} \exp \{\theta + v(a, -x)\}$$

where again

19.10.8

$$v(a, x) \sim -\frac{1}{2} \ln X + \sum_{s=1}^{\infty} (-1)^s d_{3s} / X^{3s} \quad (a < 0, x^2 + 4a \rightarrow +\infty)$$

and d_{3s} is given by 19.10.13.

(iii) a large and negative and x moderate. Write

$$19.10.9 \quad Y = \sqrt{4|a| - x^2}$$

$$\theta = 4|a|\vartheta_4(x/2\sqrt{|a|})$$

$$= \frac{1}{2} \int_0^x Y dx = \frac{1}{4} xY + |a| \arcsin \frac{x}{2\sqrt{|a|}}$$

(see Table 19.3 for $\vartheta_4 = \frac{1}{8}\pi - \vartheta_3$), then

19.10.10

$$U(a, x) = \frac{2\sqrt{\Gamma(\frac{1}{2}-a)}}{(2\pi)^{1/4}} e^{v_r} \cos \left\{ \frac{1}{4}\pi + \frac{1}{2}\pi a + \theta + v_i \right\}$$

19.10.11

$$V(a, x) =$$

$$\frac{2}{(2\pi)^{1/4} \sqrt{\Gamma(\frac{1}{2}-a)}} e^{v_r} \sin \left\{ \frac{1}{4}\pi + \frac{1}{2}\pi a + \theta + v_i \right\}$$

where

$$19.10.12 \quad v_r \sim -\frac{1}{2} \ln Y - \frac{d_6}{Y^6} + \frac{d_{12}}{Y^{12}} - \dots$$

$$v_i \sim \frac{d_3}{Y^3} - \frac{d_9}{Y^9} + \dots \quad (x^2 + 4a \rightarrow -\infty)$$

In each case the coefficients d_{3r} are given by

19.10.13

$$d_3 = \frac{1}{a} \left(\frac{x^3}{48} + \frac{1}{2} ax \right)$$

$$d_6 = \frac{3}{4} x^2 - 2a$$

$$d_9 = \frac{1}{a^3} \left(-\frac{7}{5760} x^9 - \frac{7}{320} ax^7 - \frac{49}{320} a^2 x^5 \right. \\ \left. + \frac{31}{12} a^3 x^3 - 19a^4 x \right)$$

$$d_{12} = \frac{153}{8} x^4 - 186ax^2 + 80a^2$$

See [19.11] for $d_{15}, \dots, d_{24}$, and [19.5] for an alternative form.

19.11. Modulus and Phase

When a is negative and $|x| < 2\sqrt{|a|}$, the functions U and V are oscillatory and it is sometimes convenient to write

$$19.11.1 \quad U(a, x) + i\Gamma(\frac{1}{2}-a)V(a, x) = F(a, x)e^{i\chi(a, x)}$$

$$19.11.2 \quad U'(a, x) + i\Gamma(\frac{1}{2}-a)V'(a, x) = -G(a, x)e^{i\psi(a, x)}$$

Then, when $a < 0$ and $|a| > x^2$,

19.11.3

$$F = \frac{\Gamma(\frac{1}{4}-\frac{1}{2}a)}{2^{1/2} \sqrt{\pi}} e^{v_r}, \quad \chi = (\frac{1}{2}a + \frac{1}{4})\pi + px + v_i$$

where v_r, v_i are given by 19.9.5 and $p = \sqrt{-a}$.

Alternatively, with $p = \sqrt{|a|}$, and again $-a > x^2$,

19.11.4

$$F \sim \frac{\Gamma(\frac{1}{4}-\frac{1}{2}a)}{2^{1/2} \sqrt{\pi}} \left\{ 1 + \frac{x^2}{(4p)^2} + \frac{\frac{5}{2}x^4}{(4p)^4} \right. \\ \left. + \frac{\frac{1}{2}x^6 - 144x^2}{(4p)^6} + \dots \right\}$$

$$19.11.5 \quad \chi \sim (\tfrac{1}{2}a + \tfrac{1}{4})\pi + px \left\{ 1 - \frac{\frac{2}{3}x^2}{(4p)^2} - \frac{\frac{2}{5}x^4 - 16}{(4p)^4} - \frac{\frac{4}{7}x^6 - \frac{2}{3}\frac{5}{3}x^2}{(4p)^6} - \dots \right\}$$

$$19.11.6 \quad G \sim \frac{\Gamma(\frac{3}{4} - \frac{1}{2}a)}{2^{\frac{1}{4}a - \frac{1}{4}}\sqrt{\pi}} \left\{ 1 - \frac{x^2}{(4p)^2} - \frac{\frac{3}{5}x^4}{(4p)^4} - \frac{\frac{7}{2}x^6 - 176x^2}{(4p)^6} - \dots \right\}$$

$$19.11.7 \quad \psi \sim (\tfrac{1}{2}a - \tfrac{1}{4})\pi + px \left\{ 1 - \frac{\frac{2}{3}x^2}{(4p)^2} - \frac{\frac{2}{5}x^4 + 16}{(4p)^4} - \frac{\frac{4}{7}x^6 + \frac{3}{3}\frac{9}{3}x^2}{(4p)^6} - \dots \right\}$$

Again, when $x^2 + 4a$ is large and negative, with $Y = \sqrt{4|a| - x^2}$, then

$$19.11.8 \quad F = \frac{2\sqrt{\Gamma(\frac{1}{2} - a)}}{(2\pi)^{\frac{1}{2}}} e^{v_r}, \quad \chi = \tfrac{1}{4}\pi + \tfrac{1}{2}\pi a + \theta + v_i$$

where θ , v_r and v_i are given by 19.10.9 and 19.10.12.

Another form is

$$19.11.9 \quad F \sim \frac{2\sqrt{\Gamma(\frac{1}{2} - a)}}{(2\pi)^{\frac{1}{2}}\sqrt{Y}} \left(1 + \frac{3}{4Y^4} + \frac{5a}{Y^6} + \frac{621}{32Y^8} + \dots \right) \quad (x^2 + 4a \rightarrow -\infty)$$

$$19.11.10 \quad G \sim \frac{\sqrt{Y}\sqrt{\Gamma(\frac{1}{2} - a)}}{(2\pi)^{\frac{1}{2}}} \left(1 - \frac{5}{4Y^4} - \frac{7a}{Y^6} - \frac{835}{32Y^8} - \dots \right) \quad (x^2 + 4a \rightarrow -\infty)$$

while ψ and χ are connected by

$$19.11.11 \quad \psi - \chi \sim -\tfrac{1}{2}\pi - \frac{x}{Y^3} \left(1 + \frac{47}{6Y^4} + \frac{214a}{3Y^6} + \frac{14483}{40Y^8} + \dots \right) \quad (x^2 + 4a \rightarrow -\infty)$$

Connections With Other Functions

19.12. Connection With Confluent Hypergeometric Functions (see chapter 13)

19.12.1

$$U(a, \pm x) = \frac{\sqrt{\pi}2^{-\frac{1}{2}a}x^{-\frac{1}{2}}}{\Gamma(\frac{3}{4} + \frac{1}{2}a)} M_{-\frac{1}{2}a, -\frac{1}{2}}(\tfrac{1}{2}x^2) \mp \frac{\sqrt{\pi}2^{1-\frac{1}{2}a}x^{-\frac{1}{2}}}{\Gamma(\frac{1}{4} + \frac{1}{2}a)} M_{-\frac{1}{2}a, \frac{1}{2}}(\tfrac{1}{2}x^2)$$

$$19.12.2 \quad U(a, x) = 2^{-\frac{1}{2}a}x^{-\frac{1}{2}}W_{-\frac{1}{2}a, -\frac{1}{2}}(\tfrac{1}{2}x^2)$$

19.12.3

$$U(a, \pm x) = \frac{\sqrt{\pi}2^{-\frac{1}{2}a}e^{-\frac{1}{2}x^2}}{\Gamma(\frac{3}{4} + \frac{1}{2}a)} M(\tfrac{1}{2}a + \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}x^2) \mp \frac{\sqrt{\pi}2^{1-\frac{1}{2}a}xe^{-\frac{1}{2}x^2}}{\Gamma(\frac{1}{4} + \frac{1}{2}a)} M(\tfrac{1}{2}a + \tfrac{3}{4}, \tfrac{3}{2}, \tfrac{1}{2}x^2)$$

19.12.4

$$U(a, x) = 2^{-\frac{1}{2}a}e^{-\frac{1}{2}x^2}U(\tfrac{1}{2}a + \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}x^2) = 2^{-\frac{1}{2}a}xe^{-\frac{1}{2}x^2}U(\tfrac{1}{2}a + \tfrac{3}{4}, \tfrac{3}{2}, \tfrac{1}{2}x^2)$$

Expressions for $V(a, x)$ may be obtained from these by use of 19.4.2.

19.13. Connection With Hermite Polynomials and Functions

When n is a non-negative integer

19.13.1

$$U(-n - \tfrac{1}{2}, x) = e^{-\frac{1}{2}x^2}He_n(x) = 2^{-\frac{1}{2}n}e^{-\frac{1}{2}x^2}H_n(x/\sqrt{2})$$

19.13.2

$$V(n + \tfrac{1}{2}, x) = \sqrt{2/\pi}e^{\frac{1}{2}x^2}He_n^*(x) = 2^{-\frac{1}{2}n}e^{\frac{1}{2}x^2}H_n^*(x/\sqrt{2})$$

in which $H_n(x)$ and $He_n(x)$ are Hermite polynomials (see chapter 22) while

$$19.13.3 \quad He_n^*(x) = e^{-\frac{1}{2}x^2} \frac{d^n}{dx^n} e^{\frac{1}{2}x^2} = (-i)^n He_n(ix)$$

$$19.13.4 \quad H_n^*(x) = e^{-x^2} \frac{d^n}{dx^n} e^{x^2} = (-i)^n H_n(ix)$$

This gives one elementary solution to 19.1.2 whenever $2a$ is an odd integer, positive or negative.

19.14. Connection With Probability Integrals and Dawson's Integral (see chapter 7)

If, as in [19.10]

$$19.14.1 \quad Hh_{-1}(x) = e^{-\frac{1}{2}x^2}$$

19.14.2

$$Hh_n(x) = \int_x^\infty Hh_{n-1}(t)dt = (1/n!) \int_x^\infty (t-x)^n e^{-\frac{1}{2}t^2}dt \quad (n \geq 0)$$

then

$$19.14.3 \quad U(n + \tfrac{1}{2}, x) = e^{\frac{1}{2}x^2}Hh_n(x) \quad (n \geq -1)$$

Correspondingly

$$19.14.4 \quad V(\tfrac{1}{2}, x) = \sqrt{2/\pi} e^{ix^2}$$

and

19.14.5

$$V(-n - \tfrac{1}{2}, x) = e^{-ix^2} \left\{ \int_0^x e^{-it^2} V(-n + \tfrac{1}{2}, t) dt - \frac{\sin \frac{1}{2} n \pi}{2^{1/2} \Gamma(\frac{1}{2} n + 1)} \right\} \quad (n \geq 0)$$

Here $V(-\frac{1}{2}, x)$ is closely related to Dawson's integral

$$\int_0^x e^{t^2} dt$$

These relations give a second solution of 19.1.2 whenever $2a$ is an odd integer, and a second solution is unobtainable from $U(a, x)$ by reflection in the y -axis.

19.15. Explicit Formula in Terms of Bessel Functions When $2a$ Is an Integer

Write

$$19.15.1 \quad I_{-n} - I_n = (2/\pi) \sin n\pi \cdot K_n$$

$$19.15.2 \quad I_{-n} + I_n = \cos n\pi \cdot J_n$$

where the argument of all modified Bessel functions is $\frac{1}{2}x^2$. Then

$$19.15.3 \quad U(1, x) = 2\pi^{-1/2} (\tfrac{1}{2}x)^{1/2} (-K_1 + K_1)$$

$$19.15.4 \quad U(2, x) = 2 \cdot \frac{2}{3} \pi^{-1/2} (\tfrac{1}{2}x)^{3/2} (2K_1 - 3K_1 + K_1)$$

19.15.5

$$U(3, x) = 2 \cdot \frac{2}{3} \cdot \frac{2}{5} \pi^{-1/2} (\tfrac{1}{2}x)^{5/2} (-5K_1 + 9K_1 - 5K_1 + K_1)$$

$$19.15.6 \quad V(1, x) = \tfrac{1}{2} (\tfrac{1}{2}x)^{1/2} (\mathcal{J}_1 - \mathcal{J}_1)$$

$$19.15.7 \quad V(2, x) = \tfrac{1}{2} (\tfrac{1}{2}x)^{3/2} (2\mathcal{J}_1 - 3\mathcal{J}_1 + \mathcal{J}_1)$$

$$19.15.8 \quad V(3, x) = \tfrac{1}{2} (\tfrac{1}{2}x)^{5/2} (5\mathcal{J}_1 - 9\mathcal{J}_1 + 5\mathcal{J}_1 - \mathcal{J}_1)$$

$$19.15.9 \quad U(0, x) = \pi^{-1/2} (\tfrac{1}{2}x)^{1/2} K_1$$

$$19.15.10 \quad U(-1, x) = \pi^{-1/2} (\tfrac{1}{2}x)^{3/2} (K_1 + K_1)$$

19.15.11

$$\dot{U}(-2, x) = \pi^{-1/2} (\tfrac{1}{2}x)^{5/2} (2K_1 + 3K_1 - K_1)$$

19.15.12

$$U(-3, x) = \pi^{-1/2} (\tfrac{1}{2}x)^{7/2} (5K_1 + 9K_1 - 5K_1 - K_1)$$

$$19.15.13 \quad V(0, x) = \tfrac{1}{2} (\tfrac{1}{2}x)^{1/2} \mathcal{J}_1$$

$$19.15.14 \quad V(-1, x) = (\tfrac{1}{2}x)^{3/2} (\mathcal{J}_1 + \mathcal{J}_1)$$

$$19.15.15 \quad V(-2, x) = \tfrac{2}{3} (\tfrac{1}{2}x)^{5/2} (2\mathcal{J}_1 + 3\mathcal{J}_1 - \mathcal{J}_1)$$

19.15.16

$$V(-3, x) = \tfrac{2}{3} \cdot \tfrac{2}{5} (\tfrac{1}{2}x)^{7/2} (5\mathcal{J}_1 + 9\mathcal{J}_1 - 5\mathcal{J}_1 - \mathcal{J}_1)$$

$$19.15.17 \quad U(-\tfrac{1}{2}, x) = \sqrt{2/\pi} (\tfrac{1}{2}x) K_1$$

$$19.15.18 \quad U(-\tfrac{3}{2}, x) = \sqrt{2/\pi} (\tfrac{1}{2}x)^{3/2} 2K_1$$

$$19.15.19 \quad U(-\tfrac{5}{2}, x) = \sqrt{2/\pi} (\tfrac{1}{2}x)^{5/2} (5K_1 - K_1)$$

$$19.15.20 \quad V(\tfrac{1}{2}, x) = (\tfrac{1}{2}x) (I_1 + I_{-1})$$

$$19.15.21 \quad V(\tfrac{3}{2}, x) = (\tfrac{1}{2}x)^2 (2I_1 + 2I_{-1})$$

$$19.15.22 \quad V(\tfrac{5}{2}, x) = (\tfrac{1}{2}x)^3 (5I_1 + 5I_{-1} - I_1 - I_{-1})$$

The Equation $\frac{d^2 y}{dx^2} + \left(\frac{1}{4}x^2 - a\right)y = 0$

19.16. Power Series in x

Even and odd solutions are given by 19.2.1 to 19.2.4 with $-ia$ written for a and $xe^{i\pi}$ for x ; the series involves complex quantities in which the imaginary part of the sum vanishes identically.

Alternatively,

19.16.1

$$y_1 = 1 + a \frac{x^2}{2!} + (a^2 - \tfrac{1}{2}) \frac{x^4}{4!} + (a^3 - \tfrac{3}{2}a) \frac{x^6}{6!} + (a^4 - 11a^2 + \tfrac{15}{4}) \frac{x^8}{8!} + (a^5 - 25a^3 + \tfrac{21}{4}a) \frac{x^{10}}{10!} + \dots$$

19.16.2

$$y_2 = x + a \frac{x^3}{3!} + (a^2 - \tfrac{3}{2}) \frac{x^5}{5!} + (a^3 - \tfrac{1}{2}a) \frac{x^7}{7!} + (a^4 - 17a^2 + \tfrac{63}{4}) \frac{x^9}{9!} + (a^5 - 35a^3 + \tfrac{53}{4}a) \frac{x^{11}}{11!} + \dots$$

in which non-zero coefficients a_n of $x^n/n!$ are connected by

$$19.16.3 \quad a_{n+2} = a \cdot a_n - \tfrac{1}{4}n(n-1)a_{n-2}$$

19.17. Standard Solutions (see [19.4])

$$19.17.1 \quad W(a, \pm x) = \frac{(\cosh \pi a)^{1/2}}{2\sqrt{\pi}} (G_1 y_1 \mp \sqrt{2} G_3 y_2)$$

$$19.17.2 \quad = 2^{-3/4} \left(\sqrt{\frac{G_1}{G_3}} y_1 \mp \sqrt{\frac{2G_3}{G_1}} y_2 \right)$$

where

$$19.17.3 \quad G_1 = |\Gamma(\tfrac{1}{4} + \tfrac{1}{2}ia)| \quad G_3 = |\Gamma(\tfrac{3}{4} + \tfrac{1}{2}ia)|$$

At $x=0$,

$$19.17.4 \quad W(a, 0) = \frac{1}{2i} \left| \frac{\Gamma(\frac{1}{4} + \frac{1}{2}ia)}{\Gamma(\frac{3}{4} + \frac{1}{2}ia)} \right|^{\frac{1}{2}} = \frac{1}{2i} \sqrt{\frac{G_1}{G_3}}$$

19.17.5

$$W'(a, 0) = -\frac{1}{2i} \left| \frac{\Gamma(\frac{3}{4} + \frac{1}{2}ia)}{\Gamma(\frac{1}{4} + \frac{1}{2}ia)} \right|^{\frac{1}{2}} = -\frac{1}{2i} \sqrt{\frac{G_3}{G_1}}$$

Complex Solutions

$$19.17.6 \quad E(a, x) = k^{-1} W(a, x) + ik^{\frac{1}{2}} W(a, -x)$$

$$19.17.7 \quad E^*(a, x) = k^{-1} W(a, x) - ik^{\frac{1}{2}} W(a, -x)$$

where

$$19.17.8 \quad k = \sqrt{1 + e^{2\pi a}} - e^{\pi a} \quad 1/k = \sqrt{1 + e^{2\pi a}} + e^{\pi a}$$

In terms of $U(a, x)$ of 19.3,

$$19.17.9 \quad E(a, x) = \sqrt{2} e^{i\pi a + \frac{1}{2}i\pi + \frac{1}{2}i\phi_2} U(ia, x e^{-i\pi})$$

with

$$19.17.10 \quad \phi_2 = \arg \Gamma(\frac{1}{2} + ia)$$

where the branch is defined by $\phi_2=0$ when $a=0$ and by continuity elsewhere.

Also

19.17.11

$$\sqrt{2\pi} U(ia, x e^{-i\pi}) = \Gamma(\frac{1}{2} - ia) \{ e^{\frac{1}{2}\pi a - \frac{1}{2}i\pi} U(-ia, x e^{i\pi}) + e^{-\frac{1}{2}\pi a + \frac{1}{2}i\pi} U(-ia, -x e^{i\pi}) \}$$

19.18. Wronskian and Other Relations

$$19.18.1 \quad W\{W(a, x), W(a, -x)\} = 1$$

$$19.18.2 \quad W\{E(a, x), E^*(a, x)\} = -2i$$

$$19.18.3 \quad \sqrt{1 + e^{2\pi a}} E(a, x) = e^{\pi a} E^*(a, x) + i E^*(a, -x)$$

$$19.18.4 \quad E^*(a, x) = e^{-i(\phi_2 + \frac{1}{2}\pi)} E(-a, ix)$$

19.18.5

$$\sqrt{\Gamma(\frac{1}{2} + ia)} E^*(a, x) = e^{-i\pi} \sqrt{\Gamma(\frac{1}{2} - ia)} E(-a, ix)$$

19.19. Integral Representations

These are covered for 19.1.3 as well as for 19.1.2 in 19.5 (general complex argument).

Asymptotic Expansions

19.20. Expressions in Terms of Airy Functions

When a is large and positive, write, for $0 \leq x < \infty$

$$x = 2\sqrt{a}\xi \quad t = (4a)^{\frac{1}{3}}\tau$$

19.20.1

$$\tau = -(\frac{2}{3}\vartheta_3)^{\frac{2}{3}}$$

$$\vartheta_3 = \frac{1}{2} \int_{\xi}^1 \sqrt{1-s^2} ds = \frac{1}{4} \arccos \xi - \frac{1}{4}\xi \sqrt{1-\xi^2} \quad (\xi \leq 1)$$

19.20.2

$$\tau = +(\frac{2}{3}\vartheta_2)^{\frac{2}{3}}$$

$$\vartheta_2 = \frac{1}{2} \int_1^{\xi} \sqrt{s^2-1} ds = \frac{1}{4}\xi \sqrt{\xi^2-1} - \frac{1}{4} \operatorname{arccosh} \xi \quad (\xi \geq 1)$$

Then for $x > 0$, $a \rightarrow +\infty$

19.20.3

$$W(a, x) \sim \sqrt{\pi}(4a)^{-\frac{1}{2}} e^{-\frac{1}{2}\pi a} \left(\frac{t}{\xi^2-1} \right)^{\frac{1}{2}} \operatorname{Bi}(-t)$$

19.20.4

$$W(a, -x) \sim 2\sqrt{\pi}(4a)^{-\frac{1}{2}} e^{\frac{1}{2}\pi a} \left(\frac{t}{\xi^2-1} \right)^{\frac{1}{2}} \operatorname{Ai}(-t)$$

Table 19.3 gives τ as a function of ξ . See [19.5] for further developments.

19.21. Expansions for x Large and a Moderate

When $x \gg |a|$,

19.21.1

$$E(a, x) = \sqrt{2/x} \exp \{ i(\frac{1}{4}x^2 - a \ln x + \frac{1}{2}\phi_2 + \frac{1}{4}\pi) \} s(a, x)$$

19.21.2

$$W(a, x) = \sqrt{2k/x} \{ s_1(a, x) \cos(\frac{1}{4}x^2 - a \ln x + \frac{1}{4}\pi + \frac{1}{2}\phi_2) - s_2(a, x) \sin(\frac{1}{4}x^2 - a \ln x + \frac{1}{4}\pi + \frac{1}{2}\phi_2) \}$$

19.21.3

$$W(a, -x) = \sqrt{2/kx} \{ s_1(a, x) \sin(\frac{1}{4}x^2 - a \ln x + \frac{1}{4}\pi + \frac{1}{2}\phi_2) + s_2(a, x) \cos(\frac{1}{4}x^2 - a \ln x + \frac{1}{4}\pi + \frac{1}{2}\phi_2) \}$$

where ϕ_2 is defined by 19.17.10 and

$$19.21.4 \quad s(a, x) = s_1(a, x) + i s_2(a, x)$$

19.21.5

$$s_1(a, x) \sim 1 + \frac{v_2}{1!2x^2} - \frac{u_4}{2!2^2x^4} - \frac{v_6}{3!2^3x^6} + \frac{u_8}{4!2^4x^8} + \dots$$

19.21.6

$$s_2(a, x) \sim -\frac{u_2}{1!2x^2} - \frac{v_4}{2!2^2x^4} + \frac{u_6}{3!2^3x^6} + \frac{v_8}{4!2^4x^8} - \dots$$

with

$$(x \rightarrow +\infty)$$

$$19.21.7 \quad u_r + iv_r = \Gamma(r + \frac{1}{2} + ia) / \Gamma(\frac{1}{2} + ia)$$

or

$$19.21.8 \quad s(a, x) \sim \sum_{r=0}^{\infty} (-i)^r \frac{\Gamma(2r + \frac{1}{2} + ia)}{\Gamma(\frac{1}{2} + ia)} \frac{1}{2^r r! x^{2r}}$$

19.22. Expansions for a Large With x Moderate

(i) a positiveWhen $a \gg x^2$, with $p = \sqrt{a}$, then

$$19.22.1 \quad W(a, x) = W(a, 0) \exp(-px + v_1)$$

$$19.22.2 \quad W(a, -x) = W(a, 0) \exp(px + v_2)$$

where $W(a, 0)$ is given by 19.17.4, and

19.22.3

$$v_1, v_2 \sim \pm \frac{\frac{2}{3}(\frac{1}{2}x)^3}{2p} + \frac{(\frac{1}{2}x)^2}{(2p)^2} \pm \frac{\frac{1}{2}x + \frac{2}{3}(\frac{1}{2}x)^5}{(2p)^3} \\ + \frac{2(\frac{1}{2}x)^4}{(2p)^4} \pm \frac{\frac{1}{8}(\frac{1}{2}x)^3 + \frac{4}{3}(\frac{1}{2}x)^7}{(2p)^5} + \dots \quad (a \rightarrow +\infty)$$

The upper sign gives the first function, and the lower sign the second function.

(ii) a negativeWhen $-a \gg x^2$, with $p = \sqrt{-a}$, then

19.22.4

$$W(a, x) + iW(a, -x) \\ = \sqrt{2}W(a, 0) \exp\{v_r + i(px + \frac{1}{4}\pi + v_i)\}$$

where $W(a, 0)$ is given by 19.17.4, and

19.22.5

$$v_r \sim -\frac{(\frac{1}{2}x)^2}{(2p)^2} + \frac{2(\frac{1}{2}x)^4}{(2p)^4} - \frac{9(\frac{1}{2}x)^2 + \frac{1}{3}(\frac{1}{2}x)^6}{(2p)^6} + \dots \\ v_i \sim \frac{\frac{2}{3}(\frac{1}{2}x)^3}{2p} - \frac{\frac{1}{2}x + \frac{2}{3}(\frac{1}{2}x)^5}{(2p)^3} + \frac{\frac{1}{8}(\frac{1}{2}x)^3 + \frac{4}{3}(\frac{1}{2}x)^7}{(2p)^5} - \dots \quad (a \rightarrow -\infty)$$

Further expansions of a similar type will be found in [19.3].

19.23. Darwin's Expansions

(i) a positive, $x^2 - 4a \gg 0$

Write

19.23.1

$$X = \sqrt{x^2 - 4a} \quad \theta = 4a\vartheta_2(x/2\sqrt{a}) = \frac{1}{2} \int_{2\sqrt{a}}^x X dx \\ = \frac{1}{4}xX - a \ln \frac{x+X}{2\sqrt{a}} \\ = \frac{1}{4}x\sqrt{x^2 - 4a} - a \operatorname{arccosh} \frac{x}{2\sqrt{a}}$$

(see Table 19.3 for ϑ_2), then

$$19.23.2 \quad W(a, x) = \sqrt{2ke^v} \cos(\frac{1}{4}\pi + \theta + v_i)$$

$$19.23.3 \quad W(a, -x) = \sqrt{2/ke^v} \sin(\frac{1}{4}\pi + \theta + v_i)$$

where

$$19.23.4 \quad v_r \sim -\frac{1}{2} \ln X - \frac{d_6}{X^6} + \frac{d_{12}}{X^{12}} - \dots$$

$$v_i \sim -\frac{d_3}{X^3} + \frac{d_9}{X^9} - \frac{d_{15}}{X^{15}} + \dots$$

 $(x^2 - 4a \rightarrow \infty)$ and d_{3r} is given by 19.23.12.(ii) a positive, $4a - x^2 \gg 0$

Write

19.23.5

$$Y = \sqrt{4a - x^2} \quad \theta = 4a\vartheta_4(x/2\sqrt{a}) \\ = \frac{1}{2} \int_0^x Y dx = \frac{1}{4}xY + a \arcsin \frac{x}{2\sqrt{a}}$$

(see Table 19.3 for $\vartheta_4 = \frac{1}{3}\pi - \vartheta_3$), then

$$19.23.6 \quad W(a, x) = \exp\{-\theta + v(a, x)\}$$

$$19.23.7 \quad W(a, -x) = \exp\{\theta + v(a, -x)\}$$

where

19.23.8

$$v(a, x) \sim -\frac{1}{2} \ln Y + \frac{d_3}{Y^3} + \frac{d_6}{Y^6} + \frac{d_9}{Y^9} + \dots \quad (x^2 - 4a \rightarrow -\infty)$$

and d_{3r} is again given by 19.23.12.(iii) a negative, $x^2 - 4a \gg 0$

Write

19.23.9

$$X = \sqrt{x^2 + 4|a|} \quad \theta = 4|a|\vartheta_1(x/2\sqrt{|a|}) = \frac{1}{2} \int_0^x X dx \\ = \frac{1}{4}xX - a \ln \frac{x+X}{2\sqrt{|a|}} \\ = \frac{1}{4}x\sqrt{x^2 + 4|a|} - a \operatorname{arcsinh} \frac{x}{2\sqrt{|a|}}$$

(see Table 19.3 for ϑ_1) then

$$19.23.10 \quad W(a, x) = \sqrt{2ke^v} \cos(\frac{1}{4}\pi + \theta + v_i)$$

$$19.23.11 \quad W(a, -x) = \sqrt{2/ke^v} \sin(\frac{1}{4}\pi + \theta + v_i)$$

where v_r and v_i are again given by 19.23.4. In each case the coefficients d_{3r} are given by

19.23.12

$$d_3 = -\frac{1}{a} \left(\frac{x^3}{48} - \frac{1}{2}ax \right)$$

$$d_6 = \frac{3}{4}x^2 + 2a$$

$$d_9 = \frac{1}{a^3} \left(\frac{7}{5760}x^9 - \frac{7}{320}ax^7 + \frac{49}{320}a^2x^5 + \frac{31}{12}a^3x^3 + 19a^4x \right)$$

$$d_{12} = \frac{153}{8}x^4 + 186ax^2 + 80a^2$$

See [19.11] for $d_{15}, \dots, d_{24}$, and [19.5] for an alternative form.

19.24. Modulus and Phase

When a is positive, the function $W(a, x)$ is oscillatory when $x < -2\sqrt{a}$ and when $x > 2\sqrt{a}$; when a is negative, the function is oscillatory for all x . In such cases it is sometimes convenient to write

19.24.1

$$k^{-1}W(a, x) + ik^1W(a, -x) = E(a, x) = Fe^{ix} \quad (x > 0)$$

19.24.2

$$k^{-1} \frac{dW(a, x)}{dx} + ik^1 \frac{dW(a, -x)}{dx} = E'(a, x) = -Ge^{i\psi} \quad (x > 0)$$

Then, when $x^2 \gg |a|$,

19.24.3

$$F \sim \sqrt{\frac{2}{x}} \left(1 + \frac{a}{x^2} + \frac{10a^2 - 3}{4x^4} + \frac{30a^3 - 47a}{4x^6} + \dots \right)$$

19.24.4

$$\chi \sim \frac{1}{4}x^2 - a \ln x + \frac{1}{2}\phi_2 + \frac{1}{4}\pi + \frac{4a^2 - 3}{8x^2} + \frac{4a^3 - 19a}{8x^4} + \dots$$

19.24.5

$$G \sim \sqrt{\frac{x}{2}} \left(1 - \frac{a}{x^2} - \frac{6a^2 - 5}{4x^4} - \frac{14a^3 - 63a}{4x^6} - \dots \right)$$

19.24.6

$$\psi \sim \frac{1}{4}x^2 - a \ln x + \frac{1}{2}\phi_2 - \frac{1}{4}\pi + \frac{4a^2 + 5}{8x^2} + \frac{4a^3 + 29a}{8x^4} + \dots$$

where ϕ_2 is defined by 19.17.10.

When $a < 0$, $|a| \gg x^2$

19.24.7

$$F \sim \sqrt{2}W(a, 0)e^v,$$

where v , is given by 19.22.5 with $p = \sqrt{-a}$. Also

19.24.8

$$F \sim \frac{1}{\sqrt{p}} \left(1 - \frac{x^2}{(4p)^2} + \frac{\frac{5}{2}x^4 + 8}{(4p)^4} - \frac{\frac{1}{2}x^6 + 152x^2}{(4p)^6} + \dots \right)$$

19.24.9

$$\chi \sim \frac{1}{4}\pi + px \left(1 + \frac{\frac{2}{3}x^2}{(4p)^2} - \frac{\frac{2}{3}x^4 + 16}{(4p)^4} + \frac{\frac{4}{3}x^6 + \frac{2}{3}x^2}{(4p)^6} - \dots \right)$$

19.24.10

$$G \sim \sqrt{p} \left(1 + \frac{x^2}{(4p)^2} - \frac{\frac{3}{2}x^4 + 8}{(4p)^4} + \frac{\frac{7}{2}x^6 + 168x^2}{(4p)^6} - \dots \right)$$

19.24.11

$$\psi \sim -\frac{1}{4}\pi + px \left(1 + \frac{\frac{2}{3}x^2}{(4p)^2} - \frac{\frac{2}{3}x^4 - 16}{(4p)^4} + \frac{\frac{4}{3}x^6 - \frac{3}{8}x^2}{(4p)^6} - \dots \right)$$

Again, when $a < 0$, $x^2 - 4a \gg 0$, with $X = \sqrt{x^2 + 4|a|}$, then

$$19.24.12 \quad F \sim \sqrt{2}e^{v_i}, \quad \chi = \frac{1}{4}\pi + \theta + v_i$$

where θ , v , and v_i are given by 19.23.4 and 19.23.9.

Another form also when $a > 0$, $x^2 - 4a \rightarrow \infty$ is

19.24.13

$$F \sim \sqrt{\frac{2}{X}} \left(1 - \frac{3}{4X^4} - \frac{5a}{X^6} + \frac{621}{32X^8} + \frac{1371a}{4X^{10}} - \dots \right)$$

19.24.14

$$G \sim \sqrt{\frac{X}{2}} \left(1 + \frac{5}{4X^4} + \frac{7a}{X^6} - \frac{835}{32X^8} - \frac{1729a}{4X^{10}} + \dots \right)$$

while ψ and χ are connected by

19.24.15

$$\psi - \chi \sim -\frac{1}{2}\pi + \frac{x}{X^3} \left(1 - \frac{47}{6X^4} - \frac{214a}{3X^6} + \frac{14483}{40X^8} + \dots \right)$$

19.25. Connections With Other Functions

Connection With Confluent Hypergeometric and Bessel Functions

19.25.1

$$W(a, \pm x) = 2^{-\frac{1}{2}} \left\{ \sqrt{\frac{G_1}{G_3}} H\left(-\frac{3}{4}, \frac{1}{2}a, \frac{1}{4}x^2\right) \pm \sqrt{\frac{2G_3}{G_1}} xH\left(-\frac{1}{4}, \frac{1}{2}a, \frac{1}{4}x^2\right) \right\}$$

where

19.25.2

$$H(m, n, x) = e^{-ix} {}_1F_1(m+1-in; 2m+2; 2ix)$$

19.25.3

$$= e^{-ix} M(m+1-in, 2m+2, 2ix)$$

19.25.4

$$W(0, \pm x) = 2^{-\frac{1}{2}} \sqrt{\pi x} \{ J_{-\frac{1}{2}}(\frac{1}{2}x^2) \pm J_{\frac{1}{2}}(\frac{1}{2}x^2) \} \quad (x \geq 0)$$

19.25.5

$$\frac{d}{dx} W(0, \pm x) = -2^{-\frac{1}{2}} x \sqrt{\pi x} \{ J_{\frac{1}{2}}(\frac{1}{2}x^2) \pm J_{-\frac{1}{2}}(\frac{1}{2}x^2) \} \quad (x \geq 0)$$

19.26. Zeros

Zeros of solutions $U(a, x)$, $V(a, x)$ of 19.1.2 occur only for $|x| < 2\sqrt{-a}$ when a is negative. A single exceptional zero is possible, for any a , in the general solution; neither $U(a, x)$ nor $V(a, x)$ has such a zero for $x > 0$.

Approximations may be obtained by reverting the series for ψ (or χ for zeros of derivatives) in 19.11, giving ψ (or χ) values that are multiples of $\frac{1}{2}\pi$, odd multiples for $U(a, x)$, even multiples for $V(a, x)$. Writing

$$\alpha = (\frac{1}{2}r - \frac{1}{2}a - \frac{1}{2})\pi$$

as an approximation to a zero of the function, or

$$\beta = (\frac{1}{2}r - \frac{1}{2}a + \frac{1}{2})\pi$$

as an approximation to a zero of the derivative, we obtain for the corresponding zero c or c' , with $-a = p^2$ the expressions

$$19.26.1 \quad c \approx \frac{\alpha}{p} + \frac{2\alpha^3 - 3\alpha}{48p^5} + \frac{52\alpha^5 - 240\alpha^3 + 315\alpha}{7680p^9} + \dots$$

$$19.26.2 \quad c' \approx \frac{\beta}{p} + \frac{2\beta^3 + 3\beta}{48p^5} + \frac{52\beta^5 + 280\beta^3 - 285\beta}{7680p^9} + \dots$$

These expansions, however, are of little value in the neighborhood of the turning point $x = 2\sqrt{-a}$. Here first approximations may be obtained by use of the formulas of 19.7. If a_n (negative) is a zero of $\text{Ai}(t)$, the corresponding zero c of $U(a, x)$ is obtained approximately by solving

19.26.3

$$\vartheta_3 = \frac{1}{4} \{ \arccos \xi - \xi \sqrt{1 - \xi^2} \} = \frac{(-a_n)^{\frac{1}{2}}}{6|a|} \quad c = 2\sqrt{|a|}\xi \quad (a < 0)$$

This may be done by inverse use of Table 19.3. For a zero of $V(a, x)$, a_n must be replaced by b_n , a zero of $\text{Bi}(t)$. For further developments see [19.5].

Zeros of solutions $W(a, x)$, $W(a, -x)$ of 19.1.3 occur for $|x| > 2\sqrt{a}$ when a is positive; the general solution may, however, have a single zero between $-2\sqrt{a}$ and $+2\sqrt{a}$. If a is negative, zeros are unrestricted in range.

Approximations may be obtained by reverting the series for ψ (or χ) in 19.24. With $-a = p^2$, $\alpha = (\frac{1}{2}r - \frac{1}{2})\pi$, $\beta = (\frac{1}{2}r + \frac{1}{2})\pi$, $r \geq 0$ being an odd

integer for $W(a, x)$ or its derivative, or an even integer for $W(a, -x)$ or its derivative, the zeros $\pm c$, $\pm c'$ have expansions

$$19.26.4 \quad c \approx \frac{\alpha}{p} - \frac{2\alpha^3 - 3\alpha}{48p^5} + \frac{52\alpha^5 - 240\alpha^3 + 315\alpha}{7680p^9} + \dots$$

$$19.26.5 \quad c' \approx \frac{\beta}{p} - \frac{2\beta^3 + 3\beta}{48p^5} + \frac{52\beta^5 + 280\beta^3 - 285\beta}{7680p^9} + \dots$$

When x is large and a moderate, we may solve inversely the series 19.24.4 or 19.24.6 with $\alpha = \frac{1}{2}(r\pi - \frac{1}{2}\pi - \phi_2)$, $\beta = \frac{1}{2}(r\pi + \frac{1}{2}\pi - \phi_2)$, r odd or even as above; the presence of the logarithm makes it inconvenient to revert formally.

The expansions 19.26.4 and 19.26.5 fail when x is in the neighborhood of $2\sqrt{|a|}$. When a is positive, a zero c of $W(a, -x)$ is obtained approximately by solving

19.26.6

$$\vartheta_2 = \frac{1}{4} \{ \xi \sqrt{\xi^2 - 1} - \text{arccosh } \xi \} = \frac{(-a_n)^{\frac{1}{2}}}{6a} \quad c = 2\sqrt{a}\xi \quad (a > 0)$$

with the aid of Table 19.3. For a zero of $W(a, x)$ we replace a_n by b_n . When a is negative we solve, again with the aid of Table 19.3,

19.26.7

$$\vartheta_1 = \frac{1}{4} \{ \xi \sqrt{\xi^2 + 1} + \text{arcsinh } \xi \} = \frac{(n - \frac{1}{2})\pi}{4|a|} \quad c = 2\sqrt{|a|}\xi \quad (-a > 0)$$

where $n = 1, 2, 3, \dots$ for an approximate zero of $W(a, -x)$, and $n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ for an approximate zero of $W(a, x)$. Further developments are given in [19.5].

Any of the approximations to zeros obtained above may readily be improved as follows:

Let c be a zero of y , and c' a zero of y' , where y is a solution of

$$19.26.8 \quad y'' - Iy = 0$$

Here $I = a \pm \frac{1}{2}x^2$, $I' = \pm \frac{1}{2}x$, $I'' = \pm \frac{1}{2}$; the method is general and the following formulae may be used whenever $I''' = 0$. Then if γ, γ' are approximations to the zeros c, c' and

$$19.26.9 \quad u = y(\gamma)/y'(\gamma) \quad v = y'(\gamma')/I^2 y(\gamma')$$

with $I \equiv I(\gamma)$ or $I \equiv I(\gamma')$ respectively, then

19.26.10

$$c \sim \gamma - u - \frac{1}{8} I u^3 + \frac{1}{12} I' u^4 \\ - (\frac{1}{80} I'' + \frac{1}{8} I^2) u^5 + \frac{1}{80} I I' u^6 + \dots$$

19.26.11

$$y'(c) \sim y'(\gamma) \{ 1 - \frac{1}{2} I u^2 + \frac{1}{8} I' u^3 \\ - (\frac{1}{24} I'' + \frac{1}{8} I^2) u^4 + \frac{7}{80} I I' u^5 + \dots \}$$

19.26.12

$$c' \sim \gamma' - I v - \frac{1}{2} I I' v^2 + (\frac{1}{8} I^2 I'' - \frac{1}{2} I I'^2 - \frac{1}{8} I^4) v^3 \\ + (\frac{5}{12} I^2 I' I'' - \frac{5}{8} I I'^3 - \frac{5}{12} I^4 I') v^4 + \dots$$

19.26.13

$$y(c') \sim y(\gamma') \{ 1 - \frac{1}{2} I^3 v^2 - \frac{1}{8} I^3 I' v^3 \\ - (\frac{1}{8} I^3 I'^2 - \frac{1}{24} I^4 I'' + \frac{1}{8} I^6) v^4 + \dots \}$$

The process can be repeated, if necessary, using as many terms at any stage as seems convenient.

Note the relations, holding at zeros,

$$19.26.14 \quad U'(a, c) = -\sqrt{2/\pi} V(a, c)$$

$$19.26.15 \quad V'(a, c') = \sqrt{2/\pi} U(a, c')$$

$$19.26.16 \quad W'(a, c) = -1/W(a, -c)$$

19.26.17

$$W(a, c') = 1 / \left\{ \frac{d}{dx} W(a, -x) \right\}_{x=c'} = -1/W'(a, -c')$$

19.27. Bessel Functions of Order $\pm \frac{1}{4}$, $\pm \frac{3}{4}$ as Parabolic Cylinder Functions

Most applications of these functions refer to cases where parabolic cylinder functions would be more appropriate. We have

$$19.27.1 \quad J_{\pm \frac{1}{4}}(\frac{1}{4}x^2) = \frac{2^{\frac{1}{2}}}{\sqrt{\pi x}} \{ W(0, -x) \mp W(0, x) \}$$

$$19.27.2 \quad J_{\pm \frac{1}{4}}(\frac{1}{4}x^2) = \frac{-2^{\frac{1}{2}}}{x\sqrt{\pi x}} \{ W(0, x) \pm W(0, -x) \}$$

Functions of other orders may be obtained by use of the recurrence relation 10.1.22, which here becomes

$$19.27.3 \quad \frac{1}{4}x^2 J_{\nu+1}(\frac{1}{4}x^2) - 2\nu J_{\nu}(\frac{1}{4}x^2) + \frac{1}{4}x^2 J_{\nu-1}(\frac{1}{4}x^2) = 0$$

Again

$$19.27.4 \quad I_{-\frac{1}{4}}(\frac{1}{4}x^2) + I_{\frac{1}{4}}(\frac{1}{4}x^2) = \frac{2}{\sqrt{x}} V(0, x)$$

19.27.5

$$\frac{\sqrt{2}}{\pi} K_{\frac{1}{4}}(\frac{1}{4}x^2) = I_{-\frac{1}{4}}(\frac{1}{4}x^2) - I_{\frac{1}{4}}(\frac{1}{4}x^2) = \frac{2}{\sqrt{\pi x}} U(0, x)$$

$$19.27.6 \quad I_{-\frac{1}{4}}(\frac{1}{4}x^2) + I_{\frac{1}{4}}(\frac{1}{4}x^2) = -\frac{4}{x\sqrt{x}} \frac{d}{dx} V(0, x)$$

19.27.7

$$\frac{\sqrt{2}}{\pi} K_{\frac{1}{4}}(\frac{1}{4}x^2) = I_{-\frac{1}{4}}(\frac{1}{4}x^2) - I_{\frac{1}{4}}(\frac{1}{4}x^2) \\ = -\frac{4}{x\sqrt{\pi x}} \frac{d}{dx} U(0, x)$$

As before, Bessel functions of other orders may be obtained by use of the recurrence relation 10.2.23, which here becomes

$$19.27.8 \quad \frac{1}{4}x^2 I_{\nu+1}(\frac{1}{4}x^2) + 2\nu I_{\nu}(\frac{1}{4}x^2) - \frac{1}{4}x^2 I_{\nu-1}(\frac{1}{4}x^2) = 0$$

$$19.27.9 \quad \frac{1}{4}x^2 K_{\nu+1}(\frac{1}{4}x^2) - 2\nu K_{\nu}(\frac{1}{4}x^2) - \frac{1}{4}x^2 K_{\nu-1}(\frac{1}{4}x^2) = 0$$

Numerical Methods

19.28. Use and Extension of the Tables

For $U(a, x)$, $V(a, x)$ and $W(a, x)$, interpolation x -wise may be carried out to 5-figure accuracy almost everywhere by using 5-point or 6-point Lagrangian interpolation. For $|a| \leq 1$, comparable accuracy a -wise may be obtained with 5- or 6-point interpolation.

For $|a| > 1$, $U(a, x)$ and $V(a, x)$ may be obtained by use of recurrence relations from two values, possibly obtained by interpolation, with $|a| \leq 1$; such a procedure is not available for $W(a, \pm x)$, $|a| > 1$.

In cases where straightforward use of the a -wise recurrence relation results in loss of accuracy by cancellation of leading digits, it may be worth while to remark that greater accuracy is usually attainable by use of the recurrence relation in the

reverse direction, from arbitrary starting values (often 1 and 0) for two values of a somewhat beyond the last value desired. This is because the recurrence relation is a second order homogeneous linear difference equation, and has two independent solutions. Loss of accuracy by cancellation occurs when the solution desired is diminishing as a varies, while the companion solution is increasing. By reversing the direction of progress in a , the roles of the two solutions are interchanged, and the contribution of the desired solution now increases, while the unwanted solution diminishes to the point of negligibility. By starting sufficiently beyond the last value of a for which the function is desired, we can ensure that the unwanted solution is negligible but, because the starting values were arbitrary, we have an un-

known multiple of the solution desired. The computation is then carried back until a value of a with $|a| \leq 1$ is reached, when the precise multiple that we have of the desired solution may be determined and hence removed throughout. Compare also 9.12, Example 1.

Example 1. Evaluate $U(a, 5)$ for $a=5, 6, 7, \dots$, using 19.6.4.

$$(a + \frac{1}{2})U(a+1, x) + xU(a, x) - U(a-1, x) = 0$$

a	Forward Recurrence	Backward Recurrence	Final Values
3	(-6) 5.2847*	(12) 1.59035	(-6) 5.2847**
4	(-7) 9.172*	(11) 2.76028	(-7) 9.1724
5	(-7) 1.5527	(10) 4.67131	(-7) 1.55227
6	(-8) 2.5609	(9) 7.72041	(-8) 2.5655
7	(-9) 4.1885	(9) 1.24785	(-9) 4.1466
8	(-10) 6.2220	(8) 1.97488	(-10) 6.5625
9	(-10) + 1.2676	(7) 3.06369	(-10) 1.01806
10	(-11) - 0.1221	(6) 4.66352	(-11) 1.5497
11	(-11) + 1.2654	(0) 697082	(-12) 2.3164
12	(-12) - 5.6079	102444	(-13) 3.404
13	(-12) + 3.2555	14789	(-14) 4.91
14		2111	(-15) 7.01
15		292	(-16) 9.7
16		42	
17		5	
18		1+	
19		0+	

*From tables. +Starting values.

**This value was used to obtain the constant multiplier $d = \frac{(-6)5.2847}{(12)1.59035} = (-18)3.32298$ for converting the previous column into this one.

The second column shows forward recurrence starting with values at $a=3, 4$ from Table 19.1. Backward recurrence starts with values 0 and 1 at $a=19$ and 18, containing a multiple $kU(a, 5)$ and a subsequently negligible multiple of the other solution $\Gamma(\frac{1}{2}-a)V(a, 5)$. Rounding errors convert $kU(a, x)$ into $k^*U(a, x)$ without affecting the values in the last column. The value of $1/k^*$ is identified from the known value of $U(3, 5)$, and used to obtain the final column by multiplying throughout by $1/k^*$. The improvement in $U(5, 5)$ is evident by comparison with Table 19.1.

Derivatives. These are not tabulated here. Since the functions $U(a, x)$, $V(a, x)$ and $W(a, x)$ satisfy differential equations, values of derivatives are often required.

For all these functions the equation is second order with first derivative absent, so that *second derivatives* may be readily obtained from function values by use of the differential equation.

First derivatives can be obtained for $U(a, x)$ and $V(a, x)$ by applying the appropriate recurrence

relations 19.6.1-2. If less accuracy is needed they can be found by use of mean central differences of $U(a, x)$, $V(a, x)$ and also of $W(a, x)$ with the formula

$$hu' = h \frac{du}{dx} = \mu \delta u - \frac{1}{6} \mu \delta^3 u + \frac{1}{30} \mu \delta^5 u - \dots$$

using $h=.1$; this usually gives a 3- or 4-figure value of du/dx .

If greater accuracy is needed for $dW(a, x)/dx$ it may be obtained by evaluating d^2W/dx^2 with the help of the differential equation satisfied by W and integrating this second derivative numerically. This requires one accurate value of dW/dx to start off the integration; we describe two methods for obtaining this, both making use of the difference between two fairly widely separated values of W , for example, separated by 5 or 10 tabular intervals.

(i) Write f_r, f'_r, f''_r for $W(a, x_0 + rh)$ and its first two derivatives, then f'_0 may be found from

$$hf'_0 = \frac{1}{2n} (f_n - f_{-n}) - \frac{h^2}{2n} \sum_{r=1}^{n-1} (n-r)(f''_r - f''_{-r})$$

$$- \frac{h^2}{2n} \left\{ \frac{1}{12} - \frac{1}{240} \delta^2 + \frac{3}{80480} \delta^4 - \dots \right\} (f''_n - f''_{-n})$$

$$- h^2 \left\{ \frac{1}{12} \mu \delta - \frac{1}{720} \mu \delta^3 + \frac{1}{80480} \mu \delta^5 - \dots \right\} f''_0$$

(ii) Consider a solution y of the differential equation for $W(a, x)$, namely $y'' = (-\frac{1}{4}x^2 + a)y$. If we are given values y and y' at a particular $x=x_0$ and write $T_n = H^n y^{(n)}/n!$, $T_{-1} = T_{-2} = 0$, then we may compute $T_2, T_3, T_4, \dots$ in succession by use of the recurrence relation obtained from the differential equation,

$$T_{n+2} = \frac{H^2}{(n+1)(n+2)} \left[\left(-\frac{1}{4}x_0^2 + a\right)T_n - \frac{1}{2}Hx_0T_{n-1} - \frac{1}{4}H^2T_{n-2} \right]$$

These are computed, to a fixed number of decimals until they become negligible, thus giving

$$y(x_0 \pm H) = T_0 \pm T_1 + T_2 \pm T_3 + \dots$$

This may be applied, with $H=rh$, h being the tabular interval, and r a small integer, say $r=5$, to the solutions $y=y_1, y=y_2$ having

$$\begin{aligned} y_1(x_0) &= W(a, x_0) & y'_1(x_0) &= W^{*'}(a, x_0) \\ y_2(x_0) &= 0 & y'_2(x_0) &= 1 \end{aligned}$$

in which $W^{*'}(a, x_0)$ is an approximation to $W'(a, x_0)$, not necessarily a good one; it may be

obtained from differences, for example. We thus obtain $y_1(x_0 \pm H)$ and $y_2(x_0 \pm H)$.

Now suppose

$$W'(a, x_0) = W''(a, x_0) + \lambda$$

then, for all x

$$W(a, x) = y_1(x) + \lambda y_2(x)$$

and in particular

$$W(a, x_0 \pm H) = y_1(x_0 \pm H) + \lambda y_2(x_0 \pm H)$$

The values of $W(a, x_0 \pm H)$ may be read from the tables and two independent estimates of λ obtained, whence

$$W'(a, x_0) = W''(a, x_0) + \lambda$$

to a suitable accuracy.

Example 2. Evaluate $W'(-3, 1)$ using $r=5$. From Table 19.2

$$W(-3, .5) = -.05857 \quad W(-3, 1) = -.61113$$

$$W(-3, 1.5) = -.69502$$

(i) Using the first method

x	$W(-3, x)$	$W''(-3, x)$	δ	δ^2	δ^3
0.4	+0.07298	-0.22186			
0.5	-.05857	+.17937		+131	
0.6	-.18832	.58191			
0.7	-.31226	.97503			
0.8	-.42646	1.34761	34081		
0.9	-.52722	1.68842	29775		-1095
1.0	-.61113	1.98617	24374		-1032
1.1	-.67522	2.22991	17941		
1.2	-.71706	2.40932			
1.3	-.73488	2.51513			
1.4	-.72761	2.53936			
1.5	-.69502	2.47601		-9129	
1.6	-.63774	2.32137			

The fifth decimal in $W''(-3, x)$ is only a guard figure which is hardly needed. Only the differences needed have been computed.

Then

$$\begin{aligned} \frac{1}{r^5} W'(-3, 1) &= \frac{1}{r^5} (-.69502 + .05857) - \frac{1}{1000} (10.38874) \\ &\quad - \frac{1}{1000} \left\{ \frac{1}{r^2} (2.29664) - \frac{1}{240} (-.09260) \right\} \\ &\quad - \frac{1}{100} \left\{ \frac{1}{24} (.54149) - \frac{11}{1440} (-.02127) \right\} \\ &= -.0636450 - .0103887 - .0001918 - .0002272 \\ &= -.0744527 \end{aligned}$$

Thus $W'(-3, 1) = -.74453$. This might have an error up to about $1\frac{1}{2}$ units in the last figure but is, in fact, correct to 5 decimals.

(ii) Using the second method, with

$$y_1(1) = W(-3, 1) = -.61113 \quad \text{to 5 decimals}$$

$$y_1'(1) = -.745 \quad \text{to about 3 decimals}$$

the following values result, with $H=.5$,

	y_1	y_2	$W(-3, x) = y_1 + \lambda y_2$
T_0	-.61113	.0000	At $x=1.5$
T_1	-.37250	+.5000	$x = .695223 + .4323\lambda$
			$= -.69502$
T_2	+.24827 2	.0000	$\lambda = .000203/.4323$
T_3	+ 5680 9	- 677	$= .000470$
T_4	- 1407 4	- 26	So $W'(-3, 1)$
			$= -.745 + \lambda$
			$= -.744530$
T_5	- 279 3	+ 24	At $x=.5$
T_6	+ 13 4	+ 2	$-.058363 - .4371\lambda$
			$= -.05857$
T_7	+ 5 4		$\lambda = .000207/.4371$
T_8	+ 5		$= .000474$
$y(1.5)$	-.695223	+.4323	So $W'(-3, 1)$
$y(.5)$	-.058363	-.4371	$= -.745 + \lambda$
			$= -.744526$

Thus $W'(-3, 1) = -.74453$ which is correct to 5 decimals.

Example 3. Evaluate the positive zero of $U(-3, x)$.

We use 19.7.3 to obtain a first approximation, see 19.26.3. The appropriate zero of $\text{Ai}(t)$ is at

$$t = (4|a|)^{1/3} r = -2.338$$

whence

$$\tau = -(2.338) \times (12)^{-1/3} = -.4461$$

Hence, from Table 19.3, $\xi = .3990$ and the approximate zero is $x = 2\sqrt{|a|}\xi = 1.382$.

We improve this by using 19.26.10, but take, for convenience, $x=1.4$ as an approximation, so that the value of U can be read directly from the tables. U' can be obtained as in the section following

Example 1.

We find

$$U(-3, 1.4) = .02627 \quad U'(-3, 1.4) = 2.0637$$

Then 19.26.9 gives

$$u = U/U' = .012730 \quad I = -2.51$$

$$I' = .7 \quad I'' = .5$$

and

$$c = 1.4 - .012730 + .000002 = 1.38727$$

$$y'(c) = 2.0637(1 + .000203) = 2.0641$$

which is correct to 5 decimals, while 19.26.11 gives

compared with the correct value 2.06416.

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20. Mathieu Functions

GERTRUDE BLANCH¹

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Even Solutions

$$a_r, ce_r(0, q), ce_r\left(\frac{\pi}{2}, q\right), ce'_r\left(\frac{\pi}{2}, q\right), (4q)^{\frac{r}{2}} g_{e,r}(q), (4q)^r f_{e,r}(q)$$

Odd Solutions

$$b_r, se'_r(0, q), se_r\left(\frac{\pi}{2}, q\right), se'_r\left(\frac{\pi}{2}, q\right), (4q)^{\frac{r}{2}} g_{o,r}(q), (4q)^r f_{o,r}(q)$$

$$q=0(5)25, \quad 8D \text{ or } S$$

$$a_r + 2q - (4r+2)\sqrt{q}, \quad b_r + 2q - (4r-2)\sqrt{q}$$

$$q^{-\frac{1}{2}} = .16(-.04)0, \quad 8D$$

$$r=0, 1, 2, 5, 10, 15$$

Table 20.2. Coefficients A_m and B_m	750
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$$q=5, 25; r=0, 1, 2, 5, 10, 15, \quad 9D$$

¹ Aeronautical Research Laboratories, Wright-Patterson Air Force Base, Ohio.

20. Mathieu Functions

Mathematical Properties

20.1. Mathieu's Equation

Canonical Form of the Differential Equation

$$20.1.1 \quad \frac{d^2 y}{dv^2} + (a - 2q \cos 2v)y = 0$$

Mathieu's Modified Differential Equation

$$20.1.2 \quad \frac{d^2 f}{du^2} - (a - 2q \cosh 2u)f = 0 \quad (v = iu, y = f)$$

Relation Between Mathieu's Equation and the Wave Equation for the Elliptic Cylinder

The wave equation in Cartesian coordinates is

$$20.1.3 \quad \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} + k^2 W = 0$$

A solution W is obtainable by separation of variables in elliptical coordinates. Thus, let

$$x = \rho \cosh u \cos v; \quad y = \rho \sinh u \sin v; \quad z = z;$$

ρ a positive constant; 20.1.3 becomes

$$20.1.4 \quad * \frac{\partial^2 W}{\partial z^2} + \frac{2}{\rho^2 (\cosh 2u - \cos 2v)} \left(\frac{\partial^2 W}{\partial u^2} + \frac{\partial^2 W}{\partial v^2} \right) + k^2 W = 0$$

Assuming a solution of the form

$$W = \varphi(z)f(u)g(v)$$

and substituting the above into 20.1.4 one obtains, after dividing through by W ,

$$\frac{1}{\varphi} \frac{d^2 \varphi}{dz^2} + G = 0$$

where

$$* G = \frac{2}{\rho^2 (\cosh 2u - \cos 2v)} \left\{ \frac{d^2 f}{du^2} \frac{1}{f} + \frac{d^2 g}{dv^2} \frac{1}{g} \right\} + k^2$$

Since z, u, v are independent variables, it follows that

$$20.1.5 \quad \frac{d^2 \varphi}{dz^2} + c\varphi = 0$$

where c is a constant.

Again, from the fact that $G = c$ and that u, v are independent variables, one sets

20.1.6

$$* a = \frac{d^2 f}{du^2} \frac{1}{f} + \frac{(k^2 - c)}{2} \rho^2 \cosh 2u$$

$$a = -\frac{d^2 g}{dv^2} \frac{1}{g} + \frac{(k^2 - c)}{2} \rho^2 \cos 2v$$

where a is a constant. The above are equivalent to 20.1.1 and 20.1.2. The constants c and a are often referred to as *separation constants*, due to the role they play in 20.1.5 and 20.1.6.

For some physically important solutions, the function g must be periodic, of period π or 2π . It can be shown that there exists a countably infinite set of *characteristic values* $a_r(q)$ which yield even periodic solutions of 20.1.1; there is another countably infinite sequence of *characteristic values* $b_r(q)$ which yield odd periodic solutions of 20.1.1.

It is known that there exist periodic solutions of period $k\pi$, where k is any positive integer. In what follows, however, the term *characteristic value* will be reserved for a value associated with solutions of period π or 2π only. These characteristic values are of basic importance to the general theory of the differential equation for arbitrary parameters a and q .

An Algebraic Form of Mathieu's Equation

20.1.7

$$(1-t^2) \frac{d^2 y}{dt^2} - t \frac{dy}{dt} + (a + 2q - 4qt^2)y = 0 \quad (\cos v = t)$$

Relation to Spheroidal Wave Equation

$$20.1.8 \quad (1-t^2) \frac{d^2 y}{dt^2} - 2(b+1)t \frac{dy}{dt} + (c - 4qt^2)y = 0 \quad *$$

Thus, Mathieu's equation is a special case of 20.1.8, with $b = -\frac{1}{2}$, $c = a + 2q$.

20.2. Determination of Characteristic Values

A solution of 20.1.1 with v replaced by z , having period π or 2π is of the form

$$20.2.1 \quad y = \sum_{m=0}^{\infty} (A_m \cos mz + B_m \sin mz)$$

where B_0 can be taken as zero. If the above is substituted into 20.1.1 one obtains

20.2.2

$$\sum_{m=-2}^{\infty} [(a-m^2)A_m - q(A_{m-2} + A_{m+2})] \cos mz + \sum_{m=-1}^{\infty} [(a-m^2)B_m - q(B_{m-2} + B_{m+2})] \sin mz = 0$$

$$A_{-m}, B_{-m} = 0 \quad m > 0$$

*See page II.

Equation 20.2.2 can be reduced to one of four simpler types, given in 20.2.3 and 20.2.4 below

$$20.2.3 \quad y_0 = \sum_{m=0}^{\infty} A_{2m+p} \cos (2m+p)z, \quad p=0 \text{ or } 1$$

$$20.2.4 \quad y_1 = \sum_{m=0}^{\infty} B_{2m+p} \sin (2m+p)z, \quad p=0 \text{ or } 1$$

If $p=0$, the solution is of period π ; if $p=1$, the solution is of period 2π .

Recurrence Relations Among the Coefficients

Even solutions of period π :

$$20.2.5 \quad aA_0 - qA_2 = 0$$

$$20.2.6 \quad (a-4)A_2 - q(2A_0 + A_4) = 0$$

$$20.2.7 \quad (a-m^2)A_m - q(A_{m-2} + A_{m+2}) = 0 \quad (m \geq 3)$$

Even solutions of period 2π :

$$20.2.8 \quad (a-1)A_1 - q(A_1 + A_3) = 0,$$

along with 20.2.7 for $m \geq 3$.

Odd solutions of period π :

$$20.2.9 \quad (a-4)B_2 - qB_4 = 0$$

$$* 20.2.10 \quad (a-m^2)B_m - q(B_{m-2} + B_{m+2}) = 0 \quad (m \geq 3)$$

Odd solutions of period 2π :

$$20.2.11 \quad (a-1)B_1 + q(B_1 - B_3) = 0,$$

along with 20.2.10 for $m \geq 3$.

Let

$$20.2.12 \quad Ge_m = A_m/A_{m-2}, \quad Go_m = B_m/B_{m-2};$$

$G_m = Ge_m$ or Go_m when the same operations apply to both, and no ambiguity is likely to arise. Further let

$$20.2.13 \quad V_m = (a-m^2)/q.$$

Equations 20.2.5–20.2.7 are equivalent to

$$20.2.14 \quad Ge_2 = V_0; \quad Ge_4 = V_2 - \frac{2}{Ge_2}$$

$$20.2.15 \quad G_m = 1/(V_m - G_{m+2}) \quad (m \geq 3),$$

for even solutions of period π .

Similarly

$$20.2.16 \quad V_1 - 1 = Ge_3; \text{ for even solutions of period } 2\pi, \text{ along with } 20.2.15$$

$$20.2.17 \quad V_1 + 1 = Go_3, \text{ for odd solutions of period } 2\pi, \text{ along with } 20.2.15$$

20.2.18 $V_2 = Go_4$, for odd solutions of period π , along with 20.2.15

These three-term recurrence relations among the coefficients indicate that every G_m can be developed into two types of continued fractions. Thus 20.2.15 is equivalent to

20.2.19

$$G_m = \frac{1}{V_m - G_{m+2}} = \frac{1}{V_m - \frac{1}{V_{m+2} - \frac{1}{V_{m+4} - \dots}}} \quad (m \geq 3)$$

20.2.20

$$G_{m+2} = V_m - 1/G_m \\ = V_m - \frac{1}{V_{m-2} - \frac{1}{V_{m-4} - \dots - \frac{\varphi_0}{V_{0+d} + \varphi_1}}} \quad (m \geq 3)$$

where

$$\varphi_1 = d = 0; \quad \varphi_0 = 2, \text{ if } G_{m+2} = A_{2s}/A_{2s-2}$$

$$\varphi_1 = d = \varphi_0 = 0, \text{ if } G_{m+2} = B_{2s}/B_{2s-2}$$

$$\varphi_1 = -1; \quad \varphi_0 = d = 1, \text{ if } G_{m+2} = A_{2s+1}/A_{2s-1}$$

$$\varphi_1 = d = \varphi_0 = 1, \text{ if } G_{m+2} = B_{2s+1}/B_{2s-1}$$

The four choices of the parameters φ_1 , φ_0 , d correspond to the four types of solutions 20.2.3–20.2.4. Hereafter, it will be convenient to separate the characteristic values a into two major subsets:

$a = a_r$, associated with even periodic solutions

$a = b_r$, associated with odd periodic solutions

If 20.2.19 is suitably combined with 20.2.13–20.2.18 there result four types of continued fractions, the roots of which yield the required characteristic values

$$20.2.21 \quad V_0 - \frac{2}{V_2 - \frac{1}{V_4 - \frac{1}{V_6 - \dots}}} = 0 \quad \text{Roots: } a_{2r}$$

20.2.22

$$V_1 - 1 - \frac{1}{V_3 - \frac{1}{V_5 - \frac{1}{V_7 - \dots}}} = 0 \quad \text{Roots: } a_{2r+1}$$

$$20.2.23 \quad V_2 - \frac{1}{V_4 - \frac{1}{V_6 - \frac{1}{V_8 - \dots}}} = 0 \quad \text{Roots: } b_{2r}$$

20.2.24

$$V_1 + 1 - \frac{1}{V_3 - \frac{1}{V_5 - \frac{1}{V_7 - \dots}}} = 0 \quad \text{Roots: } b_{2r+1}$$

If a is a root of 20.2.21–20.2.24, then the corresponding solution exists and is an entire function of z , for general complex values of q .

If q is real, then the Sturmian theory of second order linear differential equations yields the

*See page II.

following:

- (a) For a fixed real q , characteristic values a_r and b_r are real and distinct, if $q \neq 0$; $a_0 < b_1 < a_1 < b_2 < a_2 < \dots$, $q > 0$ and $a_r(q)$, $b_r(q)$ approach r^2 as q approaches zero.
- (b) A solution of 20.1.1 associated with a_r or b_r has r zeros in the interval $0 \leq z < \pi$, (q real).
- (c) The form of 20.2.21 and 20.2.23 shows that if a_{2r} is a root of 20.2.21 and q is different from zero, then a_{2r} cannot be a root of 20.2.23; similarly, no root of 20.2.22 can be a root of 20.2.24 if $q \neq 0$. It may be shown from other considerations that for a given point (a , q) there can be at most one periodic solution of period π or 2π if $q \neq 0$. This no longer holds for solutions of period $s\pi$, $s \geq 3$; for these all solutions are periodic, if one is.

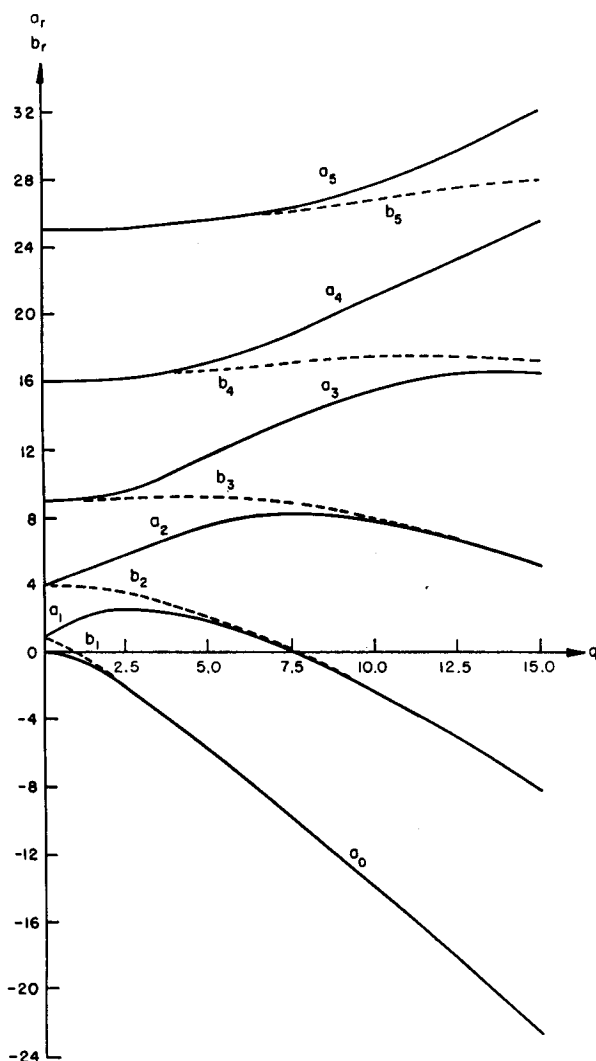


FIGURE 20.1. Characteristic Values a_r , b_r $r=0,1(1)5$

Power Series for Characteristic Values

20.2.25

$$a_0(q) = -\frac{q^2}{2} + \frac{7q^4}{128} - \frac{29q^6}{2304} + \frac{68687q^8}{18874368} + \dots$$

$$a_1(-q) = 1 - q - \frac{q^2}{8} + \frac{q^3}{64} - \frac{q^4}{1536} - \frac{11q^5}{36864} + \frac{49q^6}{589824} - \frac{55q^7}{9437184} - \frac{83q^8}{35389440} + \dots$$

$$b_2(q) = 4 - \frac{q^2}{12} + \frac{5q^4}{13824} - \frac{289q^6}{79626240} + \frac{21391q^8}{458647142400} + \dots$$

$$a_2(q) = 4 + \frac{5q^2}{12} - \frac{763q^4}{13824} + \frac{1002401q^6}{79626240} - \frac{1669068401q^8}{458647142400} + \dots$$

$$a_3(-q) = 9 + \frac{q^2}{16} - \frac{q^3}{64} + \frac{13q^4}{20480} + \frac{5q^5}{16384} - \frac{1961q^6}{23592960} + \frac{609q^7}{104857600} + \dots$$

$$b_4(q) = 16 + \frac{q^2}{30} - \frac{317q^4}{864000} + \frac{10049q^6}{2721600000} + \dots$$

$$a_4(q) = 16 + \frac{q^2}{30} + \frac{433q^4}{864000} - \frac{5701q^6}{2721600000} + \dots$$

$$a_5(-q) = 25 + \frac{q^2}{48} + \frac{11q^4}{774144} - \frac{q^5}{147456} + \frac{37q^6}{891813888} + \dots$$

$$b_6(q) = 36 + \frac{q^2}{70} + \frac{187q^4}{43904000} - \frac{5861633q^6}{92935987200000} + \dots$$

$$a_6(q) = 36 + \frac{q^2}{70} + \frac{187q^4}{43904000} + \frac{6743617q^6}{92935987200000} + \dots$$

For $r \geq 7$, and $|q|$ not too large, a_r is approximately equal to b_r , and the following approximation may be used

20.2.26

$$\left. \begin{matrix} a_r \\ b_r \end{matrix} \right\} = r^2 + \frac{q^2}{2(r^2-1)} + \frac{(5r^2+7)q^4}{32(r^2-1)^3(r^2-4)} + \frac{(9r^4+58r^2+29)q^6}{64(r^2-1)^5(r^2-4)(r^2-9)} + \dots$$

The above expansion is not limited to integral values of r , and it is a very good approximation for r of the form $n + \frac{1}{2}$ where n is an integer. In case of integral values of $r = n$, the series holds only up to terms not involving $r^2 - n^2$ in the denominator. Subsequent terms must be derived specially (as shown by Mathieu). Mulholland and Goldstein [20.38] have computed characteristic values for purely imaginary q and found that a_0 and a_2 have a common real value for $|q|$ in the neighborhood of 1.468; Bouwkamp [20.5] has computed this number as $q_0 = \pm i 1.46876852$ to 8 decimals. For values of $-iq > -iq_0$, a_0 and a_2 are conjugate complex numbers. From equation 20.2.25 it follows that the radius of convergence for the series defining a_0 is no greater than $|q_0|$. It is shown in [20.36], section 2.25 that the radius of convergence for $a_{2n}(q)$, $n \geq 2$ is greater than 3. Furthermore

$$a_r - b_r = O(q^r / r^{r-1}), \quad r \rightarrow \infty.$$

Power Series in q for the Periodic Functions (for sufficiently small $|q|$)

20.2.27

$$ce_0(z, q) = 2^{-1} \left[1 - \frac{q}{2} \cos 2z + q^2 \left(\frac{\cos 4z}{32} - \frac{1}{16} \right) - q^3 \left(\frac{\cos 6z}{1152} - \frac{11 \cos 2z}{128} \right) + \dots \right]$$

$$ce_1(z, q) = \cos z - \frac{q}{8} \cos 3z + q^2 \left[\frac{\cos 5z}{192} - \frac{\cos 3z}{64} - \frac{\cos z}{128} \right] - q^3 \left[\frac{\cos 7z}{9216} - \frac{\cos 5z}{1152} - \frac{\cos 3z}{3072} + \frac{\cos z}{512} \right] + \dots$$

$$se_1(z, q) = \sin z - \frac{q}{8} \sin 3z + q^2 \left[\frac{\sin 5z}{192} + \frac{\sin 3z}{64} - \frac{\sin z}{128} \right] - q^3 \left[\frac{\sin 7z}{9216} + \frac{\sin 5z}{1152} - \frac{\sin 3z}{3072} - \frac{\sin z}{512} \right] + \dots$$

$$ce_2(z, q) = \cos 2z - q \left(\frac{\cos 4z}{12} - \frac{1}{4} \right) + q^2 \left(\frac{\cos 6z}{384} - \frac{19 \cos 2z}{288} \right) + \dots$$

$$se_2(z, q) = \sin 2z - q \frac{\sin 4z}{12} + q^2 \left(\frac{\sin 6z}{384} - \frac{\sin 2z}{288} \right) + \dots$$

20.2.28

$$ce_r(z, q) = \cos(rz - p(\pi/2)) - q \left\{ \frac{\cos[(r+2)z - p(\pi/2)]}{4(r+1)} - \frac{\cos[(r-2)z - p(\pi/2)]}{4(r-1)} \right\} + q^2 \left\{ \frac{\cos[(r+4)z - p(\pi/2)]}{32(r+1)(r+2)} + \frac{\cos[(r-4)z - p(\pi/2)]}{32(r-1)(r-2)} - \frac{\cos[rz - p(\pi/2)]}{32} \left[\frac{2(r^2+1)}{(r^2-1)^2} \right] \right\} + \dots$$

with $p=0$ for $ce_r(z, q)$, $p=1$ for $se_r(z, q)$, $r \geq 3$.

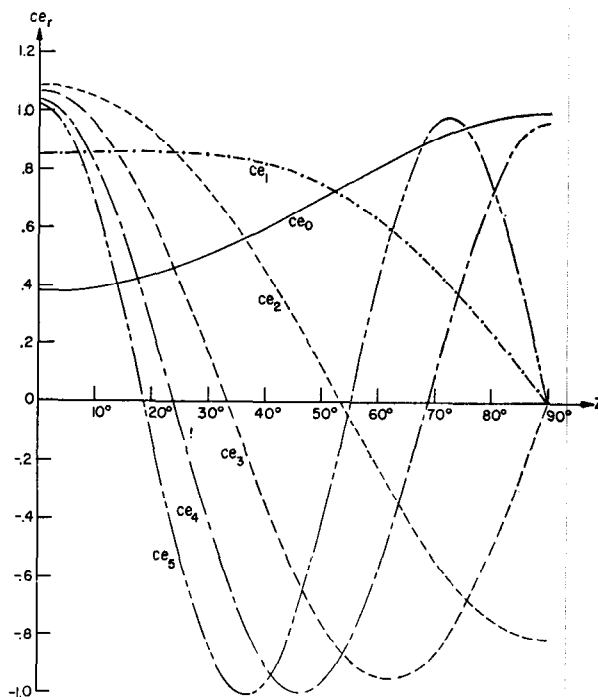


FIGURE 20.2. Even Periodic Mathieu Functions, Orders 0-5
 $q=1$.

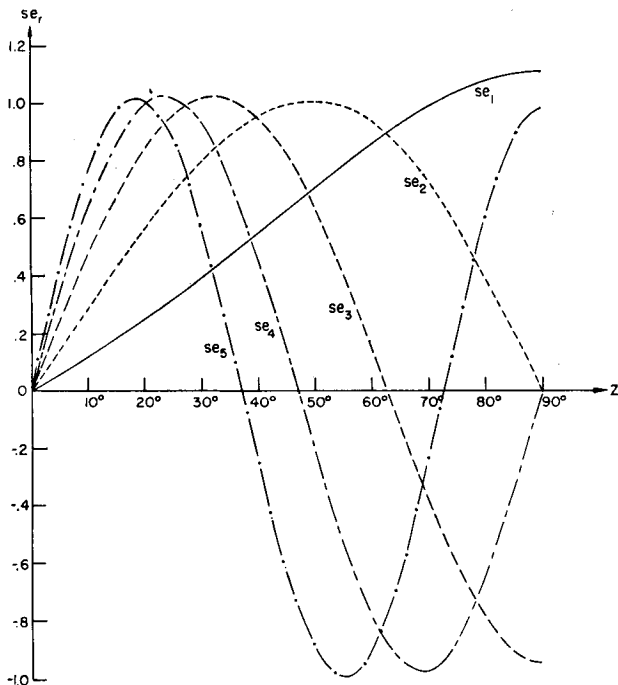


FIGURE 20.3. Odd Periodic Mathieu Functions, Orders 1-5
 $q=1$.

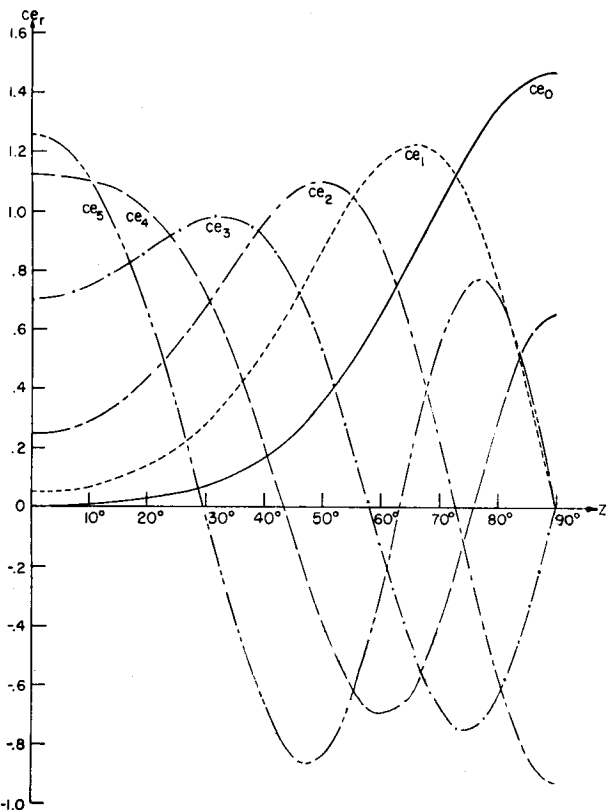


FIGURE 20.4. Even Periodic Mathieu Functions, Orders 0-5
 $q=10$.

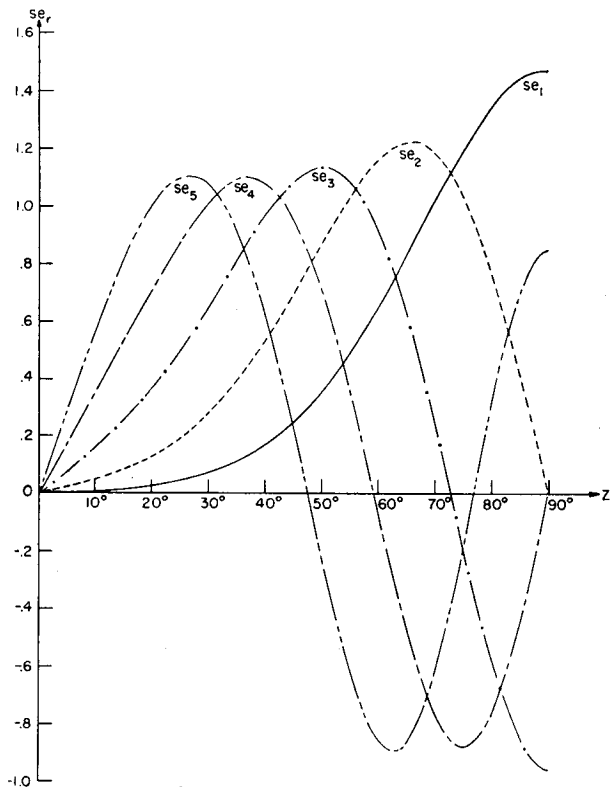


FIGURE 20.5. Odd Periodic Mathieu Functions, Orders 1-5
 $q=10$.

For coefficients associated with above functions

20.2.29

$$A_0^0(0) = 2^{-1/2}; A_r^r(0) = B_r^r(0) = 1, r > 0$$

$$A_{2s}^0 = [(-1)^s q^s / s! s! 2^{2s-1}] A_0^0 + \dots, s > 0$$

$$\begin{aligned} A_{r+2s}^r &= [(-1)^s r! q^s / 4^s (r+s)! s!] C_r^r + \dots \\ B_{r+2s}^r &= \dots \end{aligned}$$

$$rs > 0, C_r^r = A_r^r \text{ or } B_r^r$$

$$A_{r-2s}^r \text{ or } B_{r-2s}^r = \frac{(r-s-1)! q^s}{s! (r-1)! 4^s} C_r^r + \dots$$

Asymptotic Expansion for Characteristic Values, $q \gg 1$

Let $w = 2r + 1$, $q = w^4 \varphi$, φ real. Then

20.2.30

$$a \sim b_{r+1} \sim -2q + 2w\sqrt{q} - \frac{w^2 + 1}{8} - \frac{\left(w + \frac{3}{w}\right)}{2^7 \sqrt{\varphi}}$$

$$- \frac{d_1}{2^{12} \varphi} - \frac{d_2}{2^{17} \varphi^{3/2}} - \frac{d_3}{2^{20} \varphi^2} - \frac{d_4}{2^{25} \varphi^{5/2}} - \dots$$

where

$$d_1 = 5 + \frac{34}{w^2} + \frac{9}{w^4}$$

$$d_2 = \frac{33}{w} + \frac{410}{w^3} + \frac{405}{w^5}$$

$$d_3 = \frac{63}{w^2} + \frac{1260}{w^4} + \frac{2943}{w^6} + \frac{486}{w^8}$$

$$d_4 = \frac{527}{w^3} + \frac{15617}{w^5} + \frac{69001}{w^7} + \frac{41607}{w^9}$$

$$20.2.31 \quad b_{r+1} - a_r \sim 2^{4r+5} \sqrt{2/\pi} q^{4r+1} e^{-4\sqrt{q}}/r!, \quad q \rightarrow \infty$$

(given in [20.36] without proof.)

20.3. Floquet's Theorem and Its Consequences

Since the coefficients of Mathieu's equation

$$20.3.1 \quad y'' + (a - 2q \cos 2z)y = 0$$

are periodic functions of z , it follows from the known theory relating to such equations that there exists a solution of the form

$$20.3.2 \quad F_\nu(z) = e^{i\nu z} P(z),$$

where ν depends on a and q , and $P(z)$ is a periodic function, of the same period as that of the coefficients in 20.3.1, namely π . (Floquet's theorem; see [20.16] or [20.22] for its more general form.) The constant ν is called the *characteristic exponent*. Similarly

$$20.3.3 \quad F_\nu(-z) = e^{-i\nu z} P(-z)$$

satisfies 20.3.1 whenever 20.3.2 does. Both $F_\nu(z)$ and $F_\nu(-z)$ have the property

20.3.4

$$y(z+k\pi) = C^k y(z), \quad y = F_\nu(z) \text{ or } F_\nu(-z), \\ C = e^{i\nu\pi} \text{ for } F_\nu(z), \quad C = e^{-i\nu\pi} \text{ for } F_\nu(-z)$$

Solutions having the property 20.3.4 will hereafter be termed *Floquet solutions*. Whenever $F_\nu(z)$ and $F_\nu(-z)$ are linearly independent, the general solution of 20.3.1 can be put into the form

$$20.3.5 \quad y = AF_\nu(z) + BF_\nu(-z)$$

If $AB \neq 0$, the above solution will *not* be a *Floquet solution*. It will be seen later, from the method for determining ν when a and q are given, that there is some ambiguity in the definition of ν ; namely, ν can be replaced by $\nu + 2k$, where k is an arbitrary integer. This is as it should be, since the addition of the factor $\exp(2ikz)$ in 20.3.2 still leaves a periodic function of period π for the coefficient of $\exp i\nu z$.

It turns out that when a belongs to the set of characteristic values a_r and b_r of 20.2, then ν is zero or an integer. It is convenient to associate $\nu = r$ with $a_r(q)$, and $\nu = -r$ with $b_r(q)$; see [20.36]. In the special case when ν is an integer, $F_\nu(z)$ is

proportional to $F_\nu(-z)$; the second, independent solution of 20.3.1 then has the form

$$20.3.6 \quad y_2 = z c e_\nu(z, q) + \sum_{k=0}^{\infty} d_{2k+p} \sin(2k+p)z, \\ \text{associated with } c e_\nu(z, q)$$

$$20.3.7 \quad y_2 = z s e_\nu(z, q) + \sum_{k=0}^{\infty} f_{2k+p} \cos(2k+p)z, \\ \text{associated with } s e_\nu(z, q)$$

The coefficients d_{2k+p} and f_{2k+p} depend on the corresponding coefficients A_m and B_m , respectively, of 20.2, as well as on a and q . See [20.30], section (7.50)–(7.51) and [20.58], section V, for details.

If ν is not an integer, then the Floquet solutions $F_\nu(z)$ and $F_\nu(-z)$ are linearly independent. It is clear that 20.3.2 can be written in the form

$$20.3.8 \quad F_\nu(z) = \sum_{k=-\infty}^{\infty} c_{2k} e^{i(\nu+2k)z}.$$

From 20.3.8 it follows that if ν is a proper fraction m_1/m_2 , then every solution of 20.3.1 is periodic, and of period at most $2\pi m_2$. This agrees with results already noted in 20.2; i.e., both independent solutions are periodic, if one is, provided the period is different from π and 2π .

Method of Generating the Characteristic Exponent

Define two linearly independent solutions of 20.3.1, for fixed a, q by

$$20.3.9 \quad y_1(0) = 1; y_1'(0) = 0. \\ y_2(0) = 0; y_2'(0) = 1.$$

Then it can be shown that

$$20.3.10 \quad \cos \pi\nu - y_1(\pi) = 0$$

$$20.3.11 \quad \cos \pi\nu - 1 - 2y_1'\left(\frac{\pi}{2}\right) y_2\left(\frac{\pi}{2}\right) = 0$$

Thus ν may be obtained from a knowledge of $y_1(\pi)$ or from a knowledge of both $y_1'\left(\frac{\pi}{2}\right)$ and $y_2\left(\frac{\pi}{2}\right)$.

For numerical purposes 20.3.11 may be more desirable because of the shorter range of integration, and hence the lesser accumulation of round-off errors. Either ν , $-\nu$, or $\pm\nu + 2k$ (k an arbitrary integer) can be taken as the solution of 20.3.11. Once ν has been fixed, the coefficients of 20.3.8 can be determined, except for an arbitrary multiplier which is independent of z .

The characteristic exponent can also be computed from a continued fraction, in a manner analogous to developments in 20.2, if a sufficiently close first approximation to ν is available. For

systematic tabulation, this method is considerably faster than the method of numerical integration. Thus, when 20.3.8 is substituted into 20.3.1, there result the following recurrence relations:

$$20.3.12 \quad V_{2n}c_{2n} = c_{2n-2} + c_{2n+2}$$

where

$$20.3.13 \quad V_{2n} = [a - (2n + \nu)^2]/q, \quad -\infty < n < \infty.$$

When ν is complex, the coefficients V_{2n} may also be complex. As in 20.2, it is possible to generate the ratios

$$G_m = c_m/c_{m-2} \text{ and } H_{-m} = c_{-m-2}/c_{-m}$$

from the continued fractions

$$20.3.14$$

$$G_m = \frac{1}{V_m - \frac{1}{V_{m+2} - \dots}}, \quad m \geq 0$$

$$H_{-m} = \frac{1}{V_{-m-2} - \frac{1}{V_{-m-4} - \dots}}, \quad m \geq 0.$$

From the form of 20.3.13 and the known properties of continued fractions it is assured that for sufficiently large values of $|m|$ both $|G_m|$ and $|H_{-m}|$ converge. Once values of G_m and H_{-m} are available for some sufficiently large value of m , then the finite number of ratios $G_{m-2}, G_{m-4}, \dots, G_0$ can be computed in turn, if they exist. Similarly for $H_{-m+2}, \dots, H_0$. It is easy to show that ν is the correct characteristic exponent, appropriate for the point (a, q) , if and only if $H_0 G_0 = 1$. An iteration technique can be used to improve the value of ν , by the method suggested in [20.3]. One coefficient c_j can be assigned arbitrarily; the rest are then completely determined. After all the c_j become available, a multiplier (depending on q but not on z) can be found to satisfy a prescribed normalization.

It is well known that continued fractions can be converted to determinantal form. Equation 20.3.14 can in fact be written as a determinant with an infinite number of rows—a special case of Hill's determinant. See [20.19], [20.36], [20.15], or [20.30] for details. Although the determinant has actually been used in computations where high-speed computers were available, the direct use of the continued fraction seems much less laborious.

Special Cases (a, q Real)

Corresponding to $q=0$, $y_1 = \cos \sqrt{a}z$, $y_2 = \sin \sqrt{a}z$; the Floquet solutions are $\exp(iaz)$ and $\exp(-iaz)$. As a, q vary continuously in the q - a plane, ν describes curves; ν is real when (q, a) , $q \geq 0$ lies in the region between $a_r(q)$ and $b_{r+1}(q)$ and

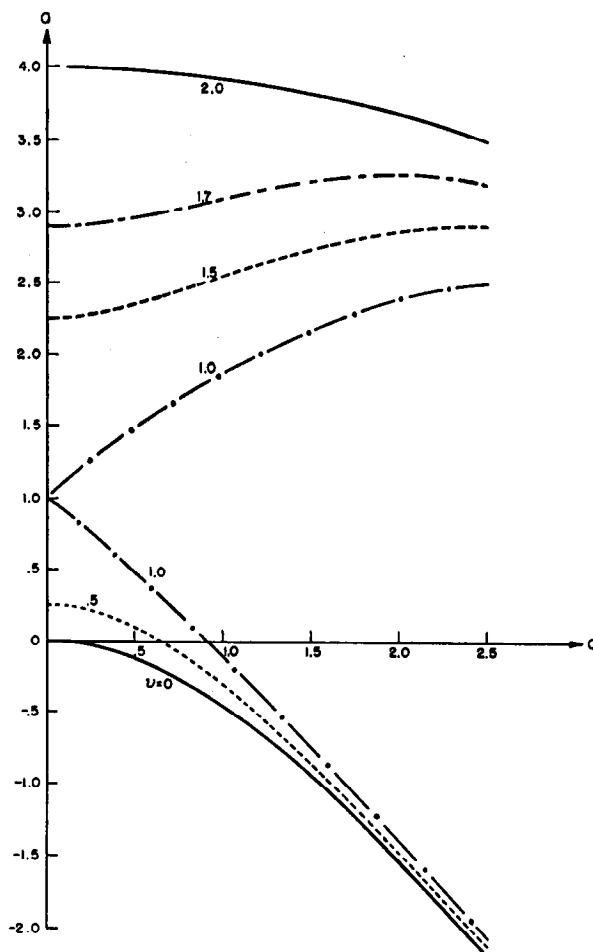


FIGURE 20.6. Characteristic Exponent-First Two Stable Regions $y = e^{i\nu z}P(x)$ where $P(x)$ is a periodic function of period π .

Definition of ν ;

In first stable region, $0 \leq \nu \leq 1$,

In second stable region, $1 \leq \nu \leq 2$.

(Constructed from tabular values supplied by T. Tamir, Brooklyn Polytechnic Institute)

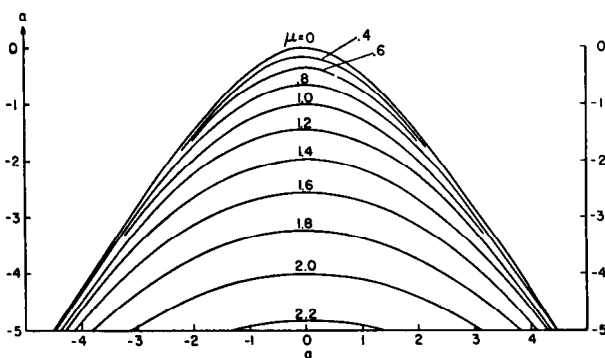


FIGURE 20.7. Characteristic Exponent in First Unstable Region. Differential equation: $y'' + (a - 2q \cos 2x)y = 0$. The Floquet solution $y = e^{i\nu z}P(x)$, where $P(x)$ is a periodic function of period π . In the first unstable region, $\nu = i\mu$; μ is given for $a \geq -5$. (Constructed at NBS.)

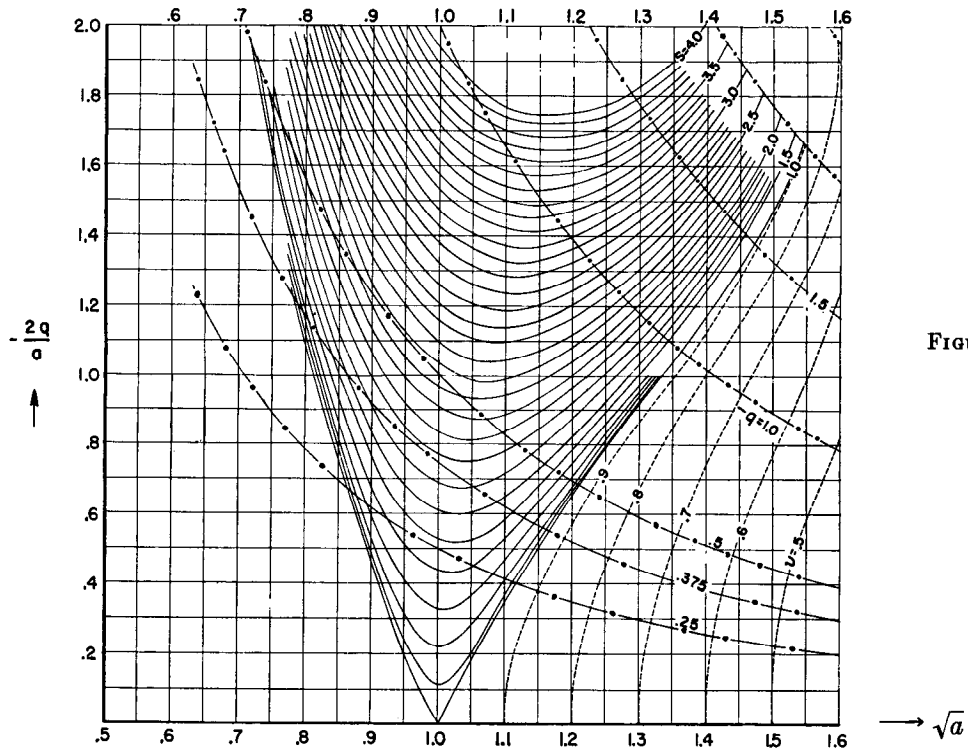


FIGURE 20.8

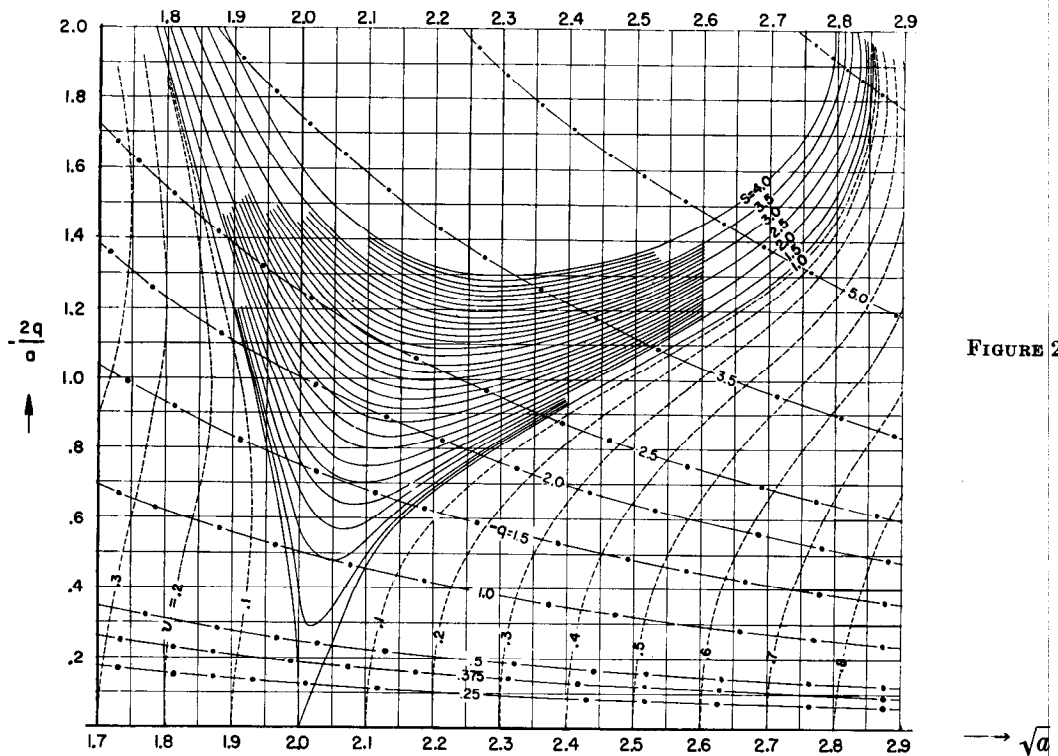


FIGURE 20.9

Charts of the Characteristic Exponent.

(From S. J. Zarodny, An elementary review of the Mathieu-Hill equation of real variable based on numerical solutions, Ballistic Research Laboratory Memo. Rept. 878, Aberdeen Proving Ground, Md., 1955, with permission.)

- $s = e^{i\pi\nu} = \text{constant}$; in unstable regions
- - - $\nu = \text{constant}$; in stable regions
- . - . Lines of constant values of $-q$.

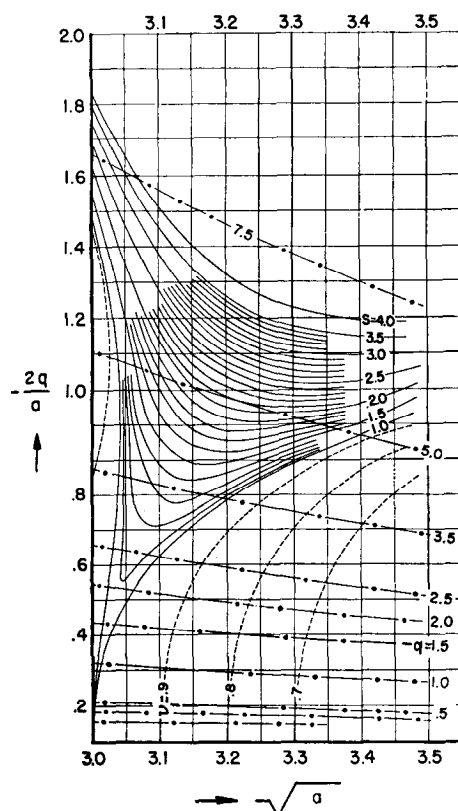


FIGURE 20.10. Chart of the Characteristic Exponent.

(From S. J. Zarodny, An elementary review of the Mathieu-Hill equation of real variable based on numerical solutions, Ballistic Research Laboratory Memo. Rept. 878, Aberdeen Proving Ground, Md., 1955, with permission)

— $s = e^{i\nu\pi} = \text{constant}$; in unstable regions
 - - - $\nu = \text{constant}$; in stable regions
 - . - . Lines of constant values of $-q$.

all solutions of 20.1.1 for real z are therefore bounded (stable); ν is complex in regions between b_r and a_r ; in these regions every solution becomes infinite at least once; hence these regions are termed "unstable regions". The characteristic curves a_r , b_r separate the regions of stability. For negative q , the stable regions are between b_{2r+1} and b_{2r+2} , a_{2r} and a_{2r+1} ; the unstable regions are between a_{2r+1} and b_{2r+1} , a_{2r} and b_{2r} .

In some problems solutions are required for real values of z only. In such cases a knowledge of the characteristic exponent ν and the periodic function $P(z)$ is sufficient for the evaluation of the required functions. For complex values of z , however, the series defining $P(z)$ converges slowly. Other solutions will be determined in the next section; they all have the remarkable property that they depend on the same coefficients c_m developed in connection with Floquet's theorem (except for an arbitrary normalization factor).

Expansions for Small q ([20.36] chapter 2)

If ν , q are fixed:

20.3.15

$$a = \nu^2 + \frac{q^2}{2(\nu^2-1)} + \frac{(5\nu^2+7)q^4}{32(\nu^2-1)^3(\nu^2-4)} + \frac{(9\nu^4+58\nu^2+29)q^6}{64(\nu^2-1)^5(\nu^2-4)(\nu^2-9)} + \dots \quad (\nu \neq 1, 2, 3).$$

For the coefficients c_2 , of 20.3.8

20.3.16

$$c_2/c_0 = \frac{-q}{4(\nu+1)} - \frac{(\nu^2+4\nu+7)q^3}{128(\nu+1)^3(\nu+2)(\nu-1)} + \dots \quad (\nu \neq 1, 2)$$

$$c_4/c_0 = q^2/32(\nu+1)(\nu+2) + \dots$$

$$c_{2s}/c_0 = (-1)^s q^s \Gamma(\nu+1)/2^{2s} s! \Gamma(\nu+s+1) + \dots$$

20.3.17

$$F_\nu(z) = c_0 \left[e^{i\nu z} - q \left\{ \frac{e^{i(\nu+2)z}}{4(\nu+1)} - \frac{e^{i(\nu-2)z}}{4(\nu-1)} \right\} \right] + \dots \quad (\nu \text{ not an integer})$$

For small values of a

20.3.18

$$\cos \nu\pi = \left(1 - \frac{a\pi^2}{2} + \frac{a^2\pi^4}{24} + \dots \right) - \frac{q^2\pi^2}{4} \left[1 + a \left(1 - \frac{\pi^2}{6} \right) + \dots \right] + q^4 \left(\frac{\pi^4}{96} - \frac{25\pi^2}{256} + \dots \right) + \dots$$

20.4. Other Solutions of Mathieu's Equation

Following Erdélyi [20.14], [20.15], define

$$20.4.1 \quad \varphi_k(z) = [e^{i\pi} \cos(z-b)/\cos(z+b)]^{1/2} J_k(f)$$

where

$$20.4.2 \quad f = 2[q \cos(z-b) \cos(z+b)]^{1/2},$$

and $J_k(f)$ is the Bessel function of order k ; b is a fixed, arbitrary complex number. By using the recurrence relations for Bessel functions the following may be verified:

20.4.3

$$\frac{d^2 \varphi_k}{dz^2} - 2q(\cos 2z) \varphi_k + q(\varphi_{k-2} + \varphi_{k+2}) + k^2 \varphi_k = 0.$$

It follows that a formal solution of 20.1.1 is given by

$$20.4.4 \quad y = \sum_{n=-\infty}^{\infty} c_{2n} \varphi_{2n+\nu}$$

where the coefficients c_{2n} are those associated with Floquet's solution. In the above, ν may be complex. Except for the special case when ν is an integer, the following holds:

$$\frac{\varphi_{2n+\nu-2}}{\varphi_{2n+\nu}} \sim \frac{\varphi_{-2n+\nu}}{\varphi_{-2n+\nu+2}} \sim \frac{-4n^2}{q[\cos(z-b)]^2} \quad (n \rightarrow \infty)$$

If ν and n are integers, $J_{-2n+\nu}(f) = (-1)^\nu J_{2n-\nu}(f)$.

$$[\varphi_{2n+\nu}/\varphi_{2n+\nu-2}] \sim -[\cos(z-b)]^2 q/4n^2$$

$$[\varphi_{-2n+\nu}/\varphi_{-2n+\nu+2}] \sim -4n^2/q[\cos(z-b)]^2$$

On the other hand

$$\frac{c_{2n}}{c_{2n-2}} \sim \frac{c_{-2n}}{c_{-2n+2}} \sim \frac{-q}{4n^2} \quad (n \rightarrow \infty)$$

It follows that **20.4.4** converges absolutely and uniformly in every closed region where

$$|\cos(z-b)| > d_1 > 1.$$

There are two such disjoint regions:

$$(I) \mathcal{J}(z-b) > d_2 > 0; \quad (|\cos(z-b)| > d_1 > 1)$$

$$(II) \mathcal{J}(z-b) < -d_2 < 0; \quad (|\cos(z-b)| > d_1 > 1)$$

If ν is an integer **20.4.4** converges for all values of z . Various representations are found by specializing b .

20.4.5

$$\text{If } b=0, y=e^{i\pi\nu/2} \sum_{n=-\infty}^{\infty} c_{2n}(-1)^n J_{2n+\nu}(2\sqrt{q} \cos z) \\ (|\cos z| > 1, |\arg 2\sqrt{q} \cos z| \leq \pi)$$

20.4.6

$$\text{If } b=\frac{\pi}{2}, y=\sum_{n=-\infty}^{\infty} c_{2n} J_{2n+\nu}(2i\sqrt{q} \sin z) \\ (|\sin z| > 1, |\arg 2\sqrt{q} \sin z| \leq \pi)$$

If $b \rightarrow \infty i$, y reduces to a multiple of the solution **20.3.8**. The fact that **20.3.8**, **20.4.5**, and **20.4.6** are special cases of **20.4.4** explains why it is that these apparently dissimilar expansions involve the same set of coefficients c_{2n} .

Since **20.4.4** results from the recurrence properties of Bessel functions, $J_k(f)$ can be replaced by $H_k^{(j)}(f)$, $j=1, 2$, where $H_k^{(j)}$ is the Hankel function, at least formally. Thus let

$$\psi_k^j = [e^{i\pi} \cos(z-b)/\cos(z+b)]^{1/2} H_k^{(j)}(f)$$

where f satisfies **20.4.2**. An examination of the ratios $\psi_{2n+\nu}/\psi_{2n+\nu-2}$ shows that

$$y = \sum_{n=-\infty}^{\infty} c_{2n} \psi_{2n+\nu}^{(j)}$$

will be a solution provided

$$|\cos(z-b)| > 1; |\cos(z+b)| > 1.$$

The above two conditions are necessary even when ν is an integer. Once b is fixed, the regions in which the solutions converge can be readily established.

Following [20.36] let

20.4.7

$$J_p(x) = Z_p^{(1)}(x); \quad Y_p(x) = Z_p^{(2)}(x); \\ H_p^{(1)}(x) = Z_p^{(3)}(x); \quad H_p^{(2)}(x) = Z_p^{(4)}(x)$$

If z is replaced by $-iz$ in **20.4.5** and **20.4.6** solutions of **20.1.2** are obtained. Thus

20.4.8

$$y_1^{(j)}(z) = \sum_{n=-\infty}^{\infty} c_{2n}(-1)^n Z_{2n+\nu}^{(j)}(2\sqrt{q} \cosh z) \\ (|\cosh z| > 1)$$

20.4.9

$$y_2^{(j)}(z) = \sum_{n=-\infty}^{\infty} c_{2n} Z_{2n+\nu}^{(j)}(2\sqrt{q} \sinh z) \\ (|\sinh z| > 1, j=1, 2, 3, 4)$$

The relation between $y_1^{(j)}(z)$ and $y_2^{(j)}(z)$ can be determined from the asymptotic properties of the Bessel functions for large values of argument. It can be shown that

20.4.10

$$y_1^{(j)}(z)/y_2^{(j)}(z) = [F_\nu(0)/F_\nu(\frac{\pi}{2})] e^{i\nu\pi/2} \quad (\Re z > 0).$$

When ν is not an integer, the above solutions do not vanish identically. See **20.6** for integral values of ν .

Solutions Involving Products of Bessel Functions

20.4.11

$$y_3^{(j)}(z) = \frac{1}{c_{2s}} \sum_{n=-\infty}^{\infty} c_{2n}(-1)^n Z_{n+\nu+s}^{(j)}(\sqrt{q}e^{iz}) J_{n-s}(\sqrt{q}e^{-iz}) \\ (j=1, 2, 3, 4)$$

satisfies **20.1.1**, where $Z_n^{(j)}(u)$ is defined in **20.4.7**, the coefficients c_{2n} belong to the Floquet solution, and s is an arbitrary integer, $c_{2s} \neq 0$. The solution converges over the entire complex z -plane if $q \neq 0$. Written with z replaced by $-iz$, one obtains solutions of **20.1.2**.

20.4.12

$$M_i^j(z, q) = \frac{1}{c_{2s}^j} \sum_{n=-\infty}^{\infty} c_{2n}^j (-1)^n Z_{n+\nu+s}^{(j)}(\sqrt{q}e^z) J_{n-s}(\sqrt{q}e^{-z})$$

It can be verified from 20.4.8 and 20.4.12 that

$$20.4.13 \quad \frac{y_1^{(j)}(z)}{M_i^j(z, q)} = F_\nu(0), \quad (\Re z > 0)$$

provided $c_{2s} \neq 0$. If $c_{2s} = 0$, the coefficient of $1/c_{2s}$ in 20.4.11 vanishes identically. For details see [20.43], [20.15], [20.36].

If s is chosen so that $|c_{2s}|$ is the largest coefficient of the set $|c_{2j}|$, then rapid convergence of 20.4.12 is obtained, when $\Re z > 0$. Even then one must be on guard against the possible loss of significant figures in the process of summing the series, especially so when q is large, and $|z|$ small. (If $j \neq 1$, then the phase of the logarithmic terms occurring in 20.4.12 must be defined, to make the functions single-valued.)

20.5. Properties of Orthogonality and Normalization

If $a(\nu+2p, q)$, $a(\nu+2s, q)$ are simple roots of 20.3.10 then

$$20.5.1 \quad \int_0^{2\pi} F_{\nu+2p}(z) F_{\nu+2s}(-z) dz = 0, \text{ if } p \neq s.$$

Define

$$20.5.2 \quad ce_\nu(z, q) = \frac{1}{2} [F_\nu(z) + F_\nu(-z)];$$

$$se_\nu(z, q) = -i \frac{1}{2} [F_\nu(z) - F_\nu(-z)]$$

$ce_\nu(z, q)$, $se_\nu(z, q)$ are thus even and odd functions of z , respectively, for all ν (when not identically zero).

If ν is an integer, then $ce_\nu(z, q)$, $se_\nu(z, q)$ are either Floquet solutions or identically zero. The solutions $ce_r(z, q)$ are associated with a_r ; $se_r(z, q)$ are associated with b_r ; r an integer.

Normalization for Integral Values of ν and Real q

$$20.5.3 \quad \int_0^{2\pi} [ce_r(z, q)]^2 dz = \int_0^{2\pi} [se_r(z, q)]^2 dz = \pi$$

For integral values of ν the summation in 20.3.8 reduces to the simpler forms 20.2.3–20.2.4; on account of 20.5.3, the coefficients A_m and B_m (for all orders r) have the property

20.5.4

$$2A_0^2 + A_2^2 + \dots = A_1^2 + A_3^2 + \dots \\ = B_1^2 + B_3^2 + \dots = B_2^2 + B_4^2 + \dots = 1.$$

20.5.5

$$A_0^2 = \frac{1}{2\pi} \int_0^{2\pi} ce_{2s}(z, q) dz; \quad A_n^2 = \frac{1}{\pi} \int_0^{2\pi} ce_r(z, q) \cos n z dz$$

$$B_n^2 = \frac{1}{\pi} \int_0^{2\pi} se_r(z, q) \sin n z dz \quad n \neq 0$$

For integral values of ν , the functions $ce_r(z, q)$ and $se_r(z, q)$ form a complete orthogonal set for the interval $0 \leq z \leq 2\pi$. Each of the four systems $ce_{2r}(z)$, $ce_{2r+1}(z)$, $se_{2r}(z)$, $se_{2r+1}(z)$ is complete in the smaller interval $0 \leq z \leq \frac{1}{2}\pi$, and each of the systems $ce_r(z)$, $se_r(z)$ is complete in $0 \leq z \leq \pi$.

If q is not real, there exist multiple roots of 20.3.10; for such special values of $a(q)$, the integrals in 20.5.3 vanish, and the normalization is therefore impossible. In applications, the particular normalization adopted is of little importance, except possibly for obtaining quantitative relations between solutions of various types. For this reason the normalization of $F_\nu(z)$, for arbitrary complex values of a , q , will not be specified here. It is worth noting, however, that solutions

$$\alpha ce_r(z, q), \quad \beta se_r(z, q)$$

defined so that

$$\alpha ce_r(0, q) = 1; \quad \left[\frac{d}{dz} \beta se_r(z, q) \right]_{z=0} = 1$$

are always possible. This normalization has in fact been used in [20.59], and also in [20.58], where the most extensive tabular material is available. The tabulated entries in [20.58] supply the conversion factors $A=1/\alpha$, $B=1/\beta$, along with the coefficients. Thus conversion from one normalization to another is rather easy.

In a similar vein, no general normalization will be imposed on the functions defined in 20.4.8.

20.6. Solutions of Mathieu's Modified Equation 20.1.2 for Integral ν (Radial Solutions)

Solutions of the first kind

20.6.1

$$Ce_{2r+p}(z, q) = ce_{2r+p}(iz, q)$$

$$= \sum_{k=0}^{\infty} A_{2k+2r+p}^2(q) \cosh(2k+p)z$$

associated with a_r

$$20.6.2 \quad Se_{2r+p}(z, q) = -ise_{2r+p}(iz, q) = \sum_{k=0}^{\infty} B_{2k+p}^{2r+p}(q) \sinh(2k+p)z, \text{ associated with } b_r,$$

writing $A_{2k+p}^{2r+p}(q) = A_{2k+p}$ for brevity; similarly for B_{2k+p} ; $p=0, 1$,

$$20.6.3 \quad Ce_{2r}(z, q) = \frac{ce_{2r}(\frac{\pi}{2}, q)}{A_0^{2r}} \sum_{k=0}^{\infty} (-1)^k A_{2k} J_{2k}(2\sqrt{q} \cosh z) = \frac{ce_{2r}(0, q)}{A_0^{2r}} \sum_{k=0}^{\infty} A_{2k} J_{2k}(2\sqrt{q} \sinh z)$$

$$20.6.4 \quad \begin{aligned} Ce_{2r+1}(z, q) &= \frac{ce'_{2r+1}(\frac{\pi}{2}, q)}{\sqrt{q} A_1^{2r+1}} \sum_{k=0}^{\infty} (-1)^{k+1} A_{2k+1} J_{2k+1}(2\sqrt{q} \cosh z) \\ &= \frac{ce'_{2r+1}(0, q)}{\sqrt{q} A_1^{2r+1}} \coth z \sum_{k=0}^{\infty} (2k+1) A_{2k+1} J_{2k+1}(2\sqrt{q} \sinh z) \end{aligned}$$

$$20.6.5 \quad \begin{aligned} Se_{2r}(z, q) &= \frac{se'_{2r}(\frac{\pi}{2}, q) \tanh z}{q B_2^{2r}} \sum_{k=1}^{\infty} (-1)^k 2k B_{2k} J_{2k}(2\sqrt{q} \cosh z) \\ &= \frac{se'_{2r}(0, q)}{q B_2^{2r}} \coth z \sum_{k=1}^{\infty} 2k B_{2k} J_{2k}(2\sqrt{q} \sinh z) \end{aligned}$$

$$20.6.6 \quad \begin{aligned} Se_{2r+1}(z, q) &= \frac{se_{2r+1}(\frac{\pi}{2}, q)}{\sqrt{q} B_1^{2r+1}} \tanh z \sum_{k=0}^{\infty} (-1)^k (2k+1) B_{2k+1} J_{2k+1}(2\sqrt{q} \cosh z) \\ &= \frac{se'_{2r+1}(0, q)}{\sqrt{q} B_1^{2r+1}} \sum_{k=0}^{\infty} B_{2k+1} J_{2k+1}(2\sqrt{q} \sinh z) \end{aligned}$$

See [20.30] for still other forms.

Solutions of the second kind, as well as solutions of the third and fourth kind (analogous to Hankel functions) are obtainable from 20.4.12.

$$20.6.7 \quad Mc_{2r}^{(j)}(z, q) = \sum_{k=0}^{\infty} (-1)^{r+k} A_{2k}^{2r}(q) [J_{k-s}(u_1) Z_{k+s}^{(j)}(u_2) + J_{k+s}(u_1) Z_{k-s}^{(j)}(u_2)] / \epsilon_s A_{2s}^{2r},$$

where $\epsilon_0=2$, $\epsilon_s=1$, for $s=1, 2, \dots$; s arbitrary, associated with a_{2r}

$$20.6.8 \quad Mc_{2r+1}^{(j)}(z, q) = \sum_{k=0}^{\infty} (-1)^{r+k} A_{2k+1}^{2r+1}(q) [J_{k-s}(u_1) Z_{k+s+1}^{(j)}(u_2) + J_{k+s+1}(u_1) Z_{k-s}^{(j)}(u_2)] / A_{2s+1}^{2r+1}$$

associated with a_{2r+1}

$$20.6.9 \quad Ms_{2r}^{(j)}(z, q) = \sum_{k=1}^{\infty} (-1)^{k+r} B_{2k}^{2r}(q) [J_{k-s}(u_1) Z_{k+s}^{(j)}(u_2) - J_{k+s}(u_1) Z_{k-s}^{(j)}(u_2)] / B_{2s}^{2r}, \text{ associated with } b_{2r},$$

$$20.6.10 \quad Ms_{2r+1}^{(j)}(z, q) = \sum_{k=0}^{\infty} (-1)^{k+r} B_{2k+1}^{2r+1}(q) [J_{k-s}(u_1) Z_{k+s+1}^{(j)}(u_2) - J_{k+s+1}(u_1) Z_{k-s}^{(j)}(u_2)] / B_{2s+1}^{2r+1}$$

associated with b_{2r+1}

where

$$u_1 = \sqrt{q} e^{-z}, \quad u_2 = \sqrt{q} e^z, \quad B_{2s+p}^{2r+p}, A_{2s+p}^{2r+p} \neq 0, \quad p=0, 1.$$

See 20.4.7 for definition of $Z_m^{(j)}(x)$.

Solutions 20.6.7–20.6.10 converge for all values of z , when $q \neq 0$. If $j=2, 3, 4$ the logarithmic terms entering into the Bessel functions $Y_m(u_2)$ must be defined, to make the functions single-valued. This can be accomplished as follows:

Define (as in [20.58])

$$20.6.11 \quad \ln(\sqrt{q} e^z) = \ln(\sqrt{q}) + z$$

See [20.15] and [20.36], section 2.75 for derivation.

Other Expressions for the Radial Functions (Valid Over More Limited Regions)

20.6.12

$$Mc_{2r}^{(j)}(z, q) = [ce_{2r}(0, q)]^{-1} \sum_{k=0}^{\infty} (-1)^{k+r} A_{2k}^{2r}(q) Z_{2k}^{(j)}(2\sqrt{q} \cosh z)$$

$$Mc_{2r+1}^{(j)}(z, q) = [ce_{2r+1}(0, q)]^{-1} \sum_{k=0}^{\infty} (-1)^{k+r} A_{2k+1}^{2r+1}(q) Z_{2k+1}^{(j)}(2\sqrt{q} \cosh z)$$

20.6.13

$$Ms_{2r}^{(j)}(z, q) = [se_{2r}'(0, q)]^{-1} \tanh z \sum_{k=1}^{\infty} (-1)^{k+r} 2k B_{2k}^{2r}(q) Z_{2k}^{(j)}(2\sqrt{q} \cosh z)$$

$$Ms_{2r+1}^{(j)}(z, q) = [se_{2r+1}'(0, q)]^{-1} \tanh z \sum_{k=0}^{\infty} (-1)^{k+r} (2k+1) B_{2k+1}^{2r+1}(q) Z_{2k+1}^{(j)}(2\sqrt{q} \cosh z)$$

Valid for $\Re z > 0$, $|\cosh z| > 1$; if $j=1$, valid for all z . They agree with 20.6.7–20.6.10 if the Bessel functions $Y_m(2q^{\frac{1}{2}} \cosh z)$ are made single-valued in a suitable way. For example, let

$$Y_m(u) = \frac{2}{\pi} (\ln u) J_m(u) + \phi(u)$$

where $\phi(u)$ is single-valued for all finite values of u . With $u = 2q^{\frac{1}{2}} \cosh z$, define

20.6.14

$$\ln(2q^{\frac{1}{2}} \cosh z) = \ln 2q^{\frac{1}{2}} + z + \ln \frac{1}{2}(1 + e^{-2z})$$

$$-\frac{\pi}{2} \leq \arg \frac{1}{2}(1 + e^{-2z}) \leq \frac{\pi}{2}$$

(If q is not positive, the phase of $\ln 2q^{\frac{1}{2}}$ must also be specified, although this specification will not affect continuity with respect to z . If $Y_m(u)$ is defined from some other expression, the definition must be compatible with 20.6.14.)

$$\sqrt{\frac{\pi}{2}} Mc_0^{(1)}(z, q)$$

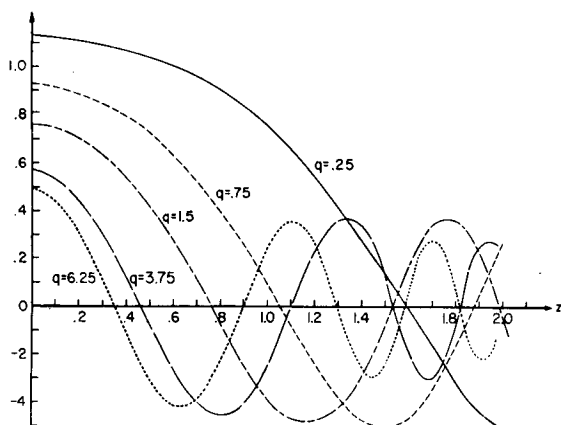


FIGURE 20.11. *Radial Mathieu Function of the First Kind.*
(From J. C. Wiltse and M. J. King, Values of the Mathieu functions, The Johns Hopkins Univ. Radiation Laboratory Tech. Rept. AF-53, 1958, with permission)

$$\sqrt{\frac{\pi}{2}} \frac{d}{dz} Mc_0^{(1)}(z, q)$$

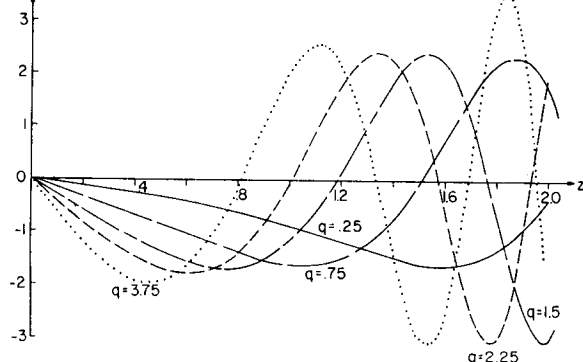


FIGURE 20.12. *Derivative of the Radial Mathieu Function of the First Kind.*

(From J. C. Wiltse and M. J. King, Derivatives, zeros, and other data pertaining to Mathieu functions, The Johns Hopkins Univ. Radiation Laboratory Tech. Rept. AF-57, 1958, with permission)

$$\sqrt{\frac{\pi}{2}} Ms_1^{(2)}(z, q)$$

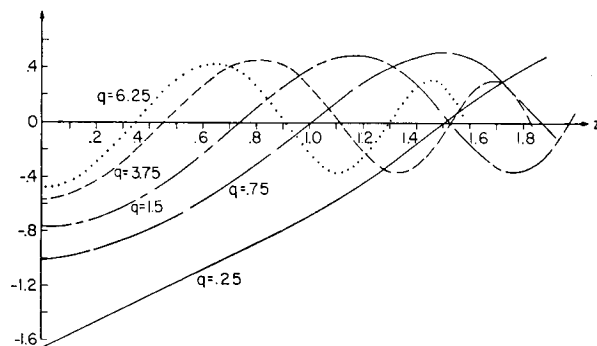


FIGURE 20.13. *Radial Mathieu Function of the Second Kind.*

(From J. C. Wiltse and M. J. King, Values of the Mathieu functions, The Johns Hopkins Univ. Radiation Laboratory Tech. Rept. AF-53, 1958, with permission)

$$\sqrt{\frac{\pi}{2}} Ms_2^{(3)}(z, -q)$$

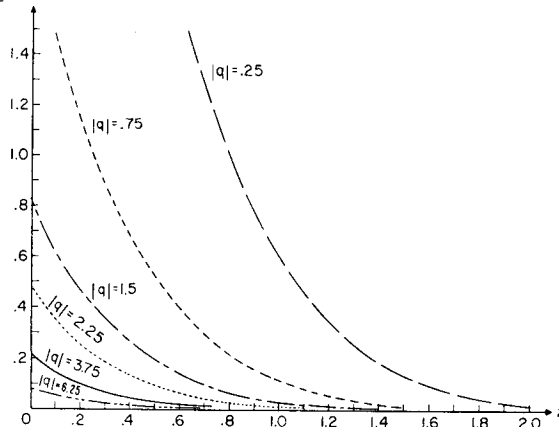


FIGURE 20.14. *Radial Mathieu Function of the Third Kind.*

(From J. C. Wiltse and M. J. King, Values of the Mathieu functions, The Johns Hopkins Univ. Radiation Laboratory Tech. Rept. AF-53, 1958, with permission)

If $j=1$, $Mc_{2r+p}^{(1)}$ and $Ms_{2r+p}^{(1)}$, $p=0, 1$ are solutions of the first kind, proportional to Ce_{2r+p} and Se_{2r+p} respectively.

Thus

20.6.15

$$Ce_{2r}(z, q) = \frac{ce_{2r}\left(\frac{\pi}{2}, q\right) ce_{2r}(0, q)}{(-1)^r A_0^{2r}} Mc_{2r}^{(1)}(z, q)$$

$$Ce_{2r+1}(z, q) = \frac{ce'_{2r+1}\left(\frac{\pi}{2}, q\right) ce_{2r+1}(0, q)}{(-1)^{r+1} \sqrt{q} A_1^{2r+1}} Mc_{2r+1}^{(1)}(z, q)$$

$$Se_{2r}(z, q) = \frac{se'_{2r}(0, q) se'_{2r}\left(\frac{\pi}{2}, q\right)}{(-1)^r q B_2^{2r}} Ms_{2r}^{(1)}(z, q)$$

$$Se_{2r+1}(z, q) = \frac{se'_{2r+1}(0, q) se_{2r+1}\left(\frac{\pi}{2}, q\right)}{(-1)^r \sqrt{q} B_1^{2r+1}} Ms_{2r+1}^{(1)}(z, q)$$

The Mathieu-Hankel functions are

20.6.16

$$M_r^{(3)}(z, q) = M_r^{(1)}(z, q) + iM_r^{(2)}(z, q)$$

$$M_r^{(4)}(z, q) = M_r^{(1)}(z, q) - iM_r^{(2)}(z, q)$$

$$M_r^{(j)} = Mc_r^{(j)} \text{ or } Ms_r^{(j)}.$$

From 20.6.7–20.6.11 and the known properties of Bessel functions one obtains

20.6.17

$$M_{2r+p}^{(2)}(z + in\pi, q) = (-1)^{np} [M_{2r+p}^{(2)}(z, q) + 2niM_{2r+p}^{(1)}(z, q)]$$

$$M_{2r+p}^{(3)}(z + in\pi, q) = (-1)^{np} [M_{2r+p}^{(3)}(z, q) - 2nM_{2r+p}^{(1)}(z, q)]$$

$$M_{2r+p}^{(4)}(z + in\pi, q) = (-1)^{np} [M_{2r+p}^{(4)}(z, q) + 2nM_{2r+p}^{(1)}(z, q)]$$

where $M = Mc$ or Ms throughout any of the above equations.

Other Properties of Characteristic Functions, q Real (Associated With a_r and b_r)

Consider

20.6.18

$$X_1 = Mc_r^{(2)}(z, q) + Mc_r^{(2)}(-z, q);$$

$$X_2 = Ms_r^{(2)}(z, q) - Ms_r^{(2)}(-z, q)$$

Since X_1 is an even solution it must be proportional to $Mc_r^{(1)}(z, q)$; for 20.1.2 admits of only one even solution (aside from an arbitrary constant factor). Similarly, X_2 is proportional to $Ms_r^{(1)}(z, q)$. The proportionality factors can be found by considering values of the functions at $z=0$. Define, therefore,

20.6.19

$$Mc_r^{(2)}(-z, q) = -Mc_r^{(2)}(z, q) - 2f_{e,r} Mc_r^{(1)}(z, q)$$

20.6.20

$$Ms_r^{(2)}(-z, q) = Ms_r^{(2)}(z, q) - 2f_{o,r} Ms_r^{(1)}(z, q)$$

where

20.6.21

$$f_{e,r} = -Mc_r^{(2)}(0, q)/Mc_r^{(1)}(0, q)$$

$$f_{o,r} = \left[\frac{d}{dz} Ms_r^{(2)}(z, q) / \frac{d}{dz} Ms_r^{(1)}(z, q) \right]_{z=0}$$

See [20.58].

In particular the above equations can be used to extend solutions of 20.6.12–20.6.13 when $\Re z < 0$. For although the latter converge for $\Re z < 0$, provided only $|\cosh z| > 1$, they do not represent the same functions as 20.6.9–20.6.10.

20.7. Representations by Integrals and Some Integral Equations

Let

20.7.1

$$G(u) = \oint_C K(u, t) V(t) dt$$

be defined for u in a domain U and let the contour C belong to the region T of the complex t -plane, with $t=\gamma_0$ as the starting point of the contour and $t=\gamma_1$ as its end-point. The kernel $K(u, t)$ and the function $V(t)$ satisfy 20.7.3 and the hypotheses in 20.7.2.

20.7.2 $K(u, t)$ and its first two partial derivatives with respect to u and t are continuous for t on C and u in U ; V and $\frac{dV}{dt}$ are continuous in t .

20.7.3

$$\left[\frac{\partial K}{\partial t} V - \frac{dV}{dt} K \right]_{\gamma_0}^{\gamma_1} = 0; \frac{d^2 V}{dt^2} + (a - 2q \cos 2t) V = 0.$$

If K satisfies

20.7.4

$$\frac{\partial^2 K}{\partial u^2} + \frac{\partial^2 K}{\partial t^2} + 2q(\cosh 2u - \cos 2t) K = 0$$

then $G(u)$ is a solution of Mathieu's modified equation 20.1.2.

If $K(u, t)$ satisfies

20.7.5

$$\frac{\partial^2 K}{\partial u^2} + \frac{\partial^2 K}{\partial t^2} + 2q(\cos 2u - \cos 2t) K = 0$$

then $G(u)$ is a solution of Mathieu's equation 20.1.1, with u replacing v .

Kernels $K_1(z, t)$ and $K_2(z, t)$

$$20.7.6 \quad K_1(z, t) = Z_{\nu}^{(j)}(u) [M(z, t)]^{-\nu/2}, \quad (\Re z > 0)$$

where

$$20.7.7 \quad u = \sqrt{2q(\cosh 2z + \cos 2t)}$$

$$20.7.8 \quad M(z, t) = \cosh(z + it) / \cosh(z - it)$$

To make $M^{-1\nu}$ single-valued, define

20.7.9

$$\cosh(z + i\pi) = e^{i\pi} \cosh z$$

$$\cosh(z - i\pi) = e^{-i\pi} \cosh z$$

$$M(z, 0) = 1$$

$$[M(z, \pi)]^{-1\nu} = e^{-i\nu\pi} M(z, 0)$$

Let

$$20.7.10 \quad G(z, q) = \frac{1}{\pi} \int_0^\pi K_1(u, t) F_\nu(t) dt, \quad (\Re z > 0)$$

where $F_\nu(t)$ is defined in 20.3.8. It may be verified that $K_1 F_\nu$ satisfies 20.7.3, K satisfies 20.7.2 and 20.7.4. Hence G is a solution of 20.1.2 (with z replacing u). It can be shown that K_1 may be replaced by the more general function

20.7.11

$$K_2(z, t) = Z_{\nu+2s}^{(j)}(u) [M(z, t)]^{-1\nu+2s}, \quad s \text{ any integer.}$$

See 20.4.7 for definition of $Z_{\nu+2s}^{(j)}(u)$.

From the known expansions for $Z_{\nu+2s}^{(j)}(u)$ when $\Re z$ is large and positive it may be verified that

20.7.12

$$M_\nu^{(j)}(z, q) =$$

$$\frac{(-1)^s}{\pi c_{2s}} \int_0^\pi Z_{\nu+2s}^{(j)}(u) \left[\frac{\cosh z + it}{\cosh z - it} \right]^{-1\nu-2s} F_\nu(t) dt$$

($\Re z > 0, \Re(\nu + \frac{1}{2}) > 0$)

where $M_\nu^{(j)}(z, q)$ is given by 20.4.12, $s=0, 1, \dots$, $c_{2s} \neq 0$, and $F_\nu(t)$ is the Floquet solution, 20.3.8.

Kernel $K_3(z, t, a)$

$$20.7.13 \quad K_3(z, t, a) = e^{2i\sqrt{q}w}$$

where

$$20.7.14 \quad w = \cosh z \cos a \cos t + \sinh z \sin a \sin t$$

$$20.7.15 \quad G(z, q, a) = \frac{1}{\pi} \oint_C e^{2i\sqrt{q}w} F_\nu(t) dt$$

where $F_\nu(t)$ is the Floquet solution 20.3.8. The path C is chosen so that $G(z, t, a)$ exists, and 20.7.2, 20.7.3 are satisfied. Then it may be verified that $K_3(z, t, a)$, considered as a function of z and t , satisfies 20.7.4; also, considered as a function of a and t , K_3 satisfies 20.7.5. Consequently $G(z, q, a) = Y(z, q)y(a, q)$, where Y and y satisfy 20.1.2 and 20.1.1, respectively.

Choice of Path C . Three paths will be defined:

20.7.16

Path C_3 : from $-d_1 + i\infty$ to $d_2 - i\infty$, d_1, d_2 real

$$-d_1 < \arg [\sqrt{q} \{ \cosh(z + ia) \pm 1 \}] < \pi - d_1$$

$$-d_2 < \arg [\sqrt{q} \{ \cosh(z - ia) \pm 1 \}] < \pi - d_2$$

20.7.17

Path C_4 : from $d_2 - i\infty$ to $2\pi + i\infty - d_1$

(same d_1, d_2 as in 20.7.16)

20.7.18

$$F_\nu(a) M_\nu^j(z, q) = \frac{e^{-i\nu\frac{\pi}{2}}}{\pi} \oint_{C_j} e^{2i\sqrt{q}w} F_\nu(t) dt \quad j=3, 4$$

where $M_\nu^j(z, q)$ is also given by 20.4.12.

20.7.19 Path C_1 : from $-d_1 + i\infty$ to $2\pi - d_1 + i\infty$

$$F_\nu(a) M_\nu^{(1)}(z, q) = \frac{e^{-i\nu\frac{\pi}{2}}}{2\pi} \oint_{C_1} e^{2i\sqrt{q}w} F_\nu(t) dt$$

See [20.36], section 2.68.

If ν is an integer the paths can be simplified; for in that case $F_\nu(t)$ is periodic and the integrals exist when the path is taken from 0 to 2π . Still further simplifications are possible, if z is also real.

The following are among the more important integral representations for the periodic functions $ce_r(z, q)$, $se_r(z, q)$ and for the associated radial solutions.

Let $r=2s+p$, $p=0$ or 1

20.7.20

$$ce_r(z, q) = \rho_r \int_0^{\pi/2} \cos \left(2\sqrt{q} \cos z \cos t - p\frac{\pi}{2} \right) ce_r(t, q) dt$$

$$20.7.21 \quad ce_r(z, q) = \sigma_r \int_0^{\pi/2} \cosh(2\sqrt{q} \sin z \sin t) [(1-p) + p \cos z \cos t] ce_r(t, q) dt$$

$$20.7.22 \quad se_r(z, q) = \rho_r \int_0^{\pi/2} \sin \left(2\sqrt{q} \cos z \cos t + p \frac{\pi}{2} \right) \sin z \sin t se_r(t, q) dt$$

$$20.7.23 \quad se_r(z, q) = \sigma_r \int_0^{\pi/2} \sinh(2\sqrt{q} \sin z \sin t) [(1-p) \cos z \cos t + p] se_r(t, q) dt$$

where

$$20.7.24 \quad \rho_r = \frac{2}{\pi} ce_{2s} \left(\frac{\pi}{2}, q \right) / A_0^{2s}(q); p=0, \rho_r = -\frac{2}{\pi} ce'_{2s+1} \left(\frac{\pi}{2}, q \right) / \sqrt{q} A_1^{2s+1}(q) \text{ if } p=1, \text{ for functions } ce_r(z, q)$$

$$\rho_r = -\frac{4}{\pi} se'_{2s} \left(\frac{\pi}{2}, q \right) / \sqrt{q} B_2^{2s}(q); \rho_r = \frac{4}{\pi} se_{2s+1} \left(\frac{\pi}{2}, q \right) / B_1^{2s+1}(q), \text{ for functions } se_r(z, q)$$

$$\sigma_r = \frac{2}{\pi} ce_{2s}(0, q) / A_0^{2s}(q) \text{ if } p=0; \quad \sigma_r = \frac{4}{\pi} ce_{2s+1}(0, q) / A_1^{2s+1}(q), \text{ if } p=1; \text{ associated with functions } ce_r(z, q)$$

$$\sigma_r = \frac{4}{\pi} se'_{2s}(0, q) / \sqrt{q} B_2^{2s}(q), \text{ if } p=0; \quad \sigma_r = \frac{2}{\pi} se'_{2s+1}(0, q) / \sqrt{q} B_1^{2s+1}(q), \text{ if } p=1; \text{ associated with } se_r(z, q)$$

Integrals Involving Bessel Function Kernels

Let

$$20.7.25 \quad u = \sqrt{2q}(\cosh 2z + \cos 2t), (\mathcal{R} \cosh 2z > 1; \text{ if } j=1, \text{ valid also when } z=0)$$

20.7.26

$$Mc_{2r}^{(j)}(z, q) = \frac{(-1)^r 2}{\pi A_0^{2r}} \int_0^{\pi/2} Z_0^{(j)}(u) ce_{2r}(t, q) dt; Mc_{2r+1}^{(j)}(z, q) = \frac{(-1)^r 8\sqrt{q} \cosh z}{\pi A_1^{2r+1}} \int_0^{\pi/2} \frac{Z_1^{(j)}(u) \cos t}{u} ce_{2r+1}(t, q) dt$$

$$20.7.27 \quad Ms_{2r}^{(j)}(z, q) = \frac{(-1)^{r+1} 8q \sinh 2z}{\pi B_2^{2r}} \int_0^{\pi/2} \frac{Z_2^{(j)}(u) \sin 2t se_{2r}(t, q) dt}{u^2}$$

$$Ms_{2r+1}^{(j)}(z, q) = \frac{(-1)^r 8\sqrt{q} \sinh z}{\pi B_1^{2r+1}} \int_0^{\pi/2} \frac{Z_1^{(j)}(u) \sin t se_{2r+1}(t, q) dt}{u}$$

In the above the j -convention of 20.4.7 applies and the functions Mc , Ms are defined in 20.5.1-20.5.4. (These solutions are normalized so that they approach the corresponding Bessel-Hankel functions as $\mathcal{R}z \rightarrow \infty$.)

Other Integrals for $Mc_r^{(1)}(z, q)$ and $Ms_r^{(1)}(z, q)$

$$20.7.28 \quad Mc_r^{(1)}(z, q) = \frac{(-1)^{s2}}{\pi ce_r(0, q)} \int_0^{\pi/2} \cos \left(2\sqrt{q} \cosh z \cos t - p \frac{\pi}{2} \right) ce_r(t, q) dt$$

$$20.7.29 \quad Mc_r^{(1)}(z, q) = \tau_r \int_0^{\pi/2} [(1-p) + p \cosh z \cos t] \cos(2\sqrt{q} \sinh z \sin t) ce_r(t, q) dt$$

$$r=2s+p, p=0, 1; \tau_r = \frac{2}{\pi} (-1)^s / ce_{2s} \left(\frac{\pi}{2}, q \right), \text{ if } p=0; \tau_r = \frac{2}{\pi} (-1)^{s+1} 2\sqrt{q} / ce'_{2s+1} \left(\frac{\pi}{2}, q \right)$$

$$20.7.30 \quad Ms_{2r+1}^{(1)}(z, q) = \frac{2}{\pi} \frac{(-1)^r}{se_{2r+1} \left(\frac{\pi}{2}, q \right)} \int_0^{\pi/2} \sin(2\sqrt{q} \sinh z \sin t) se_{2r+1}(t, q) dt$$

$$20.7.31 \quad Ms_{2r+1}^{(1)}(z, q) = \frac{4}{\pi} \frac{\sqrt{q}(-1)^r}{se'_{2r+1}(0, q)} \int_0^{\pi/2} \sinh z \sin t \cos(2\sqrt{q} \cosh z \cos t) se_{2r+1}(t, q) dt$$

$$20.7.32 \quad Ms_{2r}^{(1)}(z, q) = \frac{4}{\pi} \sqrt{q} \frac{(-1)^{r+1}}{se'_{2r}(0, q)} \int_0^{\pi/2} \sin(2\sqrt{q} \cosh z \cos t) [\sinh z \sin t se_{2r}(t, q)] dt$$

$$20.7.33 \quad Ms_{2r}^{(1)}(z, q) = \frac{4}{\pi} \frac{(-1)^r \sqrt{q}}{se'_{2r} \left(\frac{\pi}{2}, q \right)} \int_0^{\pi/2} \sin(2\sqrt{q} \sinh z \sin t) [\cosh z \cos t se_{2r}(t, q)] dt$$

Further with $w = \cosh z \cos \alpha \cos t + \sinh z \sin \alpha \sin t$

$$20.7.34 \quad ce_r(\alpha, q) Mc_r^{(1)}(z, q) = \frac{(-1)^s(i)^{-p}}{2\pi} \int_0^{2\pi} e^{2i\sqrt{q}t} w ce_r(t, q) dt$$

$$20.7.35 \quad se_r(\alpha, q) Ms_r^{(1)}(z, q) = \frac{(-1)^s(-i)^p}{2\pi} \int_0^{2\pi} e^{2i\sqrt{q}t} w se_r(t, q) dt.$$

The above can be differentiated with respect to α , and we obtain

$$20.7.36 \quad ce'_r(\alpha, q) Mc_r^{(1)}(z, q) = \frac{(-1)^s(i)^{-p+1}\sqrt{q}}{\pi} \int_0^{2\pi} e^{2i\sqrt{q}t} w \frac{\partial w}{\partial \alpha} ce_r(t, q) dt$$

$$20.7.37 \quad se'_r(\alpha, q) Ms_r^{(1)}(z, q) = \frac{(-1)^{s+p}(i)^{-p+1}\sqrt{q}}{\pi} \int_0^{2\pi} e^{2i\sqrt{q}t} w \frac{\partial w}{\partial \alpha} se_r(t, q) dt$$

Integrals With Infinite Limits

$$r = 2s + p$$

In 20.7.38–20.7.41 below, z and q are positive.

$$20.7.38 \quad Mc_r^{(1)}(z, q) = \gamma_r \int_0^\infty \sin\left(2\sqrt{q} \cosh z \cosh t + p \frac{\pi}{2}\right) Mc_r^{(1)}(t, q) dt$$

$$\gamma_r = 2ce_{2s}\left(\frac{\pi}{2}, q\right) / \pi A_0^{2s}, \text{ if } p=0 \quad \gamma_r = 2ce'_{2s+1}\left(\frac{\pi}{2}, q\right) / \sqrt{q} \pi A_1^{2s+1}, \text{ if } p=1$$

$$20.7.39 \quad Ms_r^{(1)}(z, q) = \gamma_r \int_0^\infty \sinh z \sinh t \left[\cos\left(2\sqrt{q} \cosh z \cosh t - p \frac{\pi}{2}\right) \right] Ms_r^{(1)}(t, q) dt$$

$$\gamma_r = -4se'_{2s}\left(\frac{\pi}{2}, q\right) / \sqrt{q} \pi B_2^{2s}, \text{ if } p=0 \quad \gamma_r = -4se_{2s+1}\left(\frac{\pi}{2}, q\right) / \pi B_1^{2s+1}, \text{ if } p=1$$

$$20.7.40 \quad Mc_r^{(2)}(z, q) = \gamma_r \int_0^\infty \cos\left(2\sqrt{q} \cosh z \cosh t - p \frac{\pi}{2}\right) Mc_r^{(1)}(t, q) dt$$

$$\gamma_r = -2ce_{2s}(\tfrac{1}{2}\pi, q) / \pi A_0^{2s}, \text{ if } p=0 \quad \gamma_r = 2ce'_{2s+1}(\tfrac{1}{2}\pi, q) / \pi \sqrt{q} A_1^{2s+1}, \text{ if } p=1$$

$$20.7.41 \quad Ms_r^{(2)}(z, q) = \gamma_r \int_0^\infty \sin\left(2\sqrt{q} \cosh z \cosh t + p \frac{\pi}{2}\right) \sinh z \sinh t Ms_r^{(1)}(t, q) dt$$

$$\gamma_r = -4se'_{2s}(\tfrac{1}{2}\pi, q) / \sqrt{q} \pi B_2^{2s}, \text{ if } p=0 \quad \gamma_r = 4se_{2s+1}(\tfrac{1}{2}\pi, q) / \pi B_1^{2s+1}, \text{ if } p=1$$

Additional forms in [20.30], [20.36], [20.15].

20.8. Other Properties

Relations Between Solutions for Parameters q and $-q$

Replacing z by $\frac{1}{2}\pi - z$ in 20.1.1 one obtains

$$20.8.1 \quad y'' + (a + 2q \cos 2z)y = 0$$

Hence if $u(z)$ is a solution of 20.1.1 then $u(\frac{1}{2}\pi - z)$ satisfies 20.8.1. It can be shown that

20.8.2

$$a(-\nu, q) = a(\nu, -q) = a(\nu, q), \nu \text{ not an integer}$$

$$c_{2m}^*(-q) = \rho(-1)^m c_{2m}^*(q), \nu \text{ not an integer}$$

(c_{2m} defined in 20.3.8) and ρ depending on the normalization;

$$F_\nu(z, -q) = \rho e^{-i\nu\pi/2} F_\nu\left(z + \frac{\pi}{2}, q\right) = \rho e^{i\nu\pi/2} F_\nu\left(z - \frac{\pi}{2}, q\right)$$

20.8.3

$$a_{2r}(-q) = a_{2r}(q); b_{2r}(-q) = b_{2r}(q), \text{ for integral } \nu$$

$$a_{2r+1}(-q) = b_{2r+1}(q), b_{2r+1}(-q) = a_{2r+1}(q)$$

20.8.4

$$ce_{2r}(z, -q) = (-1)^r ce_{2r}(\frac{1}{2}\pi - z, q)$$

$$ce_{2r+1}(z, -q) = (-1)^r se_{2r+1}(\frac{1}{2}\pi - z, q)$$

$$se_{2r+1}(z, -q) = (-1)^r ce_{2r+1}(\frac{1}{2}\pi - z, q)$$

$$se_{2r}(z, -q) = (-1)^{r-1} se_{2r}(\frac{1}{2}\pi - z, q)$$

For the coefficients associated with the above solutions for integral ν :

20.8.5

$$A_{2m}^{2r}(-q) = (-1)^{m-r} A_{2m}^{2r}(q);$$

$$B_{2m}^{2r}(-q) = (-1)^{m-r} B_{2m}^{2r}(q)$$

$$A_{2m+1}^{2r+1}(-q) = (-1)^{m-r} A_{2m+1}^{2r+1}(q);$$

$$B_{2m+1}^{2r+1}(-q) = (-1)^{m-r} A_{2m+1}^{2r+1}(q).$$

For the corresponding modified equation

$$20.8.6 \quad y'' - (a + 2q \cosh 2z)y = 0$$

20.8.7

$$M_r^{(j)}(z, -q) = M_r^{(j)}\left(z + i\frac{\pi}{2}, q\right),$$

$$M_r^{(j)}(z, q) \text{ defined in } 20.4.12.$$

For integral values of ν let

20.8.8

$$I_{e_{2r}}(z, q) = \sum_{k=0}^{\infty} (-1)^{k+s} A_{2k} [I_{k-s}(u_1) I_{k+s}(u_2) + I_{k+s}(u_1) I_{k-s}(u_2)] / A_{2s} \epsilon_s$$

$$I_{o_{2r}}(z, q) = \sum_{k=1}^{\infty} (-1)^{k+s} B_{2k} [I_{k-s}(u_1) I_{k+s}(u_2) - I_{k+s}(u_1) I_{k-s}(u_2)] / B_{2s}$$

$$I_{e_{2r+1}}(z, q) = \sum_{k=0}^{\infty} (-1)^{k+s} B_{2k+1} [I_{k-s}(u_1) I_{k+s+1}(u_2) + I_{k+s+1}(u_1) I_{k-s}(u_2)] / B_{2s+1}$$

$$I_{o_{2r+1}}(z, q) = \sum_{k=0}^{\infty} (-1)^{k+s} A_{2k+1} [I_{k-s}(u_1) I_{k+s+1}(u_2) - I_{k+s+1}(u_1) I_{k-s}(u_2)] / A_{2s+1}$$

20.8.9

$$Ke_{2r}(z, q) = \sum_{k=0}^{\infty} A_{2k} [I_{k-s}(u_1) K_{k+s}(u_2) + I_{k+s}(u_1) K_{k-s}(u_2)] / A_{2s} \epsilon_s$$

$$* K_{o_{2r}}(z, q) = \sum_{k=0}^{\infty} B_{2k} [I_{k-s}(u_1) K_{k+s}(u_2) - I_{k+s}(u_1) K_{k-s}(u_2)] / B_{2s}$$

$$* Ke_{2r+1}(z, q) = \sum_{k=0}^{\infty} B_{2k+1} [I_{k-s}(u_1) K_{k+s+1}(u_2) - I_{k+s+1}(u_1) K_{k-s}(u_2)] / B_{2s+1}$$

*See page II.

$$K_{o_{2r+1}}(z, q) = \sum_{k=0}^{\infty} A_{2k+1} [I_{k-s}(u_1) K_{k+s+1}(u_2) + I_{k+s+1}(u_1) K_{k-s}(u_2)] / A_{2s+1}$$

where $I_m(x)$, $K_m(x)$ are the modified Bessel functions, u_1 , u_2 are defined below 20.6.10. Superscripts are omitted, $\epsilon_s = 2$, if $s = 0$, $\epsilon_s = 1$ if $s \neq 0$.

Then for functions of first kind:

20.8.10

$$Mc_{2r}^{(1)}(z, -q) = (-1)^r I_{e_{2r}}(z, q)$$

$$Ms_{2r}^{(1)}(z, -q) = (-1)^r I_{o_{2r}}(z, q)$$

$$Mc_{2r+1}^{(1)}(z, -q) = (-1)^r I_{e_{2r+1}}(z, q)$$

$$Ms_{2r+1}^{(1)}(z, -q) = (-1)^r I_{o_{2r+1}}(z, q)$$

For the Mathieu-Hankel function of first kind:

20.8.11

$$Mc_{2r}^{(3)}(z, -q) = (-1)^{r+1} i \frac{2}{\pi} Ke_{2r}(z, q)$$

$$Ms_{2r}^{(3)}(z, -q) = (-1)^{r+1} i \frac{2}{\pi} Ko_{2r}(z, q)$$

$$Mc_{2r+1}^{(3)}(z, -q) = (-1)^{r+1} i \frac{2}{\pi} Ke_{2r+1}(z, q)$$

$$Ms_{2r+1}^{(3)}(z, -q) = (-1)^{r+1} i \frac{2}{\pi} Ko_{2r+1}(z, q)$$

For $M_r^{(j)}(z, -q)$, $j = 2, 4$, one may use the definitions

$$M_r^{(2)} = -i(M_r^{(3)} - M_r^{(1)}); M_r = Mc_r \text{ or } Ms_r$$

also

$$M_r^{(4)}(z, -q) = 2M_r^{(1)}(z, -q) - M_r^{(3)}(z, -q)$$

$$M = Mc \text{ or } Ms; \text{ for real } z, q, M_r^{(j)}(z, -q)$$

are in general complex if $j = 2, 4$.

Zeros of the Functions for Real Values of q .

See [20.36], section 2.8 for further results.

Zeros of $ce_r(z, q)$ and $se_r(z, q)$, $Mc_r^{(1)}(z, q)$, $Ms_r^{(1)}(z, q)$.

In $0 \leq z < \pi$, $ce_r(z, q)$ and $se_r(z, q)$ have r real * zeros.

There are complex zeros if $q > 0$.

If $z_0 = x_0 + iy_0$ is any zero of $ce_r(z, q)$, $se_r(z, q)$ in

$$-\frac{\pi}{2} < x_0 < \frac{\pi}{2}, \text{ then } k\pi \pm z_0, k\pi \pm \bar{z}_0$$

are also zeros, k an integer.

In the strip $-\frac{\pi}{2} < x_0 < \frac{\pi}{2}$, the imaginary zeros of $ce_r(z, q)$, $se_r(z, q)$ are the real zeros of $Ce_r(z, q)$, $Se_r(z, q)$, hence also the real zeros of $Mc_r^{(1)}(z, q)$ and $Ms_r^{(1)}(z, q)$, respectively.

For small q , the large zeros of $Ce_r(z, q)$, $Se_r(z, q)$ approach the zeros of $J_r(2\sqrt{q} \cosh z)$.

Tabulation of Zeros

Ince [20.56] tabulates the first "non-trivial" zero (i.e. different from $0, \frac{\pi}{2}, \pi$) for $ce_r(z)$, $se_r(z)$, $r=2(1)5$ and for $se_6(z)$ to within 10^{-4} , for $q=0(1)10(2)40$. He also gives the "turning" points (zeros of the derivative) and also expansions for them for small q . Wiltse and King [20.61, 2] tabulate the first two (non-trivial) zeros of $Mc_r^{(1)}(z, q)$ and $Ms_r^{(1)}(z, q)$ and of their derivatives $r=0, 1, 2$ for 6 or 7 values of q between .25 and 10. The graphs reproduced here indicate their location.

Between two real zeros of $Mc_r^{(1)}(z, q)$, $Ms_r^{(1)}(z, q)$ there is a zero of $Mc_r^{(2)}(z, q)$, $Ms_r^{(2)}(z, q)$, respectively. No tabulation of such zeros exists yet.

Available tables are described in the References.

The most comprehensive tabulation of the characteristic values a_r , b_r (in a somewhat different notation) and of the coefficients proportional to A_m and B_m as defined in 20.5.4 and 20.5.5 can be found in [20.58]. In addition, the table contains certain important "joining factors", with the aid of which it is possible to obtain values of $Mc_r^{(j)}(z, q)$ and $Ms_r^{(j)}(z, q)$ as well as their derivatives, at $x=0$. Values of the functions $ce_r(x, q)$ and $se_r(x, q)$ for orders up to five or six can be found in [20.56]. Tabulations of less extensive character, but important in some aspects, are outlined in the other references cited. In this chapter only representative values of the various functions are given, along with several graphs.

Special Values for Arguments 0 and $\frac{\pi}{2}$

20.8.12

$$ce_{2r}\left(\frac{\pi}{2}, q\right) = (-1)^r g_{e, 2r}(q) A_0^{2r}(q) \sqrt{\frac{\pi}{2}}$$

$$ce'_{2r+1}\left(\frac{\pi}{2}, q\right) = (-1)^{r+1} g_{e, 2r+1}(q) A_1^{2r+1}(q) \sqrt{\frac{\pi}{2} q}$$

$$se'_{2r}\left(\frac{\pi}{2}, q\right) = (-1)^r g_{o, 2r}(q) B_2^{2r}(q) \cdot q \sqrt{\frac{\pi}{2}}$$

$$se_{2r+1}\left(\frac{\pi}{2}, q\right) = (-1)^r g_{o, 2r+1}(q) B_1^{2r+1}(q) \sqrt{\frac{\pi}{2} q}$$

$$Mc_r^{(1)}(0, q) = \sqrt{\frac{2}{\pi}} \frac{1}{g_{e, r}(q)}$$

$$Mc_r^{(2)}(0, q) = -\sqrt{\frac{2}{\pi}} f_{e, r}(q)/g_{e, r}(q)$$

$$\frac{d}{dz} [Mc_r^{(2)}(z, q)]_{z=0} = \sqrt{\frac{2}{\pi}} g_{e, r}(q)$$

$$\frac{d}{dz} [Ms_r^{(1)}(z, q)]_{z=0} = \sqrt{\frac{2}{\pi}} \frac{1}{g_{o, r}(q)}$$

$$\frac{d}{dz} [Ms_r^{(2)}(z, q)]_{z=0} = \sqrt{\frac{2}{\pi}} f_{o, r}(q)/g_{o, r}(q)$$

$$Ms_r^{(2)}(z, q) = -g_{o, r}(q) \sqrt{\frac{2}{\pi}}$$

The functions $f_{o, r}$, $g_{o, r}$, $f_{e, r}$, $g_{e, r}$ are tabulated in [20.58] for $q \leq 25$.

20.9. Asymptotic Representations

The representations given below are applicable to the *characteristic solutions*, for real values of q , unless otherwise noted. The Floquet exponent ν is defined below, as in [20.36] to be as follows:

In solutions associated with a_r : $\nu = r$

In solutions associated with b_r : $\nu = -r$.

For the functions defined in 20.6.7–20.6.10:

20.9.1

$$\begin{aligned} & Mc_r^{(3)}(z, q) \\ & (-1)^r Ms_r^{(3)}(z, q) \\ & \sim \frac{e^{i(2\sqrt{q} \cosh z - \frac{\pi}{2} - \frac{\pi}{4})}}{\pi^{\frac{1}{2}} q^{1/4} (\cosh z - \sigma)^{\frac{1}{2}}} \sum_{m=0}^{\infty} \frac{D_m}{[-4i\sqrt{q}(\cosh z - \sigma)]^m} \end{aligned}$$

where $D_{-1}=D_{-2}=0$; $D_0=1$, and the coefficients D_m are obtainable from the following recurrence formula:

20.9.2

$$\begin{aligned} & (m+1)D_{m+1} + \left[\left(m + \frac{1}{2}\right)^2 - \left(m + \frac{1}{4}\right) 8i\sqrt{q} \sigma \right. \\ & \left. + 2q - a \right] D_m + \left(m - \frac{1}{2}\right) [16q(1 - \sigma^2) - 8i\sqrt{q} \sigma m] D_{m-1} \\ & + 4q(2m-3)(2m-1)(1 - \sigma^2) D_{m-2} = 0 \end{aligned}$$

20.9.3

$$\begin{aligned} & Mc_r^{(4)}(z, q) \\ & (-1)^r Ms_r^{(4)}(z, q) \\ & \sim \frac{e^{-i[2\sqrt{q} \cosh z - \frac{1}{2}\pi - \frac{1}{4}\pi]}}{\pi^{\frac{1}{2}} q^{1/4} (\cosh z - \sigma)^{\frac{1}{2}}} \sum_{m=0}^{\infty} \frac{d_m}{[4i\sqrt{q}(\cosh z - \sigma)]^m} \end{aligned}$$

$d_{-1}=d_{-2}=0$; $d_0=1$, and

20.9.4

$$(m+1)d_{m+1} + \left[\left(m + \frac{1}{2}\right)^2 + \left(m + \frac{1}{4}\right) 8i\sqrt{q}\sigma + 2q - a \right] d_m + \left(m - \frac{1}{2}\right) [16q(1-\sigma^2) + 8i\sqrt{q}\sigma m] d_{m-1} + 4q(2m-3)(2m-1)(1-\sigma^2) d_{m-2} = 0.$$

In the above

$$-2\pi < \arg \sqrt{q} \cosh z < \pi \\ |\cosh z - \sigma| > |\sigma \pm 1|, \Re z > 0,$$

but σ is otherwise arbitrary. If $\sigma^2 = 1$, 20.9.2 and 20.9.4 become three-term recurrence relations.

Formulas 20.9.1 and 20.9.3 are valid for arbitrary a, q , provided ν is also known; they give multiples of 20.4.12, normalized so as to approach the corresponding Hankel functions $H_\nu^{(1)}(\sqrt{q}e^z)$, $H_\nu^{(2)}(\sqrt{q}e^z)$, as $z \rightarrow \infty$. See [20.36], section 2.63. The formula is especially useful if $|\cosh z|$ is large and q is not too large; thus if $\sigma = -1$, the absolute ratio of two successive terms in the expansion is essentially

$$\left| \left(\frac{\sqrt{q}}{m} + \frac{m}{4\sqrt{q}} + 2 \right) / (\cosh z + 1) \right|.$$

If a, q, z, ν are real, the real and imaginary components of $Mc_r^{(3)}(z, q)$ are $Mc_r^{(1)}(z, q)$ and $Mc_r^{(2)}(z, q)$, respectively; similarly for the components of $Ms_r^{(3)}(z, q)$. If the parameters are complex

$$20.9.5 \quad Mc_r^{(1)}(z, q) = \frac{1}{2} [Mc_r^{(3)}(z, q) + Mc_r^{(4)}(z, q)]$$

$$20.9.6 \quad Mc_r^{(2)}(z, q) = -\frac{i}{2} [Mc_r^{(3)}(z, q) - Mc_r^{(4)}(z, q)]$$

Replacing c by s in the above will yield corresponding relations among $Ms_r^{(j)}(z, q)$.

Formulas in which the parameter a does not enter explicitly:

Goldstein's Expansions

20.9.7

$$Mc_r^{(3)}(z, q) \sim iMs_{r+1}^{(3)}(z, q) \\ \approx [F_0(z) - iF_1(z)]e^{i\phi}/\pi^{\frac{1}{2}}q^{\frac{1}{4}}(\cosh z)^{\frac{1}{2}}$$

where

20.9.8

$$\phi = 2\sqrt{q} \sinh z - \frac{1}{2}(2r+1) \arctan \sinh z,$$

$$\Re z > 0, q > 1, w = 2r+1$$

20.9.9

$$F_0(z) \sim 1 + \frac{w}{8\sqrt{q} \cosh^2 z} \\ + \frac{1}{2048q} \left[\frac{w^4 + 86w^2 + 105}{\cosh^4 z} - \frac{w^4 + 22w^2 + 57}{\cosh^2 z} \right] \\ + \frac{1}{16384q^{3/2}} \left[\frac{-(w^5 + 14w^3 + 33w)}{\cosh^2 z} \right. \\ \left. - \frac{(2w^5 + 124w^3 + 1122w)}{\cosh^4 z} + \frac{3w^5 + 290w^3 + 1627w}{\cosh^6 z} \right] + \dots$$

20.9.10

$$F_1(z) \sim \frac{\sinh z}{\cosh^2 z} \left[\frac{w^2 + 3}{32\sqrt{q}} + \frac{1}{512q} \left(w^3 + 3w + \frac{4w^3 + 44w}{\cosh^2 z} \right) \right. \\ \left. + \frac{1}{16384q^{3/2}} \left\{ 5w^4 + 34w^2 + 9 \right. \right. \\ \left. \left. - \frac{(w^5 - 47w^4 + 667w^2 + 2835)}{12 \cosh^2 z} \right. \right. \\ \left. \left. + \frac{(w^5 + 505w^4 + 12139w^2 + 10395)}{12 \cosh^4 z} \right\} \right] + \dots$$

See [20.18] for details and an added term in $q^{-5/2}$; a correction to the latter is noted in [20.58].

The expansions 20.9.7 are especially useful when q is large and z is bounded away from zero. The order of magnitude of $Mc_r^{(2)}(0, q)$ cannot be obtained from the expansion. The expansion can also be used, with some success, for $z = ix$, when q is large, if $|\cos x| > 0$; they fail at $x = \frac{1}{2}\pi$. Thus, if q, x are real, one obtains

20.9.11

$$ce_r(x, q) \sim \frac{ce_r(0, q)2^{r-1}}{F_0(0)} \{W_1[P_0(x) - P_1(x)] \\ + W_2[P_0(x) + P_1(x)]\}$$

20.9.12

$$se_{r+1}(x, q) \sim se'_{r+1}(0, q)\tau_{r+1} \{W_1[P_0(x) - P_1(x)] \\ - W_2[P_0(x) + P_1(x)]\}$$

In the above, $P_0(x)$ and $P_1(x)$ are obtainable from $F_0(z)$, $F_1(x)$ in 20.9.9–20.9.10 by replacing $\cosh z$ with $\cos x$ and $\sinh z$ with $\sin x$. Thus $P_0(x) = F_0(ix)$; $P_1(x) = -iF_1(ix)$:

20.9.13

$$W_1 = e^{2\sqrt{q} \sin x} [\cos(\frac{1}{2}x + \frac{1}{4}\pi)]^{2r+1} / (\cos x)^{r+1}$$

$$W_2 = e^{-2\sqrt{q} \sin x} [\sin(\frac{1}{2}x + \frac{1}{4}\pi)]^{2r+1} / (\cos x)^{r+1}$$

20.9.14

$$\tau_{r+1} \sim 2^{r-1} \left/ \left[2\sqrt{q} - \frac{1}{4}w - \frac{(2w^2+3)}{64\sqrt{q}} - \frac{(7w^3+47w)}{1024q} - \dots \right] \right.$$

See 20.9.23-20.9.24 for expressions relating to $ce_r(0, q)$ and $se'_r(0, q)$. When $|\cos x| > \sqrt{4r+2}/q^{\frac{1}{2}}$, 20.9.11-20.9.12 are useful. The approximations become poorer as r increases.

Expansions in Terms of Parabolic Cylinder Functions

(Good for angles close to $\frac{1}{2}\pi$, for large values of q , especially when $|\cos x| < 2^{\frac{1}{2}}/q^{\frac{1}{2}}$.) Due to Sips [20.44-20.46].

$$20.9.15 \quad ce_r(x, q) \sim C_r[Z_0(\alpha) + Z_1(\alpha)]$$

20.9.16

$$se_{r+1}(x, q) \sim S_r[Z_0(\alpha) - Z_1(\alpha)] \sin x, \quad \alpha = 2q^{\frac{1}{2}} \cos x.$$

$$\text{Let } D_k = D_k(\alpha) = (-1)^k e^{i\alpha^2} \frac{d^k}{d\alpha^2} e^{-i\alpha^2}.$$

20.9.17

$$\begin{aligned} Z_0(\alpha) \sim & D_r + \frac{1}{4q^{\frac{1}{2}}} \left[-\frac{D_{r+4}}{16} + \frac{3}{2} \binom{r}{4} D_{r-4} \right] \\ & + \frac{1}{16q} \left[\frac{D_{r+8}}{512} - \frac{(r+2)D_{r+4}}{16} + \frac{3}{2} (r-1) \binom{r}{4} D_{r-4} \right. \\ & \left. + \frac{315}{4} \binom{r}{8} D_{r-8} \right] + \dots \end{aligned}$$

20.9.18

$$\begin{aligned} Z_1(\alpha) \sim & \frac{1}{4q^{\frac{1}{2}}} \left[-\frac{1}{4} D_{r+2} - \frac{r(r-1)}{4} D_{r-2} \right] \\ & + \frac{1}{16q} \left[\frac{D_{r+6}}{64} + \frac{(r^2-25r-36)}{64} D_{r+2} \right. \\ & \left. + \frac{r(r-1)(-r^2-27r+10)}{64} D_{r-2} - \frac{45}{4} \binom{r}{6} D_{r-6} + \dots \right] \end{aligned}$$

20.9.19

$$\begin{aligned} C_r \sim & \left(\frac{\pi}{2} \right)^{\frac{1}{2}} q^{\frac{1}{2}} / (r!)^{\frac{1}{2}} \left[1 + \frac{2r+1}{8q^{\frac{1}{2}}} \right. \\ & \left. + \frac{r^4+2r^3+263r^2+262r+108}{2048q} + \frac{f_1}{16384q^{\frac{3}{2}}} + \dots \right]^{-\frac{1}{2}} \\ & f_1 = 6r^5 + 15r^4 + 1280r^3 + 1905r^2 + 1778r + 572 \end{aligned}$$

*See page II.

20.9.20

$$\begin{aligned} S_r \sim & \left(\frac{\pi}{2} \right)^{\frac{1}{2}} q^{\frac{1}{2}} / (r!)^{\frac{1}{2}} \left[1 - \frac{2r+1}{8q^{\frac{1}{2}}} \right. \\ & \left. + \frac{r^4+2r^3-121r^2-122r-84}{2048q} + \frac{f_2}{16384q^{\frac{3}{2}}} + \dots \right]^{-\frac{1}{2}} \\ & f_2 = 2r^5 + 5r^4 - 416r^3 - 629r^2 - 1162r - 476 \end{aligned}$$

It should be noted that 20.9.15 is also valid as an approximation for $se_{r+1}(x, q)$, but 20.9.16 may give slightly better results. See [20.4.]

Explicit Expansions for Orders 0, 1, to Terms in $q^{-3/2}$ (q Large)20.9.21 For $r=0$:

$$\begin{aligned} Z_0 \sim & D_0 - \frac{D_4}{64\sqrt{q}} + \frac{1}{16q} \left(-\frac{D_4}{8} + \frac{D_8}{512} \right) * \\ & + \frac{1}{64q^{3/2}} \left(-\frac{99D_4}{256} + \frac{3D_8}{256} - \frac{D_{12}}{24576} \right) + \dots \end{aligned}$$

$$\begin{aligned} Z_1 \sim & \frac{-D_2}{16\sqrt{q}} + \frac{1}{16q} \left(-\frac{9D_2}{16} + \frac{D_6}{64} \right) \\ & + \frac{1}{64q^{3/2}} \left(-\frac{61D_2}{32} + \frac{25D_6}{256} - \frac{5D_{10}}{10240} \right) + \dots \end{aligned}$$

20.9.22 For $r=1$:

$$\begin{aligned} Z_0 \sim & D_1 - \frac{D_5}{64\sqrt{q}} + \frac{1}{16q} \left(-\frac{3D_5}{16} + \frac{D_9}{512} \right) \\ & + \frac{1}{64q^{3/2}} \left(-\frac{207D_5}{256} + \frac{D_9}{64} - \frac{D_{13}}{24576} \right) + \dots \end{aligned}$$

$$\begin{aligned} Z_1 \sim & \frac{-D_3}{16\sqrt{q}} + \frac{1}{16q} \left(-\frac{15D_3}{16} + \frac{D_7}{64} \right) \\ & + \frac{1}{64q^{3/2}} \left(-\frac{153D_3}{32} + \frac{35D_7}{256} - \frac{D_{11}}{2048} \right) + \dots \end{aligned}$$

Formulas Involving $ce_r(0, q)$ and $se_r(0, q)$

20.9.23

$$\begin{aligned} \frac{ce_0(0, q)}{ce_0(\frac{1}{2}\pi, q)} & \sim 2\sqrt{2} e^{-2\sqrt{q}} \left(1 + \frac{1}{16\sqrt{q}} + \frac{9}{256q} + \dots \right) \\ \frac{ce_2(0, q)}{ce_2(\frac{1}{2}\pi, q)} & \sim -32q\sqrt{2} e^{-2\sqrt{q}} \left(1 - \frac{1}{16\sqrt{q}} + \frac{29}{128q} + \dots \right) \end{aligned}$$

$$\frac{ce_1(0, q)}{ce'_1(\frac{1}{2}\pi, q)} \sim -4\sqrt{2}e^{-2\sqrt{q}} \left(1 + \frac{3}{16\sqrt{q}} + \frac{45}{256q} + \dots\right)$$

$$\frac{ce_3(0, q)}{ce'_3(\frac{1}{2}\pi, q)} \sim \frac{64}{3} q\sqrt{2} e^{-2\sqrt{q}} \left(1 - \frac{3}{16\sqrt{q}} + \frac{47}{128q} + \dots\right)$$

20.9.24

$$\frac{se'_1(0, q)}{se_1(\frac{1}{2}\pi, q)} \sim 4q\sqrt{2} e^{-2\sqrt{q}} \left(1 - \frac{3}{16\sqrt{q}} - \frac{11}{256q} + \dots\right)$$

$$\frac{se'_3(0, q)}{se_3(\frac{1}{2}\pi, q)} \sim -64q\sqrt{2} e^{-2\sqrt{q}} \left(1 - \frac{21}{16\sqrt{q}} - \frac{17}{128q} + \dots\right)$$

$$\frac{se'_2(0, q)}{se_2(\frac{1}{2}\pi, q)} \sim -8q\sqrt{2} e^{-2\sqrt{q}} \left(1 - \frac{9}{16\sqrt{q}} - \frac{39}{256q} + \dots\right)$$

$$\frac{se'_4(0, q)}{se_4(\frac{1}{2}\pi, q)} \sim \frac{128}{3} q\sqrt{2} e^{-2\sqrt{q}} \left(1 - \frac{31}{16\sqrt{q}} - \frac{15}{128q} + \dots\right)$$

For higher orders, these ratios are increasingly more difficult to obtain. One method of estimating values at the origin is to evaluate both **20.9.11** and **20.9.15** for some x where both expansions are satisfactory, and so to use **20.9.11** as a means to solve for $ce_r(0, q)$; similarly for $se'_r(0, q)$.

Other asymptotic expansions, valid over various regions of the complex z -plane, for real values of a, q , have been given by Langer [20.25]. It is not always easy, however, to determine the linear combinations of Langer's solutions which coincide with those defined here.

20.10. Comparative Notations

	This Volume	[20.58] NBS	[20.59] Stratton-Morse, etc.	[20.36] Meixner and Schäfer	[20.30] McLachlan	[20.15] Bateman Manuscript	Comments
Parameters in 20.1.1.....	a q a_r b_r	$b = a + 2q$ $s = 4q$ $be_r = a_r + 2q$ $bo_r = b_r + 2q$	b $c = 2\sqrt{q}$ $b_r = a_r + 2q$ $b'_r = b_r + 2q$	λ h^2 a_r b_r	a q a_r b_r	h θ a_r b_r	
Periodic Solutions, of 20.1.1:							
Even.....	$ce_r(z, q)$	$A^r Se_r(s, z)$ *	$A^r Se_r^{(1)}(c, \cos z)$ *	$ce_r(z, h^2)$ *	$ce_r(z, q)$	$ce_r(z, \theta)$	See Note 1.
Odd.....	$se_r(z, q)$	$B^r So_r(s, z)$ *	$A^r So_r^{(1)}(c, \cos z)$ *	$se_r(z, h^2)$ *	$se_r(z, q)$	$se_r(z, \theta)$	
Coefficients in Periodic Solutions:							
Even.....	$A'_m(q)$	$A^r De_m^r(s)$ *	$A^r D_m^r$ *	A'_m	A'_m	A'_m	
Odd.....	$B'_m(q)$	$B^r Do_m^r(s)$ *	$B^r F_m^r$ *	B'_m	B'_m	B'_m	
$\frac{1}{\pi} \int_0^{2\pi} y^2 dx$, y is the Standard Solution of 20.1.1.	1	$(A^r)^{-2}$ or $(B^r)^{-2}$	$(A^r)^{-2}$ or $(B^r)^{-2}$	1	1	1	See Note 1.
Floquet's Solutions 20.3.3.....	$F_r(z)$			$me_r(z, h^2)$	$\phi(z)$		
Characteristic Exponent.....	"	$\mu = i\nu$		"	$\mu = i\nu$	$\mu = i\nu$	
Normalizations of Floquet's Solutions.	Unspecified			$\frac{1}{\pi} \int_0^{2\pi} (me_r(z, h^2) me_{-r}(z, h^2) - 1)$			
Solutions of Modified Equation 20.1.2.	$Ce_r(z, q)$ $Se_r(z, q)$ $Mc_r^{(1)}(z, q)$ $Ms_r^{(1)}(z, q)$ $Mc_r^{(2)}(z, q)$ $Ms_r^{(2)}(z, q)$	$Ag_{o..r}(s) Je_r(s, q)$ $Bg_{o..r}(s) Jo_r(s, q)$ $\sqrt{\frac{2}{\pi}} Je_r(s, z)$ $\sqrt{\frac{2}{\pi}} Jo_r(s, z)$ $\sqrt{\frac{2}{\pi}} Ne_r(s, z)$ $\sqrt{\frac{2}{\pi}} No_r(s, z)$	$Ag_{o..r}(s) Je_r(c, \cosh z)$ $Bg_{o..r}(s) Jo_r(c, \cosh z)$ $\sqrt{\frac{2}{\pi}} Je_r(c, \cosh z)$ $\sqrt{\frac{2}{\pi}} Jo_r(c, \cosh z)$ $\sqrt{\frac{2}{\pi}} Ne_r(c, \cosh z)$ $\sqrt{\frac{2}{\pi}} No_r(c, \cosh z)$	$Ce_r(z, q)$ $Se_r(z, q)$ $Mc_r^{(1)}(z, h)$ $Ms_r^{(1)}(z, h)$ $Mc_r^{(2)}(z, h)$ $Ms_r^{(2)}(z, h)$	$Ce_r(z, q)$ $Se_r(z, q)$ $\sqrt{\frac{2}{\pi}} Ce_r(z, q) / Ag_{o..r}(q)$ $\sqrt{\frac{2}{\pi}} Se_r(z, q) / Bg_{o..r}(q)$ $\sqrt{\frac{2}{\pi}} Fey_r(z, q) / Ag_{o..r}(q)$ $\sqrt{\frac{2}{\pi}} Gey_r(z, q) / Bg_{o..r}(q)$	$Ce_r(z, \theta)$ $Se_r(z, \theta)$ $\sqrt{\frac{2}{\pi}} Ce_r(z, \theta) / Ag_{o..r}(q)$ $\sqrt{\frac{2}{\pi}} Se_r(z, \theta) / Bg_{o..r}(q)$ $\sqrt{\frac{2}{\pi}} Fey_r(z, \theta) / Ag_{o..r}(q)$ $\sqrt{\frac{2}{\pi}} Gey_r(z, \theta) / Bg_{o..r}(q)$	
Joining Factors.....	$\sqrt{2/\pi} / Mc_r^{(1)}(0, q)$ $\sqrt{2/\pi} / \frac{d}{dz} [Ms_r^{(1)}(z, q)]_{z=0}$ $-Mc_r^{(n)}(0, q) / Mc_r^{(1)}(0, q)$ $\left[\frac{d}{dz} \frac{Ms_r^{(n)}(z, q)}{Ms_r^{(1)}(z, q)} \right]_{z=0}$	$g_{o..r}(s)$ $g_{o..r}(s)$ $f_{o..r}(s)$ $f_{o..r}(s)$	$\sqrt{2/\pi} \lambda_r^{(e)}$ $\sqrt{2/\pi} \lambda_r^{(o)}$ $-\frac{2}{\pi} \frac{K'_1}{K_1}$ $\frac{2}{\pi} \frac{K'_1}{K_1}$	$\sqrt{2/\pi} / Mc_r^{(1)}(0, h)$ $\sqrt{2/\pi} / \frac{d}{dz} [Ms_r^{(1)}(z, h)]_{z=0}$ $-Mc_r^{(n)}(0, h) / Mc_r^{(1)}(0, h)$ Same as this volume	$(-1)^r p_r \sqrt{\frac{2}{\pi}} / A$ $(-1)^r s_r \sqrt{\frac{2}{\pi}} / B$ $-\frac{Fey_r(0, q)}{Ce_r(0, q)}$ $\left[\frac{d}{dz} \frac{Gey_r(z, q)}{Se_r(z, q)} \right]_{z=0}$	Same as [20.30] Same as [20.30] Same as [20.30] Same as [20.30]	See Note 2. See Note 3.

- NOTE: 1. The conversion factors A^r and B^r are tabulated in [20.58] along with the coefficients.
 2. The multipliers p_r and s_r are defined in [20.30], Appendix 1, section 3, equations 3, 4, 5, 6.
 3. See [20.59], sections (5.3) and (5.5). In eq. (316) of (5.5), the first term should have a minus sign.

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21. Spheroidal Wave Functions

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21. Spheroidal Wave Functions

Mathematical Properties

21.1. Definition of Elliptical Coordinates

$$21.1.1 \quad \xi = \frac{r_1 + r_2}{2f}; \quad \eta = \frac{r_1 - r_2}{2f}$$

r_1 and r_2 are the distances to the foci of a family of confocal ellipses and hyperbolas; $2f$ is the distance between foci.

$$21.1.2 \quad a = f\xi, \quad b = f\sqrt{\xi^2 - 1}, \quad e = \frac{f}{a}$$

a =semi-major axis; b =semi-minor axis; e =eccentricity.

Equation of Family of Confocal Ellipses

$$21.1.3 \quad \frac{x^2}{\xi^2} + \frac{y^2}{\xi^2 - 1} = f^2 \quad (1 < \xi < \infty)$$

Equation of Family of Confocal Hyperbolas

$$21.1.4 \quad \frac{x^2}{\eta^2} - \frac{y^2}{1 - \eta^2} = f^2 \quad (-1 < \eta < 1)$$

Relations Between Cartesian and Elliptical Coordinates

$$21.1.5 \quad x = f\xi\eta; \quad y = f\sqrt{(\xi^2 - 1)(1 - \eta^2)}$$

21.2. Definition of Prolate Spheroidal Coordinates

If the system of confocal ellipses and hyperbolas referred to in 21.1.3 and 21.1.4 revolves around the major axis, then

$$21.2.1 \quad \frac{x^2}{\xi^2} + \frac{r^2}{\xi^2 - 1} = f^2; \quad \frac{x^2}{\eta^2} - \frac{r^2}{1 - \eta^2} = f^2$$

$$y = r \cos \phi; \quad z = r \sin \phi; \quad 0 \leq \phi \leq 2\pi$$

where ξ , η and ϕ are prolate spheroidal coordinates.

Relations Between Cartesian and Prolate Spheroidal Coordinates

21.2.2

$$x = f\xi\eta; \quad y = f\sqrt{(\xi^2 - 1)(1 - \eta^2)} \cos \phi; \\ z = f\sqrt{(\xi^2 - 1)(1 - \eta^2)} \sin \phi$$

21.3. Definition of Oblate Spheroidal Coordinates

If the system of confocal ellipses and hyperbolas referred to in 21.1.3 and 21.1.4 revolves around the minor axis, then

$$21.3.1 \quad \frac{r^2}{\xi^2} + \frac{y^2}{\xi^2 - 1} = f^2; \quad \frac{r^2}{\eta^2} - \frac{y^2}{1 - \eta^2} = f^2$$

$$z = r \cos \phi; \quad x = r \sin \phi; \quad 0 \leq \phi \leq 2\pi$$

where ξ , η and ϕ are oblate spheroidal coordinates.

Relations Between Cartesian and Oblate Spheroidal Coordinates

21.3.2

$$x = f\xi\eta \sin \phi; \quad y = f\sqrt{(\xi^2 - 1)(1 - \eta^2)}; \quad z = f\xi\eta \cos \phi$$

21.4. Laplacian in Spheroidal Coordinates

21.4.1

$$\nabla^2 = \frac{1}{h_\xi h_\eta h_\phi} \left[\frac{\partial}{\partial \xi} \left(\frac{h_\eta h_\phi}{h_\xi} \frac{\partial}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left(\frac{h_\xi h_\phi}{h_\eta} \frac{\partial}{\partial \eta} \right) + \frac{\partial}{\partial \phi} \left(\frac{h_\xi h_\eta}{h_\phi} \frac{\partial}{\partial \phi} \right) \right]$$

$$h_\xi^2 = \left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2 + \left(\frac{\partial z}{\partial \xi} \right)^2$$

$$h_\eta^2 = \left(\frac{\partial x}{\partial \eta} \right)^2 + \left(\frac{\partial y}{\partial \eta} \right)^2 + \left(\frac{\partial z}{\partial \eta} \right)^2$$

$$h_\phi^2 = \left(\frac{\partial x}{\partial \phi} \right)^2 + \left(\frac{\partial y}{\partial \phi} \right)^2 + \left(\frac{\partial z}{\partial \phi} \right)^2$$

Metric Coefficients for Prolate Spheroidal Coordinates

21.4.2

$$h_\xi = f\sqrt{\frac{\xi^2 - \eta^2}{\xi^2 - 1}}; \quad h_\eta = f\sqrt{\frac{\xi^2 - \eta^2}{1 - \eta^2}}; \quad h_\phi = f\sqrt{(\xi^2 - 1)(1 - \eta^2)} \quad *$$

Metric Coefficients for Oblate Spheroidal Coordinates

21.4.3

$$h_\xi = f\sqrt{\frac{\xi^2 - \eta^2}{\xi^2 - 1}}; \quad h_\eta = f\sqrt{\frac{\xi^2 - \eta^2}{1 - \eta^2}}; \quad h_\phi = f\xi\eta \quad *$$

21.5. Wave Equation in Prolate and Oblate Spheroidal Coordinates

Wave Equation in Prolate Spheroidal Coordinates

21.5.1

$$\nabla^2 \Phi + k^2 \Phi = \frac{\partial}{\partial \xi} \left[(\xi^2 - 1) \frac{\partial \Phi}{\partial \xi} \right] + \frac{\partial}{\partial \eta} \left[(1 - \eta^2) \frac{\partial \Phi}{\partial \eta} \right] \\ + \frac{\xi^2 - \eta^2}{(\xi^2 - 1)(1 - \eta^2)} \frac{\partial^2 \Phi}{\partial \phi^2} + c^2 (\xi^2 - \eta^2) \Phi = 0$$

$$(c = \frac{1}{2}fk)$$

*See page II.

Wave Equation in Oblate Spheroidal Coordinates

21.5.2

$$\nabla^2 \Phi + k^2 \Phi = \frac{\partial}{\partial \xi} \left[(\xi^2 + 1) \frac{\partial \Phi}{\partial \xi} \right] + \frac{\partial}{\partial \eta} \left[(1 - \eta^2) \frac{\partial \Phi}{\partial \eta} \right] + \frac{\xi^2 + \eta^2}{(\xi^2 + 1)(1 - \eta^2)} \frac{\partial^2 \Phi}{\partial \phi^2} + c^2 (\xi^2 + \eta^2) \Phi = 0$$

$$\left(c = \frac{1}{2} f k \right)$$

21.5.2 may be obtained from 21.5.1 by the transformations

$$\xi \rightarrow \pm i\xi, c \rightarrow \mp ic.$$

21.6. Differential Equations for Radial and Angular Prolate Spheroidal Wave Functions

If in 21.5.1 we put

$$\Phi = R_{mn}(c, \xi) S_{mn}(c, \eta) \frac{\cos m\phi}{\sin m\phi}$$

then the "radial solution" $R_{mn}(c, \xi)$ and the "angular solution" $S_{mn}(c, \eta)$ satisfy the differential equations

21.6.1

$$\frac{d}{d\xi} \left[(\xi^2 - 1) \frac{d}{d\xi} R_{mn}(c, \xi) \right] - \left(\lambda_{mn} - c^2 \xi^2 + \frac{m^2}{\xi^2 - 1} \right) R_{mn}(c, \xi) = 0$$

21.6.2

$$\frac{d}{d\eta} \left[(1 - \eta^2) \frac{d}{d\eta} S_{mn}(c, \eta) \right] + \left(\lambda_{mn} - c^2 \eta^2 - \frac{m^2}{1 - \eta^2} \right) S_{mn}(c, \eta) = 0$$

where the separation constants (or eigenvalues) λ_{mn} are to be determined so that $R_{mn}(c, \xi)$ and $S_{mn}(c, \eta)$ are finite at $\xi = \pm 1$ and $\eta = \pm 1$ respectively.

(21.6.1 and 21.6.2 are identical. Radial and angular prolate spheroidal functions satisfy the same differential equation over different ranges of the variable.)

Differential Equations for Radial and Angular Oblate Spheroidal Functions

21.6.3

$$\frac{d}{d\xi} \left[(\xi^2 + 1) \frac{d}{d\xi} R_{mn}(c, \xi) \right] - \left(\lambda_{mn} - c^2 \xi^2 - \frac{m^2}{\xi^2 + 1} \right) R_{mn}(c, \xi) = 0$$

21.6.4

$$\frac{d}{d\eta} \left[(1 - \eta^2) \frac{d}{d\eta} S_{mn}(c, \eta) \right] + \left(\lambda_{mn} + c^2 \eta^2 - \frac{m^2}{1 - \eta^2} \right) S_{mn}(c, \eta) = 0$$

(21.6.3 may be obtained from 21.6.1 by the transformations $\xi \rightarrow \pm i\xi, c \rightarrow \mp ic$; 21.6.4 may be obtained from 21.6.2 by the transformation $c \rightarrow \mp ic$.)

21.7. Prolate Angular Functions

21.7.1

$$S_{mn}^{(1)}(c, \eta) = \sum_{r=0,1}^{\infty} d_r^{mn}(c) P_{m+r}^m(\eta)$$

= Prolate angular function of the first kind

21.7.2

$$S_{mn}^{(2)}(c, \eta) = \sum_{r=-\infty}^{\infty} d_r^{mn}(c) Q_{m+r}^m(\eta)$$

= Prolate angular function of the second kind

($P_n^m(\eta)$ and $Q_n^m(\eta)$ are associated Legendre functions of the first and second kinds respectively. However, for $-1 \leq z \leq 1$, $P_n^m(z) = (1 - z^2)^{m/2} d^m P_n(z) / dz^m$ (see 8.6.6). The summation is extended over even values or odd values of r .)

Recurrence Relations Between the Coefficients

21.7.3

$$\alpha_k d_{k+2} + (\beta_k - \lambda_{mn}) d_k + \gamma_k d_{k-2} = 0$$

$$\alpha_k = \frac{(2m+k+2)(2m+k+1)c^2}{(2m+2k+3)(2m+2k+5)}$$

$$\beta_k = (m+k)(m+k+1)$$

$$+ \frac{2(m+k)(m+k+1) - 2m^2 - 1}{(2m+2k-1)(2m+2k+3)} c^2$$

$$\gamma_k = \frac{k(k-1)c^2}{(2m+2k-3)(2m+2k-1)}$$

Transcendental Equation for λ_{mn}

21.7.4

$$U(\lambda_{mn}) = U_1(\lambda_{mn}) + U_2(\lambda_{mn}) = 0$$

$$U_1(\lambda_{mn}) = \gamma_r^m - \lambda_{mn} - \frac{\beta_r^m}{\gamma_{r-2}^m - \lambda_{mn}} - \frac{\beta_{r-2}^m}{\gamma_{r-4}^m - \lambda_{mn}} - \dots$$

$$U_2(\lambda_{mn}) = -\frac{\beta_{r+2}^m}{\gamma_{r+2}^m - \lambda_{mn}} - \frac{\beta_{r+4}^m}{\gamma_{r+4}^m - \lambda_{mn}} - \dots$$

$$\beta_k^m = \frac{k(k-1)(2m+k)(2m+k-1)c^4}{(2m+2k-1)^2(2m+2k+1)(2m+2k-3)}$$

$$(k \geq 2)$$

$$\gamma_k^m = (m+k)(m+k+1)$$

$$+ \frac{1}{2} c^2 \left[1 - \frac{4m^2 - 1}{(2m+2k-1)(2m+2k+3)} \right] (k \geq 0)$$

(The choice of r in 21.7.4 is arbitrary.)

Power Series Expansion for λ_{mn}

21.7.5

$$\lambda_{mn} = \sum_{k=0}^{\infty} l_{2k} c^{2k}$$

$$l_0 = n(n+1)$$

$$l_2 = \frac{1}{2} \left[1 - \frac{(2m-1)(2m+1)}{(2n-1)(2n+3)} \right]$$

$$l_4 = \frac{-(n-m+1)(n-m+2)(n+m+1)(n+m+2)}{2(2n+1)(2n+3)^3(2n+5)} + \frac{(n-m-1)(n-m)(n+m-1)(n+m)}{2(2n-3)(2n-1)^3(2n+1)}$$

$$l_6 = (4m^2-1) \left[\frac{(n-m+1)(n-m+2)(n+m+1)(n+m+2)}{(2n-1)(2n+1)(2n+3)^5(2n+5)(2n+7)} - \frac{(n-m-1)(n-m)(n+m-1)(n+m)}{(2n-5)(2n-3)(2n-1)^5(2n+1)(2n+3)} \right]$$

$$l_8 = 2(4m^2-1)^2 A + \frac{1}{16} B + \frac{1}{8} C + \frac{1}{2} D$$

$$A = \frac{(n-m-1)(n-m)(n+m-1)(n+m)}{(2n-5)^2(2n-3)(2n-1)^7(2n+1)(2n+3)^2} - \frac{(n-m+1)(n-m+2)(n+m+1)(n+m+2)}{(2n-1)^2(2n+1)(2n+3)^7(2n+5)(2n+7)^2}$$

$$B = \frac{(n-m-3)(n-m-2)(n-m-1)(n-m)(n+m-3)(n+m-2)(n+m-1)(n+m)}{(2n-7)(2n-5)^2(2n-3)^3(2n-1)^4(2n+1)} \\ - \frac{(n-m+1)(n-m+2)(n-m+3)(n-m+4)(n+m+1)(n+m+2)(n+m+3)(n+m+4)}{(2n+1)(2n+3)^4(2n+5)^3(2n+7)^2(2n+9)}$$

$$C = \frac{(n-m+1)^2(n-m+2)^2(n+m+1)^2(n+m+2)^2}{(2n+1)^2(2n+3)^7(2n+5)^2} - \frac{(n-m-1)^2(n-m)^2(n+m-1)^2(n+m)^2}{(2n-3)^2(2n-1)^7(2n+1)^2}$$

$$D = \frac{(n-m-1)(n-m)(n-m+1)(n-m+2)(n+m-1)(n+m)(n+m+1)(n+m+2)}{(2n-3)(2n-1)^4(2n+1)^2(2n+3)^4(2n+5)}$$

Asymptotic Expansion for λ_{mn}

21.7.6

$$\lambda_{mn}(c) = cq + m^2 - \frac{1}{8}(q^2+5) - \frac{q}{64c}(q^2+11-32m^2) \\ - \frac{1}{1024c^2}[5(q^4+26q^2+21)-384m^2(q^2+1)] \\ - \frac{1}{c^3} \left[\frac{1}{128^2}(33q^5+1594q^3+5621q) \right. \\ \left. - \frac{m^2}{128}(37q^3+167q) + \frac{m^4}{8}q \right] \\ - \frac{1}{c^4} \left[\frac{1}{256^2}(63q^6+4940q^4+43327q^2+22470) \right. \\ \left. - \frac{m^2}{512}(115q^4+1310q^2+735) + \frac{3m^4}{8}(q^2+1) \right] \\ - \frac{1}{c^5} \left[\frac{1}{1024^2}(527q^7+61529q^5+1043961q^3 \right. \\ \left. + 2241599q) - \frac{m^2}{32 \cdot 1024}(5739q^5+127550q^3 \right. \\ \left. + 298951q) + \frac{m^4}{512}(355q^3+1505q) - \frac{m^6q}{16} \right] + O(c^{-6}) \\ q = 2(n-m) + 1$$

Refinement of Approximate Values of λ_{mn}

If $\lambda_{mn}^{(1)}$ is an approximation to λ_{mn} obtained either from 21.7.5 or 21.7.6 then

21.7.7

$$\lambda_{mn} = \lambda_{mn}^{(1)} + \delta\lambda_{mn}$$

$$\delta\lambda_{mn} = \frac{U_1(\lambda_{mn}^{(1)}) + U_2(\lambda_{mn}^{(1)})}{\Delta_1 + \Delta_2}$$

$$\Delta_1 = 1 + \frac{\beta_r^m}{(N_r^m)^2} + \frac{\beta_r^m \beta_{r-2}^m}{(N_r^m N_{r-2}^m)^2} + \frac{\beta_r^m \beta_{r-2}^m \beta_{r-4}^m}{(N_r^m N_{r-2}^m N_{r-4}^m)^2} + \dots$$

$$\Delta_2 = \frac{(N_{r+2}^m)^2}{\beta_{r+2}^m} + \frac{(N_{r+2}^m N_{r+4}^m)^2}{\beta_{r+2}^m \beta_{r+4}^m} + \frac{(N_{r+2}^m N_{r+4}^m N_{r+6}^m)^2}{\beta_{r+2}^m \beta_{r+4}^m \beta_{r+6}^m} + \dots$$

$$N_r^m = \frac{(2m+r)(2m+r-1)c^2}{(2m+2r-1)(2m+2r+1)} \frac{d_r}{d_{r-2}} \quad (r \geq 2)$$

$$\beta_r^m = \frac{r(r-1)(2m+r)(2m+r-1)c^4}{(2m+2r-1)^2(2m+2r+1)(2m+2r-3)} \quad (r \geq 2)$$

Evaluation of Coefficients

Step 1. Calculate N_r^m 's from

21.7.8

$$N_{r+2}^m = \gamma_r^m - \lambda_{mn} - \frac{\beta_r^m}{N_r^m} \quad (r \geq 2)$$

$$N_2^m = \gamma_0^m - \lambda_{mn}; N_3^m = \gamma_1^m - \lambda_{mn}$$

$$\gamma_r^m = (m+r)(m+r+1)$$

$$+ \frac{1}{2} c^2 \left[1 - \frac{4m^2 - 1}{(2m+2r-1)(2m+2r+3)} \right] \quad (r \geq 0)$$

Step 2. Calculate ratios $\frac{d_0}{d_{2r}}$ and $\frac{d_1}{d_{2p+1}}$ from

$$21.7.9 \quad \frac{d_0}{d_{2r}} = \left(\frac{d_0}{d_2} \right) \left(\frac{d_2}{d_4} \right) \cdots \left(\frac{d_{2r-2}}{d_{2r}} \right)$$

$$21.7.10 \quad \frac{d_1}{d_{2p+1}} = \left(\frac{d_1}{d_3} \right) \left(\frac{d_3}{d_5} \right) \cdots \left(\frac{d_{2p-1}}{d_{2p+1}} \right)$$

and the formula for N_r^m in 21.7.7.

The coefficients d_r^m are determined to within the arbitrary factor d_0 for r even and d_1 for r odd. The choice of these factors depends on the normalization scheme adopted.

Normalization of Angular Functions

Meixner-Schärfke Scheme

$$21.7.11 \quad \int_{-1}^1 [S_{mn}(c, \eta)]^2 d\eta = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}$$

Stratton-Morse-Chu-Little-Corbató Scheme

$$21.7.12 \quad \sum_{r=0,1} \frac{(r+2m)!}{r!} d_r = \frac{(n+m)!}{(n-m)!}$$

(This normalization has the effect that $S_{mn}(c, \eta) \rightarrow P_n^m(\eta)$ as $\eta \rightarrow 1$.)

Flammer Scheme [21.4]

21.7.13

$$S_{mn}(c, 0) = P_n^m(0) = \frac{(-1)^{\frac{n-m}{2}} (n+m)!}{2^n \left(\frac{n-m}{2} \right)! \left(\frac{n+m}{2} \right)!} \quad (n-m) \text{ even}$$

21.7.14

$$S'_{mn}(c, 0) = P_n^m(0) = \frac{(-1)^{\frac{n-m-1}{2}} (n+m+1)!}{2^n \left(\frac{n-m-1}{2} \right)! \left(\frac{n+m+1}{2} \right)!} \quad (n-m) \text{ odd}$$

The above lead to the following conditions for d_r^m

21.7.15

$$\sum_{r=0}^{\infty} \frac{(-1)^{r/2} (r+2m)!}{2^r \left(\frac{r}{2} \right)! \left(\frac{r+2m}{2} \right)!} d_r^m = \frac{(-1)^{\frac{n-m}{2}} (n+m)!}{2^{n-m} \left(\frac{n-m}{2} \right)! \left(\frac{n+m}{2} \right)!} \quad (n-m) \text{ even}$$

21.7.16

$$\sum_{r=1}^{\infty} \frac{(-1)^{\frac{r-1}{2}} (r+2m+1)!}{2^r \left(\frac{r-1}{2} \right)! \left(\frac{r+2m+1}{2} \right)!} d_r^m = \frac{(-1)^{\frac{n-m-1}{2}} (n+m+1)!}{2^{n-m} \left(\frac{n-m-1}{2} \right)! \left(\frac{n+m+1}{2} \right)!} \quad (n-m) \text{ odd}$$

(The normalization scheme 21.7.13 and 21.7.14 is also used in [21.10].)

Asymptotic Expansions for $S_{mn}(c, \eta)$

21.7.17

$$S_{mn}(c, \eta) = (1-\eta^2)^{\frac{1}{2}} U_{mn}(c, \eta) \quad (c \rightarrow \infty)$$

$$U_{mn}(x) = \sum_{l=-\infty}^{\infty} h_l^i D_{l+r}(x) \quad l = n-m$$

where the $D_r(x)$'s are the parabolic cylinder functions (see chapter 19).

$$D_r(x) = (-1)^r e^{x^2/4} \frac{d^2}{dx^2} e^{-x^2/2} = 2^{-r/2} e^{-x^2/4} H_r \left(\frac{x}{\sqrt{2}} \right)$$

and the $H_r(x)$ are the Hermite polynomials (see chapter 22). (For tables of $h_{\pm}^i/h_0^i$ see [21.4].)

Expansion of $S_{mn}(c, \eta)$ in Powers of η

21.7.18

$$S_{mn}(c, \eta) = (1-\eta^2)^{m/2} \sum_{r=0,1}^{\infty} p_r^{mn}(c) \eta^r$$

$$(r+1)(r+2)p_{r+2}^{mn}(c) - [r(r+2m+1) + m(m+1) - \lambda_{mn}(c)] p_r^{mn}(c) - c^2 p_{r-2}^{mn}(c) = 0$$

(The derivation of the transcendental equation for λ_{mn} is similar to the derivation of 21.7.4 from 21.7.3.)

Expansion of $S_{mn}(c, \eta)$ in Powers of $(1-\eta^2)$

21.7.19

$$S_{mn}(c, \eta) = (1-\eta^2)^{m/2} \sum_{k=0}^{\infty} c_{2k}^{mn} (1-\eta^2)^k \quad (n-m) \text{ even}$$

21.7.20

$$S_{mn}(c, \eta) = \eta(1-\eta^2)^{m/2} \sum_{k=0}^{\infty} c_{2k}^{mn} (1-\eta^2)^k \quad (n-m) \text{ odd}$$

$$c_{2k}^{mn} = \frac{1}{2^m k! (m+k)!} \sum_{r=k}^{\infty} \frac{(2m+2r)!}{(2r)!} (-r)_k \left(m+r+\frac{1}{2}\right)_k d_{2r}^{mn} \quad (n-m) \text{ even}$$

$$c_{2k}^{mn} = \frac{1}{2^m k! (m+k)!} \sum_{r=k}^{\infty} \frac{(2m+2r+1)!}{(2r+1)!} (-r)_k \left(m+r+\frac{3}{2}\right)_k d_{2r+1}^{mn} \quad (n-m) \text{ odd}$$

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2) \dots (\alpha+k+1)$$

(The d_r^{mn} 's are the coefficients in 21.7.1.)

Prolate Angular Functions—Second Kind

Expansion 21.7.2 ultimately leads to

21.7.21

$$S_{mn}^{(2)}(c, \eta) = \sum_{r=-2m, -2m+1}^{\infty} d_r^{mn} Q_{m+r}^{mn}(\eta) + \sum_{r=2m+2, 2m+1}^{\infty} d_{r-1}^{mn} P_{r-m-1}^{mn}(\eta)$$

(The coefficients d_r^{mn} are the same as in 21.7.1; the coefficients $d_{\rho|r}^{mn}$ are tabulated in [21.4].)

21.8. Oblate Angular Functions

Power Series Expansion for Eigenvalues

$$21.8.1 \quad \lambda_{mn} = \sum_{k=0}^{\infty} (-1)^k l_k c^{2k}$$

where the l_k 's are the same as in 21.7.5.

Asymptotic Expansion for Eigenvalues [21.4]

21.8.2

$$\lambda_{mn} = -c^2 + 2c(2\nu+m+1) - 2\nu(\nu+m+1) - (m+1) + \Lambda_{mn}$$

$$\nu = \frac{1}{2}(n-m) \text{ for } (n-m) \text{ even;}$$

$$\nu = \frac{1}{2}(n-m-1) \text{ for } (n-m) \text{ odd}$$

$$\Lambda_{mn} = \sum_{k=1}^{\infty} \beta_k^{mn} c^{-k}$$

$$\beta_1^{mn} = -2^{-3}q(q^2+1-m^2)$$

$$\beta_2^{mn} = -2^{-6}[5q^4+10q^2+1-2m^2(3q^2+1)+m^4]$$

$$\beta_3^{mn} = -2^{-9}q[33q^4+114q^2+37-2m^2(23q^2+25) + 13m^4]$$

$$\beta_4^{mn} = -2^{-10}[63q^6+340q^4+239q^2+14 - 10m^2(10q^4+23q^2+3)+m^4(39q^2-18)-2m^6]$$

$$\beta_k^{mn} = \nu(\nu+m)a_k^{-1} + (\nu+1)(\nu+m+1)a_k^{-1}$$

$$q = n+1 \text{ for } (n-m) \text{ even; } q = n \text{ for } (n-m) \text{ odd}$$

(For the definition of $a_k^{\pm r}$ see 21.8.3.)

Asymptotic Expansion for Oblate Angular Functions

21.8.3

$$S_{mn}(-ic, \eta) \sim (1-\eta^2)^{m/2} \sum_{s=-\nu}^{\infty} A_s^{mn} \{ e^{-c(1-\eta)} L_{\nu+s}^{(m)}[2c(1-\eta)] + (-1)^{n-m} e^{-c(1+\eta)} L_{\nu+s}^{(m)}[2c(1+\eta)] \}$$

where the $L_{\nu}^{(m)}(x)$ are Laguerre polynomials (see chapter 22) and

$$\frac{A_{\pm r}^{mn}}{A_0^{mn}} = \sum_{k=r}^{\infty} a_k^{\pm r}(m, n) c^{-k}$$

(Expressions of $a_k^{\pm r}$ are given in [21.4].)

21.9. Radial Spheroidal Wave Functions

21.9.1

$$R_{mn}^{(p)}(c, \xi) = \left\{ \sum_{r=0,1}^{\infty} \frac{(2m+r)!}{r!} d_r^{mn} \right\}^{-1} \left(\frac{\xi^2-1}{\xi^2} \right)^{m/2} \cdot \sum_{r=0,1}^{\infty} i^{r+m-n} \frac{(2m+r)!}{r!} d_r^{mn} Z_{m+r}^{(p)}(c\xi)^*$$

$$Z_n^{(p)}(z) = \sqrt{\frac{\pi}{2z}} J_{n+\frac{1}{2}}(z) \quad (p=1)$$

$$= \sqrt{\frac{\pi}{2z}} Y_{n+\frac{1}{2}}(z) \quad (p=2)$$

($J_{n+\frac{1}{2}}(z)$ and $Y_{n+\frac{1}{2}}(z)$ are Bessel functions, order $n+\frac{1}{2}$, of the first and second kind respectively (see chapter 10).)

$$21.9.2 \quad R_{mn}^{(3)}(c, \xi) = R_{mn}^{(1)}(c, \xi) + iR_{mn}^{(2)}(c, \xi)$$

$$21.9.3 \quad R_{mn}^{(4)}(c, \xi) = R_{mn}^{(1)}(c, \xi) - iR_{mn}^{(2)}(c, \xi)$$

Asymptotic Behavior of $R_{mn}^{(1)}(c, \xi)$ and $R_{mn}^{(2)}(c, \xi)$

$$21.9.4 \quad R_{mn}^{(1)}(c, \xi) \xrightarrow{c\xi \rightarrow \infty} \frac{1}{c\xi} \cos[c\xi - \frac{1}{2}(n+1)\pi]$$

$$21.9.5 \quad R_{mn}^{(2)}(c, \xi) \xrightarrow{c\xi \rightarrow \infty} \frac{1}{c\xi} \sin[c\xi - \frac{1}{2}(n+1)\pi]$$

*See page II.

21.10. Joining Factors for Prolate Spheroidal Wave Functions

21.10.1

$$S_{mn}^{(1)}(c, \xi) = \kappa_{mn}^{(1)}(c) R_{mn}^{(1)}(c, \xi)$$

$$\kappa_{mn}^{(1)}(c) = \frac{(2m+1)(n+m)! \sum_{r=0}^{\infty} d_r^{mn} (2m+r)!/r!}{2^{n+m} d_0^{mn}(c) c^m m! \left(\frac{n-m}{2}\right)! \left(\frac{n+m}{2}\right)!} \quad (n-m) \text{ even}$$

$$= \frac{(2m+3)(n+m+1)! \sum_{r=1}^{\infty} d_r^{mn} (2m+r)!/r!}{2^{n+m} d_1^{mn}(c) c^{m+1} m! \left(\frac{n-m-1}{2}\right)! \left(\frac{n+m+1}{2}\right)!} \quad (n-m) \text{ odd}$$

21.10.2

$$S_{mn}^{(2)}(c, \xi) = \kappa_{mn}^{(2)}(c) R_{mn}^{(2)}(c, \xi)$$

$$\kappa_{mn}^{(2)}(c) = \frac{2^{n-m} (2m)! \left(\frac{n-m}{2}\right)! \left(\frac{n+m}{2}\right)! d_{-2m}^{mn}(c)}{(2m-1)m!(n+m)! c^{m-1}} \sum_{r=0}^{\infty} \frac{(2m+r)!}{r!} d_r^{mn}(c) \quad (n-m) \text{ even}$$

$$= - \frac{2^{n-m} (2m)! \left(\frac{n-m-1}{2}\right)! \left(\frac{n+m+1}{2}\right)! d_{-2m+1}^{mn}(c)}{(2m-3)(2m-1)m!(n+m+1)! c^{m-2}} \sum_{r=1}^{\infty} \frac{(2m+r)!}{r!} d_r^{mn}(c) \quad (n-m) \text{ odd}$$

(The expression for joining factors appropriate to the oblate case may be obtained from the above formulas by the transformation $c \rightarrow -ic$.)

21.11. Notation

Notation for Prolate Spheroidal Wave Functions

	Ang. coord.	Rad. coord.	Independent variable	Ang. wave function	Rad. wave function	Eigenvalue	Normalization of angular functions	Remarks
Stratton, Morse, Chu, Little and Corbató	η	ξ	h	$S_{ml}(h, \eta)$	$j_{eml}(h, \xi)$ $ne_{ml}(h, \xi)$ $he_{ml}(h, \xi)$	$A_{ml}(h)$	$S_{ml}(h, 1) = P_l^m(1)$	$l = \text{Flammer's } n$ $A_{ml} = \lambda_{mn}$
Flammer and this chapter	η	ξ	c	$S_{mn}(c, \eta)$	$R_{mn}^{(i)}(c, \xi)$	$\lambda_{mn}(c)$	$S_{mn}(c, 0) = P_n^m(0)$ $(n-m)$ even $S_{mn}(c, 0) = P_n^{m'}(0)$ $(n-m)$ odd	
Chu and Stratton	η	ξ	c	$S_{ml}^{(i)}(c, \eta)$	$R_{ml}^{(i)}(c, \xi)$	A_{ml}	$S_{ml}^{(i)}(c, 0) = P_{n+l}^{m+i}(0)$ $(l \text{ even})$ $S_{ml}^{(i)}(c, 0) = P_{n+l}^{m+i}(0)$ $(l \text{ odd})$	$l = \text{Flammer's } n-m$ $A_{ml} = -\lambda_{m, n-m}$
Meixner and Schäfke	η	ξ	γ	$PS_n^m(\eta, \gamma^2)$	$S_n^{m(i)}(\xi, \gamma^2)$	$\lambda_n^m(\gamma^2)$	$\int_{-1}^1 [PS_n^m(\eta, \gamma^2)]^2 d\eta$ $= \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}$	$\lambda_n^m(\gamma^2) = \lambda_{mn}(c) - c^2$
Morse and Feshbach	$\eta = \cos \vartheta$	$\xi = \cosh \mu$	h	$S_{ml}(h, \eta)$	$j_{eml}(h, \xi)$ $ne_{ml}(h, \xi)$ $he_{ml}(h, \xi)$	A_{ml}	$[(1-\eta^2)^{-m/2} S_{ml}(h, \eta)]_{\eta=1} = [(1-\eta^2)^{-m/2} P_l^m(\eta)]_{\eta=1}$	$l = \text{Flammer's } n$ $A_{ml} = \lambda_{mn}$
Page	ξ	η	ϵ	$U_{lm}(\xi)$	$v_{lm}(\eta)$ $p_{lm}(\eta)$ $q_{lm}(\eta)$	α_{lm}	$[(1-\xi^2)^{-m/2} U_{lm}(\xi)]_{\xi=1} = 1$	$l = \text{Flammer's } n$ $\alpha_{lm} = \lambda_{mn} - c^2$

Notation for Oblate Spheroidal Wave Functions

Stratton, Morse, Chu, Little and Corbató	η	ξ	g	$S_{ml}(ig, \eta)$	$j_{eml}(ig, -i\xi)$	A_{ml}	$S_{ml}(ig, 1) = P_l^m(1)$	$l = \text{Flammer's } n$ $A_{ml} = \lambda_{mn}$
Flammer and this chapter	η	ξ	c	$S_{mn}(-ic, \eta)$	$R_{mn}^{(i)}(-ic, i\xi)$	$\lambda_{mn}(-ic)$	$S_{mn}(-ic, 0) = P_n^m(0)$ $(n-m)$ even $S_{mn}(-ic, 0) = P_n^{m'}(0)$ $(n-m)$ odd	
Chu and Stratton	η	ξ	c	$S_{ml}^{(i)}(-ic, \eta)$	$R_{ml}^{(i)}(-ic, i\xi)$	B_{ml}	$S_{ml}^{(i)}(-ic, 0) = P_{n+l}^{m+i}(0)$ $(l \text{ even})$ $S_{ml}^{(i)}(-ic, 0) = P_{n+l}^{m+i}(0)$ $(l \text{ odd})$	$l = \text{Flammer's } n-m$ $B_{lm} = -\lambda_{m, n-m}$
Meixner and Schäfke	η	ξ	γ	$ps_n^m(\eta, -\gamma^2)$	$S_n^{m(i)}(-i\xi, i\gamma^2)$	$\lambda_n^m(-\gamma^2)$	$\int_{-1}^1 [ps_n^m(\eta, -\gamma^2)]^2 d\eta$ $= \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}$	$\lambda_n^m(-\gamma^2) = \lambda_{mn}(-ic) + c^2$
Morse and Feshbach	$\eta = \cos \vartheta$	$\xi = \sinh \mu$	g	$S_{ml}(ig, \eta)$	$j_{eml}(ig, -i\xi)$ $ne_{ml}(ig, -i\xi)$ $he_{ml}(ig, -i\xi)$	A_{ml}	$[(1-\eta^2)^{-m/2} S_{ml}(ig, \eta)]_{\eta=1} = [(1-\eta^2)^{-m/2} P_l^m(\eta)]_{\eta=1}$	$l = \text{Flammer's } n$ $A_{ml} = \lambda_{mn}$
Leitner and Spence	η	ξ	ϵ	$U_{lm}(\eta)$	(i) $v_{lm}(\xi)$	α_{lm}	$[(1-\eta^2)^{-m/2} U_{lm}(\eta)]_{\eta=1} = 1$	$l = \text{Flammer's } n$ $\alpha_{lm} = \lambda_{mn} + c^2$

The notation in this chapter closely follows the notation in [21.4].

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22. Orthogonal Polynomials

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22. Orthogonal Polynomials

Mathematical Properties

22.1. Definition of Orthogonal Polynomials

A system of polynomials $f_n(x)$, degree $[f_n(x)] = n$, is called orthogonal on the interval $a \leq x \leq b$, with respect to the weight function $w(x)$, if

22.1.1

$$\int_a^b w(x) f_n(x) f_m(x) dx = 0 \quad (n \neq m; n, m = 0, 1, 2, \dots)$$

The weight function $w(x)[w(x) \geq 0]$ determines the system $f_n(x)$ up to a constant factor in each polynomial. The specification of these factors is referred to as standardization. For suitably standardized orthogonal polynomials we set

22.1.2

$$\int_a^b w(x) f_n^2(x) dx = h_n, \quad f_n(x) = k_n x^n + k'_n x^{n-1} + \dots \quad (n = 0, 1, 2, \dots)$$

These polynomials satisfy a number of relationships of the same general form. The most important ones are:

Differential Equation

$$22.1.3 \quad g_2(x) f_n'' + g_1(x) f_n' + a_n f_n = 0$$

where $g_2(x)$, $g_1(x)$ are independent of n and a_n a constant depending only on n .

Recurrence Relation

$$22.1.4 \quad f_{n+1} = (a_n + x b_n) f_n - c_n f_{n-1}$$

where

22.1.5

$$b_n = \frac{k_{n+1}}{k_n}, \quad a_n = b_n \left(\frac{k'_{n+1}}{k_{n+1}} - \frac{k'_n}{k_n} \right), \quad c_n = \frac{k_{n+1} k_{n-1} h_n}{k_n^2 h_{n-1}}$$

Rodrigues' Formula

$$22.1.6 \quad f_n = \frac{1}{e_n w(x)} \frac{d^n}{dx^n} \{ w(x) [g(x)]^n \}$$

where $g(x)$ is a polynomial in x independent of n .

The system $\left\{ \frac{df_n}{dx} \right\}$ consists again of orthogonal polynomials.

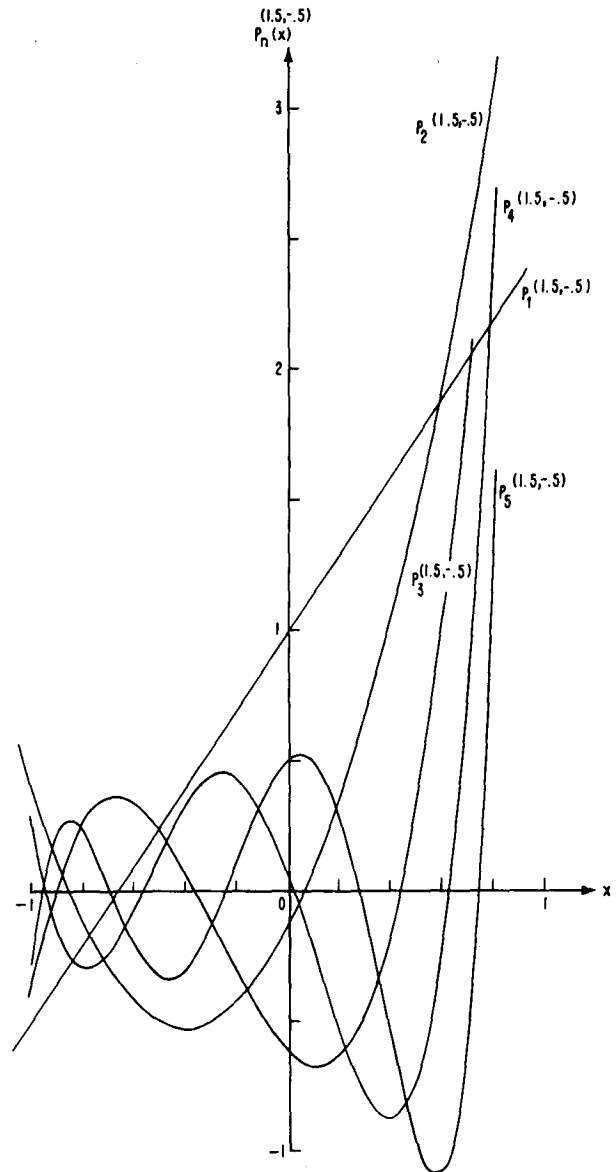


FIGURE 22.1. *Jacobi Polynomials* $P_n^{(\alpha, \beta)}(x)$, $\alpha = 1.5$, $\beta = -.5$, $n = 1(1)5$.

22.2. Orthogonality Relations

	$f_n(x)$	Name of Polynomial	a	b	$w(x)$	Standardization	h_n	Remarks
22.2.1	$P_n^{(\alpha, \beta)}(x)$	Jacobi	-1	1	$(1-x)^\alpha(1+x)^\beta$	$P_n^{(\alpha, \beta)}(1) = \binom{n+\alpha}{n}$	$\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}$	$\alpha > -1, \beta > -1$
22.2.2	$G_n(p, q, x)$	Jacobi	0	1	$(1-x)^{p-1}x^{q-1}$	$k_n = 1$	$\frac{n!\Gamma(n+q)\Gamma(n+p)\Gamma(n+p-q+1)}{(2n+p)\Gamma^2(2n+p)}$	$p-q > -1, q > 0$
22.2.3	$C_n^{(\alpha)}(x)$	Ultraspherical (Gegenbauer)	-1	1	$(1-x^2)^{\alpha-\frac{1}{2}}$	$C_n^{(\alpha)}(1) = \binom{n+2\alpha-1}{n}$ ($\alpha \neq 0$)	$\frac{\pi 2^{1-2\alpha} \Gamma(n+2\alpha)}{n!(n+\alpha)[\Gamma(\alpha)]^2} \quad \alpha \neq 0$	$\alpha > -\frac{1}{2}$
						$C_n^{(0)}(1) = \frac{2}{n},$ $C_0^{(0)}(1) = 1$	$\frac{2\pi}{n^2} \quad \alpha = 0$	
22.2.4	$T_n(x)$	Chebyshev of the first kind	-1	1	$(1-x^2)^{-\frac{1}{2}}$	$T_n(1) = 1$	$\begin{cases} \frac{\pi}{2} & n \neq 0 \\ \pi & n = 0 \end{cases}$	
22.2.5	$U_n(x)$	Chebyshev of the second kind	-1	1	$(1-x^2)^{\frac{1}{2}}$	$U_n(1) = n+1$	$\frac{\pi}{2}$	8
22.2.6	$C_n(x)$	Chebyshev of the first kind	-2	2	$\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}$	$C_n(2) = 2$	$\begin{cases} 4\pi & n \neq 0 \\ 8\pi & n = 0 \end{cases}$	
22.2.7	$S_n(x)$	Chebyshev of the second kind	-2	2	$\left(1-\frac{x^2}{4}\right)^{\frac{1}{2}}$	$S_n(2) = n+1$	π	
22.2.8	$T_n^*(x)$	Shifted Chebyshev of the first kind	0	1	$(x-x^2)^{-\frac{1}{2}}$	$T_n^*(1) = 1$	$\begin{cases} \frac{\pi}{2} & n \neq 0 \\ \pi & n = 0 \end{cases}$	
22.2.9	$U_n^*(x)$	Shifted Chebyshev of the second kind	0	1	$(x-x^2)^{\frac{1}{2}}$	$U_n^*(1) = n+1$	$\frac{\pi}{8} \quad *$	
22.2.10	$P_n(x)$	Legendre (Spherical)	-1	1	1	$P_n(1) = 1$	$\frac{2}{2n+1}$	
22.2.11	$P_n^*(x)$	Shifted Legendre	0	1	1		$\frac{1}{2n+1}$	

*See page 11.

22.2. Orthogonality Relations—Continued

22.2.12	$L_n^{(\alpha)}(x)$	Generalized Laguerre	0	∞	$e^{-x}x^\alpha$	$k_n = \frac{(-1)^n}{n!}$	$\frac{\Gamma(\alpha+n+1)}{n!}$	$\alpha > -1$
22.2.13	$L_n(x)$	Laguerre	0	∞	e^{-x}	$k_n = \frac{(-1)^n}{n!}$	1	
* 22.2.14	$H_n(x)$	Hermite	$-\infty$	∞	e^{-x^2}	$e_n = (-1)^n$	$\sqrt{\pi}2^n n!$	
* 22.2.15	$He_n(x)$	Hermite	$-\infty$	∞	$e^{-\frac{x^2}{2}}$	$e_n = (-1)^n$	$\sqrt{2\pi}n!$	

*See page II.

22.3. Explicit Expressions

$$f_n(x) = d_n \sum_{m=0}^N c_m g_m(x)$$

	$f_n(x)$	N	d_n	c_m	$g_m(x)$	k_n	Remarks
22.3.1	$P_n^{(\alpha, \beta)}(x)$	n	$\frac{1}{2^n}$	$\binom{n+\alpha}{m} \binom{n+\beta}{n-m}$	$(x-1)^{n-m}(x+1)^m$	$\frac{1}{2^n} \binom{2n+\alpha+\beta}{n}$	$\alpha > -1, \beta > -1$
22.3.2	$P_n^{(\alpha, \beta)}(x)$	n	$\frac{\Gamma(\alpha+n+1)}{n! \Gamma(\alpha+\beta+n+1)}$	$\binom{n}{m} \frac{\Gamma(\alpha+\beta+n+m+1)}{2^m \Gamma(\alpha+m+1)}$	$(x-1)^m$	$\frac{1}{2^n} \binom{2n+\alpha+\beta}{n}$	$\alpha > -1, \beta > -1$
22.3.3	$G_n(p, q, x)$	n	$\frac{\Gamma(q+n)}{\Gamma(p+2n)}$	$(-1)^m \binom{n}{m} \frac{\Gamma(p+2n-m)}{\Gamma(q+n-m)}$	x^{n-m}	1	$p-q > -1, q > 0$
22.3.4	$C_n^{(\alpha)}(x)$	$\left[\frac{n}{2} \right]$	$\frac{1}{\Gamma(\alpha)}$	$(-1)^m \frac{\Gamma(\alpha+n-m)}{m!(n-2m)!}$	$(2x)^{n-2m}$	$\frac{2^n}{n!} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$	$\alpha > -\frac{1}{2}, \alpha \neq 0$
22.3.5	$C_n^{(0)}(x)$	$\left[\frac{n}{2} \right]$	1	$(-1)^m \frac{(n-m-1)!}{m!(n-2m)!}$	$(2x)^{n-2m}$	$\frac{2^n}{n} \quad n \neq 0$	$n \neq 0, C_0^{(0)}(1) = 1$
22.3.6	$T_n(x)$	$\left[\frac{n}{2} \right]$	$\frac{n}{2}$	$(-1)^m \frac{(n-m-1)!}{m!(n-2m)!}$	$(2x)^{n-2m}$	2^{n-1}	
22.3.7	$U_n(x)$	$\left[\frac{n}{2} \right]$	1	$(-1)^m \frac{(n-m)!}{m!(n-2m)!}$	$(2x)^{n-2m}$	2^n	
22.3.8	$P_n(x)$	$\left[\frac{n}{2} \right]$	$\frac{1}{2^n}$	$(-1)^m \binom{n}{m} \binom{2n-2m}{n}$	x^{n-2m}	$\frac{(2n)!}{2^n (n!)^2}$	
22.3.9	$L_n^{(\alpha)}(x)$	n	1	$(-1)^m \binom{n+\alpha}{n-m} \frac{1}{m!}$	x^m	$\frac{(-1)^n}{n!}$	$\alpha > -1$
22.3.10	$H_n(x)$	$\left[\frac{n}{2} \right]$	$n!$	$(-1)^m \frac{1}{m!(n-2m)!}$	$(2x)^{n-2m}$	2^n	see 22.11
22.3.11	$He_n(x)$	$\left[\frac{n}{2} \right]$	$n!$	$(-1)^m \frac{1}{m! 2^m (n-2m)!}$	x^{n-2m}	1	

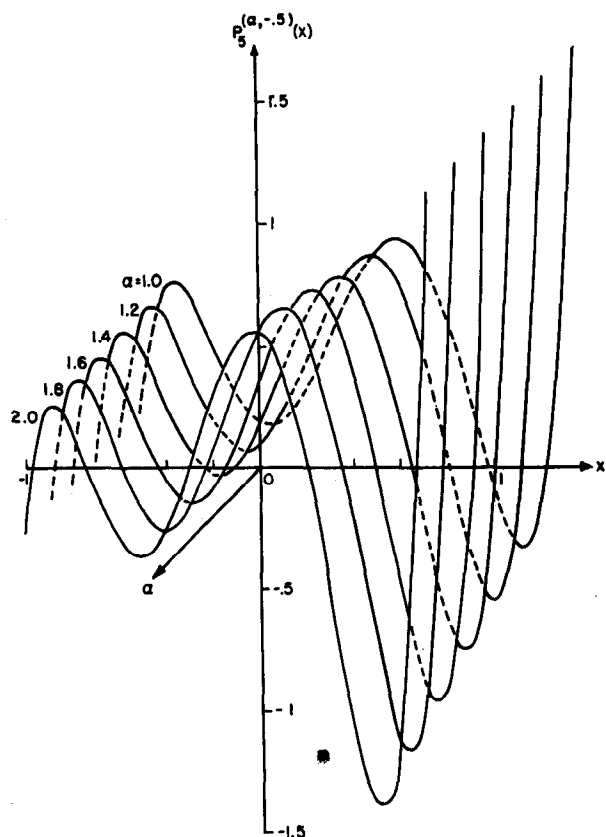


FIGURE 22.2. *Jacobi Polynomials* $P_n^{(\alpha, \beta)}(x)$,
 $\alpha=1(.2)2$, $\beta= -.5$, $n=5$.

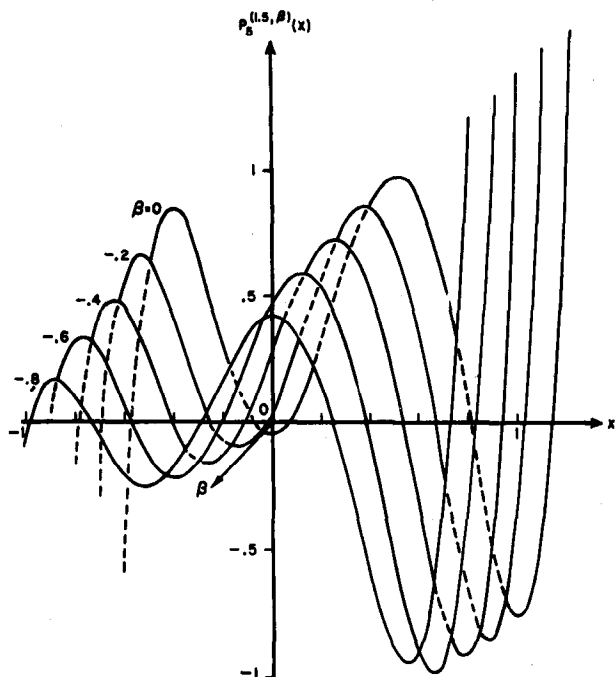


FIGURE 22.3. *Jacobi Polynomials* $P_n^{(\alpha, \beta)}(x)$,
 $\alpha=1.5$, $\beta= -.8(.2)0$, $n=5$.

Explicit Expressions Involving Trigonometric Functions

$$f_n(\cos \theta) = \sum_{m=0}^n a_m \cos(n-2m)\theta$$

	$f_n(\cos \theta)$	a_m	Remarks
22.3.12	$C_n^{(\alpha)}(\cos \theta)$	$\frac{\Gamma(\alpha+m)\Gamma(\alpha+n-m)}{m!(n-m)![\Gamma(\alpha)]^2}$	$\alpha \neq 0$
22.3.13	$P_n(\cos \theta)$	$\frac{1}{4^n} \binom{2m}{m} \binom{2n-2m}{n-m}$	

$$22.3.14 \quad C_n^{(0)}(\cos \theta) = \frac{2}{n} \cos n\theta$$

$$22.3.15 \quad T_n(\cos \theta) = \cos n\theta$$

$$22.3.16 \quad U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}$$

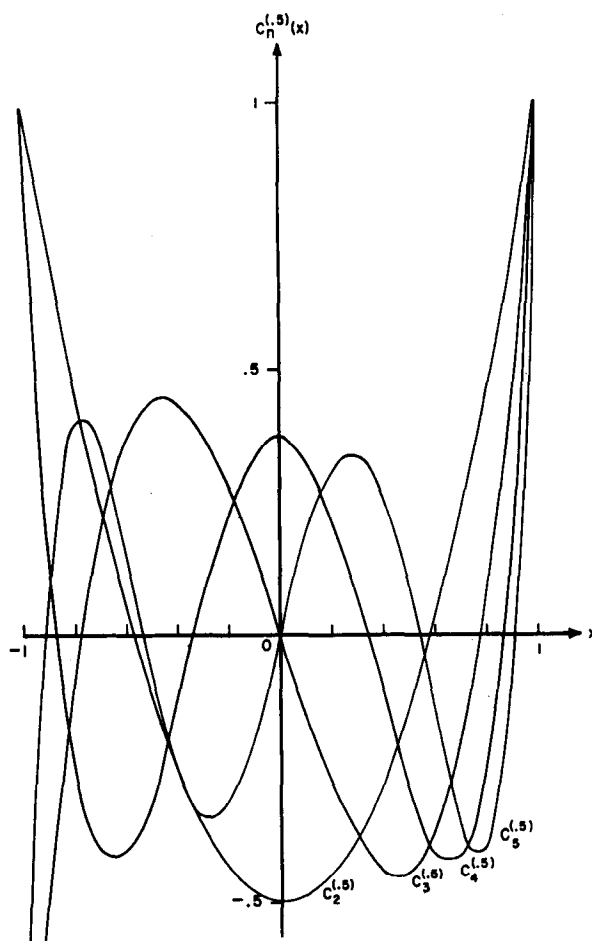
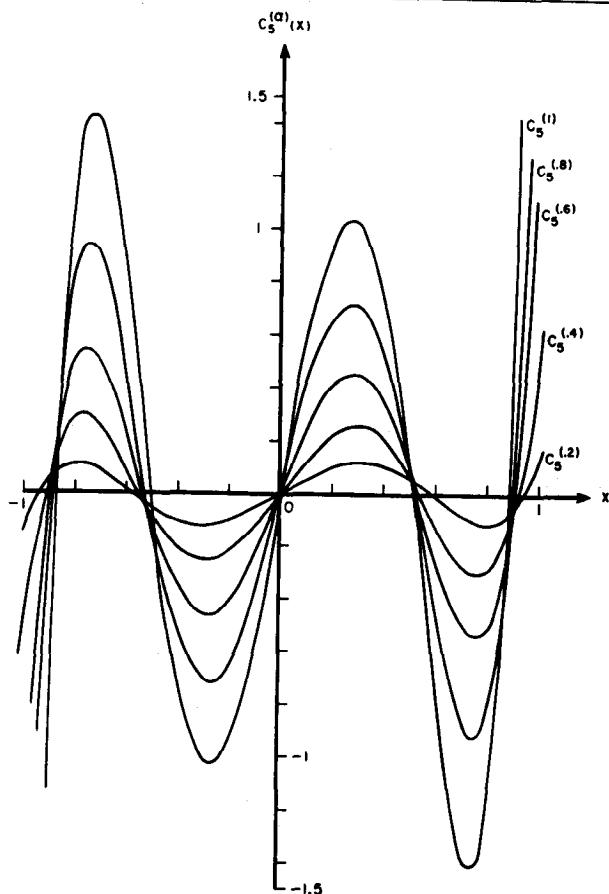


FIGURE 22.4. *Gegenbauer (Ultraspherical) Polynomials* $C_n^{(\alpha)}(x)$, $\alpha=.5$, $n=2(1)5$.

22.4. Special Values

	$f_n(x)$	$f_n(-x)$	$f_n(1)$	$f_n(0)$	$f_0(x)$	$f_1(x)$
22.4.1	$P_n^{(\alpha, \beta)}(x)$	$(-1)^n P_n^{(\beta, \alpha)}(x)$	$\binom{n+\alpha}{n}^*$		1	$\frac{1}{2}[\alpha - \beta + (\alpha + \beta + 2)x]$
22.4.2	$C_n^{(\alpha)}(x)$ $\alpha \neq 0$	$(-1)^n C_n^{(\alpha)}(x)$	$\binom{n+2\alpha-1}{n}$	$\begin{cases} 0, n=2m+1 \\ (-1)^{n/2} \frac{\Gamma(\alpha+n/2)}{\Gamma(\alpha)(n/2)!}, n=2m \end{cases}$	1	$2\alpha x$
22.4.3	$C_n^{(0)}(x)$	$(-1)^n C_n^{(0)}(x)$	$\frac{2}{n}, n \neq 0$	$\begin{cases} \frac{(-1)^m}{m}, n=2m \neq 0 \\ 0, n=2m+1 \end{cases}$	1	$2x$
22.4.4	$T_n(x)$	$(-1)^n T_n(x)$	1	$\begin{cases} (-1)^m, n=2m \\ 0, n=2m+1 \end{cases}$	1	x
22.4.5	$U_n(x)$	$(-1)^n U_n(x)$	$n+1$	$\begin{cases} (-1)^m, n=2m \\ 0, n=2m+1 \end{cases}$	1	$2x$
22.4.6	$P_n(x)$	$(-1)^n P_n(x)$	1	$\begin{cases} \frac{(-1)^m}{4^m} \binom{2m}{m}, n=2m^* \\ 0, n=2m+1 \end{cases}$	1	x
22.4.7	$L_n^{(\alpha)}(x)$		$\binom{n+\alpha}{n}$		1	$-x + \alpha + 1$
22.4.8	$H_n(x)$	$(-1)^n H_n(x)$		$\begin{cases} (-1)^m \frac{(2m)!}{m!}, n=2m \\ 0, n=2m+1 \end{cases}$	1	$2x$


 FIGURE 22.5. Gegenbauer (Ultraspherical) Polynomials $C_n^{(\alpha)}(x)$, $\alpha = .2(.2)1$, $n = 5$.

22.5. Interrelations

Interrelations Between Orthogonal Polynomials of the Same Family

Jacobi Polynomials

22.5.1

$$P_n^{(\alpha, \beta)}(x) = \frac{\Gamma(2n + \alpha + \beta + 1)}{n! \Gamma(n + \alpha + \beta + 1)} G_n\left(\alpha + \beta + 1, \beta + 1, \frac{x+1}{2}\right)$$

22.5.2

$$G_n(p, q, x) = \frac{n! \Gamma(n+p)}{\Gamma(2n+p)} P_n^{(p-q, q-1)}(2x-1)$$

(see [22.21]).

22.5.3

$$F_n(p, q, x) = (-1)^n n! \frac{\Gamma(q)}{\Gamma(q+n)} P_n^{(p-q, q-1)}(2x-1)$$

(see [22.13]).

Ultraspherical Polynomials

$$22.5.4 \quad C_n^{(0)}(x) = \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} C_n^{(\alpha)}(x)$$

Chebyshev Polynomials

$$22.5.5 \quad T_n(x) = \frac{1}{2} C_n(2x) = T_n^*\left(\frac{1+x}{2}\right)$$

$$22.5.6 \quad T_n(x) = U_n(x) - x U_{n-1}(x)$$

*See page II.

$$22.5.7 \quad T_n(x) = xU_{n-1}(x) - U_{n-2}(x)$$

$$22.5.8 \quad T_n(x) = \frac{1}{2} [U_n(x) - U_{n-2}(x)]$$

$$22.5.9 \quad U_n(x) = S_n(2x) = U_n^*\left(\frac{1+x}{2}\right)$$

$$22.5.10 \quad U_{n-1}(x) = \frac{1}{1-x^2} [xT_n(x) - T_{n+1}(x)]$$

$$22.5.11 \quad C_n(x) = 2T_n\left(\frac{x}{2}\right) = 2T_n^*\left(\frac{x+2}{4}\right)$$

$$22.5.12 \quad C_n(x) = S_n(x) - S_{n-2}(x)$$

$$22.5.13 \quad S_n(x) = U_n\left(\frac{x}{2}\right) = U_n^*\left(\frac{x+2}{4}\right)$$

$$22.5.14 \quad T_n^*(x) = T_n(2x-1) = \frac{1}{2} C_n(4x-2)$$

(see [22.22]).

$$22.5.15 \quad U_n^*(x) = S_n(4x-2) = U_n(2x-1)$$

(see [22.22]).

Generalized Laguerre Polynomials

$$22.5.16 \quad L_n^{(0)}(x) = L_n(x)$$

$$22.5.17 \quad L_n^{(m)}(x) = (-1)^m \frac{d^m}{dx^m} [L_{n+m}(x)]$$

Hermite Polynomials

$$22.5.18 \quad He_n(x) = 2^{-n/2} H_n\left(\frac{x}{\sqrt{2}}\right)$$

(see [22.20]).

$$22.5.19 \quad H_n(x) = 2^{n/2} He_n(x\sqrt{2})$$

(see [22.13], [22.20]).

Interrelations Between Orthogonal Polynomials of Different Families

Jacobi Polynomials

22.5.20

$$P_n^{(\alpha-\frac{1}{2}, \alpha-\frac{1}{2})}(x) = \frac{\Gamma(2\alpha)\Gamma(\alpha+n+\frac{1}{2})}{\Gamma(2\alpha+n)\Gamma(\alpha+\frac{1}{2})} C_n^{(\alpha)}(x)$$

22.5.21

$$P_n^{(\alpha, \frac{1}{2})}(x) = \frac{(\frac{1}{2})_{n+1}}{\sqrt{\frac{x+1}{2}} (\alpha+\frac{1}{2})_{n+1}} C_{2n+1}^{(\alpha+\frac{1}{2})}\left(\sqrt{\frac{x+1}{2}}\right)$$

$$22.5.22 \quad P_n^{(\alpha, -\frac{1}{2})}(x) = \frac{(\frac{1}{2})_n}{(\alpha+\frac{1}{2})_n} C_{2n}^{(\alpha+\frac{1}{2})}\left(\sqrt{\frac{x+1}{2}}\right)$$

$$22.5.23 \quad P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x) = \frac{1}{4^n} \binom{2n}{n} T_n(x)$$

$$22.5.24 \quad P_n^{(0,0)}(x) = P_n(x)$$

Ultraspherical Polynomials

22.5.25

$$C_{2n}^{(\alpha)}(x) = \frac{\Gamma(\alpha+n)n!2^{2n}}{\Gamma(\alpha)(2n)!} P_n^{(\alpha-\frac{1}{2}, -\frac{1}{2})}(2x^2-1) \quad (\alpha \neq 0)$$

22.5.26

$$C_{2n+1}^{(\alpha)}(x) = \frac{\Gamma(\alpha+n+1)n!2^{2n+1}}{\Gamma(\alpha)(2n+1)!} x P_n^{(\alpha-\frac{1}{2}, \frac{1}{2})}(2x^2-1) \quad (\alpha \neq 0)$$

22.5.27

$$C_n^{(\alpha)}(x) = \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(2\alpha+n)}{\Gamma(2\alpha)\Gamma(\alpha+n+\frac{1}{2})} P_n^{(\alpha-\frac{1}{2}, \alpha-\frac{1}{2})}(x) \quad (\alpha \neq 0)$$

22.5.28

$$C_n^{(0)}(x) = \frac{2}{n} T_n(x) = 2 \frac{(n-1)!}{\Gamma(n+\frac{1}{2})} \sqrt{\pi} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x) \quad *$$

Chebyshev Polynomials

$$22.5.29 \quad T_{2n+1}(x) = \frac{n!\sqrt{\pi}}{\Gamma(n+\frac{1}{2})} x P_n^{(-\frac{1}{2}, \frac{1}{2})}(2x^2-1)$$

$$22.5.30 \quad U_{2n}(x) = \frac{n!\sqrt{\pi}}{\Gamma(n+\frac{1}{2})} P_n^{(\frac{1}{2}, -\frac{1}{2})}(2x^2-1)$$

$$22.5.31 \quad T_n(x) = \frac{n!\sqrt{\pi}}{\Gamma(n+\frac{1}{2})} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x)$$

$$22.5.32 \quad U_n(x) = \frac{(n+1)!\sqrt{\pi}}{2\Gamma(n+\frac{3}{2})} P_n^{(\frac{1}{2}, \frac{1}{2})}(x)$$

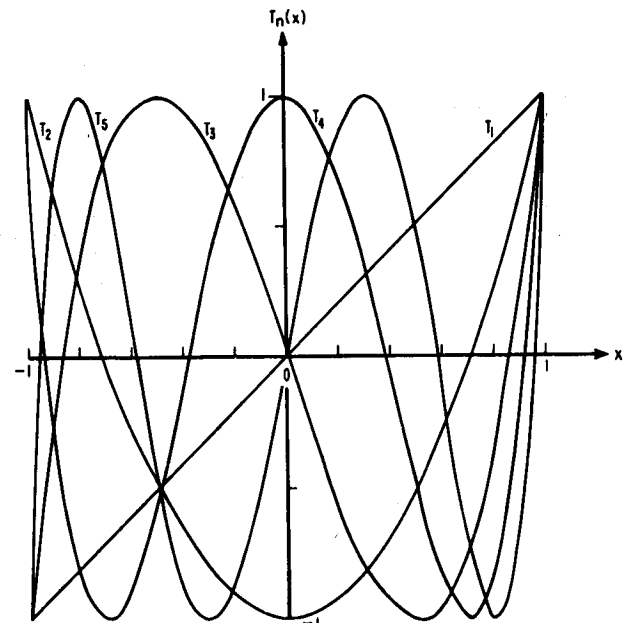


FIGURE 22.6. Chebyshev Polynomials $T_n(x)$, $n=1(1)5$.

*See page II.

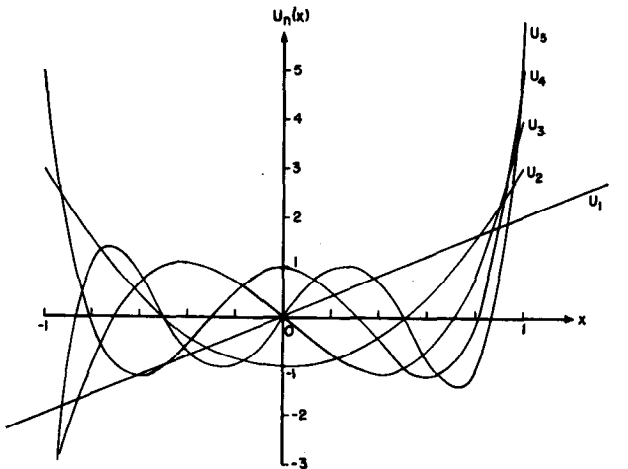


FIGURE 22.7. Chebyshev Polynomials $U_n(x)$, $n=1(1)5$.

$$22.5.33 \quad T_n(x) = \frac{n}{2} C_n^{(0)}(x)$$

$$22.5.34 \quad U_n(x) = C_n^{(1)}(x)$$

Legendre Polynomials

$$22.5.35 \quad P_n(x) = P_n^{(0,0)}(x)$$

$$22.5.36 \quad P_n(x) = C_n^{(1/2)}(x)$$

$$22.5.37$$

$$\frac{d^m}{dx^m} [P_n(x)] = 1 \cdot 3 \cdots (2m-1) C_{n-m}^{(m+1)}(x) \quad (m \leq n)$$

Generalized Laguerre Polynomials

$$22.5.38 \quad L_n^{(-1/2)}(x) = \frac{(-1)^n}{n! 2^{2n}} H_{2n}(\sqrt{x})$$

$$22.5.39 \quad L_n^{(1/2)}(x) = \frac{(-1)^n}{n! 2^{2n+1} \sqrt{x}} H_{2n+1}(\sqrt{x})$$

Hermite Polynomials

$$22.5.40 \quad H_{2m}(x) = (-1)^m 2^{2m} m! L_m^{(-1/2)}(x^2)$$

$$22.5.41 \quad H_{2m+1}(x) = (-1)^m 2^{2m+1} m! x L_m^{(1/2)}(x^2)$$

Orthogonal Polynomials as Hypergeometric Functions (see chapter 15)

$$f_n(x) = {}_dF(a, b; c; g(x))$$

For each of the listed polynomials there are numerous other representations in terms of hypergeometric functions.

	$f_n(x)$	d	a	b	c	$g(x)$
22.5.42	$P_n^{(\alpha, \beta)}(x)$	$\binom{n+\alpha}{n}$	$-n$	$n+\alpha+\beta+1$	$\alpha+1$	$\frac{1-x}{2}$
22.5.43	$P_n^{(\alpha, \beta)}(x)$	$\binom{2n+\alpha+\beta}{n} \left(\frac{x-1}{2}\right)^n$	$-n$	$-n-\alpha$	$-2n-\alpha-\beta$	$\frac{2}{1-x}$
22.5.44	$P_n^{(\alpha, \beta)}(x)$	$\binom{n+\alpha}{n} \left(\frac{1+x}{2}\right)^n$	$-n$	$-n-\beta$	$\alpha+1$	$\frac{x-1}{x+1}$
22.5.45	$P_n^{(\alpha, \beta)}(x)$	$\binom{n+\beta}{n} \left(\frac{x-1}{2}\right)^n$	$-n$	$-n-\alpha$	$\beta+1$	$\frac{x+1}{x-1}$
22.5.46	$C_n^{(\alpha)}(x)$	$\frac{\Gamma(n+2\alpha)}{n! \Gamma(2\alpha)}$	$-n$	$n+2\alpha$	$\alpha+\frac{1}{2}$	$\frac{1-x}{2}$
22.5.47	$T_n(x)$	1	$-n$	n	$\frac{1}{2}$	$\frac{1-x}{2}$
22.5.48	$U_n(x)$	$n+1$	$-n$	$n+2$	$\frac{3}{2}$	$\frac{1-x}{2}$
22.5.49	$P_n(x)$	1	$-n$	$n+1$	1	$\frac{1-x}{2}$
22.5.50	$P_n(x)$	$\binom{2n}{n} \left(\frac{x-1}{2}\right)^n$	$-n$	$-n$	$-2n$	$\frac{2}{1-x}$
22.5.51	$P_n(x)$	$\binom{2n}{n} \left(\frac{x}{2}\right)^n$	$-\frac{n}{2}$	$\frac{1-n}{2}$	$\frac{1}{2}-n$	$\frac{1}{x^2}$
22.5.52	$P_{2n}(x)$	$(-1)^n \frac{(2n)!}{2^{2n} (n!)^2}$	$-n$	$n+\frac{1}{2}$	$\frac{1}{2}$	x^2
22.5.53	$P_{2n+1}(x)$	$(-1)^n \frac{(2n+1)!}{2^{2n} (n!)^2} x$	$-n$	$n+\frac{3}{2}$	$\frac{3}{2}$	x^2

*See page II.

Orthogonal Polynomials as Confluent Hypergeometric Functions (see chapter 13)

$$22.5.54 \quad L_n^{(\alpha)}(x) = \binom{n+\alpha}{n} M(-n, \alpha+1, x)$$

Orthogonal Polynomials as Parabolic Cylinder Functions (see chapter 19)

$$22.5.55 \quad H_n(x) = 2^n U\left(\frac{1}{2} - \frac{1}{2}n, \frac{3}{2}, x^2\right)$$

$$22.5.56 \quad H_{2m}(x) = (-1)^m \frac{(2m)!}{m!} M\left(-m, \frac{1}{2}, x^2\right)$$

22.5.57

$$* H_{2m+1}(x) = (-1)^m \frac{(2m+1)!}{m!} 2x M\left(-m, \frac{3}{2}, x^2\right)$$

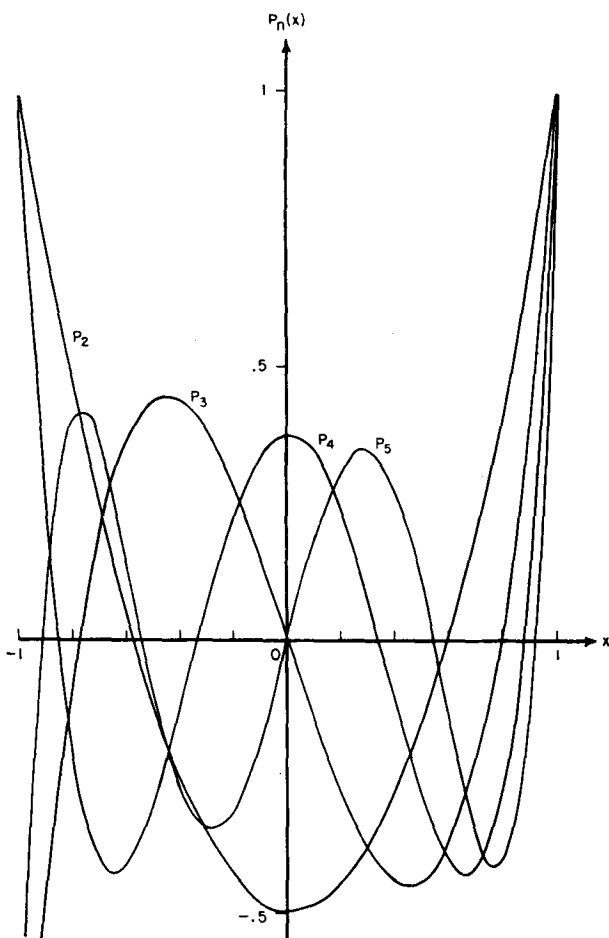


FIGURE 22.8. Legendre Polynomials $P_n(x)$, $n=2(1)5$.

*See page II.

22.5.58

$$H_n(x) = 2^{n/2} e^{x^2/2} D_n(\sqrt{2}x) = 2^{n/2} e^{x^2/2} U\left(-n - \frac{1}{2}, \sqrt{2}x\right)$$

$$22.5.59 \quad He_n(x) = e^{x^2/4} D_n(x) = e^{x^2/4} U\left(-n - \frac{1}{2}, x\right)$$

Orthogonal Polynomials as Legendre Functions (see chapter 8)

22.5.60

$$C_n^{(\alpha)}(x) =$$

$$\frac{\Gamma(\alpha + \frac{1}{2}) \Gamma(2\alpha + n)}{n! \Gamma(2\alpha)} \left[\frac{1}{4} (x^2 - 1) \right]^{1-\frac{\alpha}{2}} P_{n+\alpha-\frac{1}{2}}^{(1-\alpha)}(x) \quad (\alpha \neq 0)$$

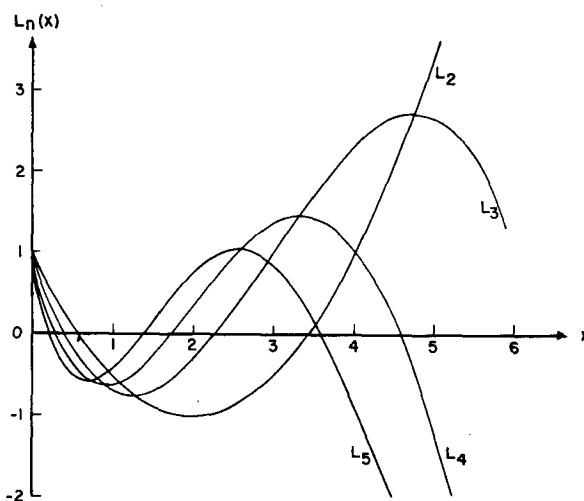


FIGURE 22.9. Laguerre Polynomials $L_n(x)$, $n=2(1)5$.

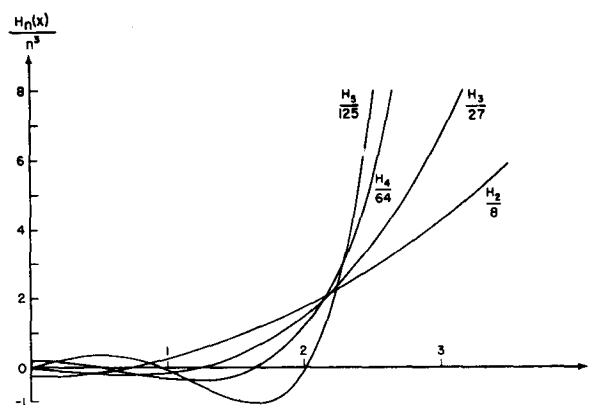


FIGURE 22.10. Hermite Polynomials $\frac{H_n(x)}{n!}$, $n=2(1)5$.

22.6. Differential Equations

$$g_2(x)y'' + g_1(x)y' + g_0(x)y = 0$$

	y	$g_2(x)$	$g_1(x)$	$g_0(x)$
22.6.1	$P_n^{(\alpha, \beta)}(x)$	$1-x^2$	$\beta - \alpha - (\alpha + \beta + 2)x$	$n(n + \alpha + \beta + 1)$
22.6.2	$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha, \beta)}(x)$	$1-x^2$	$\alpha - \beta + (\alpha + \beta - 2)x$	$(n+1)(n + \alpha + \beta)$
22.6.3	$(1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta+1}{2}} P_n^{(\alpha, \beta)}(x)$	1	0	$\frac{1}{4} \frac{1-\alpha^2}{(1-x)^2} + \frac{1}{4} \frac{1-\beta^2}{(1+x)^2}$ $+ \frac{2n(n + \alpha + \beta + 1) + (\alpha+1)(\beta+1)}{2(1-x^2)}$
22.6.4	$\left(\sin \frac{x}{2}\right)^{\alpha+\frac{1}{2}} \left(\cos \frac{x}{2}\right)^{\beta+\frac{1}{2}} P_n^{(\alpha, \beta)}(\cos x)$	1	0	$\frac{1-4\alpha^2}{16 \sin^2 \frac{x}{2}} + \frac{1-4\beta^2}{16 \cos^2 \frac{x}{2}}$ $+ \left(n + \frac{\alpha + \beta + 1}{2}\right)^2$
22.6.5	$C_n^{(\alpha)}(x)$	$1-x^2$	$-(2\alpha+1)x$	$n(n+2\alpha)$
22.6.6	$(1-x^2)^{\alpha-\frac{1}{2}} C_n^{(\alpha)}(x)$	$1-x^2$	$(2\alpha-3)x$	$(n+1)(n+2\alpha-1)$
22.6.7	$(1-x^2)^{\frac{\alpha}{2}+\frac{1}{4}} C_n^{(\alpha)}(x)$	1	0	$\frac{(n+\alpha)^2}{1-x^2} + \frac{2+4\alpha-4\alpha^2+x^2}{4(1-x^2)^2}$
22.6.8	$(\sin x)^\alpha C_n^{(\alpha)}(\cos x)$	1	0	$(n+\alpha)^2 + \frac{\alpha(1-\alpha)}{\sin^2 x}$
22.6.9	$T_n(x)$	$1-x^2$	$-x$	n^2
22.6.10	$T_n(\cos x)$	1	0	n^2
22.6.11	$\frac{1}{\sqrt{1-x^2}} T_n(x); U_{n-1}(x)$	$1-x^2$	$-3x$	n^2-1
22.6.12	$U_n(x)$	$1-x^2$	$-3x$	$n(n+2)$
22.6.13	$P_n(x)$	$1-x^2$	$-2x$	$n(n+1)$
22.6.14	$\sqrt{1-x^2} P_n(x)$	1	0	$\frac{n(n+1)}{1-x^2} + \frac{1}{(1-x^2)^2}$
22.6.15	$L_n^{(\alpha)}(x)$	x	$\alpha+1-x$	n
22.6.16	$e^{-x} x^{\alpha/2} L_n^{(\alpha)}(x)$	x	$\alpha+1$	$n + \frac{\alpha}{2} + 1 - \frac{\alpha^2}{4x}$
22.6.17	$e^{-x/2} x^{(\alpha+1)/2} L_n^{(\alpha)}(x)$	1	0	$\frac{2n+\alpha+1}{2x} + \frac{1-\alpha^2}{4x^2} - \frac{1}{4}$
22.6.18	$e^{-x^2/2} x^{\alpha+\frac{1}{2}} L_n^{(\alpha)}(x^2)$	1	0	$4n+2\alpha+2-x^2 + \frac{1-4\alpha^2}{4x^2}$
22.6.19	$H_n(x)$	1	$-2x$	$2n$
22.6.20	$e^{-\frac{x^2}{2}} H_n(x)$	1	0	$2n+1-x^2$
22.6.21	$He_n(x)$	1	$-x$	n

*See page II.

22.7. Recurrence Relations

Recurrence Relations With Respect to the Degree n

$$a_{1n}f_{n+1}(x) = (a_{2n} + a_{3n}x)f_n(x) - a_{4n}f_{n-1}(x)$$

	f_n	a_{1n}	a_{2n}	a_{3n}	a_{4n}
22.7.1	$P_n^{(\alpha, \beta)}(x)$	$2(n+1)(n+\alpha+\beta+1)$ $(2n+\alpha+\beta)$	$(2n+\alpha+\beta+1)(\alpha^2-\beta^2)$	$(2n+\alpha+\beta)_3$	$2(n+\alpha)(n+\beta)$ $(2n+\alpha+\beta+2)$
22.7.2	$G_n(p, q, x)$	$(2n+p-2)_4(2n+p-1)$	$-[2n(n+p)+q(p-1)]$ $(2n+p-2)_3$	$(2n+p-2)_4$ $(2n+p-1)$	$n(n+q-1)(n+p-1)$ $(n+p-q)(2n+p+1)$
22.7.3	$C_n^{(\alpha)}(x)$	$n+1$	0	$2(n+\alpha)$	$n+2\alpha-1$
22.7.4	$T_n(x)$	1	0	2	1
22.7.5	$U_n(x)$	1	0	2	1
22.7.6	$S_n(x)$	1	0	1	1
22.7.7	$C_n(x)$	1	0	1	1
22.7.8	$T_n^*(x)$	1	-2	4	1
22.7.9	$U_n^*(x)$	1	-2	4	1
22.7.10	$P_n(x)$	$n+1$	0	$2n+1$	n
22.7.11	$P_n^*(x)$	$n+1$	$-2n-1$	$4n+2$	n
22.7.12	$L_n^{(\alpha)}(x)$	$n+1$	$2n+\alpha+1$	-1	$n+\alpha$
22.7.13	$H_n(x)$	1	0	2	$2n$
22.7.14	$He_n(x)$	1	0	1	n

Miscellaneous Recurrence Relations

Jacobi Polynomials

22.7.15

$$\left(n + \frac{\alpha}{2} + \frac{\beta}{2} + 1\right) (1-x)P_n^{(\alpha+1, \beta)}(x) \\ = (n+\alpha+1)P_n^{(\alpha, \beta)}(x) - (n+1)P_{n+1}^{(\alpha, \beta)}(x)$$

22.7.16

$$\left(n + \frac{\alpha}{2} + \frac{\beta}{2} + 1\right) (1+x)P_n^{(\alpha, \beta+1)}(x) \\ = (n+\beta+1)P_n^{(\alpha, \beta)}(x) + (n+1)P_{n+1}^{(\alpha, \beta)}(x)$$

22.7.17

$$(1-x)P_n^{(\alpha+1, \beta)}(x) + (1+x)P_n^{(\alpha, \beta+1)}(x) = 2P_n^{(\alpha, \beta)}(x)$$

22.7.18

$$(2n+\alpha+\beta)P_n^{(\alpha-1, \beta)}(x) = (n+\alpha+\beta)P_n^{(\alpha, \beta)}(x) \\ - (n+\beta)P_{n-1}^{(\alpha, \beta)}(x)$$

22.7.19

$$(2n+\alpha+\beta)P_n^{(\alpha, \beta-1)}(x) = (n+\alpha+\beta)P_n^{(\alpha, \beta)}(x) \\ + (n+\alpha)P_{n-1}^{(\alpha, \beta)}(x)$$

$$22.7.20 \quad P_n^{(\alpha, \beta-1)}(x) - P_n^{(\alpha-1, \beta)}(x) = P_{n-1}^{(\alpha, \beta)}(x)$$

Ultraspherical Polynomials

22.7.21

$$2\alpha(1-x^2)C_{n-1}^{(\alpha+1)}(x) = (2\alpha+n-1)C_{n-1}^{(\alpha)}(x) - nx C_n^{(\alpha)}(x)$$

22.7.22

$$= (n+2\alpha)x C_n^{(\alpha)}(x) \\ - (n+1)C_{n+1}^{(\alpha)}(x)$$

$$22.7.23 \quad (n+\alpha)C_{n+1}^{(\alpha-1)}(x) = (\alpha-1)[C_{n+1}^{(\alpha)}(x) - C_{n-1}^{(\alpha)}(x)]$$

Chebyshev Polynomials

22.7.24

$$2T_m(x)T_n(x) = T_{n+m}(x) + T_{n-m}(x) \quad (n \geq m) \quad *$$

22.7.25

$$2(x^2-1)U_{m-1}(x)U_{n-1}(x) = T_{n+m}(x) - T_{n-m}(x) \quad (n \geq m)$$

22.7.26

$$2T_m(x)U_{n-1}(x) = U_{n+m-1}(x) + U_{n-m-1}(x) \quad (n > m)$$

22.7.27

$$2T_n(x)U_{m-1}(x) = U_{n+m-1}(x) - U_{n-m-1}(x) \quad (n > m)$$

$$22.7.28 \quad 2T_n(x)U_{n-1}(x) = U_{2n-1}(x)$$

*See page II.

Generalized Laguerre Polynomials

22.7.29

$$L_n^{(\alpha+1)}(x) = \frac{1}{x} [(x-n)L_n^{(\alpha)}(x) + (\alpha+n)L_{n-1}^{(\alpha)}(x)]$$

22.7.30

$$L_n^{(\alpha-1)}(x) = L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x)$$

22.7.31

$$L_n^{(\alpha+1)}(x) = \frac{1}{x} [(n+\alpha+1)L_n^{(\alpha)}(x) - (n+1)L_{n+1}^{(\alpha)}(x)]$$

22.7.32

$$L_n^{(\alpha-1)}(x) = \frac{1}{n+\alpha} [(n+1)L_{n+1}^{(\alpha)}(x) - (n+1-x)L_n^{(\alpha)}(x)]$$

22.8. Differential Relations

$$g_2(x) \frac{d}{dx} f_n(x) = g_1(x) f_n(x) + g_0(x) f_{n-1}(x)$$

	f_n	g_2	g_1	g_0
22.8.1	$P_n^{(\alpha, \beta)}(x)$	$(2n+\alpha+\beta)(1-x^2)$	$n[\alpha-\beta-(2n+\alpha+\beta)x]$	$2(n+\alpha)(n+\beta)$
22.8.2	$C_n^{(\alpha)}(x)$	$1-x^2$	$-nx$	$n+2\alpha-1$
22.8.3	$T_n(x)$	$1-x^2$	$-nx$	n
22.8.4	$U_n(x)$	$1-x^2$	$-nx$	$n+1$
22.8.5	$P_n(x)$	$1-x^2$	$-nx$	n
22.8.6	$L_n^{(\alpha)}(x)$	x	n	$-(n+\alpha)$
22.8.7	$H_n(x)$	1	0	$2n$
22.8.8	$He_n(x)$	1	0	n

22.9. Generating Functions

$$g(x, z) = \sum_{n=0}^{\infty} a_n f_n(x) z^n$$

$$R = \sqrt{1-2xz+z^2}$$

	$f_n(x)$	a_n	$g(x, z)$	Remarks
22.9.1	$P_n^{(\alpha, \beta)}(x)$	$2^{-\alpha-\beta}$	$R^{-1}(1-z+R)^{-\alpha}(1+z+R)^{-\beta}$	$ z < 1$
22.9.2	$C_n^{(\alpha)}(x)$	$\frac{2^{1-\alpha} \Gamma(\alpha + \frac{1}{2} + n) \Gamma(2\alpha)}{\Gamma(\alpha + \frac{1}{2}) \Gamma(2\alpha + n)}$	$R^{-1}(1-xz+R)^{1-\alpha}$	$ z < 1, \alpha \neq 0$
22.9.3	$C_n^{(\alpha)}(x)$	1	$R^{-2\alpha}$	$ z < 1, \alpha \neq 0$
22.9.4	$C_n^{(0)}(x)$	1	$-\ln R^2$	$ z < 1$
22.9.5	$C_n^{(\alpha)}(x)$	$\frac{\Gamma(2\alpha)}{\Gamma(\alpha + \frac{1}{2}) \Gamma(2\alpha + n)}$	$e^{z \cos \theta} \left(\frac{z}{2} \sin \theta \right)^{1-\alpha} J_{\alpha-1/2}(z \sin \theta)$	$x = \cos \theta$
22.9.6	$T_n(x)$	2	$\left(\frac{1-z^2}{R^2} + 1 \right)$	$-1 < x < 1, z < 1$
22.9.7	$T_n(x)$	$\frac{\sqrt{2}}{4^n} \binom{2n}{n}$	$R^{-1}(1-xz+R)^{1/2}$	$-1 < x < 1, z < 1$
22.9.8	$T_n(x)$	$\frac{1}{n}$	$1 - \frac{1}{2} \ln R^2$	$a_0 = 1, -1 < x < 1, z < 1$
22.9.9	$T_n(x)$	1	$\frac{1-xz}{R^2}$	$-1 < x < 1, z < 1$
22.9.10	$U_n(x)$	1	R^{-2}	$-1 < x < 1, z < 1$
22.9.11	$U_n(x)$	$\frac{\sqrt{2}}{4^{n+1}} \binom{2n+2}{n+1}$	$\frac{1}{R} (1-xz+R)^{-1/2}$	$-1 < x < 1, z < 1$

*See page II.

22.9. Generating Functions—Continued

$$g(x, z) = \sum_{n=0}^{\infty} a_n f_n(x) z^n \quad R = \sqrt{1 - 2xz + z^2}$$

	$f_n(x)$	a_n	$g(x, z)$	Remarks
22.9.12	$P_n(x)$	1	R^{-1}	$-1 < x < 1$ $ z < 1$
22.9.13	$P_n(x)$	$\frac{1}{n!}$	$e^x \cos {}^0J_0(z \sin \theta)$	$x = \cos \theta$
22.9.14	$S_n(x)$	1	$(1 - xz + z^2)^{-1}$	$-2 < x < 2$ $ z < 1$
22.9.15	$L_n^{(\alpha)}(x)$	1	$(1 - z)^{-\alpha-1} \exp\left(\frac{xz}{z-1}\right)$	$ z < 1$
22.9.16	$L_n^{(\alpha)}(x)$	$\frac{1}{\Gamma(n+\alpha+1)}$	$(xz)^{-\frac{1}{2}\alpha} e^x J_\alpha[2(xz)^{1/2}]$	
22.9.17	$H_n(x)$	$\frac{1}{n!}$	$e^{x^2 - z^2}$	
22.9.18	$H_{2n}(x)$	$\frac{(-1)^n}{(2n)!}$	$e^z \cos(2x\sqrt{z})$ *	
22.9.19	$H_{2n+1}(x)$	$\frac{(-1)^n}{(2n+1)!}$	$z^{-1/2} e^z \sin(2x\sqrt{z})$ *	

22.10. Integral Representations

Contour Integral Representations

$f_n(x) = \frac{g_0(x)}{2\pi i} \int_C [g_1(z, x)]^n g_2(z, x) dz$ where C is a closed contour taken around $z=a$ in the positive sense

	$f_n(x)$	$g_0(x)$	$g_1(z, x)$	$g_2(z, x)$	a	Remarks
22.10.1	$P_n^{(\alpha, \beta)}(x)$	$\frac{1}{(1-x)^\alpha (1+x)^\beta}$	$\frac{z^2-1}{2(z-x)}$	$\frac{(1-z)^\alpha (1+z)^\beta}{z-x}$	x	± 1 outside C
22.10.2	$C_n^{(\alpha)}(x)$	1	$1/z$	$(1-2xz+z^2)^{-\alpha} z^{-1}$	0	Both zeros of $1-2xz+z^2$ outside C , $\alpha > 0$
22.10.3	$T_n(x)$	$1/2$	$1/z$	$\frac{1-z^2}{z(1-2xz+z^2)}$	0	Both zeros of $1-2xz+z^2$ outside C
22.10.4	$U_n(x)$	1	$1/z$	$\frac{1}{z(1-2xz+z^2)}$	0	Both zeros of $1-2xz+z^2$ outside C
22.10.5	$P_n(x)$	1	$1/z$	$\frac{1}{z} (1-2xz+z^2)^{-1/2}$	0	Both zeros of $1-2xz+z^2$ outside C
22.10.6	$P_n(x)$	$\frac{1}{2^n}$	$\frac{z^2-1}{z-x}$	$\frac{1}{z-x}$	x	
22.10.7	$L_n^{(\alpha)}(x)$	e^{xz-x^α}	$\frac{z}{z-x}$	$\frac{z^\alpha}{z-x} e^{-x}$	x	Zero outside C
22.10.8	$L_n^{(\alpha)}(x)$	1	$1 + \frac{x}{z}$	$e^{-x} \left(1 + \frac{x}{z}\right)^\alpha 1/z$	0	$z = -x$ outside C
22.10.9	$H_n(x)$	$n!$	$1/z$	$\frac{e^{x^2-z^2}}{z}$	0	

Miscellaneous Integral Representations

$$22.10.10 \quad C_n^{(\alpha)}(x) = \frac{2^{(1-2\alpha)} \Gamma(n+2\alpha)}{n! [\Gamma(\alpha)]^2} \int_0^\pi [x + \sqrt{x^2-1} \cos \phi]^n (\sin \phi)^{2\alpha-1} d\phi \quad (\alpha > 0)$$

$$22.10.11 \quad C_n^{(\alpha)}(\cos \theta) = \frac{2^{1-\alpha} \Gamma(n+2\alpha)}{n! [\Gamma(\alpha)]^2} (\sin \theta)^{1-2\alpha} \int_0^\theta \frac{\cos(n+\alpha)\phi}{(\cos \phi - \cos \theta)^{1-\alpha}} d\phi \quad (\alpha > 0)$$

*See page II.

$$22.10.12 \quad P_n(\cos \theta) = \frac{1}{\pi} \int_0^\pi (\cos \theta + i \sin \theta \cos \phi)^n d\phi$$

$$22.10.13 \quad P_n(\cos \theta) = \frac{\sqrt{2}}{\pi} \int_0^\pi \frac{\sin(n + \frac{1}{2})\phi d\phi}{(\cos \theta - \cos \phi)^{\frac{1}{2}}}$$

$$22.10.14 \quad L_n^{(\alpha)}(x) = \frac{e^x x^{-\frac{\alpha}{2}}}{n!} \int_0^\infty e^{-t} t^{n+\frac{\alpha}{2}} J_\alpha(2\sqrt{tx}) dt$$

22.10.15

$$H_n(x) = e^{x^2} \frac{2^{n+1}}{\sqrt{\pi}} \int_0^\infty e^{-t^2} t^n \cos\left(2xt - \frac{n}{2}\pi\right) dt$$

22.11. Rodrigues' Formula

$$f_n(x) = \frac{1}{a_n \rho(x)} \frac{d^n}{dx^n} \{ \rho(x) (g(x))^n \}$$

The polynomials given in the following table are the only orthogonal polynomials which satisfy this formula.

	$f_n(x)$	a_n	$\rho(x)$	$g(x)$
22.11.1	$P_n^{(\alpha, \beta)}(x)$	$(-1)^n 2^n n!$	$(1-x)^\alpha (1+x)^\beta$	$1-x^2$
22.11.2	$C_n^{(\alpha)}(x)$	$(-1)^n 2^n n! \frac{\Gamma(2\alpha)\Gamma(\alpha+n+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})\Gamma(n+2\alpha)}$	$(1-x^2)^{\alpha-\frac{1}{2}}$	$1-x^2$
22.11.3	$T_n(x)$	$(-1)^n 2^n \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}$	$(1-x^2)^{-\frac{1}{2}}$	$1-x^2$
22.11.4	$U_n(x)$	$(-1)^n 2^{n+1} \frac{\Gamma(n+\frac{1}{2})}{(n+1)\sqrt{\pi}}$	$(1-x^2)^{\frac{1}{2}}$	$1-x^2$
22.11.5	$P_n(x)$	$(-1)^n 2^n n!$	1	$1-x^2$
22.11.6	$L_n^{(\alpha)}(x)$	$n!$	$e^{-x} x^\alpha$	x
22.11.7	$H_n(x)$	$(-1)^n$	e^{-x^2}	1
22.11.8	$He_n(x)$	$(-1)^n$	$e^{-x^2/2}$	1

22.12. Sum Formulas

Christoffel-Darboux Formula

22.12.1

$$\sum_{m=0}^n \frac{1}{h_m} f_m(x) f_m(y) = \frac{k_n}{k_{n+1} h_n} \frac{f_{n+1}(x) f_n(y) - f_n(x) f_{n+1}(y)}{x-y}$$

Miscellaneous Sum Formulas (Only a Limited Selection Is Given Here.)

$$22.12.2 \quad \sum_{m=0}^n T_{2m}(x) = \frac{1}{2} [1 + U_{2n}(x)]$$

$$22.12.3 \quad \sum_{m=0}^{n-1} T_{2m+1}(x) = \frac{1}{2} U_{2n-1}(x)$$

$$22.12.4 \quad \sum_{m=0}^n U_{2m}(x) = \frac{1 - T_{2n+2}(x)}{2(1-x^2)}$$

$$22.12.5 \quad \sum_{m=0}^{n-1} U_{2m+1}(x) = \frac{x - T_{2n+1}(x)}{2(1-x^2)}$$

$$22.12.6 \quad \sum_{m=0}^n L_m^{(\alpha)}(x) L_n^{(\beta)}(y) = L_n^{(\alpha+\beta+1)}(x+y)$$

$$22.12.7 \quad \sum_{m=0}^n \binom{n+\alpha}{m} \mu^{n-m} (1-\mu)^m L_n^{(\alpha)}(x) = L_n^{(\alpha)}(\mu x)$$

22.12.8

$$H_n(x+y) = \frac{1}{2^{n/2}} \sum_{k=0}^n \binom{n}{k} H_k(\sqrt{2}x) H_{n-k}(\sqrt{2}y)$$

22.13. Integrals Involving Orthogonal Polynomials

22.13.1

$$2n \int_0^x (1-y)^\alpha (1+y)^\beta P_n^{(\alpha, \beta)}(y) dy = P_{n-1}^{(\alpha+1, \beta+1)}(0) - (1-x)^{\alpha+1} (1+x)^{\beta+1} P_{n-1}^{(\alpha+1, \beta+1)}(x)$$

22.13.2

$$\frac{n(2\alpha+n)}{2\alpha} \int_0^x (1-y^2)^{\alpha-\frac{1}{2}} C_n^{(\alpha)}(y) dy = C_{n-1}^{(\alpha+1)}(0) - (1-x^2)^{\alpha+\frac{1}{2}} C_{n-1}^{(\alpha+1)}(x)$$

22.13.3

$$\int_{-1}^1 \frac{T_n(y) dy}{(y-x)\sqrt{1-y^2}} = \pi U_{n-1}(x)$$

22.13.4

$$\int_{-1}^1 \frac{\sqrt{1-y^2} U_{n-1}(y) dy}{(y-x)} = -\pi T_n(x) \quad *$$

22.13.5

$$\int_{-1}^1 (1-x)^{-1/2} P_n(x) dx = \frac{2^{3/2}}{2n+1} \quad *$$

22.13.6

$$\int_0^\pi P_{2n}(\cos \theta) d\theta = \frac{\pi}{16^n} \binom{2n}{n}^2$$

22.13.7

$$\int_0^\pi P_{2n+1}(\cos \theta) \cos \theta d\theta = \frac{\pi}{4^{2n+1}} \binom{2n}{n} \binom{2n+2}{n+1}$$

*See page 11.

22.13.8

$$\int_0^1 x^\lambda P_{2n}(x) dx = \frac{(-1)^n \Gamma\left(n - \frac{\lambda}{2}\right) \Gamma\left(\frac{1}{2} + \frac{\lambda}{2}\right)}{2\Gamma\left(-\frac{\lambda}{2}\right) \Gamma\left(n + \frac{3}{2} + \frac{\lambda}{2}\right)} \quad (\lambda > -1)$$

22.13.9

$$\int_0^1 x^\lambda P_{2n+1}(x) dx = \frac{(-1)^n \Gamma\left(n + \frac{1}{2} - \frac{\lambda}{2}\right) \Gamma\left(1 + \frac{\lambda}{2}\right)}{2\Gamma\left(n + 2 + \frac{\lambda}{2}\right) \Gamma\left(\frac{1}{2} - \frac{\lambda}{2}\right)} \quad (\lambda > -2)$$

22.13.10

$$\int_{-1}^x \frac{P_n(t) dt}{\sqrt{x-t}} = \frac{1}{(n+\frac{1}{2})\sqrt{1+x}} [T_n(x) + T_{n+1}(x)]$$

22.13.11

$$\int_x^1 \frac{P_n(t) dt}{\sqrt{t-x}} = \frac{1}{(n+\frac{1}{2})\sqrt{1-x}} [T_n(x) - T_{n+1}(x)]$$

$$22.13.12 \quad \int_x^\infty e^{-t} L_n^{(\alpha)}(t) dt = e^{-x} [L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x)]$$

22.13.13

$$\Gamma(\alpha + \beta + n + 1) \int_0^x (x-t)^{\beta-1} t^\alpha L_n^{(\alpha)}(t) dt = \Gamma(\alpha + n + 1) \Gamma(\beta) x^{\alpha+\beta} L_n^{(\alpha+\beta)}(x) \quad (\Re \alpha > -1, \Re \beta > 0)$$

22.13.14

$$\int_0^x L_m(t) L_n(x-t) dt = \int_0^x L_{m+n}(t) dt = L_{m+n}(x) - L_{m+n+1}(x)$$

$$22.13.15 \quad \int_0^x e^{-t^2} H_n(t) dt = H_{n-1}(0) - e^{-x^2} H_{n-1}(x)$$

$$22.13.16 \quad \int_0^x H_n(t) dt = \frac{1}{2(n+1)} [H_{n+1}(x) - H_{n+1}(0)]$$

$$22.13.17 \quad \int_{-\infty}^\infty e^{-t^2} H_{2m}(tx) dt = \sqrt{\pi} \frac{(2m)!}{m!} (x^2 - 1)^m$$

22.13.18

$$\int_{-\infty}^\infty e^{-t^2} t H_{2m+1}(tx) dt = \sqrt{\pi} \frac{(2m+1)!}{m!} x(x^2 - 1)^m$$

$$22.13.19 \quad \int_{-\infty}^\infty e^{-t^2} t^n H_n(xt) dt = \sqrt{\pi} n! P_n(x)$$

22.13.20

$$\int_0^\infty e^{-t^2} [H_n(t)]^2 \cos(xt) dt = \sqrt{\pi} 2^{n-1} n! e^{-\frac{1}{2}x^2} L_n\left(\frac{x^2}{2}\right)$$

22.14. Inequalities

22.14.1

$$|P_n^{(\alpha, \beta)}(x)| \leq \begin{cases} \binom{n+q}{n} \approx n^q, & \text{if } q = \max(\alpha, \beta) \geq -1/2 \\ & (\alpha > -1, \beta > -1) \\ |P_n^{(\alpha, \beta)}(x')| \approx \sqrt{\frac{1}{n}}, & \text{if } q < -\frac{1}{2} \end{cases}$$

x' maximum point nearest to $\frac{\beta - \alpha}{\alpha + \beta + 1}$

22.14.2

$$|C_n^{(\alpha)}(x)| \leq \begin{cases} \binom{n+2\alpha-1}{n} & (\alpha > 0) \\ |C_n^{(\alpha)}(x')| & (-\frac{1}{2} < \alpha < 0) \end{cases}$$

$x' = 0$ if $n = 2m$; $x' =$ maximum point nearest zero if $n = 2m + 1$

22.14.3

$$|C_n^{(\alpha)}(\cos \theta)| < 2^{1-\alpha} \frac{n^{\alpha-1}}{(\sin \theta)^\alpha \Gamma(\alpha)} \quad (0 < \alpha < 1, 0 < \theta < \pi)$$

$$22.14.4 \quad |T_n(x)| \leq 1 \quad (-1 \leq x \leq 1)$$

$$22.14.5 \quad \left| \frac{dT_n(x)}{dx} \right| \leq n^2 \quad (-1 \leq x \leq 1)$$

$$22.14.6 \quad |U_n(x)| \leq n+1 \quad (-1 \leq x \leq 1)$$

$$22.14.7 \quad |P_n(x)| \leq 1 \quad (-1 \leq x \leq 1)$$

$$22.14.8 \quad \left| \frac{dP_n(x)}{dx} \right| \leq \frac{1}{2} n(n+1) \quad (-1 \leq x \leq 1)$$

$$22.14.9 \quad |P_n(x)| \leq \sqrt{\frac{2}{\pi n}} \frac{1}{4\sqrt{1-x^2}} \quad (-1 < x \leq 1)$$

22.14.10

$$P_n^2(x) - P_{n-1}(x)P_{n+1}(x) < \frac{2n+1}{3n(n+1)} \quad (-1 \leq x \leq 1)$$

22.14.11

$$P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \geq \frac{1 - P_n^2(x)}{(2n-1)(n+1)} \quad (-1 \leq x \leq 1)$$

$$22.14.12 \quad |L_n(x)| \leq e^{x/2} \quad (x \geq 0)$$

$$22.14.13 \quad |L_n^{(\alpha)}(x)| \leq \frac{\Gamma(\alpha + n + 1)}{n! \Gamma(\alpha + 1)} e^{x/2} \quad (\alpha \geq 0, x \geq 0)$$

22.14.14

$$|L_n^{(\alpha)}(x)| \leq \left[2 - \frac{\Gamma(\alpha + n + 1)}{n! \Gamma(\alpha + 1)} \right] e^{x/2} \quad (-1 < \alpha < 0, x \geq 0)$$

$$22.14.15 \quad |H_{2m}(x)| \leq e^{x^2/2} 2^{2m} m! \left[2 - \frac{1}{2^{2m}} \binom{2m}{m} \right]$$

$$22.14.16 \quad |H_{2m+1}(x)| \leq x e^{x^2/2} \frac{(2m+2)!}{(m+1)!} \quad (x \geq 0)$$

$$22.14.17 \quad |H_n(x)| < e^{x^2/2} k 2^{n/2} \sqrt{n!} \quad k \approx 1.086435$$

22.15. Limit Relations

22.15.1

$$\lim_{n \rightarrow \infty} \left[\frac{1}{n^\alpha} P_n^{(\alpha, \beta)} \left(\cos \frac{x}{n} \right) \right] \\ = \lim_{n \rightarrow \infty} \frac{1}{n^\alpha} P_n^{(\alpha, \beta)} \left(1 - \frac{x^2}{2n^2} \right) = \left(\frac{2}{x} \right)^\alpha J_\alpha(x)$$

$$22.15.2 \quad \lim_{n \rightarrow \infty} \left[\frac{1}{n^\alpha} L_n^{(\alpha)} \left(\frac{x}{n} \right) \right] = x^{-\alpha/2} J_\alpha(2\sqrt{x})$$

$$22.15.3 \quad \lim_{n \rightarrow \infty} \left[\frac{(-1)^n \sqrt{n}}{4^n n!} H_{2n} \left(\frac{x}{2\sqrt{n}} \right) \right] = \frac{1}{\sqrt{\pi}} \cos x$$

$$22.15.4 \quad \lim_{n \rightarrow \infty} \left[\frac{(-1)^n}{4^n n!} H_{2n+1} \left(\frac{x}{2\sqrt{n}} \right) \right] = \frac{2}{\sqrt{\pi}} \sin x$$

$$22.15.5 \quad \lim_{\beta \rightarrow \infty} P_n^{(\alpha, \beta)} \left(1 - \frac{2x}{\beta} \right) = L_n^{(\alpha)}(x)$$

$$22.15.6 \quad \lim_{\alpha \rightarrow \infty} \frac{1}{\alpha^{n/2}} C_n^{(\alpha)} \left(\frac{x}{\sqrt{\alpha}} \right) = \frac{1}{n!} H_n(x)$$

For asymptotic expansions, see [22.5] and [22.17].

22.16. Zeros

For tables of the zeros and associated weight factors necessary for the Gaussian-type quadrature formulas see chapter 25. All the zeros of the orthogonal polynomials are real, simple and located in the interior of the interval of orthogonality.

Explicit and Asymptotic Formulas and Inequalities

Notations:

$$x_m^{(n)} \text{ mth zero of } f_n(x) (x_1^{(n)} < x_2^{(n)} < \dots < x_n^{(n)}) \quad \left| \quad j_{\alpha, m}, \text{ mth positive zero of the Bessel function } J_\alpha(x) \right. \\ \theta_m^{(n)} = \arccos x_{n-m+1}^{(n)} (0 < \theta_1^{(n)} < \theta_2^{(n)} < \dots < \theta_n^{(n)} < \pi) \quad \left| \quad 0 < j_{\alpha, 1} < j_{\alpha, 2} < \dots \right.$$

	$f_n(x)$	Relation
22.16.1	$P_n^{(\alpha, \beta)}(\cos \theta)$	$\lim_{n \rightarrow \infty} n \theta_m^{(n)} = j_{\alpha, m} \quad (\alpha > -1, \beta > -1)$
22.16.2	$C_n^{(\alpha)}(x)$	$x_m^{(n)} = 1 - \frac{j_{\alpha-1/2, m}^2}{2n^2} \left[1 - \frac{2\alpha}{n} + O\left(\frac{1}{n^2}\right) \right]$
22.16.3	$C_n^{(\alpha)}(\cos \theta)$	$\frac{(m+\alpha-1)\pi}{n+\alpha} \leq \theta_m^{(n)} \leq \frac{m\pi}{n+\alpha} \quad (0 \leq \alpha \leq 1)$
22.16.4	$T_n(x)$	$x_m^{(n)} = \cos \frac{2m-1}{2n} \pi$
22.16.5	$U_n(x)$	$x_m^{(n)} = \cos \frac{m}{n+1} \pi$
22.16.6	$P_n(\cos \theta)$	$\begin{cases} \frac{2m-1}{2n+1} \pi \leq \theta_m^{(n)} \leq \frac{2m}{2n+1} \pi \\ \theta_m^{(n)} = \frac{4m-1}{4n+2} \pi + \frac{1}{8n^2} \cot \frac{4m-1}{4n+2} \pi + O(n^{-3}) \end{cases}$
22.16.7	$P_n(x)$	$\begin{cases} x_m^{(n)} = 1 - \frac{j_{0, m}^2}{2n^2} \left[1 - \frac{1}{n} + O(n^{-2}) \right] \\ x_m^{(n)} = 1 - \frac{4\xi_m^{(n)}}{2n+1+\xi_m^{(n)}}; \xi_m^{(n)} = \frac{j_{0, m}^2}{4n+2} \left[1 + \frac{j_{0, m-2}^2}{12(2n+1)^2} \right] + O\left(\frac{1}{n^5}\right) \end{cases}$
22.16.8	$L_n^{(\alpha)}(x)$	$\begin{cases} x_m^{(n)} > \frac{j_{\alpha, m}^2}{4k_n} \\ x_m^{(n)} < \frac{k_m}{k_n} (2k_m + \sqrt{4k_m^2 + \frac{1}{4} - \alpha^2}) \\ x_m^{(n)} = \frac{j_{\alpha, m}^2}{4k_n} \left(1 + \frac{2(\alpha^2-1) + j_{\alpha, m}^2}{48k_n^2} \right) + O(n^{-5}) \end{cases} \quad \left. \vphantom{\begin{cases} x_m^{(n)} > \frac{j_{\alpha, m}^2}{4k_n} \\ x_m^{(n)} < \frac{k_m}{k_n} (2k_m + \sqrt{4k_m^2 + \frac{1}{4} - \alpha^2}) \\ x_m^{(n)} = \frac{j_{\alpha, m}^2}{4k_n} \left(1 + \frac{2(\alpha^2-1) + j_{\alpha, m}^2}{48k_n^2} \right) + O(n^{-5}) \end{cases}} \right\} k_r = r + \frac{\alpha+1}{2}$

For error estimates see [22.6].

22.17. Orthogonal Polynomials of a Discrete Variable

In this section some polynomials $f_n(x)$ are listed which are orthogonal with respect to the scalar product

$$22.17.1 \quad (f_n, f_m) = \sum_i w^*(x_i) f_n(x_i) f_m(x_i).$$

The x_i are the integers in the interval $a \leq x_i \leq b$ and $w^*(x_i)$ is a positive function such that

$\sum_i w^*(x_i)$ is finite. The constant factor which is still free in each polynomial when only the orthogonality condition is given is defined here by the explicit representation (which corresponds to the Rodrigues' formula)

$$22.17.2 \quad f_n(x) = \frac{1}{r_n w^*(x)} \Delta^n [w^*(x) g(x, n)]$$

where $g(x, n) = g(x)g(x-1) \dots g(x-n+1)$ and $g(x)$ is a polynomial in x independent of n .

Name	a	b	$w^*(x)$	r_n	$g(x, n)$	Remarks
Chebyshev	0	$N-1$	1	$1/n!$	$\binom{x}{n} \binom{x-N}{n}$	
Krawtchouk	0	N	$p^x q^{N-x} \binom{N}{x}$	$(-1)^n n!$	$\frac{q^n x!}{(x-n)!}$	$p, q > 0;$ $p+q=1$
Charlier	0	∞	$\frac{e^{-a} a^x}{x!}$	$(-1)^n \sqrt{a^n n!}$	$\frac{x!}{(x-n)!}$	$a > 0$
Meixner	0	∞	$\frac{c^x \Gamma(b+x)}{\Gamma(b)x!}$	c^n	$\frac{x!}{(x-n)!}$	$b > 0, 0 < c < 1$
Hahn	0	∞	$\frac{\Gamma(b)\Gamma(c+x)\Gamma(d+x)}{x!\Gamma(b+x)\Gamma(c)\Gamma(d)}$	$n!$	$\frac{x!\Gamma(b+x)}{(x-n)!\Gamma(b+x-n)}$	

For a more complete list of the properties of these polynomials see [22.5] and [22.17].

Numerical Methods

22.18. Use and Extension of the Tables

Evaluation of an orthogonal polynomial for which the coefficients are given numerically.

Example 1. Evaluate $L_6(1.5)$ and its first and second derivative using Table 22.10 and the Horner scheme.

	1	-36	450	-2400	5400	-4320	720
$x=1.5$		1.5	-51.75	597.375	-2703.9375	4044.09375	-413.859375
	1	-34.5	398.25	-1802.625	2696.0625	-275.90625	306.140625
1.5		1.5	-49.5	523.125	-1919.25	1165.21875	$L_6 = \frac{306.140625}{720}$ $= .42519\ 53$
	1	-33.0	348.75	-1279.500	776.8125	889.3125	
1.5		1.5	-47.25	452.250	-1240.875		$L'_6 = \frac{889.3125}{720}$ $= 1.23515\ 625$
	1	-31.5	301.50	-827.250	-464.0625		$L''_6 = 2 \frac{[-464.0625]}{720}$ $= -1.28906\ 25$

Evaluation of an orthogonal polynomial using the explicit representation when the coefficients are not given numerically.

If an isolated value of the orthogonal polynomial $f_n(x)$ is to be computed, use the proper explicit expression rewritten in the form

$$f_n(x) = d_n(x)a_0(x)$$

and generate $a_0(x)$ recursively, where

$$a_{m-1}(x) = 1 - \frac{b_m}{c_m} f(x) a_m(x) \quad (m=n, n-1, \dots, 2, 1, a_n(x)=1).$$

The $d_n(x)$, b_m , c_m , $f(x)$ for the polynomials of this chapter are listed in the following table:

$f_n(x)$	$d_n(x)$	b_m	c_m	$f(x)$
$P_n^{(\alpha, \beta)}$	$\binom{n+\alpha}{n}$	$(n-m+1)(\alpha+\beta+n+m)$	$2m(\alpha+m)$	$1-x$
$C_{2n}^{(\alpha)}$	$(-1)^n \frac{(\alpha)_n}{n!}$	$2(n-m+1)(\alpha+n+m-1)$	$m(2m-1)$	x^2
$C_{2n+1}^{(\alpha)}$	$(-1)^n \frac{(\alpha)_{n+1}}{n!} 2x$	$2(n-m+1)(\alpha+n+m)$	$m(2m+1)$	x^2
T_{2n}	$(-1)^n$	$2(n-m+1)(n+m-1)$	$m(2m-1)$	x^2
T_{2n+1}	$(-1)^n (2n+1)x$	$2(n-m+1)(n+m)$	$m(2m+1)$	x^2
U_{2n}	$(-1)^n$	$2(n-m+1)(n+m)$	$m(2m-1)$	x^2
U_{2n+1}	$(-1)^n 2(n+1)x$	$2(n-m+1)(n+m+1)$	$m(2m+1)$	x^2
P_{2n}	$\frac{(-1)^n}{4^n} \binom{2n}{n}$	$(n-m+1)(2n+2m-1)$	$m(2m-1)$	x^2
P_{2n+1}	$\frac{(-1)^n}{4^n} \binom{2n+1}{n} (n+1)x$	$(n-m+1)(2n+2m+1)$	$m(2m+1)$	x^2
$L_n^{(\alpha)}$	$\binom{n+\alpha}{n}$	$n-m+1$	$m(\alpha+m)$	x
H_{2n}	$(-1)^n \frac{(2n)!}{n!}$	$2(n-m+1)$	$m(2m-1)$	x^2
H_{2n+1}	$(-1)^n \frac{(2n+1)!}{n!} 2x$	$2(n-m+1)$	$m(2m+1)$	x^2

Example 2. Compute $P_8^{(1/2, 3/2)}(2)$. Here $d_8 = \binom{8.5}{8} = 3.33847$, $f(2) = -1$.

m	8	7	6	5	4	3	2	1	0
a_m	1	1.132353	1.366667	1.841026	3.008392	6.849651	26.44156	223.1091	6545.533
b_m	18	34	48	60	70	78	84	88	90
c_m	136	105	78	55	36	21	10	3	0

$$P_8^{(1/2, 3/2)}(2) = d_8 a_0(2) = (3.33847)(6545.533) = 21852.07$$

Evaluation of orthogonal polynomials by means of their recurrence relations

Example 3. Compute $C_n^{(1)}(2.5)$ for $n=2, 3, 4, 5, 6$.

From Table 22.2 $C_0^{(1)}=1$, $C_1^{(1)}=1.25$ and from 22.7 the recurrence relation is

$$C_{n+1}^{(1)}(2.5) = [5(n+\frac{1}{2})C_n^{(1)}(2.5) - (n-\frac{1}{2})C_{n-1}^{(1)}(2.5)] \frac{1}{n+1}.$$

n	2	3	4	5	6
$C_n^{(1)}(2.5)$	3.65625	13.08594	50.87648	207.0649	867.7516

Check: Compute $C_6^{(1)}(2.5)$ by the method of Example 2.

Change of Interval of Orthogonality

In some applications it is more convenient to use polynomials orthogonal on the interval $[0, 1]$. One can obtain the new polynomials from the ones given in this chapter by the substitution $x=2\bar{x}-1$. The coefficients of the new polynomial can be computed from the old by the following recursive scheme, provided the standardization is not changed. If

$$f_n(x) = \sum_{m=0}^n a_m x^m, \quad f_n^*(x) = f_n(2x-1) = \sum_{m=0}^n a_m^* x^m$$

then the a_m^* are given recursively by the a_m through the relations

$$a_m^{(j)} = 2a_m^{(j-1)} - a_{m+1}^{(j-1)}; \quad m=n-1, n-2, \dots, j; \quad j=0, 1, 2, \dots, n$$

$$a_m^{(-1)} = a_m/2, \quad m=0, 1, 2, \dots, n$$

$$a_n^{(j)} = 2^j a_n, \quad j=0, 1, 2, \dots, n \text{ and } a_m^{(m)} = a_m^*; \quad m=0, 1, 2, \dots, n.$$

Example 4. Given $T_5(x) = 5x - 20x^3 + 16x^5$, find $T_5^*(x)$.

$m \backslash j$	5	4	3	2	1	0
-1	$8 = a_5^{(-1)}$	0	$-10 = a_3^{(-1)}$	0	$2.5 = a_1^{(-1)}$	0
0	16	-16	-4	4	1	-1 = a_0^*
1	32	-64	56	-48	$50 = a_1^*$	
2	64	-192	304	-400 = a_2^*		
3	128	-512	1120 = a_3^*			
4	256	-1280 = a_4^*				
5	512 = a_5^*					

Hence, $T_5^*(x) = 512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1$.

22.19. Least Square Approximations

Problem: Given a function $f(x)$ (analytically or in form of a table) in a domain D (which may be a continuous interval or a set of discrete points).² Approximate $f(x)$ by a polynomial $F_n(x)$ of given degree n such that a weighted sum of the squares of the errors in D is least.

Solution: Let $w(x) \geq 0$ be the weight function chosen according to the relative importance of the errors in different parts of D . Let $f_m(x)$ be orthogonal polynomials in D relative to $w(x)$, i.e. $(f_m, f_n) = 0$ for $m \neq n$, where

$$(f, g) = \begin{cases} \int_D w(x) f(x) g(x) dx & \text{if } D \text{ is a continuous interval} \\ \sum_{m=1}^N w(x_m) f(x_m) g(x_m) & \text{if } D \text{ is a set of } N \text{ discrete points } x_m. \end{cases}$$

Then

$$F_n(x) = \sum_{m=0}^n a_m f_m(x)$$

where

$$a_m = (f, f_m) / (f_m, f_m).$$

² $f(x)$ has to be square integrable, see e.g. [22.17].

*See page II.

D a Continuous Interval

Example 5. Find a least square polynomial of degree 5 for $f(x) = \frac{1}{1+x}$, in the interval $2 \leq x \leq 5$, using the weight function

$$w(x) = \frac{1}{\sqrt{(x-2)(5-x)}}$$

which stresses the importance of the errors at the ends of the interval.

Reduction to interval $[-1, 1]$, $t = \frac{2x-7}{3}$

$$w(x(t)) = \frac{2}{3} \frac{1}{\sqrt{1-t^2}}$$

From 22.2, $f_m(t) = T_m(t)$ and

$$a_m = \frac{4}{3\pi} \int_{-1}^1 \frac{1}{\sqrt{1-t^2}} \frac{1}{t+3} T_m(t) dt \quad (m \neq 0)$$

$$a_0 = \frac{2}{3\pi} \int_{-1}^1 \frac{1}{\sqrt{1-t^2}} \frac{dt}{t+3}$$

Evaluating the integrals numerically we get

$$\frac{1}{1+x} \sim .235703 - .080880T_1\left(\frac{2x-7}{3}\right) + .013876T_2\left(\frac{2x-7}{3}\right) - .002380T_3\left(\frac{2x-7}{3}\right) \\ + .000408T_4\left(\frac{2x-7}{3}\right) - .000070T_5\left(\frac{2x-7}{3}\right)$$

D a Set of Discrete Points

If $x_m = m (m=0, 1, 2, \dots, N)$ and $w(x)=1$, use the Chebyshev polynomials in the discrete range 22.17. It is convenient to introduce here a slightly different standardization such that

$$f_n(x) = \sum_{m=0}^n (-1)^m \binom{n}{m} \binom{n+m}{m} \frac{x!(N-m)!}{(x-m)!N!} \\ (f_n, f_n) = \frac{(N+n+1)!(N-n)!}{(2n+1)(N!)^2}$$

$$\text{Recurrence relation: } f_0(x)=1, f_1(x)=1-\frac{2x}{N}$$

$$(n+1)(N-n)f_{n+1}(x) = (2n+1)(N-2x)f_n(x) - n(N+n+1)f_{n-1}(x)$$

Example 6. Approximate in the least square sense the function $f(x)$ given in the following table by a third degree polynomial.

x	$f(x)$	$\bar{x} = \frac{x-10}{2}$	$f_0(\bar{x})$	$f_1(\bar{x})$	$f_2(\bar{x})$	$f_3(\bar{x})$
10	.3162	0	1	1	1	1
12	.2887	1	1	1/2	-1/2	-2
14	.2673	2	1	0	-1	0
16	.2500	3	1	-1/2	-1/2	2
18	.2357	4	1	-1	1	-1

	$f_0(\bar{x})$	$f_1(\bar{x})$	$f_2(\bar{x})$	$f_3(\bar{x})$
$(f_n, f_n) = \sum_{\bar{x}=0}^4 f_n^2(\bar{x})$	5	2.5	3.5	10
$(f, f_n) = \sum_{\bar{x}=0}^4 f_n(\bar{x})f(2\bar{x}+10)$	1.3579	.09985	.01525	.0031
$a_n = \frac{(f, f_n)}{(f_n, f_n)}$	.271580	.039940	.0043571	.000310
$f(x) \sim .27158 + .03994(3.5 - .25x) + .0043571(23.5 - 3.5x + .125x^2) + .00031(266 - 59.8333x \\ + 4.375x^2 - .10417x^3)$				
$f(x) \sim .59447 - .043658x + .0019009x^2 - .000032292x^3$				

22.20. Economization of Series

Problem: Given $f(x) = \sum_{m=0}^n a_m x^m$ in the interval $-1 \leq x \leq 1$ and $R > 0$. Find $\bar{f}(x) = \sum_{m=0}^N b_m x^m$ with N as small as possible, such that $|\bar{f}(x) - f(x)| < R$.

Solution: Express $f(x)$ in terms of Chebyshev polynomials using Table 22.3,

$$f(x) = \sum_{m=0}^n b_m T_m(x)$$

Then, since $|T_m(x)| \leq 1 (-1 \leq x \leq 1)$

$$\bar{f}(x) = \sum_{m=0}^N b_m T_m(x)$$

within the desired accuracy if

$$\sum_{m=N+1}^n |b_m| < R$$

$\bar{f}(x)$ is evaluated most conveniently by using the recurrence relation (see 22.7).

Example 7. Economize $f(x)=1+x/2+x^2/3+x^3/4+x^4/5+x^5/6$ with $R=.05$.

From Table 22.3

$$f(x) = \frac{1}{120} [149T_0(x) + 32T_2(x) + 3T_4(x)] \\ + \frac{1}{96} [76T_1(x) + 11T_3(x) + T_5(x)]$$

so

$$\bar{f}(x) = \frac{1}{120} [149T_0(x) + 32T_2(x)] + \frac{1}{96} [76T_1(x) + 11T_3(x)]$$

since

$$|\bar{f}(x) - f(x)| \leq \frac{1}{40} + \frac{1}{96} < .05$$

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Tables

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Coefficients for the Jacobi Polynomials $P_n^{(\alpha, \beta)}(x) = a_n^{-1} \sum_{m=0}^n c_m (x-1)^m$

Table 22.1

	a_n	$(x-1)^0$	$(x-1)^1$	$(x-1)^2$	$(x-1)^3$	$(x-1)^4$	$(x-1)^5$	$(x-1)^6$
$P_0^{(\alpha, \beta)}$	1	1						
$P_1^{(\alpha, \beta)}$	2	$2(\alpha+1)$	$\alpha+\beta+2$					
$P_2^{(\alpha, \beta)}$	8	$4(\alpha+1)_2$	$4(\alpha+\beta+3)(\alpha+2)$	$(\alpha+\beta+3)_2$				
$P_3^{(\alpha, \beta)}$	48	$8(\alpha+1)_3$	$12(\alpha+\beta+4)(\alpha+2)_2$	$6(\alpha+\beta+4)_2(\alpha+3)$	$(\alpha+\beta+4)_3$			
$P_4^{(\alpha, \beta)}$	384	$16(\alpha+1)_4$	$32(\alpha+\beta+5)(\alpha+2)_3$	$24(\alpha+\beta+5)_2(\alpha+3)_2$	$8(\alpha+\beta+5)_3(\alpha+4)$	$(\alpha+\beta+5)_4$		
$P_5^{(\alpha, \beta)}$	3840	$32(\alpha+1)_5$	$80(\alpha+\beta+6)(\alpha+2)_4$	$80(\alpha+\beta+6)_2(\alpha+3)_3$	$40(\alpha+\beta+6)_3(\alpha+4)_2$	$10(\alpha+\beta+6)_4(\alpha+5)$	$(\alpha+\beta+6)_5$	
$P_6^{(\alpha, \beta)}$	46080	$64(\alpha+1)_6$	$192(\alpha+\beta+7)(\alpha+2)_5$	$240(\alpha+\beta+7)_2(\alpha+3)_4$	$160(\alpha+\beta+7)_3(\alpha+4)_3$	$60(\alpha+\beta+7)_4(\alpha+5)_2$	$12(\alpha+\beta+7)_5(\alpha+6)$	$(\alpha+\beta+7)_6$

$$(m)_n = m(m+1)(m+2) \dots (m+n-1)$$

$$P_6^{(1,1)}(x) = \frac{1}{3840} [(8)_6(x-1)^6 + 10(8)_4(6)(x-1)^4 + 40(8)_3(5)_2(x-1)^3 + 80(8)_2(4)_3(x-1)^2 + 80(8)(3)_4(x-1) + 32(2)_6]$$

$$P_6^{(1,1)}(x) = \frac{1}{3840} [95040(x-1)^6 + 475200(x-1)^4 + 864000(x-1)^3 + 691200(x-1)^2 + 230400(x-1) + 23040]$$

Table 22.2

Coefficients for the Ultraspherical Polynomials $C_n^{(\alpha)}(x)$ and for x^n in terms of $C_m^{(\alpha)}(x)$

$$C_n^{(\alpha)}(x) = a_n^{-1} \sum_{m=0}^n c_m x^m \text{ and } x^n = b_n^{-1} \sum_{m=0}^n d_m C_m^{(\alpha)}(x) \quad (\alpha \neq 0)$$

		x^0	x^1	x^2	x^3	x^4	x^5	x^6	
	b_n	1	2α	$2(\alpha)_2$	$4(\alpha)_3$	$4(\alpha)_4$	$8(\alpha)_5$	$8(\alpha)_6$	
$C_0^{(\alpha)}$	a_n 1	1	1	α		$3\alpha(\alpha+3)$		$15\alpha(\alpha+4)(\alpha+5)$	$C_0^{(\alpha)}$
$C_1^{(\alpha)}$	1		2α	1	$3(\alpha+1)$		$15(\alpha+1)(\alpha+4)$		$C_1^{(\alpha)}$
$C_2^{(\alpha)}$	1	$-\alpha$		$2(\alpha)_2$	1	$6(\alpha+2)$		$45(\alpha+2)(\alpha+5)$	$C_2^{(\alpha)}$
$C_3^{(\alpha)}$	3		$-6(\alpha)_2$		$4(\alpha)_3$	3	$30(\alpha+3)$		$C_3^{(\alpha)}$
$C_4^{(\alpha)}$	6	$3(\alpha)_2$		$-12(\alpha)_3$		$4(\alpha)_4$	6	$90(\alpha+4)$	$C_4^{(\alpha)}$
$C_5^{(\alpha)}$	15		$15(\alpha)_3$		$-20(\alpha)_4$		$4(\alpha)_5$	30	$C_5^{(\alpha)}$
$C_6^{(\alpha)}$	90	$-15(\alpha)_3$		$90(\alpha)_4$		$-60(\alpha)_5$		$8(\alpha)_6$	$C_6^{(\alpha)}$
		x^0	x^1	x^2	x^3	x^4	x^5	x^6	

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2) \dots (\alpha+n-1)$$

$$C_3^{(2)}(x) = \frac{1}{3} [4(2)_2 x^3 - 6(2)_2 x] \quad x^3 = \frac{1}{4(2)_3} [3(3) C_1^{(2)}(x) + 3 C_3^{(2)}(x)]$$

$$C_3^{(2)}(x) = \frac{1}{3} [96x^3 - 36x] \quad x^3 = \frac{1}{96} [9 C_1^{(2)}(x) + 3 C_3^{(2)}(x)]$$

Table 22.3

Coefficients for the Chebyshev Polynomials $T_n(x)$ and for x^n in terms of $T_m(x)$

$$T_n(x) = \sum_{m=0}^n c_m x^m \quad x^n = b_n^{-1} \sum_{m=0}^n d_m T_m(x)$$

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	
b_n	1	1	2	4	8	16	32	64	128	256	512	1024	2048	
T_0	1	1		1	3		10		35		126		462	T_0
T_1		1	1		3		10		35		126		462	T_1
T_2	-1		2	1		4		15		56		210		T_2
T_3		-3		4	1		5		21		84		330	T_3
T_4	1		-8		8	1		6		28		120		T_4
T_5		5		-20		16	1		7		36		165	T_5
T_6	-1		18		-48		32	1		8		45		T_6
T_7		-7		56		-112		64	1		9		55	T_7
T_8	1		-32		160		-256		128	1		10		T_8
T_9		9		-120		432		-576		256	1		11	T_9
T_{10}	-1		50		-400		1120		-1280		512	1		T_{10}
T_{11}		-11		220		-1232		2816		-2816		1024	1	T_{11}
T_{12}	1		-72		840		-3584		6912		-6144		2048	1 T_{12}
	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	

$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1 \quad x^6 = \frac{1}{32} [10T_6 + 15T_2 + 6T_4 + T_0]$$

Chebyshev Polynomials $T_n(x)$

Table 22.4

$n \backslash x$	0.2	0.4	0.6	0.8	1.0
0	+1.00000 00000	+1.00000 00000	+1.00000 00000	+1.00000 00000	1
1	+0.20000 00000	+0.40000 00000	+0.60000 00000	+0.80000 00000	1
2	-0.92000 00000	-0.68000 00000	-0.28000 00000	+0.28000 00000	1
3	-0.56800 00000	-0.94400 00000	-0.93600 00000	-0.35200 00000	1
4	+0.69280 00000	-0.07520 00000	-0.84320 00000	-0.84320 00000	1
5	+0.84512 00000	+0.88384 00000	-0.07584 00000	-0.99712 00000	1
6	-0.35475 20000	+0.78227 20000	+0.75219 20000	-0.75219 20000	1
7	-0.98702 08000	-0.25802 24000	+0.97847 04000	-0.20638 72000	1
8	-0.04005 63200	-0.98868 99200	+0.42197 24800	+0.42197 24800	1
9	+0.97099 82720	-0.53292 95360	-0.47210 34240	+0.88154 31680	1
10	+0.42845 56288	+0.56234 62912	-0.98849 65888	+0.98849 65888	1
11	-0.79961 60205	+0.98280 65690	-0.71409 24826	+0.70005 13741	1
12	-0.74830 20370	+0.22389 89640	+0.13158 56097	+0.13158 56097	1

Table 22.5

Coefficients for the Chebyshev Polynomials $U_n(x)$ and for x^n in terms of $U_m(x)$

$$U_n(x) = \sum_{m=0}^n c_m x^m \quad x^n = b_n^{-1} \sum_{m=0}^n d_m U_m(x)$$

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	
b_n	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	
U_0	1	1		1	2		5		14		42		132	U_0
U_1		2	1		2	5		14		42		132		U_1
U_2	-1		4	1	3		9		28		90		297	U_2
U_3		-4		8	1	4		14		48		165		U_3
U_4	1		-12		16	1	5		20		75		275	U_4
U_5		6		-32		32	1	6		27		110		U_5
U_6	-1		24		-80		64	1	7		35		154	U_6
U_7		-8		80		-192		128	1	8		44		U_7
U_8	1		-40		240		-448		256	1	9		54	U_8
U_9		10		-160		672		-1024		512	1	10		U_9
U_{10}	-1		60		-560		1792		-2304		1024	1	11	U_{10}
U_{11}		-12		280		-1792		4608		-5120		2048	1	U_{11}
U_{12}	1		-84		1120		-5376		11520		-11264		4096	1 U_{12}
	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	

$$U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1 \quad x^6 = \frac{1}{64} [5U_0 + 9U_2 + 5U_4 + U_6]$$

Table 22.6

Chebyshev Polynomials $U_n(x)$

$n \backslash x$	0.2	0.4	0.6	0.8	1.0
0	+1.00000 00000	+1.00000 00000	+1.00000 00000	+1.00000 00000	1
1	+0.40000 00000	+0.80000 00000	+1.20000 00000	+1.60000 00000	2
2	-0.84000 00000	-0.36000 00000	+0.44000 00000	+1.56000 00000	3
3	-0.73600 00000	-1.08800 00000	-0.67200 00000	+0.89600 00000	4
4	+0.54560 00000	-0.51040 00000	-1.24640 00000	-0.12640 00000	5
5	+0.95424 00000	+0.67968 00000	-0.82368 00000	-1.09824 00000	6
6	-0.16390 40000	+1.05414 40000	+0.25798 40000	-1.63078 40000	7
7	-1.01980 16000	+0.16363 52000	+1.13326 08000	-1.51101 44000	8
8	-0.24401 66400	-0.92323 58400	+1.10192 89600	-0.78683 90400	9
9	+0.92219 49440	-0.90222 38720	+0.18905 39520	+0.25207 19360	10
10	+0.61289 46176	+0.20145 67424	-0.87506 42176	+1.19015 41376	11
11	-0.67703 70970	+1.06338 92659	-1.23913 10131	+1.65217 46842	12
12	-0.88370 94564	+0.64925 46703	-0.61189 29981	+1.45332 53571	13

Table 22.7

Coefficients for the Chebyshev Polynomials $C_n(x)$ and for x^n in terms of $C_m(x)$

$$C_n(x) = \sum_{m=0}^n c_m x^m \quad x^n = b_n^{-1} \sum_{m=0}^n d_m C_m(x)$$

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	
b_n	2	1	1	1	1	1	1	1	1	1	1	1	1	
C_0	2	1		1		3		10		35		126		462
C_1		1	1		3		10		35		126		462	
C_2	-2		1	1		4		15		56		210		792
C_3		-3		1	1		5		21		84		330	
C_4	2		-4		1	1		6		28		120		495
C_5^*		5		-5		1	1		7		36		165	
C_6	-2		9		-6		1	1		8		45		220
C_7		-7		14		-7		1	1		9		55	
C_8	2		-16		20		-8		1	1		10		66
C_9		9		-30		27		-9		1	1		11	
C_{10}	-2		25		-50		35		-10		1	1		12
C_{11}		-11		55		-77		44		-11		1	1	
C_{12}	2		-36		105		-112		54		-12		1	1
	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	

*See page II.

$$C_6(x) = x^6 - 6x^4 + 9x^2 - 2 \quad x^6 = 10C_0 + 15C_2 + 6C_4 + C_6$$

Table 22.8

Coefficients for the Chebyshev Polynomials $S_n(x)$ and for x^n in terms of $S_m(x)$

$$S_n(x) = \sum_{m=0}^n c_m x^m \quad x^n = \sum_{m=0}^n d_m S_m(x)$$

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	
S_0	1	1		1		2		5		14		42		132
S_1		1	1		2		5		14		42		132	
S_2	-1		1	1		3		9		28		90		297
S_3		-2		1	1		4		14		48		165	
S_4	1		-3		1	1		5		20		75		275
S_5		3		-4		1	1		6		27		110	
S_6	-1		6		-5		1	1		7		35		154
S_7		-4		10		-6		1	1		8		44	
S_8	1		-10		15		-7		1	1		9		54
S_9		5		-20		21		-8		1	1		10	
S_{10}	-1		15		-35		28		-9		1	1		11
S_{11}		-6		35		-56		36		-10		1	1	
S_{12}	1		-21		70		-84		45		-11		1	1
	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	

$$S_6(x) = x^6 - 5x^4 + 6x^2 - 1 \quad x^6 = 5S_0 + 9S_2 + 5S_4 + S_6$$

*See page II.

Table 22.9

Coefficients for the Legendre Polynomials $P_n(x)$ and for x^n in terms of $P_m(x)$

$$P_n(x) = a_n^{-1} \sum_{m=0}^n c_m x^m \quad x^n = b_n^{-1} \sum_{m=0}^n d_m P_m(x)$$

		x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}		
	$a_n \ b_n$	1	1	3	5	35	63	231	429	6435	12155	46189	88179	676039	b_n	
P_0	1	1	1	1		7		33		715		4199		52003	P_0	
P_1	1		1	1		3		27		143		3315		20349	P_1	
P_2	2	-1		3	2		20		110		2600		16150		208012	P_2
P_3	2		-3		5	2		28		182		4760		31654		P_3
P_4	8	3		-30		35	8		72		2160		15504		220248	P_4
P_5	8		15		-70		63	8		88		2992		23408		P_5
P_6	16	-5		105		-315		231	16		832		7904		133952	P_6
P_7	16		-35		315		-693		429	16		960		10080		P_7
P_8	128	35		-1260		6930		-12012		6435	128		2176		50048	P_8
P_9	128		315		-4620		18018		-25740		12155	128		2432		P_9
P_{10}	256	-63		3465		-30030		90090		-109395		46189	256		10752	P_{10}
P_{11}	256		-693		15015		-90090		218790		-230945		88179	256		P_{11}
P_{12}	1024	231		-18018		225225		-1021020		2078505		-1939938		676039	1024	P_{12}
		x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}		

$$P_6(x) = \frac{1}{16} [231x^6 - 315x^4 + 105x^2 - 5] \quad x^6 = \frac{1}{231} [33P_0 + 110P_2 + 72P_4 + 16P_6]$$

For values of $P_n(x)$, see chapter 8.

Coefficients for the Laguerre Polynomials $L_n(x)$ and for x^n in terms of $L_m(x)$

Table 22.10

$$L_n(x) = a_n^{-1} \sum_{m=0}^n c_m x^m \quad x^n = \sum_{m=0}^n d_m L_m(x)$$

	a_n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}		
L_0	1	1	1	1	2	6	24	120	720	5040	40320	362880	3628800	39916800	479001600	L_0
L_1	1	1	-1	-1	-4	-18	-96	-600	-4320	-35280	-322560	-3265920	-36288000	-439084800	-5748019200	L_1
L_2	2	2	-4	1	2	18	144	1200	10800	105840	1128960	13063680	163296000	2195424000	31614105600	L_2
L_3	6	6	-18	9	-1	-6	-96	-1200	-14400	-176400	-2257920	-30481920	-435456000	-6586272000	-105380352000	L_3
L_4	24	24	-96	72	-16	1	24	600	10800	176400	2822400	45722880	762048000	13172544000	237105792000	L_4
L_5	120	120	-600	600	-200	25	-1	-120	-4320	-105840	-2257920	-45722880	-914457600	-18441561600	-379369267200	L_5
L_6	720	720	-4320	5400	-2400	450	-36	1	720	35280	1128960	30481920	762048000	18441561600	442597478400	L_6
L_7	5040	5040	-35280	52920	-29400	7350	-882	49	-1	-5040	-322560	-13063680	-435456000	-13172544000	-379369267200	L_7
L_8	40320	40320	-322560	564480	-376320	117600	-18816	1568	-64	1	40320	3269520	163296000	6586272000	237105792000	L_8
L_9	362880	362880	-3265920	6631840	-5080320	1905120	-381024	42336	-2592	81	-1	-362880	-36288000	-2195424000	-105380352000	L_9
L_{10}	3628800	3628800	-36288000	81048000	-72576000	31752000	-7620480	1058400	-86400	4050	-100	1	3628800	439084800	31614105600	L_{10}
L_{11}	39916800	39916800	-439084800	1097712000	-1097712000	548856000	-153679680	25613280	-2613600	163350	-6050	121	-1	-39916800	-5748019200	L_{11}
L_{12}	479001600	479001600	-5748019200	15807052800	-17563392000	9879408000	-3161410560	614718720	-75271680	5880600	-290400	8712	-144	1	479001600	L_{12}
	a_n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}		

$$L_6(x) = \frac{1}{720} [x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720]$$

$$x^6 = 720L_0 - 4320L_1 + 10800L_2 - 14400L_3 + 10800L_4 - 4320L_5 + 720L_6$$

Coefficients for the Hermite Polynomials $H_n(x)$ and for x^n in terms of $H_m(x)$

Table 22.12

$$H_n(x) = \sum_{m=0}^n c_m x^m \quad x^n = b_n^{-1} \sum_{m=0}^n d_m H_m(x)$$

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	
b_n	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	b_n
H_0	1 1		2		12		120		1680		* 30240		665280	H_0
H_1		2 1		6		60		840		15120		332640		H_1
H_2	-2		4 1		12		180		3360		75600		1995840	H_2
H_3		-12		8 1		20		420		10080		277200		H_3
H_4	12		-48		16 1		30		840		25200		831600	H_4
H_5		120		-160		32 1		42		1512		55440		H_5
H_6	-120		720		-480		64 1		56		2520		110880	H_6
H_7		-1680		3360		-1344		128 1		72		3960		H_7
H_8	1680		-13440		13440		-3584		256 1		90		5940	H_8
H_9		30240		-80640		48384		-9216		512 1		110		H_9
H_{10}	-30240		302400		-403200		161280		-23040		1024 1		132	H_{10}
H_{11}		-665280		2217600		-1774080		506880		-56320		2048 1		H_{11}
H_{12}	665280		-7983360		13305600		-7096320		1520640		-135168		4096 1	H_{12}
	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	

$$H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$$

$$x^6 = \frac{1}{64} (120H_0 + 180H_2 + 30H_4 + H_6)$$

*See page II.

23. Bernoulli and Euler Polynomials— Riemann Zeta Function

EMILIE V. HAYNSWORTH¹ AND KARL GOLDBERG²

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$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}, \quad 20D$$

$$\eta(n) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^n}, \quad 20D$$

$$\lambda(n) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^n}, \quad 20D$$

$$\beta(n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^n}, \quad 18D$$

$$n=1(1)42$$

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$$\sum_{k=1}^m k^n, \quad n=1(1)10, \quad m=1(1)100$$

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23. Bernoulli and Euler Polynomials—Riemann Zeta Function

Mathematical Properties

23.1. Bernoulli and Euler Polynomials and the Euler-Maclaurin Formula

Generating Functions

$$23.1.1 \quad \frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad |t| < 2\pi \quad \left| \frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \right. \quad |t| < \pi$$

Bernoulli and Euler Numbers

$$23.1.2 \quad B_n = B_n(0) \quad n=0, 1, \dots \quad \left| \quad E_n = 2^n E_n\left(\frac{1}{2}\right) = \text{integer} \quad n=0, 1, \dots \right.$$

$$23.1.3 \quad B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_4=-\frac{1}{30} \quad \left| \quad E_0=1, E_2=-1, E_4=5 \right.$$

(For occurrence of B_n and E_n in series expansions of circular functions, see chapter 4.)

Sums of Powers

$$23.1.4 \quad \sum_{k=1}^m k^n = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1} \quad \left| \quad \sum_{k=1}^m (-1)^{m-k} k^n = \frac{E_n(m+1) + (-1)^m E_n(0)}{2} \right. \quad m, n=1, 2, \dots$$

Derivatives and Differences

$$23.1.5 \quad B'_n(x) = nB_{n-1}(x) \quad n=1, 2, \dots \quad \left| \quad E'_n(x) = nE_{n-1}(x) \quad n=1, 2, \dots \right.$$

$$23.1.6 \quad B_n(x+1) - B_n(x) = nx^{n-1} \quad n=0, 1, \dots \quad \left| \quad E_n(x+1) + E_n(x) = 2x^n \quad n=0, 1, \dots \right.$$

Expansions

$$23.1.7 \quad B_n(x+h) = \sum_{k=0}^n \binom{n}{k} B_k(x) h^{n-k} \quad n=0, 1, \dots \quad \left| \quad E_n(x+h) = \sum_{k=0}^n \binom{n}{k} E_k(x) h^{n-k} \quad n=0, 1, \dots \right.$$

$$E_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k} \quad n=0, 1, \dots$$

Symmetry

$$23.1.8 \quad B_n(1-x) = (-1)^n B_n(x) \quad n=0, 1, \dots \quad \left| \quad E_n(1-x) = (-1)^n E_n(x) \quad n=0, 1, \dots \right.$$

$$23.1.9 \quad (-1)^n B_n(-x) = B_n(x) + nx^{n-1} \quad n=0, 1, \dots \quad \left| \quad (-1)^{n+1} E_n(-x) = E_n(x) - 2x^n \quad n=0, 1, \dots \right.$$

Multiplication Theorem

$$23.1.10 \quad B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right) \quad \left. \begin{array}{l} n=0, 1, \dots \\ m=1, 2, \dots \end{array} \right| \quad \begin{array}{l} E_n(mx) = m^n \sum_{k=0}^{m-1} (-1)^k E_n\left(x + \frac{k}{m}\right) \quad n=0, 1, \dots \\ E_n(mx) = -\frac{2}{n+1} m^n \sum_{k=0}^{m-1} (-1)^k B_{n+1}\left(x + \frac{k}{m}\right) \quad m=1, 3, \dots \end{array}$$

$$E_n(mx) = -\frac{2}{n+1} m^n \sum_{k=0}^{m-1} (-1)^k B_{n+1}\left(x + \frac{k}{m}\right) \quad n=0, 1, \dots$$

$$m=2, 4, \dots$$

Integrals

$$23.1.11 \quad \int_a^x B_n(t) dt = \frac{B_{n+1}(x) - B_{n+1}(a)}{n+1}$$

$$23.1.12 \quad \int_0^1 B_n(t) B_m(t) dt = (-1)^{n-1} \frac{m!n!}{(m+n)!} B_{m+n}$$

$$m, n = 1, 2, \dots$$

$$\int_a^x E_n(t) dt = \frac{E_{n+1}(x) - E_{n+1}(a)}{n+1}$$

$$\int_0^1 E_n(t) E_m(t) dt = (-1)^n 4^{2^{m+n+2}-1} \frac{m!n!}{(m+n+2)!} B_{m+n+2}$$

$$m, n = 0, 1, \dots$$

(The polynomials are orthogonal for $m+n$ odd.)

Inequalities

$$23.1.13 \quad |B_{2n}| > |B_{2n}(x)| \quad n = 1, 2, \dots, \quad 1 > x > 0$$

$$23.1.14$$

$$\frac{2(2n+1)!}{(2\pi)^{2n+1}} \left(\frac{1}{1-2^{-2n}} \right) > (-1)^{n+1} B_{2n+1}(x) > 0$$

$$n = 1, 2, \dots, \quad \frac{1}{2} > x > 0$$

$$23.1.15$$

$$\frac{2(2n)!}{(2\pi)^{2n}} \left(\frac{1}{1-2^{1-2n}} \right) > (-1)^{n+1} B_{2n} > \frac{2(2n)!}{(2\pi)^{2n}}$$

$$n = 1, 2, \dots$$

$$4^{-n} |E_{2n}| > (-1)^n E_{2n}(x) > 0 \quad n = 1, 2, \dots, \quad \frac{1}{2} > x > 0$$

$$\frac{4(2n-1)!}{\pi^{2n}} \left(1 + \frac{1}{2^{2n}-2} \right) > (-1)^n E_{2n-1}(x) > 0$$

$$n = 1, 2, \dots, \quad \frac{1}{2} > x > 0$$

$$\frac{4^{n+1}(2n)!}{\pi^{2n+1}} > (-1)^n E_{2n} > \frac{4^{n+1}(2n)!}{\pi^{2n+1}} \left(\frac{1}{1+3^{-1-2n}} \right)$$

$$n = 0, 1, \dots$$

Fourier Expansions

$$23.1.16$$

$$B_n(x) = -2 \frac{n!}{(2\pi)^n} \sum_{k=1}^{\infty} \frac{\cos(2\pi kx - \frac{1}{2}\pi n)}{k^n}$$

$$n > 1, 1 \geq x \geq 0$$

$$n = 1, 1 > x > 0$$

$$23.1.17$$

$$B_{2n-1}(x) = \frac{(-1)^n 2(2n-1)!}{(2\pi)^{2n-1}} \sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k^{2n-1}}$$

$$n > 1, 1 \geq x \geq 0$$

$$n = 1, 1 > x > 0$$

$$23.1.18$$

$$B_{2n}(x) = \frac{(-1)^{n-1} 2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{\cos 2k\pi x}{k^{2n}}$$

$$n = 1, 2, \dots, \quad 1 \geq x \geq 0$$

$$E_n(x) = 4 \frac{n!}{\pi^{n+1}} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x - \frac{1}{2}\pi n)}{(2k+1)^{n+1}}$$

$$n > 0, 1 \geq x \geq 0$$

$$n = 0, 1 > x > 0$$

$$E_{2n-1}(x) = \frac{(-1)^n 4(2n-1)!}{\pi^{2n}} \sum_{k=0}^{\infty} \frac{\cos(2k+1)\pi x}{(2k+1)^{2n}}$$

$$n = 1, 2, \dots, \quad 1 \geq x \geq 0$$

$$E_{2n}(x) = \frac{(-1)^n 4(2n)!}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\pi x}{(2k+1)^{2n+1}}$$

$$n > 0, 1 \geq x \geq 0$$

$$n = 0, 1 > x > 0$$

Special Values

$$23.1.19 \quad B_{2n+1} = 0 \quad n = 1, 2, \dots \quad E_{2n+1} = 0 \quad n = 0, 1, \dots$$

$$23.1.20 \quad B_n(0) = (-1)^n B_n(1) = B_n$$

$$n = 0, 1, \dots \quad E_n(0) = -E_n(1) = -2(n+1)^{-1}(2^{n+1}-1)B_{n+1} \quad n = 1, 2, \dots$$

$$23.1.21 \quad B_n\left(\frac{1}{2}\right) = -(1-2^{1-n})B_n \quad n = 0, 1, \dots \quad E_n\left(\frac{1}{2}\right) = 2^{-n}E_n \quad n = 0, 1, \dots$$

$$\begin{aligned} 23.1.22 \quad B_n\left(\frac{1}{4}\right) &= (-1)^n B_n\left(\frac{3}{4}\right) \\ &= -2^{-n}(1-2^{1-n})B_n - n4^{-n}E_{n-1} \\ &\quad n=1, 2, \dots \end{aligned}$$

$$\begin{aligned} E_{2n-1}\left(\frac{1}{3}\right) &= -E_{2n-1}\left(\frac{2}{3}\right) \\ &= -(2n)^{-1}(1-3^{1-2n})(2^{2n}-1)B_{2n} \\ &\quad n=1, 2, \dots \end{aligned}$$

$$\begin{aligned} 23.1.23 \quad B_{2n}\left(\frac{1}{3}\right) &= B_{2n}\left(\frac{2}{3}\right) \\ &= -2^{-1}(1-3^{1-2n})B_{2n} \quad n=0, 1, \dots \end{aligned}$$

$$\begin{aligned} 23.1.24 \quad B_{2n}\left(\frac{1}{6}\right) &= B_{2n}\left(\frac{5}{6}\right) \\ &= 2^{-1}(1-2^{1-2n})(1-3^{1-2n})B_{2n} \\ &\quad n=0, 1, \dots \end{aligned}$$

Symbolic Operations

$$\begin{aligned} 23.1.25 \quad p(B(x)+1) - p(B(x)) &= p'(x) & p(E(x)+1) + p(E(x)) &= 2p(x) \\ 23.1.26 \quad B_n(x+h) &= (B(x)+h)^n \quad n=0, 1, \dots & E_n(x+h) &= (E(x)+h)^n \quad n=0, 1, \dots \end{aligned}$$

Here $p(x)$ denotes a polynomial in x and after expanding we set $\{B(x)\}^n = B_n(x)$ and $\{E(x)\}^n = E_n(x)$.

Relations Between the Polynomials

$$\begin{aligned} 23.1.27 \quad E_{n-1}(x) &= \frac{2^n}{n} \left\{ B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right) \right\} \\ &= \frac{2}{n} \left\{ B_n(x) - 2^n B_n\left(\frac{x}{2}\right) \right\} \quad n=1, 2, \dots \end{aligned}$$

$$\begin{aligned} 23.1.28 \quad E_{n-2}(x) &= 2 \binom{n}{2}^{-1} \sum_{k=0}^{n-2} \binom{n}{k} (2^{n-k}-1) B_{n-k} B_k(x) \\ &\quad n=2, 3, \dots \end{aligned}$$

$$\begin{aligned} 23.1.29 \quad B_n(x) &= 2^{-n} \sum_{k=0}^n \binom{n}{k} B_{n-k} E_k(2x) \quad n=0, 1, \dots \end{aligned}$$

Euler-Maclaurin Formulas

Let $F(x)$ have its first $2n$ derivatives continuous on an interval (a, b) . Divide the interval into m equal parts and let $h=(b-a)/m$. Then for some θ , $1 > \theta > 0$, depending on $F^{(2n)}(x)$ on (a, b) , we have

23.1.30

$$\begin{aligned} \sum_{k=0}^m F(a+kh) &= \frac{1}{h} \int_a^b F(t) dt + \frac{1}{2} \{F(b) + F(a)\} \\ &+ \sum_{k=1}^{n-1} \frac{h^{2k-1}}{(2k)!} B_{2k} \{F^{(2k-1)}(b) - F^{(2k-1)}(a)\} \\ &+ \frac{h^{2n}}{(2n)!} B_{2n} \sum_{k=0}^{m-1} F^{(2n)}(a+kh+\theta h) \end{aligned}$$

Equivalent to this is

23.1.31

$$\begin{aligned} \frac{1}{h} \int_x^{x+h} F(t) dt &= \frac{1}{2} \{F(x+h) + F(x)\} \\ &- \sum_{k=1}^{n-1} \frac{h^{2k-1}}{(2k)!} B_{2k} \{F^{(2k-1)}(x+h) - F^{(2k-1)}(x)\} \\ &- \frac{h^{2n}}{(2n)!} B_{2n} F^{(2n)}(x+\theta h) \quad b-h \geq x \geq a \end{aligned}$$

Let $\hat{B}_n(x) = B_n(x-[x])$. The Euler Summation Formula is

23.1.32

$$\begin{aligned} \sum_{k=0}^{m-1} F(a+kh+\omega h) &= \frac{1}{h} \int_a^b F(t) dt \\ &+ \sum_{k=1}^p \frac{h^{k-1}}{k!} B_k(\omega) \{F^{(k-1)}(b) - F^{(k-1)}(a)\} \\ &- \frac{h^p}{p!} \int_0^1 \hat{B}_p(\omega-t) \left\{ \sum_{k=0}^{m-1} F^{(p)}(a+kh+th) \right\} dt \\ &\quad p \leq 2n, 1 \geq \omega \geq 0 \end{aligned}$$

23.2. Riemann Zeta Function and Other Sums of Reciprocal Powers

$$23.2.1 \quad \zeta(s) = \sum_{k=1}^{\infty} k^{-s} \quad \Re s > 1$$

$$23.2.2 \quad = \prod_p (1 - p^{-s})^{-1} \quad \Re s > 1$$

(product over all primes p).

$$23.2.3 \quad = \frac{1}{s-1} + \frac{1}{2} + \sum_{k=1}^n \frac{B_{2k}}{2k} \binom{s+2k-2}{2k-1} - \frac{(s+2n)}{(2n+1)} \int_1^{\infty} \frac{B_{2n+1}(x-[x])}{x^{s+2n+1}} dx$$

$$s \neq 1, n=1, 2, \dots, \Re s > -2n$$

$$* 23.2.4 \quad = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1} dz$$

$$23.2.5 \quad = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n$$

where

$$\gamma_n = \lim_{m \rightarrow \infty} \left\{ \sum_{k=1}^m \frac{(\ln k)^n}{k} - \frac{(\ln m)^{n+1}}{n+1} \right\} \quad \Re s > 0$$

$$23.2.6 \quad = 2^s \pi^{s-1} \sin\left(\frac{1}{2}\pi s\right) \Gamma(1-s) \zeta(1-s)$$

$$23.2.7 \quad = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \quad \Re s > 1$$

$$23.2.8 \quad = \frac{1}{(1-2^{1-s})\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x + 1} dx$$

$$23.2.9 \quad = \sum_{k=1}^n k^{-s} + (s-1)^{-1} n^{1-s} - s \int_n^{\infty} \frac{x-[x]}{x^{s+1}} dx$$

$$n=1, 2, \dots, \Re s > 0$$

$$23.2.10 \quad = \frac{\exp(\ln 2\pi - 1 - \frac{1}{2}\gamma)s}{2(s-1)\Gamma(\frac{1}{2}s+1)} \prod_p \left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}$$

product over all zeros ρ of $\zeta(s)$ with $\Re \rho > 0$.

The contour C in the fourth formula starts at infinity on the positive real axis, circles the origin once in the positive direction excluding the points $\pm 2ni\pi$ for $n=1, 2, \dots$, and returns to the starting point. Therefore $\zeta(s)$ is regular for all values of s except for a simple pole at $s=1$ with residue 1.

Special Values

$$23.2.11 \quad \zeta(0) = -\frac{1}{2}$$

$$23.2.12 \quad \zeta(1) = \infty$$

$$23.2.13 \quad \zeta'(0) = -\frac{1}{2} \ln 2\pi$$

$$23.2.14 \quad \zeta(-2n) = 0 \quad n=1, 2, \dots$$

$$23.2.15 \quad \zeta(1-2n) = -\frac{B_{2n}}{2n} \quad n=1, 2, \dots$$

$$23.2.16 \quad \zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} |B_{2n}| \quad n=1, 2, \dots$$

$$23.2.17$$

$$\zeta(2n+1) = \frac{(-1)^{n+1} (2\pi)^{2n+1}}{2(2n+1)!} \int_0^1 B_{2n+1}(x) \cot(\pi x) dx$$

$$n=1, 2, \dots$$

Sums of Reciprocal Powers

The sums referred to are

$$23.2.18 \quad \zeta(n) = \sum_{k=1}^{\infty} k^{-n} \quad n=2, 3, \dots$$

$$23.2.19$$

$$\eta(n) = \sum_{k=1}^{\infty} (-1)^{k-1} k^{-n} = (1-2^{1-n}) \zeta(n) \quad n=1, 2, \dots$$

$$23.2.20$$

$$\lambda(n) = \sum_{k=0}^{\infty} (2k+1)^{-n} = (1-2^{-n}) \zeta(n) \quad n=2, 3, \dots$$

$$23.2.21$$

$$\beta(n) = \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-n} \quad n=1, 2, \dots$$

These sums can be calculated from the Bernoulli and Euler polynomials by means of the last two formulas for special values of the zeta function (note that $\eta(1) = \ln 2$), and

$$23.2.22 \quad \beta(2n+1) = \frac{(\pi/2)^{2n+1}}{2(2n)!} |E_{2n}| \quad n=0, 1, \dots$$

$$23.2.23$$

$$\beta(2n) = \frac{(-1)^n \pi^{2n}}{4(2n-1)!} \int_0^1 E_{2n-1}(x) \sec(\pi x) dx$$

$$n=1, 2, \dots$$

$\beta(2)$ is known as Catalan's constant. Some other special values are

$$23.2.24 \quad \zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

$$23.2.25 \quad \zeta(4) = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$$

$$23.2.26 \quad \eta(2) = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$$

$$23.2.27 \quad \eta(4) = 1 - \frac{1}{2^4} + \frac{1}{3^4} - \dots = \frac{7\pi^4}{720}$$

$$23.2.28 \quad \lambda(2) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

$$23.2.29 \quad \lambda(4) = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$$

$$23.2.30 \quad \beta(1) = 1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}$$

$$23.2.31 \quad \beta(3) = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \dots = \frac{\pi^3}{32}$$

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COEFFICIENTS b_k OF THE BERNOULLI POLYNOMIALS $B_n(x) = \sum_{k=0}^n b_k x^k$

Table 23.1

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1															
1	$-\frac{1}{2}$	1														
2	$\frac{1}{6}$	-1	1													
3	0	$\frac{1}{2}$	$-\frac{3}{2}$	1												
4	$-\frac{1}{30}$	0	1	-2	1											
5	0	$-\frac{1}{6}$	0	$\frac{5}{3}$	$-\frac{5}{2}$	1										
6	$\frac{1}{42}$	0	$-\frac{1}{2}$	0	$-\frac{5}{2}$	-3	1									
7	0	$\frac{1}{6}$	0	$-\frac{7}{6}$	0	$\frac{7}{2}$	$-\frac{7}{2}$	1								
8	$-\frac{1}{30}$	0	$\frac{2}{3}$	0	$-\frac{7}{3}$	0	$\frac{14}{3}$	-4	1							
9	0	$-\frac{3}{10}$	0	2	0	$-\frac{21}{5}$	0	6	$-\frac{9}{2}$	1						
10	$\frac{5}{66}$	0	$-\frac{3}{2}$	0	5	0	-7	0	$\frac{15}{2}$	-5	1					
11	0	$\frac{5}{6}$	0	$-\frac{11}{2}$	0	11	0	-11	0	$\frac{55}{6}$	$-\frac{11}{2}$	1				
12	$-\frac{691}{2730}$	0	5	0	$-\frac{33}{2}$	0	22	0	$-\frac{33}{2}$	0	11	-6	1			
13	0	$-\frac{691}{210}$	0	$\frac{65}{3}$	0	$-\frac{429}{10}$	0	$\frac{286}{7}$	0	$-\frac{143}{6}$	0	13	$-\frac{13}{2}$	1		
14	$\frac{7}{6}$	0	$-\frac{691}{30}$	0	$\frac{455}{6}$	0	$-\frac{1001}{10}$	0	$\frac{143}{2}$	0	$-\frac{1001}{30}$	0	$\frac{91}{6}$	-7	1	
15	0	$\frac{35}{2}$	0	$-\frac{691}{6}$	0	$\frac{455}{2}$	0	$-\frac{429}{2}$	0	$\frac{715}{6}$	0	$-\frac{91}{2}$	0	$\frac{35}{2}$	$-\frac{15}{2}$	1

$$B_n = B_n(0)$$

COEFFICIENTS e_k OF THE EULER POLYNOMIALS $E_n(x) = \sum_{k=0}^n e_k x^k$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1															
1	$-\frac{1}{2}$	1														
2	0	-1	1													
3	$\frac{1}{4}$	0	$-\frac{3}{2}$	1												
4	0	1	0	-2	1											
5	$-\frac{1}{2}$	0	$\frac{5}{2}$	0	$-\frac{5}{2}$	1										
6	0	-3	0	5	0	-3	1									
7	$\frac{17}{8}$	0	$-\frac{21}{2}$	0	$\frac{35}{4}$	0	$-\frac{7}{2}$	1								
8	0	17	0	-28	0	14	0	-4	1							
9	$-\frac{31}{2}$	0	$\frac{153}{2}$	0	-63	0	21	0	$-\frac{9}{2}$	1						
10	0	-155	0	255	0	-126	0	30	0	-5	1					
11	$\frac{691}{4}$	0	$-\frac{1705}{2}$	0	$\frac{2805}{4}$	0	-231	0	$\frac{165}{4}$	0	$-\frac{11}{2}$	1				
12	0	2073	0	-3410	0	1683	0	-396	0	55	0	-6	1			
13	$-\frac{5461}{2}$	0	$\frac{26949}{2}$	0	$-\frac{22165}{2}$	0	$\frac{7293}{2}$	0	$-\frac{1287}{2}$	0	$\frac{143}{2}$	0	$-\frac{13}{2}$	1		
14	0	-38227	0	62881	0	-31031	0	7293	0	-1001	0	91	0	-7	1	
15	$\frac{929569}{16}$	0	$-\frac{573405}{2}$	0	$\frac{943215}{4}$	0	$-\frac{155155}{2}$	0	$\frac{109395}{8}$	0	$-\frac{3003}{2}$	0	$\frac{455}{4}$	0	$-\frac{15}{2}$	1

24. Combinatorial Analysis

K. GOLDBERG,¹ M. NEWMAN,² E. HAYNSWORTH³

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24. Combinatorial Analysis

Mathematical Properties

In each sub-section of this chapter we use a fixed format which emphasizes the use and methods of extending the accompanying tables. The format follows this form:

I. Definitions

- A. Combinatorial
- B. Generating functions
- C. Closed form

II. Relations

- A. Recurrences
- B. Checks in computing
- C. Basic use in numerical analysis

III. Asymptotic and Special Values

In general the notations used are standard. This includes the difference operator Δ defined on functions of x by $\Delta f(x) = f(x+1) - f(x)$, $\Delta^{n+1}f(x) = \Delta(\Delta^n f(x))$, the Kronecker delta δ_{ij} , the Riemann zeta function $\zeta(s)$ and the greatest common divisor symbol (m, n) . The range of the summands for a summation sign without limits is explained to the right of the formula.

The notations which are not standard are those for the multinomials which are arbitrary shorthand for use in this chapter, and those for the Stirling numbers which have never been standardized. A short table of various notations for these numbers follows:

Notations for the Stirling Numbers

Reference	First Kind	Second Kind
This chapter	$S_n^{(m)}$	$\mathfrak{S}_n^{(m)}$
[24.2] Fort	$S_n^{(m)}$	$\mathcal{S}_n^{(m)}$ *
[24.7] Jordan	$S_n^{(m)}$	$\mathfrak{S}_n^m$ *
[24.10] Moser and Wyman	S_n^m	σ_n^m
[24.9] Milne-Thomson	$\binom{n-1}{m-1} B_{n-m}^{(n)}$	$\binom{n}{m} B_{n-m}^{(-m)}$
[24.15] Riordan	$s(n, m)$	$S(n, m)$
[24.1] Carlitz }	$(-1)^{n-m} S_1(n-1, n-m)$	$S_2(m, n-m)$
[24.3] Gould }		
Miksa	$S(n-m+1, n)$	${}_m S_n$
(Unpublished tables)		
[24.17] Gupta		$u(n, m)$

We feel that a capital S is natural for Stirling numbers of the first kind; it is infrequently used for other notation in this context. But once it is used we have difficulty finding a suitable symbol for Stirling numbers of the second kind. The numbers are sufficiently important to warrant

a special and easily recognizable symbol, and yet that symbol must be easy to write. We have settled on a script capital $\mathfrak{S}$ without any certainty that we have settled this question permanently.

We feel that the subscript-superscript notation emphasizes the generating functions (which are powers of mutually inverse functions) from which most of the important relations flow.

24.1. Basic Numbers

24.1.1 Binomial Coefficients

I. Definitions

A. $\binom{n}{m}$ is the number of ways of choosing m objects from a collection of n distinct objects without regard to order.

B. Generating functions

$$* (1+x)^n = \sum_{m=0}^n \binom{n}{m} x^m \quad n=0, 1, \dots$$

$$(1-x)^{-m-1} = \sum_{n=m}^{\infty} \binom{n}{m} x^{n-m} \quad |x| < 1$$

C. Closed form

$$\begin{aligned} \binom{n}{m} &= \frac{n!}{m!(n-m)!} = \binom{n}{n-m} \\ &= \frac{n(n-1) \dots (n-m+1)}{m!} \quad n \geq m \end{aligned}$$

II. Relations

A. Recurrences

$$\begin{aligned} \binom{n+1}{m} &= \binom{n}{m} + \binom{n}{m-1} \quad n \geq m \geq 1 \\ &= \binom{n}{m} + \binom{n-1}{m-1} + \dots + \binom{n-m}{0} \quad n \geq m \end{aligned}$$

B. Checks

$$\sum_{m=0}^n \binom{r}{m} \binom{s}{n-m} = \binom{r+s}{n} \quad r+s \geq n$$

$$\sum_{m=0}^n (-1)^{n-m} \binom{r}{m} = \binom{r-1}{n} \quad r \geq n+1$$

$$\binom{n}{m} \equiv \binom{n_0}{m_0} \binom{n_1}{m_1} \dots \pmod{p} \quad p \text{ a prime}$$

where

$$n = \sum_{k=0}^{\infty} n_k p^k, \quad m = \sum_{k=0}^{\infty} m_k p^k \quad p > m_k, n_k \geq 0$$

C. Numerical analysis

$$\begin{aligned} \Delta^n f(x) &= \sum_{m=0}^n (-1)^{n-m} \binom{n}{m} f(x+m) \\ &= \sum_{k=0}^r \binom{r}{k} \Delta^{n+k} f(x-r) \end{aligned}$$

$$\begin{aligned} \sum_{m=0}^s (-1)^m \binom{n}{m} f(x-m) \\ = \sum_{k=0}^s (-1)^{s-k} \binom{n-k-1}{s-k} \Delta^k f(x-s) \quad s < n \end{aligned}$$

III. Special Values

$$\begin{aligned} \binom{n}{0} &= \binom{n}{n} = 1 \\ \binom{2n}{n} &= \frac{2^n (2n-1)(2n-3) \dots 3 \cdot 1}{n!} \end{aligned}$$

24.1.2 Multinomial Coefficients

I. Definitions

A. $(n; n_1, n_2, \dots, n_m)$ is the number of ways of putting $n = n_1 + n_2 + \dots + n_m$ different objects into m different boxes with n_k in the k -th box, $k = 1, 2, \dots, m$.

$(n; a_1, a_2, \dots, a_n)^*$ is the number of permutations of $n = a_1 + 2a_2 + \dots + na_n$ symbols composed of a_k cycles of length k for $k = 1, 2, \dots, n$.

$(n; a_1, a_2, \dots, a_n)'$ is the number of ways of partitioning a set of $n = a_1 + 2a_2 + \dots + na_n$ different objects into a_k subsets containing k objects for $k = 1, 2, \dots, n$.

B. Generating functions

$$(x_1 + x_2 + \dots + x_m)^n = \sum (n; n_1, n_2, \dots, n_m) x_1^{n_1} x_2^{n_2} \dots x_m^{n_m} \quad \text{summed over } n_1 + n_2 + \dots + n_m = n$$

$$\left(\sum_{k=1}^{\infty} \frac{x_k}{k} t^k \right)^m = m! \sum_{n=m}^{\infty} \frac{t^n}{n!} \sum (n; a_1, a_2, \dots, a_n)^* x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \quad \text{summed over } a_1 + 2a_2 + \dots + na_n = n$$

$$\left(\sum_{k=1}^{\infty} \frac{x_k}{k!} t^k \right)^m = m! \sum_{n=m}^{\infty} \frac{t^n}{n!} \sum (n; a_1, a_2, \dots, a_n)' x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} \quad \text{and } a_1 + a_2 + \dots + a_n = m$$

C. Closed forms

$$(n; n_1, n_2, \dots, n_m) = n! / n_1! n_2! \dots n_m! \quad n_1 + n_2 + \dots + n_m = n$$

$$(n; a_1, a_2, \dots, a_n)^* = n! / 1^{a_1} a_1! 2^{a_2} a_2! \dots n^{a_n} a_n! \quad a_1 + 2a_2 + \dots + na_n = n$$

$$(n; a_1, a_2, \dots, a_n)' = n! / (1!)^{a_1} a_1! (2!)^{a_2} a_2! \dots (n!)^{a_n} a_n! \quad a_1 + a_2 + \dots + a_n = n$$

II. Relations

A. Recurrence

$$(n+m; n_1+1, n_2+1, \dots, n_m+1) = \sum_{k=1}^m (n+m-1; n_1+1, \dots, n_{k-1}+1, n_k, n_{k+1}+1, \dots, n_m+1)$$

B. Checks

$$* \sum (n; n_1, n_2, \dots, n_m) = \begin{cases} m^n & \text{all } n_i \geq 1 \\ m! S_n^{(m)} & \end{cases} \quad \text{summed over } n_1 + n_2 + \dots + n_m = n$$

$$\sum (n; a_1, a_2, \dots, a_n)^* = (-1)^{n-m} S_n^{(m)} \quad \text{summed over } a_1 + 2a_2 + \dots + na_n = n \text{ and } a_1 + a_2 + \dots + a_n = m$$

$$\sum (n; a_1, a_2, \dots, a_n)' = S_n^{(m)}$$

C. Numerical analysis (Faà di Bruno's formula)

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{m=0}^n f^{(m)}(g(x)) \sum (n; a_1, a_2, \dots, a_n)' \{g'(x)\}^{a_1} \{g''(x)\}^{a_2} \dots \{g^{(n)}(x)\}^{a_n}$$

summed over $a_1 + 2a_2 + \dots + na_n = n$ and $a_1 + a_2 + \dots + a_n = m$.

*See page II.

$$\begin{vmatrix}
 P_1 & 1 & 0 & \dots & 0 \\
 P_2 & P_1 & 2 & \dots & . \\
 P_3 & P_2 & P_1 & \dots & . \\
 . & . & . & \dots & . \\
 . & . & . & \dots & 0 \\
 . & . & . & \dots & n-1 \\
 P_n & P_{n-1} & P_{n-2} & \dots & P_1
 \end{vmatrix} = \Sigma (-1)^{n-\Sigma a_i} (n; a_1, a_2, \dots, a_n) * P_1^{a_1} P_2^{a_2} \dots P_n^{a_n}$$

summed over $a_1 + 2a_2 + \dots + na_n = n$; e.g. if $P_k = \Sigma_{j=1}^n x_j^k$ for $k=1, 2, \dots, n$ then the determinant and sum equal $n! \Sigma x_1 x_2 \dots x_n$, the latter sum denoting the n -th elementary symmetric function of $x_1, x_2, \dots, x_r$.

24.1.3 Stirling Numbers of the First Kind

I. Definitions

A. $(-1)^{n-m} S_n^{(m)}$ is the number of permutations of n symbols which have exactly m cycles.

B. Generating functions

$$x(x-1) \dots (x-n+1) = \sum_{m=0}^n S_n^{(m)} x^m$$

$$\{\ln(1+x)\}^m = m! \sum_{n=m}^{\infty} S_n^{(m)} \frac{x^n}{n!} \quad |x| < 1$$

C. Closed form (see closed form for $\mathfrak{S}_n^{(m)}$)

$$S_n^{(m)} = \sum_{k=0}^{n-m} (-1)^k \binom{n-1+k}{n-m+k} \binom{2n-m}{n-m-k} \mathfrak{S}_{n-m+k}^{(k)}$$

II. Relations

A. Recurrences

$$S_{n+1}^{(m)} = S_n^{(m-1)} - n S_n^{(m)} \quad n \geq m \geq 1$$

$$\binom{m}{r} S_n^{(m)} = \sum_{k=m-r}^{n-r} \binom{n}{k} S_{n-k}^{(r)} S_k^{(m-r)} \quad n \geq m \geq r$$

B. Checks

$$\sum_{m=1}^n S_n^{(m)} = 0 \quad n > 1$$

$$\sum_{m=0}^n (-1)^{n-m} S_n^{(m)} = n!$$

$$\sum_{k=m}^n S_{n+1}^{(k+1)} n^{k-m} = S_n^{(m)}$$

C. Numerical analysis

$$\frac{d^m}{dx^m} f(x) = m! \sum_{n=m}^{\infty} \frac{S_n^{(m)}}{n!} \Delta^n f(x)$$

if convergent.

III. Asymptotics and Special Values

$$|S_n^{(m)}| \sim (n-1)! (\gamma + \ln n)^{m-1} / (m-1)!$$

for $m = o(\ln n)$

$$\lim_{m \rightarrow \infty} \frac{S_{n+m}^{(m)}}{m^{2n}} = \frac{(-1)^n}{2^n n!}$$

$$\lim_{n \rightarrow \infty} \frac{S_{n+1}^{(m)}}{n S_n^{(m)}} = -1$$

$$S_n^{(0)} = \delta_{0n}$$

$$S_n^{(1)} = (-1)^{n-1} (n-1)!$$

$$S_n^{(n-1)} = -\binom{n}{2}$$

$$S_n^{(n)} = 1$$

24.1.4 Stirling Numbers of the Second Kind

I. Definitions

A. $\mathfrak{S}_n^{(m)}$ is the number of ways of partitioning a set of n elements into m non-empty subsets.

B. Generating functions

$$x^n = \sum_{m=0}^n \mathfrak{S}_n^{(m)} x(x-1) \dots (x-m+1)$$

$$(e^x - 1)^m = m! \sum_{n=m}^{\infty} \mathfrak{S}_n^{(m)} \frac{x^n}{n!}$$

$$(1-x)^{-1} (1-2x)^{-1} \dots (1-mx)^{-1} = \sum_{n=m}^{\infty} \mathfrak{S}_n^{(m)} x^{n-m} \quad |x| < m^{-1}$$

C. Closed form

$$\mathfrak{S}_n^{(m)} = \frac{1}{m!} \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} k^n$$

II. Relations

A. Recurrences

$$\mathcal{S}_{n+1}^{(m)} = m \mathcal{S}_n^{(m)} + \mathcal{S}_n^{(m-1)} \quad n \geq m \geq 1$$

$$\binom{m}{r} \mathcal{S}_n^{(m)} = \sum_{k=m-r}^{n-r} \binom{n}{k} \mathcal{S}_{n-k}^{(r)} \mathcal{S}_k^{(m-r)} \quad n \geq m \geq r$$

B. Checks

$$\sum_{m=0}^n (-1)^{n-m} m! \mathcal{S}_n^{(m)} = 1$$

$$\sum_{k=m}^n \mathcal{S}_{k-1}^{(m-1)} m^{n-k} = \mathcal{S}_n^{(m)}$$

$$\mathcal{S}_n^{(m)} = \sum_{k=0}^{n-m} (-1)^k \binom{n-1+k}{n-m+k} \binom{2n-m}{n-m-k} \mathcal{S}_{n-m+k}^{(k)}$$

$$\sum_{k=m}^n \mathcal{S}_k^{(m)} \mathcal{S}_n^{(k)} = \sum_{k=m}^n \mathcal{S}_n^{(k)} \mathcal{S}_k^{(m)} = \delta_{mn}$$

C. Numerical analysis

$$\Delta^m f(x) = m! \sum_{n=m}^{\infty} \frac{\mathcal{S}_n^{(m)}}{n!} f^{(n)}(x) \quad \text{if convergent}$$

$$\sum_{k=0}^n k^m = \sum_{k=0}^m k! \mathcal{S}_n^{(k)} \binom{n+1}{k+1}$$

$$\sum_{k=0}^n k^m x^k = \sum_{j=0}^m \mathcal{S}_m^{(j)} x^j \frac{d^j}{dx^j} \left\{ \frac{1-x^{n+1}}{1-x} \right\}$$

III. Asymptotics and Special Values

$$\lim_{n \rightarrow \infty} m^{-n} \mathcal{S}_n^{(m)} = m!^{-1}$$

$$\mathcal{S}_{n+m}^{(m)} \sim \frac{m^{2n}}{2^n n!} \quad \text{for } n = o(m^{\frac{1}{2}})$$

$$\lim_{n \rightarrow \infty} \frac{\mathcal{S}_{n+1}^{(m)}}{\mathcal{S}_n^{(m)}} = m$$

$$\mathcal{S}_n^{(0)} = \delta_{0n}$$

$$\mathcal{S}_n^{(1)} = \mathcal{S}_n^{(n)} = 1$$

$$\mathcal{S}_n^{(n-1)} = \binom{n}{2}$$

24.2. Partitions

24.2.1 Unrestricted Partitions

I. Definitions

A. $p(n)$ is the number of decompositions of n into integer summands without regard to order. E.g., $5 = 1 + 4 = 2 + 3 = 1 + 1 + 3 = 1 + 2 + 2 = 1 + 1 + 1 + 2 = 1 + 1 + 1 + 1 + 1$ so that $p(5) = 7$.

B. Generating function

$$\sum_{n=0}^{\infty} p(n) x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-1} = \left\{ \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{3n^2+n}{2}} \right\}^{-1} \quad |x| < 1$$

C. Closed form

$$p(n) = \frac{1}{\pi \sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} A_k(n) \frac{d}{dn} \frac{\sinh \left\{ \frac{\pi}{k} \sqrt{\frac{2}{3}} \sqrt{n - \frac{1}{24}} \right\}}{\sqrt{n - \frac{1}{24}}}$$

where

$$A_k(n) = \sum_{\substack{0 < h \leq k \\ (h, k) = 1}} e^{\pi i s(h, k)} e^{-\frac{2\pi i h n}{k}}$$

$$s(h, k) = \sum_{j=1}^{k-1} \frac{j}{k} \left(\left(\frac{hj}{k} \right) \right)$$

$$\begin{aligned} ((x)) &= x - [x] - \frac{1}{2} \quad \text{if } x \text{ is not an integer} \\ &= 0 \quad \text{if } x \text{ is an integer} \end{aligned}$$

II. Relations

A. Recurrence

$$\begin{aligned} p(n) &= \sum_{1 \leq \frac{3k^2 \pm k}{2} \leq n} (-1)^{k-1} p\left(n - \frac{3k^2 \pm k}{2}\right) \quad p(0) = 1 \\ &= \frac{1}{n} \sum_{k=1}^n \sigma_1(k) p(n-k) \end{aligned}$$

B. Check

$$p(n) + \sum_{1 \leq \frac{3k^2 \pm k}{2} \leq n} (-1)^k \frac{3k^2 \pm k}{2} p\left(n - \frac{3k^2 \pm k}{2}\right) = \sigma_1(n)$$

III. Asymptotics

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi \sqrt{\frac{2}{3}} \sqrt{n}}$$

24.2.2 Partitions Into Distinct Parts

I. Definitions

A. $q(n)$ is the number of decompositions of n into distinct integer summands without regard to order. E.g., $5 = 1 + 4 = 2 + 3$ so that $q(5) = 3$.

B. Generating function

$$\sum_{n=0}^{\infty} q(n) x^n = \prod_{n=1}^{\infty} (1 + x^n) = \prod_{n=1}^{\infty} (1 - x^{2n-1})^{-1} \quad |x| < 1$$

C. Closed form

$$q(n) = \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} A_{2k-1}(n) \frac{d}{dn} J_0 \left(\frac{\pi i}{2k-1} \sqrt{\frac{1}{3}} \sqrt{n + \frac{1}{24}} \right)$$

where $J_0(x)$ is the Bessel function of order 0 and $A_{2k-1}(n)$ was defined in part I.C. of the previous subsection.

II. Relations

A. Recurrences

$$\sum_{0 \leq \frac{3k^2 \pm k}{2} \leq n} (-1)^k q\left(n - \frac{3k^2 \pm k}{2}\right) = (-1)^r \text{ if } n = 3r^2 \pm r$$

$$q(0) = 1$$

$$= 0 \text{ otherwise}$$

$$q(n) = \frac{1}{n} \sum_{k=1}^n \left\{ \sigma_1(k) - 2\sigma_1\left(\frac{k}{2}\right) \right\} q(n-k)$$

B. Check

$$\sum_{0 \leq 3k^2 \pm k \leq n} (-1)^k q(n - (3k^2 \pm k)) = 1 \text{ if } n = \frac{r^2 - r}{2}$$

$$= 0 \text{ otherwise.}$$

III. Asymptotics

$$q(n) \sim \frac{1}{4 \cdot 3^{1/4} \cdot n^{3/4}} e^{\pi \sqrt{1/3} \sqrt{n}}$$

24.3. Number Theoretic Functions

24.3.1 The Möbius Function

I. Definitions

A. $\mu(n) = 1$ if $n = 1$
 $= (-1)^k$ if n is the product of k distinct primes
 $= 0$ if n is divisible by a square > 1 .

B. Generating functions

$$\sum_{n=1}^{\infty} \mu(n) n^{-s} = 1/\zeta(s) \quad \Re s > 1$$

$$\sum_{n=1}^{\infty} \frac{\mu(n) x^n}{1 - x^n} = x \quad |x| < 1$$

II. Relations

A. Recurrence

$$\mu(mn) = \mu(m)\mu(n) \text{ if } (m, n) = 1$$

$$= 0 \text{ if } (m, n) > 1$$

B. Check

$$\sum_{d|n} \mu(d) = \delta_{n1}$$

C. Numerical analysis

$$g(n) = \sum_{d|n} f(d) \text{ for all } n \text{ if and only if}$$

$$f(n) = \sum_{d|n} \mu(d) g(n/d) \text{ for all } n$$

$$g(n) = \prod_{d|n} f(d) \text{ for all } n \text{ if and only if}$$

$$f(n) = \prod_{d|n} g(n/d)^{\mu(d)} \text{ for all } n$$

$$g(x) = \sum_{n=1}^{\lfloor x \rfloor} f(x/n) \text{ for all } x > 0 \text{ if and only if}$$

$$f(x) = \sum_{n=1}^{\lfloor x \rfloor} \mu(n) g(x/n) \text{ for all } x > 0$$

$$g(x) = \sum_{n=1}^{\infty} f(nx) \text{ for all } x > 0 \text{ if and only if}$$

$$f(x) = \sum_{n=1}^{\infty} \mu(n) g(nx) \text{ for all } x > 0$$

$$\text{and if } \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |f(mnx)| = \sum_{n=1}^{\infty} \sigma_0(n) |f(nx)| \text{ converges.}$$

The cyclotomic polynomial of order n is

$$\prod_{d|n} (x^d - 1)^{\mu(n/d)}$$

III. Asymptotics

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \ln n = -1$$

$$\sum_{n \leq x} \mu(n) = O(xe^{-c\sqrt{\ln x}})$$

24.3.2 The Euler Totient Function

I. Definitions

A. $\varphi(n)$ is the number of integers not exceeding and relatively prime to n .

B. Generating functions

$$\sum_{n=1}^{\infty} \varphi(n) n^{-s} = \frac{\zeta(s-1)}{\zeta(s)} \quad \Re s > 2$$

$$\sum_{n=1}^{\infty} \frac{\varphi(n) x^n}{1 - x^n} = \frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} k x^k \quad |x| < 1$$

C. Closed form

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

over distinct primes p dividing n .

II. Relations

A. Recurrence

$$\varphi(mn) = \varphi(m)\varphi(n) \quad (m, n) = 1$$

B. Checks

$$\sum_{d|n} \varphi(d) = n$$

$$\varphi(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d$$

$$a^{\varphi(n)} \equiv 1 \pmod{n} \quad (a, n) = 1$$

III. Asymptotics

$$\frac{1}{n^2} \sum_{k=1}^n \varphi(k) = \frac{3}{\pi^2} + O\left(\frac{\ln n}{n}\right)$$

24.3.3 Divisor Functions

I. Definitions

A. $\sigma_k(n)$ is the sum of the k -th powers of the divisors of n . Often $\sigma_0(n)$ is denoted by $d(n)$, and $\sigma_1(n)$ by $\sigma(n)$.

B. Generating functions

$$\sum_{n=1}^{\infty} \sigma_k(n) n^{-s} = \zeta(s) \zeta(s-k) \quad \Re s > k+1$$

$$\sum_{n=1}^{\infty} \sigma_k(n) x^n = \sum_{n=1}^{\infty} \frac{n^k x^n}{1-x^n} \quad |x| < 1$$

C. Closed form

$$\sigma_k(n) = \sum_{d|n} d^k = \prod_{i=1}^s \frac{p_i^{k(a_i+1)} - 1}{p_i^k - 1} \quad n = p_1^{a_1} p_2^{a_2} \dots p_s^{a_s}$$

II. Relations

A. Recurrences

$$\sigma_k(mn) = \sigma_k(m) \sigma_k(n) \quad (m, n) = 1$$

$$\sigma_k(np) = \sigma_k(n) \sigma_k(p) - p^k \sigma_k(n/p) \quad p \text{ prime}$$

III. Asymptotics

$$\frac{1}{n} \sum_{m=1}^n \sigma_0(m) = \ln n + 2\gamma - 1 + O(n^{-1})$$

(γ = Euler's constant)

24.3.4 Primitive Roots

I. Definitions

The integers not exceeding and relatively prime to a fixed integer n form a group; the group is cyclic if and only if $n=2, 4$ or n is of the form p^k or $2p^k$ where p is an odd prime. Then g is a primitive root of n if it generates that group; i.e., if $g, g^2, \dots, g^{\varphi(n)}$ are distinct modulo n . There are $\varphi(\varphi(n))$ primitive roots of n .

II. Relations

A. Recurrences. If g is a primitive root of a prime p and $g^{p-1} \not\equiv 1 \pmod{p^2}$ then g is a primitive root of p^k for all k . If $g^{p-1} \equiv 1 \pmod{p^2}$ then $g+p$ is a primitive root of p^k for all k .

If g is a primitive root of p^k then either g or $g+p^k$, whichever is odd, is a primitive root of $2p^k$.

B. Checks. If g is a primitive root of n then g^k is a primitive root of n if and only if $(k, \varphi(n)) = 1$, and each primitive root of n is of this form.

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Multinomials and Partitions

Table 24.2

$$\pi = 1^{a_1}, 2^{a_2}, \dots, n^{a_n}, n = a_1 + 2a_2 + \dots + na_n, m = a_1 + a_2 + \dots + a_n$$

$$M_1 = (n; n_1, n_2, \dots, n_m) = n! / (1!)^{a_1} (2!)^{a_2} \dots (n!)^{a_n}$$

$$M_2 = (n; a_1, a_2, \dots, a_n)^* = n! / 1^{a_1} a_1! 2^{a_2} a_2! \dots n^{a_n} a_n!$$

$$M_3 = (n; a_1, a_2, \dots, a_n)' = n! / (1!)^{a_1} a_1! (2!)^{a_2} a_2! \dots (n!)^{a_n} a_n!$$

n	m	π	M_1	M_2	M_3	n	m	π	M_1	M_2	M_3
1	1	1	1	1	1	8	1	8	1	5040	1
							2	1, 7	8	5760	8
2	1	2	1	1	1			2, 6	28	3360	28
	2	1 ²	2	1	1			3, 5	56	2688	56
								4 ²	70	1260	35
3	1	3	1	2	1		3	1 ² , 6	56	3360	28
	2	1, 2	3	3	3			1, 2, 5	168	4032	168
	3	1 ³	6	1	1			1, 3, 4	280	3360	280
								2 ² , 4	420	1260	210
4	1	4	1	6	1		4	2, 3 ²	560	1120	280
	2	1, 3	4	8	4			1 ³ , 5	336	1344	56
		2 ²	6	3	3			1 ² , 2, 4	840	2520	420
	3	1 ² , 2	12	6	6			1 ² , 3 ²	1120	1120	280
	4	1 ⁴	24	1	1			1, 2 ² , 3	1680	1680	840
								2 ⁴	2520	105	105
5	1	5	1	24	1		5	1 ⁴ , 4	1680	420	70
	2	1, 4	5	30	5			1 ³ , 2, 3	3360	1120	560
		2, 3	10	20	10			1 ² , 2 ³	5040	420	420
	3	1 ² , 3	20	20	10		6	1 ⁵ , 3	6720	112	56
		1, 2 ²	30	15	15			1 ⁴ , 2 ²	10080	210	210
	4	1 ³ , 2	60	10	10		7	1 ⁶ , 2	20160	28	28
	5	1 ⁵	120	1	1		8	1 ⁸	40320	1	1
						9	1	9	1	40320	1
6	1	6	1	120	1		2	1, 8	9	45360	9
	2	1, 5	6	144	6			2, 7	36	25920	36
		2, 4	15	90	15			3, 6	84	20160	84
		3 ²	20	40	10			4, 5	126	18144	126
	3	1 ² , 4	30	90	15		3	1 ² , 7	72	25920	36
		1, 2, 3	60	120	60			1, 2, 6	252	30240	252
		2 ³	90	15	15			1, 3, 5	504	24192	504
	4	1 ³ , 3	120	40	20			1, 4 ²	630	11340	315
		1 ² , 2 ²	180	45	45			2 ² , 5	756	9072	378
	5	1 ⁴ , 2	360	15	15			2, 3, 4	1260	15120	1260
	6	1 ⁶	720	1	1			3 ³	1680	2240	280
							4	1 ³ , 6	504	10080	84
7	1	7	1	720	1			1 ² , 2, 5	1512	18144	756
	2	1, 6	7	840	7			1 ² , 3, 4	2520	15120	1260
		2, 5	21	504	21			1, 2 ² , 4	3780	11340	1890
		3, 4	35	420	35			1, 2, 3 ²	5040	10080	2520
	3	1 ² , 5	42	504	21		5	2 ³ , 3	7560	2520	1260
		1, 2, 4	105	630	105			1 ⁴ , 5	3024	3024	126
		1, 3 ²	140	280	70			1 ³ , 2, 4	7560	7560	1260
	4	2 ² , 3	210	210	105			1 ³ , 3 ²	10080	3360	840
		1 ³ , 4	210	210	35			1 ² , 2 ² , 3	15120	7560	3780
		1 ² , 2, 3	420	420	210		6	1, 2 ⁴	22680	945	945
		1, 2 ³	630	105	105			1 ⁵ , 4	15120	756	126
	5	1 ⁴ , 3	840	70	35			1 ⁴ , 2, 3	30240	2520	1260
		1 ³ , 2 ²	1260	105	105		7	1 ³ , 2 ³	45360	1260	1260
	6	1 ⁶ , 2	2520	21	21			1 ⁶ , 3	60480	168	84
	7	1 ⁷	5040	1	1			1 ⁵ , 2 ²	90720	378	378
							8	1 ⁷ , 2	181440	36	36
							9	1 ⁹	362880	1	1

Table 24.2

Multinomials and Partitions

n	m	π	M_1	M_2	M_3	n	m	π	M_1	M_2	M_3
10	1	10	1	362880	1	10		$2^3, 4$	18900	18900	3150
	2	1, 9	10	403200	10			$2^2, 3^2$	25200	25200	6300
		2, 8	45	226800	45		5	$1^4, 6$	5040	25200	210
		3, 7	120	172800	120			$1^3, 2, 5$	15120	60480	2520
		4, 6	210	151200	210			$1^3, 3, 4$	25200	50400	4200
		5^2	252	72576	126			$1^2, 2^2, 4$	*37800	*56700	9450
	3	$1^2, 8$	90	226800	45			$1^2, 2, 3^2$	50400	50400	12600
		1, 2, 7	360	259200	360			$1, 2^3, 3$	75600	25200	12600
		1, 3, 6	840	201600	840			2^5	113400	945	945
		1, 4, 5	1260	181440	1260		6	$1^5, 5$	30240	6048	252
		$2^2, 6$	1260	75600	630			$1^4, 2, 4$	75600	18900	3150
		2, 3, 5	2520	*120960	2520			$1^4, 3^2$	100800	8400	2100
		$2, 4^2$	3150	56700	1575			$1^3, 2^2, 3$	151200	25200	12600
		$3^2, 4$	4200	50400	2100			$1^2, 2^4$	226800	4725	4725
	4	$1^3, 7$	720	86400	120		7	$1^6, 4$	151200	1260	210
		$1^2, 2, 6$	2520	151200	1260			$1^5, 2, 3$	302400	5040	2520
		$1^2, 3, 5$	5040	120960	2520			$1^4, 2^3$	453600	3150	3150
		$1^2, 4^2$	6300	56700	1575		8	$1^7, 3$	604800	240	120
		1, $2^2, 5$	7560	90720	3780			$1^6, 2^2$	907200	630	630
		1, 2, 3, 4	12600	151200	12600		9	$1^8, 2$	1814400	45	45
		$1, 3^3$	16800	22400	2800		10	1^{10}	3628800	1	1

*See page II.

STIRLING NUMBERS OF THE FIRST KIND $s_n^{(m)}$

Table 24.3

$n \backslash m$	1	2	3
1	1		
2	-1	1	
3	2	-3	1
4	-6	11	-6
5	24	-50	35
6	-120	274	-225
7	720	-1764	1624
8	-5040	13068	-13132
9	40320	-1 09584	1 18124
10	-3 62880	10 26576	-11 72700
11	36 28800	-106 28640	127 53576
12	-399 16800	1205 43840	-1509 17976
13	4790 01600	-14864 42880	19315 59552
14	-62270 20800	1 98027 59040	-2 65967 17056
15	8 71782 91200	-28 34656 47360	39 21567 97824
16	-130 76743 68000	433 91630 01600	-616 58176 14720
17	2092 27898 88000	-7073 42823 93600	10299 22448 37120
18	-35568 74280 96000	1 22340 55905 79200	-1 82160 24446 24640
19	6 40237 37057 28000	-22 37698 80585 21600	34 01224 95938 22720
20	-121 64510 04088 32000	431 56514 68176 38400	-668 60973 03411 53280
21	2432 90200 81766 40000	-8752 94803 67616 00000	13803 75975 36407 04000
22	-51090 94217 17094 40000	1 86244 81078 01702 40000	-2 98631 90286 32163 84000
23	11 24000 72777 76076 80000	-41 48476 77933 54547 20000	67 56146 67377 09306 88000
24	-258 52016 73888 49766 40000	965 38966 65249 30662 40000	-1595 39850 27606 68605 44000
25	6204 48401 73323 94393 60000	-23427 87216 39871 85664 00000	39254 95373 27809 77192 96000

$\sum_{m=1}^{\infty} S_n^{(m)} = e^{-1}$
 $\sum_{m=1}^{\infty} S_n^{(2)} = 0,7407167843$
 $A \phi 142$
 $A \phi 254$

$n \backslash m$	4	5	6
4	1		
5	-10	1	
6	85	-15	1
7	-735	175	-21
8	6769	-1960	322
9	-67284	22449	-4536
10	7 23680	-2 69325	63273
11	-84 09500	34 16930	-9 02055
12	1052 58076	-459 95730	133 39535
13	-14140 14888	6572 06836	-2060 70150
14	2 03137 53096	-99577 03756	33361 18786
15	-31 09892 60400	15 97216 05680	-5 66633 66760
16	505 69957 03824	-270 68133 45600	100 96721 07080
17	-8707 77488 75904	4836 60092 33424	-1886 15670 58880
18	1 58331 39757 27488	-90929 99058 44112	36901 26492 34384
19	-30 32125 40077 19424	17 95071 22809 21504	-7 55152 75920 63024
20	610 11607 57404 91776	-371 38478 73452 28000	161 42973 65301 18960
21	-12870 93124 51509 88800	8037 81182 26450 51776	-3599 97951 79476 07200
22	2 84093 31590 18114 68800	-1 81664 97952 06970 76096	83637 38169 95448 02976
23	-65 48684 85270 30686 97600	42 80722 86535 71471 42912	-20 21687 37691 06827 41568
24	1573 75898 28594 15107 32800	-1050 05310 75591 74529 84576	507 79532 53430 28501 98976
25	-39365 61409 13866 31181 31200	26775 03356 42796 03823 62624	-13237 14091 57918 58577 60000

From unpublished tables of Francis L. Miksa, with permission.

Table 24.3 STIRLING NUMBERS OF THE FIRST KIND $s_n^{(m)}$

$n \backslash m$	7	8	9
7	1		
8	-28	1	
9	546	-36	1
10	-9450	870	-45
11	1 57773	- 18150	1320
12	-26 37558	3 57423	- 32670
13	449 90231	-69 26634	7 49463
14	-7909 43153	1350 36473	-166 69653
15	1 44093 22928	- 26814 53775	3684 11615
16	-27 28032 10680	5 46311 29553	- 82076 28000
17	537 45234 77960	-114 69012 83528	18 59531 77553
18	- 11022 84661 84200	2487 18452 97936	-430 81053 01929
19	2 35312 50405 49984	- 55792 16815 47048	10241 77407 32658
20	-52 26090 33625 12720	12 95363 69899 43896	-2 50385 87554 67550
21	1206 64780 37803 73360	-311 33364 31613 90640	63 03081 20992 94896
22	- 28939 58339 73354 47760	7744 65431 01695 76800	-1634 98069 72465 83456
23	7 20308 21644 09246 53696	-1 99321 97822 10661 37360	43714 22964 95944 12832
24	-185 88776 35505 19497 76576	53 04713 71552 54458 12976	-12 04749 26016 17376 32496
25	4969 10165 05554 96448 36800	-1459 01905 52766 26492 88000	342 18695 95940 71489 92880

$n \backslash m$	10	11	12
10	1		
11	-55	1	
12	1925	-66	1
13	- 55770	2717	-78
14	14 74473	- 91091	3731
15	-373 12275	27 49747	-1 43325
16	9280 95740	-785 58480	48 99622
17	-2 30571 59840	21850 31420	-1569 52432
18	57 79248 94833	-6 02026 93980	48532 22764
19	-1471 07534 08923	166 15733 86473	-14 75607 03732
20	38192 20555 02195	-4628 06477 51910	446 52267 57381
21	-10 14229 98655 11450	1 30753 50105 40395	- 13558 51828 99530
22	276 01910 92750 35346	-37 60053 50868 59745	4 15482 38514 30525
23	-7707 40110 12973 61068	1103 23088 11859 49736	-129 00665 98183 31295
24	2 20984 45497 94337 17396	- 33081 71136 85742 04996	4070 38405 70075 69521
25	-65 08376 17966 81468 50000	10 14945 52782 52146 37300	-1 30770 92873 67558 73500

$n \backslash m$	13	14	15	16
13	1			
14	-91	1		
15	5005	-105	1	
16	-2 18400	6580	-120	1
17	83 94022	-3 23680	8500	-136
18	-2996 50806	138 96582	-4 68180	10812
19	1 02469 37272	-5497 89282	223 23822	-6 62796
20	-34 22525 11900	2 06929 33630	-9739 41900	349 16946
21	1131 02769 95381	-75 61111 84500	4 01717 71630	- 16722 80820
22	- 37310 09998 02531	2718 86118 69881	-159 97183 88730	7 52896 68850
23	12 36304 58470 86207	- 97125 04609 39913	6238 24164 21941	-325 60911 03430
24	-413 35671 43013 14056	34 70180 64487 04206	-2 40604 60386 44556	13727 25118 00831
25	13990 94520 02391 06865	-1246 20006 90702 15000	92 44691 13761 73550	-5 70058 63218 64500

$n \backslash m$	17	18	19	20	21	22	23	24	25
17	1								
18	-153	1							
19	13566	-171	1						
20	-9 20550	16815	-190	1					
21	533 27946	-12 56850	20615	-210	1				
22	- 27921 67686	797 21796	-16 89765	25025	-231	1			
23	13 67173 57942	- 45460 47198	1168 96626	-22 40315	30107	-253	1		
24	-640 05903 36096	24 12764 43496	- 72346 69596	1684 23871	-29 32776	35926	-276	1	
25	29088 66798 67135	-1219 12249 80000	41 49085 13800	-1 12768 42500	2388 10495	-37 95000	42550	-300	1

COMBINATORIAL ANALYSIS

STIRLING NUMBERS OF THE SECOND KIND $S_n^{(m)}$ Table 24.1

$n \backslash m$	2	3	4	5	6
1	1				
2	1				
3	3	1			
4	7	6	1		
5	15	25	10	1	
6	31	90	65	15	1
7	63	301	350	140	21
8	127	966	1701	1050	266
9	255	3025	7770	6951	2646
10	511	9330	34105	42525	22827
11	1023	28501	1 45750	2 46730	1 79487
12	2047	86526	6 11501	13 79400	13 23652
13	4095	2 61625	25 32530	75 08501	93 21312
14	8191	7 88970	103 91745	400 75035	634 36373
15	16383	23 75101	423 55950	2107 66920	4206 93273
16	32767	71 41686	1717 98901	10961 90550	27349 26558
17	65535	214 57825	6943 37290	56527 51651	1 75057 49898
18	1 31071	644 39010	27988 06985	2 89580 95545	11 06872 51039
19	2 62143	1934 48101	1 12596 66950	14 75892 84710	69 30816 01779
20	5 24287	5806 06446	4 52321 15901	74 92060 90500	430 60788 95384
21	10 48575	17423 43625	18 15090 70050	379 12625 68401	2658 56794 62804
22	20 97151	52280 79450	72 77786 23825	1913 78219 12055	16330 53393 45225
23	41 94303	1 56863 35501	291 63425 74750	9641 68881 84100	99896 98579 83405
24	83 88607	4 70632 00806	1168 10566 34501	48500 07834 95250	6 09023 60360 84530
25	167 77215	14 11979 91025	4677 12897 38810	2 43668 49741 10751	37 02641 70000 02430
26	334 54431	47 23958 18101	16514 10566 10566	1167 92144 10929 73005	1203 16339 21753 57500
27	668 108861	148 35916 54652	54652 21417 64053	12327 24764 65204	7118 71322 91275
28	1336 217722	367 71832 20505	1 14239 90799 91620	1 24196 33035 33920	83514 37993 77954
29	2672 435444	915 14364 00400	9 74195 50199 00400	12 32006 88117 96900	9 59340 12973 13460
30	5344 870888	2236 28918 20505	82 31828 21583 20505	120 62257 43260 72500	108 25408 17849 31500
31	10688 1741776	5572 57836 68580	690 22372 11183 68580	1167 92144 10929 73005	1203 16339 21753 57500
32	21376 3483552	14144 115712 20505	1 14239 90799 91620	1 24196 33035 33920	83514 37993 77954
33	42752 6967104	36368 229424 20505	9 74195 50199 00400	12 32006 88117 96900	9 59340 12973 13460
34	85504 13934208	90916 458848 20505	82 31828 21583 20505	120 62257 43260 72500	108 25408 17849 31500
35	171008 27868416	227232 917696 20505	690 22372 11183 68580	1167 92144 10929 73005	1203 16339 21753 57500
36	342016 55736832	564464 1835392 20505	1 14239 90799 91620	1 24196 33035 33920	83514 37993 77954
37	684032 111473664	1411136 3678784 20505	9 74195 50199 00400	12 32006 88117 96900	9 59340 12973 13460
38	1368064 222947328	3628272 9036736 20505	82 31828 21583 20505	120 62257 43260 72500	108 25408 17849 31500
39	2736128 445894656	9070544 21693472 20505	690 22372 11183 68580	1167 92144 10929 73005	1203 16339 21753 57500
40	5472256 891789312	22741088 54186944 20505	1 14239 90799 91620	1 24196 33035 33920	83514 37993 77954
41	10944512 1783578624	56482176 136373888 20505	9 74195 50199 00400	12 32006 88117 96900	9 59340 12973 13460
42	21889024 3567157248	141164352 332747776 20505	82 31828 21583 20505	120 62257 43260 72500	108 25408 17849 31500
43	43778048 7134314496	362850704 846495552 20505	690 22372 11183 68580	1167 92144 10929 73005	1203 16339 21753 57500
44	87556096 14268628992	907361408 2168991104 20505	1 14239 90799 91620	1 24196 33035 33920	83514 37993 77954
45	175112192 28537257984	2274282816 5418982208 20505	9 74195 50199 00400	12 32006 88117 96900	9 59340 12973 13460
46	350224384 57074515968	5648425632 13639964416 20505	82 31828 21583 20505	120 62257 43260 72500	108 25408 17849 31500
47	700448768 114149031936	14117251264 33279928832 20505	690 22372 11183 68580	1167 92144 10929 73005	1203 16339 21753 57500
48	1400897536 228298063872	36288502528 84659844160 20505	1 14239 90799 91620	1 24196 33035 33920	83514 37993 77954
49	2801795072 456596127744	90737005056 216899288320 20505	9 74195 50199 00400	12 32006 88117 96900	9 59340 12973 13460
50	5603590144 913192255488	227434010112 541899576640 20505	82 31828 21583 20505	120 62257 43260 72500	108 25408 17849 31500
51	11207180288 1826384510976	564868020224 1363999153280 20505	690 22372 11183 68580	1167 92144 10929 73005	1203 16339 21753 57500
52	22414360576 3652769021952	1411736040448 3327995766400 20505	1 14239 90799 91620	1 24196 33035 33920	83514 37993 77954
53	44828721152 7305538043904	3628872080896 8465991532800 20505	9 74195 50199 00400	12 32006 88117 96900	9 59340 12973 13460
54	89657442304 14611076087808	9073740161792 21689957664000 20505	82 31828 21583 20505	120 62257 43260 72500	108 25408 17849 31500
55	179314884608 29222152175616	22743680323904 54190000000000 20505	690 22372 11183 68580	1167 92144 10929 73005	1203 16339 21753 57500
56	358629769216 58444304351232	56487360647808 136400000000000 20505	1 14239 90799 91620	1 24196 33035 33920	83514 37993 77954
57	717259538432 116888608702464	141174721297632 332800000000000 20505	9 74195 50199 00400	12 32006 88117 96900	9 59340 12973 13460
58	1434519076864 233777217404928	362889442595264 846600000000000 20505	82 31828 21583 20505	120 62257 43260 72500	108 25408 17849 31500
59	2869038153728 467554434809856	907378885190528 2169000000000000 20505	690 22372 11183 68580	1167 92144 10929 73005	1203 16339 21753 57500
60	5738076307456 935108869619712	2274396803980800 5419000000000000 20505	1 14239 90799 91620	1 24196 33035 33920	83514 37993 77954
61	11476152614912 1870217739239424	5648793607961600 13640000000000000 20505	9 74195 50199 00400	12 32006 88117 96900	9 59340 12973 13460
62	22952305229824 3740435478478848	1411754421592320 33280000000000000 20505	82 31828 21583 20505	120 62257 43260 72500	108 25408 17849 31500
63	45904610459648 7480870956957696	3628897843184640 84660000000000000 20505	690 22372 11183 68580	1167 92144 10929 73005	1203 16339 21753 57500
64	91809220919296 14961741913915392	9073795686369280 216900000000000000 20505	1 14239 90799 91620	1 24196 33035 33920	83514 37993 77954
65	183618441838592 29923483827830784	22743991372738560 541900000000000000 20505	9 74195 50199 00400	12 32006 88117 96900	9 59340 12973 13460
66	367236883677184 59846967655661568	56488002745477120 1364000000000000000 20505	82 31828 21583 20505	120 62257 43260 72500	108 25408 17849 31500
67	734473767354368 119693935311323136	14117576450914240 3328000000000000000 20505	690 22372 11183 68580	1167 92144 10929 73005	1203 16339 21753 57500
68	1468947534708736 239387870622646272	36289004901828480 8466000000000000000 20505	1 14239 90799 91620	1 24196 33035 33920	83514 37993 77954
69	2937895069417472 478775741245292544	90738019803656960 21690000000000000000 20505	9 74195 50199 00400	12 32006 88117 96900	9 59340 12973 13460
70	5875790138834944 957551482490585088	227440039607313920 54190000000000000000 20505	82 31828 21583 20505	120 62257 43260 72500	108 25408 17849 31500
71	11751580277669888 1915102964981170176	56488019803656960 136400000000000000000 20505	690 22372 11183 68580	1167 92144 10929 73005	1203 16339 21753 57500
72	23503160555339776 3830205929962340352	14117581121462720 332800000000000000000 20505	1 14239 90799 91620	1 24196 33035 33920	83514 37993 77954
73	47006321110679552 7660411859924680704	36289023803656960 846600000000000000000 20505	9 74195 50199 00400	12 32006 88117 96900	9 59340 12973 13460
74	94012642221359104 15320823719849361408	90738039603656960 2169000000000000000000 20505	82 31828 21583 20505	120 62257 43260 72500	108 25408 17849 31500
75	188025284442718208 30641647439698722816	227440079207313920 5419000000000000000000 20505	690 22372 11183 68580	1167 92144 10929 73005	1203 16339 21753 57500
76	376050568885436416 61283294879397445632	56488039603656960 13640000000000000000000 20505	1 14239 90799 91620	1 24196 33035 33920	83514 37993 77954
77	752101137770872832 122566589758794891264	14117582242925440 33280000000000000000000 20505	9 74195 50199 00400	12 32006 88117 96900	9 59340 12973 13460
78	1504202275541745664 245133179517589782528	36289027603656960 84660000000000000000000 20505	82 31828 21583 20505	120 62257 43260 72500	108 25408 17849 31500
79	3008404551083491328 490266359035179565056	90738047203656960 216900000000000000000000 20505	690 22372 11183 68580	1167 92144 10929 73005	1203 16339 21753 57500
80	6016809102166982656 980532718070359130112	227440098407313920 541900000000000000000000 20505	1 14239 90799 91620	1 24196 33035 33920	83514 37993 77954
81	12033618204333965312 1961065436140718260224	56488049603656960 1364000000000000000000000 20505	9 74195 50199 00400	12 32006 88117 96900	9 59340 12973 13460
82	24067236408667930624 3922130872281436520448	14117582485850880 3328000000000000000000000 20505	82 31828 21583 20505	120 62257 43260 72500	108 25408 17849 31500
83	48134472817335861248 7844261744562873040896	36289029603656960 8466000000000000000000000 20505	690 22372 11183 68580	1167 92144 10929 73005	1203 16339 21753 57500
84	96268945634671722496 15688523489125746081792	90738059203656960 21690000000000000000000000 20505	1 14239 90799 91620	1 24196 33035 33920	83514 37993 77954
85	192537891269343444992 31377046978251492163584	227440118407313920 54190000000000000000000000 20505	9 74195 50199 00400	12 32006 88117 96900	9 59340 12973 13460
86	385075782538686889984 62754093956502984327168	56488059403656960 136400000000000000000000000 20505	82 31828 21583 20505	120 62257 43260 72500	108 25408 17849 31500
87	770151565077373779968 125508187913005968654336	14117582571660800 332800000000000000000000000 20505	690 22372 11183 68580	1167 92144 10929 73005	1203 16339 21753 57500
88	1540303130154747559936 251016375826011937308672	36289030403656960 846600000000000000000000000 20505	1 14239 90799 91620	1 24196 33035 33920	83514 37993 77954
89	3080606260309495119872 502032751652023874617344	90738060803656960 2169000000000000000000000000 20505	9 74195 50199 00400	12 32006 88117 96900	9 59340 12973 13460
90	6161212520618990239744 904065503304047749234688	227440138407313920 5419000000000000000000000000 20505	82 31828 21583 20505	120 62257 4326	

25. Numerical Interpolation, Differentiation, and Integration

PHILIP J. DAVIS¹ AND IVAN POLONSKY²

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Table 25.1. n -Point Lagrangian Interpolation Coefficients ($3 \leq n \leq 8$) . .	900
$n=3, 4, p=-\left[\frac{n-1}{2}\right] (.01) \left[\frac{n}{2}\right],$ Exact	
$n=5, 6, p=-\left[\frac{n-1}{2}\right] (.01) \left[\frac{n}{2}\right],$ 10D	
$n=7, 8, p=-\left[\frac{n-1}{2}\right] (.1) \left[\frac{n}{2}\right],$ 10D	
Table 25.2. n -Point Coefficients for k -th Order Differentiation ($1 \leq k \leq 5$)	914
$k=1, \quad n=3(1)6, \quad$ Exact	
$k=2(1)5, \quad n=k+1(1)6, \quad$ Exact	
Table 25.3. n -Point Lagrangian Integration Coefficients ($3 \leq n \leq 10$) . .	915
$n=3(1)10, \quad$ Exact	
Table 25.4. Abscissas and Weight Factors for Gaussian Integration ($2 \leq n \leq 96$).	916
$n=2(1)10, 12, \quad$ 15D	
$n=16(4)24(8)48(16)96, \quad$ 21D	
Table 25.5. Abscissas for Equal Weight Chebyshev Integration ($2 \leq n \leq 9$)	920
$n=2(1)7, 9, \quad$ 10D	
Table 25.6. Abscissas and Weight Factors for Lobatto Integration ($3 \leq n \leq 10$).	920
$n=3(1)10, \quad$ 8-10D	
Table 25.7. Abscissas and Weight Factors for Gaussian Integration for Integrands with a Logarithmic Singularity ($2 \leq n \leq 4$)	920
$n=2(1)4, \quad$ 6D	

¹ National Bureau of Standards.

² National Bureau of Standards. (Presently, Bell Tel. Labs., Whippany, N.J.)

Table 25.8. Abscissas and Weight Factors for Gaussian Integration of Moments ($1 \leq n \leq 8$)	Page 921
$k=0(1)5, \quad n=1(1)8, \quad 10D$	
Table 25.9. Abscissas and Weight Factors for Laguerre Integration ($2 \leq n \leq 15$)	923
$n=2(1)10, 12, 15, \quad 12D \text{ or } S$	
Table 25.10. Abscissas and Weight Factors for Hermite Integration ($2 \leq n \leq 20$)	924
$n=2(1)10, 12, 16, 20, \quad 13-15D \text{ or } S$	
Table 25.11. Coefficients for Filon's Quadrature Formula ($0 \leq \theta \leq 1$) . .	924
$\theta=0(.01).1(.1)1, \quad 8D$	

25. Numerical Interpolation, Differentiation, and Integration

Numerical analysts have a tendency to accumulate a multiplicity of tools each designed for highly specialized operations and each requiring special knowledge to use properly. From the vast stock of formulas available we have culled the present selection. We hope that it will be useful. As with all such compendia, the reader may miss his favorites and find others whose utility he thinks is marginal.

We would have liked to give examples to illuminate the formulas, but this has not been feasible. Numerical analysis is partially a science and partially an art, and short of writing a textbook on the subject it has been impossible to indicate where and under what circumstances the various formulas are useful or accurate, or to elucidate the numerical difficulties to which one might be led by uncritical use. The formulas are therefore issued together with a caveat against their blind application.

Formulas

Notation: Abscissas: $x_0 < x_1 < \dots$; functions: $f, g, \dots$; values: $f(x_i) = f_i, f'(x_i) = f'_i, f'', f^{(2)}, \dots$ indicate 1st, 2^d, $\dots$ derivatives. If abscissas are equally spaced, $x_{i+1} - x_i = h$ and $f_p = f(x_0 + ph)$ (p not necessarily integral). R, R_n indicate remainders.

25.1. Differences

Forward Differences

25.1.1

$$\Delta(f_n) = \Delta_n = \Delta_n^1 = f_{n+1} - f_n$$

$$\Delta_n^2 = \Delta_{n+1}^1 - \Delta_n^1 = f_{n+2} - 2f_{n+1} + f_n$$

$$\Delta_n^3 = \Delta_{n+1}^2 - \Delta_n^2 = f_{n+3} - 3f_{n+2} + 3f_{n+1} - f_n$$

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1} = \sum_{j=0}^k (-1)^j \binom{k}{j} f_{n+k-j}$$

Central Differences

25.1.2

$$\delta(f_{n+\frac{1}{2}}) = \delta_{n+\frac{1}{2}} = \delta_{n+\frac{1}{2}}^1 = f_{n+1} - f_n$$

$$\delta_n^2 = \delta_{n+\frac{1}{2}}^1 - \delta_{n-\frac{1}{2}}^1 = f_{n+1} - 2f_n + f_{n-1}$$

$$\delta_{n+\frac{1}{2}}^3 = \delta_{n+1}^2 - \delta_n^2 = f_{n+2} - 3f_{n+1} + 3f_n - f_{n-1}$$

$$\delta_n^{2k} = \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} f_{n+k-j}$$

$$\delta_{n+\frac{1}{2}}^{2k+1} = \sum_{j=0}^{2k+1} (-1)^j \binom{2k+1}{j} f_{n+k+\frac{1}{2}-j}$$

$$\delta_{\frac{1}{2}n}^k = \Delta_{\frac{1}{2}(n-k)}^k \text{ if } n \text{ and } k \text{ are of same parity.}$$

Forward Differences

$$\begin{array}{ccc} x_0 & f_0 & \\ & \Delta_0 & \\ x_1 & f_1 & \Delta_0^2 \\ & \Delta_1 & \Delta_0^3 \\ x_2 & f_2 & \Delta_1^2 \\ & \Delta_2 & \\ x_3 & f_3 & \end{array}$$

Central Differences

$$\begin{array}{ccc} x_{-1} & f_{-1} & \\ & \delta_{-1} & \\ x_0 & f_0 & \delta_0^2 \\ & \delta_1 & \delta_0^3 \\ x_1 & f_1 & \delta_1^2 \\ & \delta_{3/2} & \\ x_2 & f_2 & \end{array}$$

Mean Differences

25.1.3

$$\mu(f_n) = \frac{1}{2}(f_{n+\frac{1}{2}} + f_{n-\frac{1}{2}})$$

Divided Differences

25.1.4

$$[x_0, x_1] = \frac{f_0 - f_1}{x_0 - x_1} = [x_1, x_0]$$

$$[x_0, x_1, x_2] = \frac{[x_0, x_1] - [x_1, x_2]}{x_0 - x_2}$$

$$[x_0, x_1, \dots, x_k] = \frac{[x_0, \dots, x_{k-1}] - [x_1, \dots, x_k]}{x_0 - x_k}$$

Divided Differences in Terms of Functional Values

25.1.5

$$[x_0, x_1, \dots, x_n] = \sum_{k=0}^n \frac{f_k}{\pi'_n(x_k)}$$

25.1.6 where $\pi_n(x) = (x-x_0)(x-x_1)\dots(x-x_n)$ and $\pi'_n(x)$ is its derivative:

25.1.7

$$\pi'_n(x_k) = (x_k - x_0) \dots (x_k - x_{k-1})(x_k - x_{k+1}) \dots (x_k - x_n)$$

Let D be a simply connected domain with a piecewise smooth boundary C and contain the points $z_0, \dots, z_n$ in its interior. Let $f(z)$ be analytic in D and continuous in $D+C$. Then,

$$\mathbf{25.1.8} \quad [z_0, z_1, \dots, z_n] = \frac{1}{2\pi i} \int_C \frac{f(z)}{\prod_{k=0}^n (z - z_k)} dz$$

$$\mathbf{25.1.9} \quad \Delta_0^n = h^n f^{(n)}(\xi) \quad (x_0 < \xi < x_n)$$

25.1.10

$$[x_0, x_1, \dots, x_n] = \frac{\Delta_0^n}{n! h^n} = \frac{f^{(n)}(\xi)}{n!} \quad (x_0 < \xi < x_n)$$

25.1.11

$$[x_{-n}, x_{-n+1}, \dots, x_0, \dots, x_n] = \frac{\delta_0^{2n}}{h^{2n}(2n)!}$$

Reciprocal Differences

25.1.12

$$\rho(x_0, x_1) = \frac{x_0 - x_1}{f_0 - f_1}$$

$$\rho_2(x_0, x_1, x_2) = \frac{x_0 - x_2}{\rho(x_0, x_1) - \rho(x_1, x_2)} + f_1$$

$$\rho_3(x_0, x_1, x_2, x_3) = \frac{x_0 - x_3}{\rho_2(x_0, x_1, x_2) - \rho_2(x_1, x_2, x_3)} + \rho(x_1, x_2)$$

...

$$\rho_n(x_0, x_1, \dots, x_n) = \frac{x_0 - x_n}{\rho_{n-1}(x_0, \dots, x_{n-1}) - \rho_{n-1}(x_1, \dots, x_n)} + \rho_{n-2}(x_1, \dots, x_{n-1})$$

25.2. Interpolation

Lagrange Interpolation Formulas

$$\mathbf{25.2.1} \quad f(x) = \sum_{i=0}^n l_i(x) f_i + R_n(x)$$

25.2.2

$$l_i(x) = \frac{\pi_n(x)}{(x-x_i)\pi'_n(x_i)} = \frac{(x-x_0)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)}$$

Remainder in Lagrange Interpolation Formula

25.2.3

$$R_n(x) = \pi_n(x) \cdot [x_0, x_1, \dots, x_n, x] = \pi_n(x) \cdot \frac{f^{(n+1)}(\xi)}{(n+1)!} \quad (x_0 < \xi < x_n)$$

25.2.4

$$|R_n(x)| \leq \frac{(x_n - x_0)^{n+1}}{(n+1)!} \max_{x_0 \leq x \leq x_n} |f^{(n+1)}(x)|$$

25.2.5

$$R_n(z) = \frac{\pi_n(z)}{2\pi i} \int_C \frac{f(t)}{(t-z)(t-z_0)\dots(t-z_n)} dt$$

The conditions of **25.1.8** are assumed here.

Lagrange Interpolation, Equally Spaced Abscissas

n Point Formula

$$\mathbf{25.2.6} \quad f(x_0 + ph) = \sum_k A_k^n(p) f_k + R_{n-1}$$

$$\text{For } n \text{ even,} \quad \left(-\frac{1}{2}(n-2) \leq k \leq \frac{1}{2}n\right).$$

$$\text{For } n \text{ odd,} \quad \left(-\frac{1}{2}(n-1) \leq k \leq \frac{1}{2}(n-1)\right).$$

25.2.7

$$A_k^n(p) = \frac{(-1)^{\frac{1}{2}n+k}}{\left(\frac{n-2}{2}+k\right)!\left(\frac{1}{2}n-k\right)!(p-k)} \prod_{t=1}^n \left(p + \frac{1}{2}n - t\right) \quad n \text{ even.}$$

$$A_k^n(p) = \frac{(-1)^{\frac{1}{2}(n-1)+k}}{\left(\frac{n-1}{2}+k\right)!\left(\frac{n-1}{2}-k\right)!(p-k)} \prod_{t=0}^{n-1} \left(p + \frac{n-1}{2} - t\right), \quad n \text{ odd.}$$

25.2.8

$$R_{n-1} = \frac{1}{n!} \prod_k (p-k) h^n f^{(n)}(\xi) \approx \frac{1}{n!} \prod_k (p-k) \Delta_0^n \quad (x_0 < \xi < x_n)$$

k has the same range as in **25.2.6**.

Lagrange Two Point Interpolation Formula (Linear Interpolation)

$$\mathbf{25.2.9} \quad f(x_0 + ph) = (1-p)f_0 + pf_1 + R_1$$

$$\mathbf{25.2.10} \quad R_1(p) \approx .125h^2 f^{(2)}(\xi) \approx .125\Delta^2$$

Lagrange Three Point Interpolation Formula**25.2.11**

$$f(x_0 + ph) = A_{-1}f_{-1} + A_0f_0 + A_1f_1 + R_2$$

$$\approx \frac{p(p-1)}{2}f_{-1} + (1-p^2)f_0 + \frac{p(p+1)}{2}f_1$$

25.2.12

$$R_2(p) \approx .065h^3f^{(3)}(\xi) \approx .065\Delta^3 \quad (|p| \leq 1)$$

Lagrange Four Point Interpolation Formula**25.2.13**

$$f(x_0 + ph) = A_{-1}f_{-1} + A_0f_0 + A_1f_1 + A_2f_2 + R_3$$

$$\approx \frac{-p(p-1)(p-2)}{6}f_{-1} + \frac{(p^2-1)(p-2)}{2}f_0$$

$$- \frac{p(p+1)(p-2)}{2}f_1 + \frac{p(p^2-1)}{6}f_2$$

25.2.14

$$R_3(p) \approx$$

$$.024h^4f^{(4)}(\xi) \approx .024\Delta^4 \quad (0 < p < 1)$$

$$.042h^4f^{(4)}(\xi) \approx .042\Delta^4 \quad (-1 < p < 0, 1 < p < 2)$$

$$(x_{-1} < \xi < x_2)$$

Lagrange Five Point Interpolation Formula**25.2.15**

$$f(x_0 + ph) = \sum_{i=-2}^2 A_i f_i + R_4$$

$$\approx \frac{(p^2-1)p(p-2)}{24}f_{-2} - \frac{(p-1)p(p^2-4)}{6}f_{-1}$$

$$+ \frac{(p^2-1)(p^2-4)}{4}f_0 - \frac{(p+1)p(p^2-4)}{6}f_1$$

$$+ \frac{(p^2-1)p(p+2)}{24}f_2$$

25.2.16

$$R_4(p) \approx$$

$$.012h^5f^{(5)}(\xi) \approx .012\Delta^5 \quad (|p| < 1)$$

$$.031h^5f^{(5)}(\xi) \approx .031\Delta^5 \quad (1 < |p| < 2) \quad (x_{-2} < \xi < x_2)$$

Lagrange Six Point Interpolation Formula**25.2.17**

$$f(x_0 + ph) = \sum_{i=-2}^3 A_i f_i + R_5$$

$$\approx \frac{-p(p^2-1)(p-2)(p-3)}{120}f_{-2}$$

$$+ \frac{p(p-1)(p^2-4)(p-3)}{24}f_{-1}$$

$$- \frac{(p^2-1)(p^2-4)(p-3)}{12}f_0$$

$$+ \frac{p(p+1)(p^2-4)(p-3)}{12}f_1 - \frac{p(p^2-1)(p+2)(p-3)}{24}f_2$$

$$+ \frac{p(p^2-1)(p^2-4)}{120}f_3$$

25.2.18

$$R_5(p) \approx$$

$$.0049h^6f^{(6)}(\xi) \approx .0049\Delta^6 \quad (0 < p < 1)$$

$$.0071h^6f^{(6)}(\xi) \approx .0071\Delta^6 \quad (-1 < p < 0, 1 < p < 2)$$

$$.024h^6f^{(6)}(\xi) \approx .024\Delta^6 \quad (-2 < p < -1, 2 < p < 3)$$

$$(x_{-2} < \xi < x_3)$$

Lagrange Seven Point Interpolation Formula

$$25.2.19 \quad f(x_0 + ph) = \sum_{i=-3}^3 A_i f_i + R_6$$

25.2.20

$$R_6(p) \approx \begin{cases} .0025h^7f^{(7)}(\xi) \approx .0025\Delta^7 & (|p| < 1) \\ .0046h^7f^{(7)}(\xi) \approx .0046\Delta^7 & (1 < |p| < 2) \\ .019h^7f^{(7)}(\xi) \approx .019\Delta^7 & (2 < |p| < 3) \end{cases}$$

$$(x_{-3} < \xi < x_3)$$

Lagrange Eight Point Interpolation Formula

$$25.2.21 \quad f(x_0 + ph) = \sum_{i=-3}^4 A_i f_i + R_7$$

25.2.22

$$R_7(p) \approx \begin{cases} .0011h^8f^{(8)}(\xi) \approx .0011\Delta^8 & (0 < p < 1) \\ .0014h^8f^{(8)}(\xi) \approx .0014\Delta^8 & (-1 < p < 0) \\ & (1 < p < 2) \\ .0033h^8f^{(8)}(\xi) \approx .0033\Delta^8 & (-2 < p < -1) \\ & (2 < p < 3) \\ .016h^8f^{(8)}(\xi) \approx .016\Delta^8 & (-3 < p < -2) \\ & (3 < p < 4) \end{cases}$$

$$(x_{-3} < \xi < x_4)$$

Aitken's Iteration Method

Let $f(x|x_0, x_1, \dots, x_k)$ denote the unique polynomial of k^{th} degree which coincides in value with $f(x)$ at $x_0, \dots, x_k$.

25.2.23

$$f(x|x_0, x_1) = \frac{1}{x_1 - x_0} \begin{vmatrix} f_0 & x_0 - x \\ f_1 & x_1 - x \end{vmatrix}$$

$$f(x|x_0, x_2) = \frac{1}{x_2 - x_0} \begin{vmatrix} f_0 & x_0 - x \\ f_2 & x_2 - x \end{vmatrix}$$

$$f(x|x_0, x_1, x_2) = \frac{1}{x_2 - x_1} \begin{vmatrix} f(x|x_0, x_1) & x_1 - x \\ f(x|x_0, x_2) & x_2 - x \end{vmatrix}$$

$$f(x|x_0, x_1, x_2, x_3) = \frac{1}{x_3 - x_2} \begin{vmatrix} f(x|x_0, x_1, x_2) & x_2 - x \\ f(x|x_0, x_1, x_3) & x_3 - x \end{vmatrix}$$

Taylor Expansion**25.2.24**

$$f(x) = f_0 + (x-x_0)f'_0 + \frac{(x-x_0)^2}{2!}f''_0 + \dots + \frac{(x-x_0)^n}{n!}f^{(n)}_0 + R_n$$

$$\begin{aligned} 25.2.25 \quad R_n &= \int_{x_0}^x f^{(n+1)}(t) \frac{(x-t)^n}{n!} dt \\ &= \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \quad (x_0 < \xi < x) \end{aligned}$$

Newton's Divided Difference Interpolation Formula**25.2.26**

$$f(x) = f_0 + \sum_{k=1}^n \pi_{k-1}(x) [x_0, x_1, \dots, x_k] + R_n$$

x_0	f_0	$[x_0, x_1]$	
x_1	f_1	$[x_0, x_1, x_2]$	
x_2	f_2	$[x_1, x_2, x_3]$	$[x_0, x_1, x_2, x_3]$
x_3	f_3	$[x_2, x_3]$	

25.2.27

$$R_n(x) = \pi_n(x) [x_0, \dots, x_n, x] = \pi_n(x) \frac{f^{(n+1)}(\xi)}{(n+1)!} \quad (x_0 < \xi < x_n)$$

(For π_n see 25.1.6.)**Newton's Forward Difference Formula****25.2.28**

$$f(x_0 + ph) = f_0 + p\Delta_0 + \binom{p}{2}\Delta_0^2 + \dots + \binom{p}{n}\Delta_0^n + R_n$$

x_0	f_0	Δ_0	Δ_0^2	Δ_0^3
x_1	f_1	Δ_1	Δ_1^2	
x_2	f_2	Δ_2	Δ_2^2	
x_3	f_3			

25.2.29

$$R_n = h^{n+1} \binom{p}{n+1} f^{(n+1)}(\xi) \approx \binom{p}{n+1} \Delta_0^{n+1} \quad (x_0 < \xi < x_n)$$

Relation Between Newton and Lagrange Coefficients**25.2.30**

$$\begin{aligned} \binom{p}{2} &= A_{-1}^3(p) & \binom{p}{3} &= -A_{-1}^4(p) & \binom{p}{4} &= A_2^5(1-p) \\ & & & & \binom{p}{5} &= A_3^6(2-p) \end{aligned}$$

Everett's Formula**25.2.31**

$$\begin{aligned} f(x_0 + ph) &= (1-p)f_0 + pf_1 - \frac{p(p-1)(p-2)}{3!}\delta_0^3 \\ &\quad + \frac{(p+1)p(p-1)}{3!}\delta_1^3 + \dots - \binom{p+n-1}{2n+1}\delta_0^{2n} \\ &\quad + \binom{p+n}{2n+1}\delta_1^{2n} + R_{2n} \\ &= (1-p)f_0 + pf_1 + E_2\delta_0^2 + F_2\delta_1^2 + E_4\delta_0^4 \\ &\quad + F_4\delta_1^4 + \dots + R_{2n} \end{aligned}$$

x_0	f_0	δ_0^2	δ_0^4
		δ_1^2	δ_1^4
x_1	f_1	δ_1^2	δ_1^4

25.2.32

$$\begin{aligned} R_{2n} &= h^{2n+2} \binom{p+n}{2n+2} f^{(2n+2)}(\xi) \\ &\approx \binom{p+n}{2n+2} \left[\frac{\Delta_{-n-1}^{2n+2} + \Delta_{-n}^{2n+2}}{2} \right] \quad (x_{-n} < \xi < x_{n+1}) \end{aligned}$$

Relation Between Everett and Lagrange Coefficients**25.2.33**

$$\begin{aligned} E_2 &= A_{-1}^4 & E_4 &= A_{-2}^6 & E_6 &= A_{-3}^8 \\ F_2 &= A_2^4 & F_4 &= A_3^6 & F_6 &= A_4^8 \end{aligned}$$

Everett's Formula With Throwback (Modified Central Difference)**25.2.34**

$$f(x_0 + ph) = (1-p)f_0 + pf_1 + E_2\delta_{m,0}^2 + F_2\delta_{m,1}^2 + R$$

$$25.2.35 \quad \delta_m^2 = \delta^2 - .184\delta^4$$

$$25.2.36 \quad R \approx .00045|\mu\delta_1^4| + .00061|\delta_1^6|$$

25.2.37

$$\begin{aligned} f(x_0 + ph) &= (1-p)f_0 + pf_1 + E_2\delta_0^2 + F_2\delta_1^2 \\ &\quad + E_4\delta_{m,0}^4 + F_4\delta_{m,1}^4 + R \end{aligned}$$

$$25.2.38 \quad \delta_m^4 = \delta^4 - .207\delta^6 + \dots$$

$$25.2.39 \quad R \approx .000032|\mu\delta_1^6| + .000052|\delta_1^8|$$

25.2.40

$$\begin{aligned} f(x_0 + ph) &= (1-p)f_0 + pf_1 + E_2\delta_0^2 + F_2\delta_1^2 \\ &\quad + E_4\delta_0^4 + F_4\delta_1^4 + E_6\delta_{m,0}^6 + F_6\delta_{m,1}^6 + R \end{aligned}$$

$$25.2.41 \quad \delta_m^6 = \delta^6 - .218\delta^8 + .049\delta^{10} + \dots$$

$$25.2.42 \quad R \approx .0000037|\mu\delta_1^8| + \dots$$

Simultaneous Throwback**25.2.43**

$$f(x_0 + ph) = (1-p)f_0 + pf_1 + E_2\delta_{m,0}^2 + F_2\delta_{m,1}^2 + E_4\delta_{m,0}^4 + F_4\delta_{m,1}^4 + R$$

$$25.2.44 \quad \delta_m^2 = \delta^2 - .01312\delta^6 + .0043\delta^8 - .001\delta^{10}$$

$$25.2.45 \quad \delta_m^4 = \delta^4 - .27827\delta^6 + .0685\delta^8 - .016\delta^{10}$$

$$25.2.46 \quad R \approx .00000083|\mu\delta_1^6| + .0000094\delta^7$$

Bessel's Formula With Throwback**25.2.47**

$$f(x_0 + ph) = (1-p)f_0 + pf_1 + B_2(\delta_{m,0}^2 + \delta_{m,1}^2) + B_3\delta_1^3 + R, \quad B_2 = \frac{p(p-1)}{4}, \quad B_3 = \frac{p(p-1)(p-\frac{1}{2})}{6}$$

$$25.2.48 \quad \delta_m^2 = \delta^2 - .184\delta^4$$

$$25.2.49 \quad R \approx .00045|\mu\delta_1^4| + .00087|\delta_1^5|$$

Thiele's Interpolation Formula**25.2.50**

$$f(x) = f(x_1) + \frac{\frac{x-x_1}{\rho(x_1, x_2) + x - x_2}}{\frac{\rho_2(x_1, x_2, x_3) - f(x_1) + x - x_3}{\left(\frac{\rho_3(x_1, x_2, x_3, x_4)}{-\rho(x_1, x_2) + \dots}\right)}}$$

(For reciprocal differences, ρ , see 25.1.12.)**Trigonometric Interpolation****Gauss' Formula**

$$25.2.51 \quad f(x) \approx \sum_{k=0}^{2n} f_k \zeta_k(x) = t_n(x)$$

25.2.52

$$\zeta_k(x) = \frac{\sin \frac{1}{2}(x-x_0) \dots \sin \frac{1}{2}(x-x_{k-1})}{\sin \frac{1}{2}(x_k-x_0) \dots \sin \frac{1}{2}(x_k-x_{k-1})} \frac{\sin \frac{1}{2}(x-x_{k+1}) \dots \sin \frac{1}{2}(x-x_{2n})}{\sin \frac{1}{2}(x_k-x_{k+1}) \dots \sin \frac{1}{2}(x_k-x_{2n})}$$

$t_n(x)$ is a trigonometric polynomial of degree n such that $t_n(x_k) = f_k \quad (k=0, 1, \dots, 2n)$

Harmonic Analysis

Equally spaced abscissas

$$x_0 = 0, \quad x_1, \dots, x_{m-1}, x_m = 2\pi$$

25.2.53

$$f(x) \approx \frac{1}{2} a_0 + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx)$$

25.2.54

$$m = 2n + 1$$

$$a_k = \frac{2}{2n+1} \sum_{r=0}^{2n} f_r \cos kx_r; \quad b_k = \frac{2}{2n+1} \sum_{r=0}^{2n} f_r \sin kx_r \quad (k=0, 1, \dots, n)$$

25.2.55

$$m = 2n$$

$$a_k = \frac{1}{n} \sum_{r=0}^{2n-1} f_r \cos kx_r; \quad b_k = \frac{1}{n} \sum_{r=0}^{2n-1} f_r \sin kx_r \quad (k=0, 1, \dots, n) \quad (k=0, 1, \dots, n-1)$$

 b_n is arbitrary.**Subtabulation**

Let $f(x)$ be tabulated initially in intervals of width h . It is desired to subtabulate $f(x)$ in intervals of width h/m . Let Δ and $\bar{\Delta}$ designate differences with respect to the original and the final intervals respectively. Thus $\bar{\Delta}_0 = f\left(x_0 + \frac{h}{m}\right) - f(x_0)$. Assuming that the original 5th order differences are zero,

25.2.56

$$\bar{\Delta}_0 = \frac{1}{m} \Delta_0 + \frac{1-m}{2m^2} \Delta_0^2 + \frac{(1-m)(1-2m)}{6m^3} \Delta_0^3 + \frac{(1-m)(1-2m)(1-3m)}{24m^4} \Delta_0^4$$

$$\bar{\Delta}_0^2 = \frac{1}{m^2} \Delta_0^2 + \frac{1-m}{m^3} \Delta_0^3 + \frac{(1-m)(7-11m)}{12m^4} \Delta_0^4$$

$$\bar{\Delta}_0^3 = \frac{1}{m^3} \Delta_0^3 + \frac{3(1-m)}{2m^4} \Delta_0^4$$

$$\bar{\Delta}_0^4 = \frac{1}{m^4} \Delta_0^4$$

From this information we may construct the final tabulation by addition. For $m=10$,

25.2.57

$$\bar{\Delta}_0 = .1\Delta_0 - .045\Delta_0^2 + .0285\Delta_0^3 - .02066\Delta_0^4$$

$$\bar{\Delta}_0^2 = .01\Delta_0^2 - .009\Delta_0^3 + .007725\Delta_0^4$$

$$\bar{\Delta}_0^3 = .001\Delta_0^3 - .00135\Delta_0^4$$

$$\bar{\Delta}_0^4 = .0001\Delta_0^4$$

Linear Inverse InterpolationFind p , given $f_p (= f(x_0 + ph))$.**Linear****25.2.58**

$$p \approx \frac{f_p - f_0}{f_1 - f_0}$$

Quadratic Inverse Interpolation

25.2.59

$$(f_1 - 2f_0 + f_{-1})p^2 + (f_1 - f_{-1})p + 2(f_0 - f_p) \approx 0$$

Inverse Interpolation by Reversion of Series

$$25.2.60 \quad \text{Given } f(x_0 + ph) = f_p = \sum_{k=0}^{\infty} a_k p^k$$

25.2.61

$$p = \lambda + c_2 \lambda^2 + c_3 \lambda^3 + \dots, \quad \lambda = (f_p - a_0)/a_1$$

25.2.62

$$c_2 = -a_2/a_1$$

$$c_3 = \frac{-a_3}{a_1} + 2 \left(\frac{a_2}{a_1} \right)^2$$

$$c_4 = \frac{-a_4}{a_1} + \frac{5a_2 a_3}{a_1^2} - \frac{5a_2^3}{a_1^3}$$

$$c_5 = \frac{-a_5}{a_1} + \frac{6a_2 a_4}{a_1^2} + \frac{3a_3^2}{a_1^2} - \frac{21a_2^2 a_3}{a_1^3} + \frac{14a_2^4}{a_1^4}$$

Inversion of Newton's Forward Difference Formula

25.2.63

$$a_0 = f_0$$

$$a_1 = \Delta_0 - \frac{\Delta_0^2}{2} + \frac{\Delta_0^3}{3} - \frac{\Delta_0^4}{4} + \dots$$

$$a_2 = \frac{\Delta_0^2}{2} - \frac{\Delta_0^3}{2} + \frac{11\Delta_0^4}{24} + \dots$$

$$a_3 = \frac{\Delta_0^3}{6} - \frac{\Delta_0^4}{4} + \dots$$

$$a_4 = \frac{\Delta_0^4}{24} + \dots$$

(Used in conjunction with 25.2.62.)

Inversion of Everett's Formula

25.2.64

$$a_0 = f_0$$

$$a_1 = \delta_1 - \frac{\delta_0^2}{3} - \frac{\delta_1^2}{6} + \frac{\delta_0^4}{20} + \frac{\delta_1^4}{30} + \dots$$

$$a_2 = \frac{\delta_0^2}{2} - \frac{\delta_0^4}{24} + \dots$$

$$a_3 = \frac{-\delta_0^2 + \delta_1^2}{6} - \frac{\delta_0^4 + \delta_1^4}{24} + \dots$$

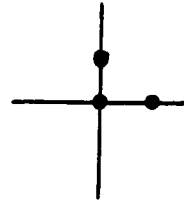
$$a_4 = \frac{\delta_0^4}{24} + \dots$$

$$a_5 = \frac{-\delta_0^4 + \delta_1^4}{120} + \dots$$

(Used in conjunction with 25.2.62.)

Bivariate Interpolation**Three Point Formula (Linear)**

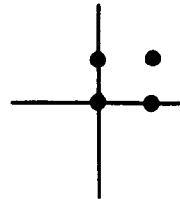
25.2.65



$$f(x_0 + ph, y_0 + qk) = (1-p-q)f_{0,0} + pf_{1,0} + qf_{0,1} + O(h^2)$$

Four Point Formula

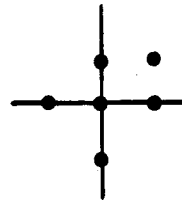
25.2.66



$$f(x_0 + ph, y_0 + qk) = (1-p)(1-q)f_{0,0} + p(1-q)f_{1,0} + q(1-p)f_{0,1} + pqf_{1,1} + O(h^2)$$

Six Point Formula

25.2.67



$$f(x_0 + ph, y_0 + qk) = \frac{q(q-1)}{2} f_{0,-1} + \frac{p(p-1)}{2} f_{-1,0}$$

$$+ (1 + pq - p^2 - q^2) f_{0,0}$$

$$+ \frac{p(p-2q+1)}{2} f_{1,0}$$

$$+ \frac{q(q-2p+1)}{2} f_{0,1} + pqf_{1,1} + O(h^3)$$

25.3. Differentiation**Lagrange's Formula**

$$25.3.1 \quad f'(x) = \sum_{k=0}^n l'_k(x) f_k + R'_n(x)$$

(See 25.2.1.)

$$25.3.2 \quad l'_k(x) = \sum_{j \neq k} \frac{\pi_n(x)}{(x-x_k)(x-x_j)\pi'_n(x_k)}$$

25.3.3

$$R'_n(x) = \frac{f^{(n+1)}}{(n+1)!}(\xi) \pi'_n(x) + \frac{\pi_n(x)}{(n+1)!} \frac{d}{dx} f^{(n+1)}(\xi)$$

$$\xi = \xi(x) \quad (x_0 < \xi < x_n)$$

Equally Spaced Abscissas

Three Points

25.3.4

$$f'_p = f'(x_0 + ph)$$

$$= \frac{1}{h} \left\{ (p - \frac{1}{2})f_{-1} - 2pf_0 + (p + \frac{1}{2})f_1 \right\} + R'_2$$

Four Points

25.3.5

$$f'_p = f'(x_0 + ph) = \frac{1}{h} \left\{ -\frac{3p^2 - 6p + 2}{6} f_{-1} \right.$$

$$+ \frac{3p^2 - 4p - 1}{2} f_0 - \frac{3p^2 - 2p - 2}{2} f_1$$

$$\left. + \frac{3p^2 - 1}{6} f_2 \right\} + R'_3$$

Five Points

25.3.6

$$f'_p = f'(x_0 + ph) = \frac{1}{h} \left\{ \frac{2p^3 - 3p^2 - p + 1}{12} f_{-2} \right.$$

$$- \frac{4p^3 - 3p^2 - 8p + 4}{6} f_{-1} + \frac{2p^3 - 5p}{2} f_0$$

$$- \frac{4p^3 + 3p^2 - 8p - 4}{6} f_1$$

$$\left. + \frac{2p^3 + 3p^2 - p - 1}{12} f_2 \right\} + R'_4$$

For numerical values of differentiation coefficients see Table 25.2.

Markoff's Formulas

(Newton's Forward Difference Formula Differentiated)

25.3.7

$$f'(a_0 + ph) = \frac{1}{h} \left[\Delta_0 + \frac{2p-1}{2} \Delta_0^2 \right.$$

$$+ \frac{3p^2 - 6p + 2}{6} \Delta_0^3 + \dots + \frac{d}{dp} \binom{p}{n} \Delta_0^n \left. \right] + R'_n$$

25.3.8

$$R'_n = h^n f^{(n+1)}(\xi) \frac{d}{dp} \binom{p}{n+1} + h^{n+1} \binom{p}{n+1} \frac{d}{dx} f^{(n+1)}(\xi)$$

$$(a_0 < \xi < a_n)$$

$$25.3.9 \quad hf'_0 = \Delta_0 - \frac{1}{2} \Delta_0^2 + \frac{1}{3} \Delta_0^3 - \frac{1}{4} \Delta_0^4 + \dots$$

$$25.3.10 \quad h^2 f''_0 = \Delta_0^2 - \Delta_0^3 + \frac{11}{12} \Delta_0^4 - \frac{5}{6} \Delta_0^5 + \dots$$

25.3.11

$$h^3 f'''_0 = \Delta_0^3 - \frac{3}{2} \Delta_0^4 + \frac{7}{4} \Delta_0^5 - \frac{15}{8} \Delta_0^6 + \dots$$

25.3.12

$$h^4 f^{(4)}_0 = \Delta_0^4 - 2\Delta_0^5 + \frac{17}{6} \Delta_0^6 - \frac{7}{2} \Delta_0^7 + \dots$$

25.3.13

$$h^5 f^{(5)}_0 = \Delta_0^5 - \frac{5}{2} \Delta_0^6 + \frac{25}{6} \Delta_0^7 - \frac{35}{6} \Delta_0^8 + \dots$$

Everett's Formula

25.3.14

$$hf'(x_0 + ph) \approx -f_0 + f_1 - \frac{3p^2 - 6p + 2}{6} \delta_0^2 + \frac{3p^2 - 1}{6} \delta_1^2$$

$$- \frac{5p^4 - 20p^3 + 15p^2 + 10p - 6}{120} \delta_0^4 + \frac{5p^4 - 15p^2 + 4}{120} \delta_1^4$$

$$+ \dots - \left[\binom{p+n-1}{2n+1} \right]' \delta_0^{2n} + \left[\binom{p+n}{2n+1} \right]' \delta_1^{2n}$$

25.3.15

$$hf'_0 \approx -f_0 + f_1 - \frac{1}{3} \delta_0^2 - \frac{1}{6} \delta_1^2 + \frac{1}{20} \delta_0^4 + \frac{1}{30} \delta_1^4$$

Differences in Terms of Derivatives

25.3.16

$$\Delta_0 \approx hf'_0 + \frac{h^2}{2!} f''_0 + \frac{h^3}{3!} f'''_0 + \frac{h^4}{4!} f^{(4)}_0 + \frac{h^5}{5!} f^{(5)}_0$$

25.3.17

$$\Delta_0^2 \approx h^2 f''_0 + h^3 f'''_0 + \frac{7}{12} h^4 f^{(4)}_0 + \frac{1}{4} h^5 f^{(5)}_0$$

25.3.18

$$\Delta_0^3 \approx h^3 f'''_0 + \frac{3}{2} h^4 f^{(4)}_0 + \frac{5}{4} h^5 f^{(5)}_0$$

25.3.19

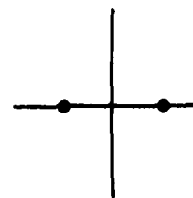
$$\Delta_0^4 \approx h^4 f^{(4)}_0 + 2h^5 f^{(5)}_0$$

25.3.20

$$\Delta_0^5 \approx h^5 f^{(5)}_0$$

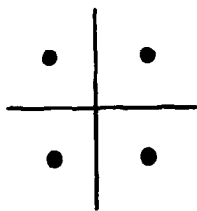
Partial Derivatives

25.3.21



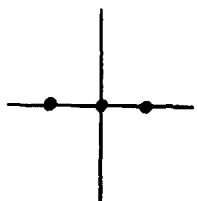
$$\frac{\partial f_{0,0}}{\partial x} = \frac{1}{2h} (f_{1,0} - f_{-1,0}) + O(h^2)$$

25.3.22



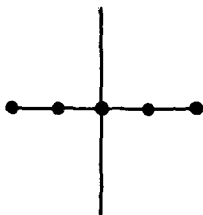
$$\frac{\partial f_{0,0}}{\partial x} = \frac{1}{4h} (f_{1,1} - f_{-1,1} + f_{1,-1} - f_{-1,-1}) + O(h^2)$$

25.3.23



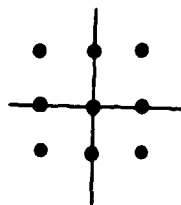
$$\frac{\partial^2 f_{0,0}}{\partial x^2} = \frac{1}{h^2} (f_{1,0} - 2f_{0,0} + f_{-1,0}) + O(h^2)$$

25.3.24



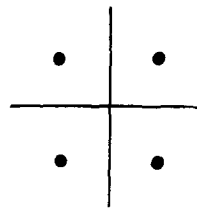
$$\frac{\partial^3 f_{0,0}}{\partial x^3} = \frac{1}{12h^3} (-f_{2,0} + 16f_{1,0} - 30f_{0,0} + 16f_{-1,0} - f_{-2,0}) + O(h^4)$$

25.3.25



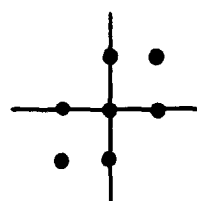
$$\frac{\partial^2 f_{0,0}}{\partial x^2} = \frac{1}{3h^2} (f_{1,1} - 2f_{0,1} + f_{-1,1} + f_{1,0} - 2f_{0,0} + f_{-1,0} + f_{1,-1} - 2f_{0,-1} + f_{-1,-1}) + O(h^2)$$

25.3.26



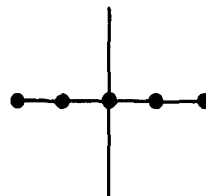
$$\frac{\partial^2 f_{0,0}}{\partial x \partial y} = \frac{1}{4h^2} (f_{1,1} - f_{1,-1} - f_{-1,1} + f_{-1,-1}) + O(h^2)$$

25.3.27



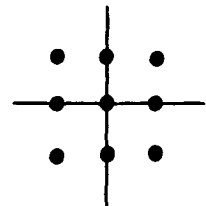
$$\frac{\partial^2 f_{0,0}}{\partial x \partial y} = \frac{-1}{2h^2} (f_{1,0} + f_{-1,0} + f_{0,1} + f_{0,-1} - 2f_{0,0} - f_{1,1} - f_{-1,-1}) + O(h^2)$$

25.3.28



$$\frac{\partial^4 f_{0,0}}{\partial x^4} = \frac{1}{h^4} (f_{2,0} - 4f_{1,0} + 6f_{0,0} - 4f_{-1,0} + f_{-2,0}) + O(h^2)$$

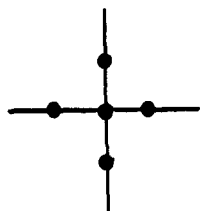
25.3.29



$$\frac{\partial^4 f_{0,0}}{\partial x^2 \partial y^2} = \frac{1}{h^4} (f_{1,1} + f_{-1,1} + f_{1,-1} + f_{-1,-1} - 2f_{1,0} - 2f_{-1,0} - 2f_{0,1} - 2f_{0,-1} + 4f_{0,0}) + O(h^2)$$

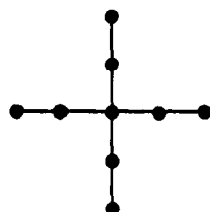
Laplacian

25.3.30



$$\begin{aligned}\nabla^2 u_{0,0} &= \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)_{0,0} \\ &= \frac{1}{h^2} (u_{1,0} + u_{0,1} + u_{-1,0} + u_{0,-1} - 4u_{0,0}) + O(h^2)\end{aligned}$$

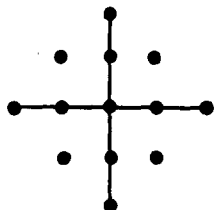
25.3.31



$$\begin{aligned}\nabla^2 u_{0,0} &= \frac{1}{12h^2} [-60u_{0,0} + 16(u_{1,0} + u_{0,1} + u_{-1,0} + u_{0,-1}) \\ &\quad - (u_{2,0} + u_{0,2} + u_{-2,0} + u_{0,-2})] + O(h^4)\end{aligned}$$

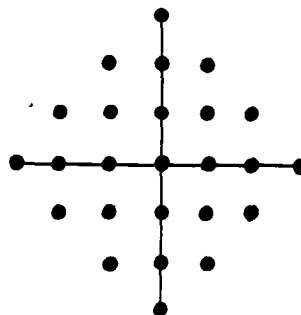
Biharmonic Operator

25.3.32



$$\begin{aligned}\nabla^4 u_{0,0} &= \left(\frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} \right)_{0,0} \\ &= \frac{1}{h^4} [20u_{0,0} - 8(u_{1,0} + u_{0,1} + u_{-1,0} + u_{0,-1}) \\ &\quad + 2(u_{1,1} + u_{1,-1} + u_{-1,1} + u_{-1,-1}) \\ &\quad + (u_{0,2} + u_{2,0} + u_{-2,0} + u_{0,-2})] + O(h^2)\end{aligned}$$

25.3.33



$$\begin{aligned}\nabla^4 u_{0,0} &= \frac{1}{6h^4} [- (u_{0,3} + u_{0,-3} + u_{3,0} + u_{-3,0}) \\ &\quad + 14(u_{0,2} + u_{0,-2} + u_{2,0} + u_{-2,0}) \\ &\quad - 77(u_{0,1} + u_{0,-1} + u_{1,0} + u_{-1,0}) \\ &\quad + 184u_{0,0} + 20(u_{1,1} + u_{1,-1} + u_{-1,1} + u_{-1,-1}) \\ &\quad - (u_{1,2} + u_{2,1} + u_{1,-2} + u_{2,-1} + u_{-1,2} + u_{-2,1} \\ &\quad + u_{-1,-2} + u_{-2,-1})] + O(h^4)\end{aligned}$$

25.4. Integration

Trapezoidal Rule

25.4.1

$$\begin{aligned}\int_{x_0}^{x_1} f(x) dx &= \frac{h}{2} (f_0 + f_1) - \frac{1}{2} \int_{x_0}^{x_1} (t - x_0)(x_1 - t) f''(t) dt \\ &= \frac{h}{2} (f_0 + f_1) - \frac{h^3}{12} f''(\xi) \quad (x_0 < \xi < x_1)\end{aligned}$$

Extended Trapezoidal Rule

25.4.2

$$\begin{aligned}\int_{x_0}^{x_m} f(x) dx &= h \left[\frac{f_0}{2} + f_1 + \dots + f_{m-1} + \frac{f_m}{2} \right] \\ &\quad - \frac{mh^3}{12} f''(\xi)\end{aligned}$$

Error Term in Trapezoidal Formula for Periodic Functions

If $f(x)$ is periodic and has a continuous k^{th} derivative, and if the integral is taken over a period, then

$$25.4.3 \quad |\text{Error}| \leq \frac{\text{constant}}{m^k}$$

Modified Trapezoidal Rule

25.4.4

$$\begin{aligned}\int_{x_0}^{x_m} f(x) dx &= h \left[\frac{f_0}{2} + f_1 + \dots + f_{m-1} + \frac{f_m}{2} \right] \\ &\quad + \frac{h}{24} [-f_{-1} + f_1 + f_{m-1} - f_{m+1}] + \frac{11m}{720} h^5 f^{(4)}(\xi)\end{aligned}$$

Simpson's Rule

25.4.5

$$\begin{aligned}\int_{x_0}^{x_2} f(x) dx &= \frac{h}{3} [f_0 + 4f_1 + f_2] \\ &+ \frac{1}{6} \int_{x_0}^{x_1} (x_0 - t)^2 (x_1 - t) f^{(3)}(t) dt \\ &+ \frac{1}{6} \int_{x_1}^{x_2} (x_2 - t)^2 (x_1 - t) f^{(3)}(t) dt \\ &= \frac{h}{3} [f_0 + 4f_1 + f_2] - \frac{h^5}{90} f^{(4)}(\xi)\end{aligned}$$

Extended Simpson's Rule

25.4.6

$$\begin{aligned}\int_{x_0}^{x_{2n}} f(x) dx &= \frac{h}{3} [f_0 + 4(f_1 + f_3 + \dots + f_{2n-1}) \\ &+ 2(f_2 + f_4 + \dots + f_{2n-2}) + f_{2n}] - \frac{nh^5}{90} f^{(4)}(\xi)\end{aligned}$$

Euler-Maclaurin Summation Formula

25.4.7

$$\begin{aligned}\int_{x_0}^{x_n} f(x) dx &= h \left[\frac{f_0}{2} + f_1 + f_2 + \dots + f_{n-1} + \frac{f_n}{2} \right] \\ &- \frac{B_2}{2!} h^2 (f'_n - f'_0) - \dots - \frac{B_{2k} h^{2k}}{(2k)!} [f_n^{(2k-1)} - f_0^{(2k-1)}] + R_{2k} \\ R_{2k} &= \frac{\theta n B_{2k+2} h^{2k+3}}{(2k+2)!} \max_{x_0 \leq x \leq x_n} |f^{(2k+2)}(x)|, \quad (-1 \leq \theta \leq 1)\end{aligned}$$

(For B_{2k} , Bernoulli numbers, see chapter 23.)

If $f^{(2k+2)}(x)$ and $f^{(2k+4)}(x)$ do not change sign for $x_0 < x < x_n$ then $|R_{2k}|$ is less than the first neglected term. If $f^{(2k+2)}(x)$ does not change sign for $x_0 < x < x_n$, $|R_{2k}|$ is less than twice the first neglected term.

Lagrange Formula

25.4.8

$$\int_a^b f(x) dx = \sum_{i=0}^n (L_i^{(n)}(b) - L_i^{(n)}(a)) f_i + R_n$$

(See 25.2.1.)

25.4.9

$$L_i^{(n)}(x) = \frac{1}{\pi'_n(x_i)} \int_{x_0}^x \frac{\pi_n(t)}{t - x_i} dt = \int_{x_0}^x l_i(t) dt$$

$$25.4.10 \quad R_n = \frac{1}{(n+1)!} \int_a^b \pi_n(x) f^{(n+1)}(\xi(x)) dx$$

Equally Spaced Abscissas

25.4.11

$$\int_{x_0}^{x_k} f(x) dx = \frac{1}{h^n} \sum_{i=0}^n f_i \frac{(-1)^{n-i}}{i!(n-i)!} \int_{x_0}^{x_k} \frac{\pi_n(x)}{x - x_i} dx + R_n$$

$$25.4.12 \quad \int_{x_m}^{x_{m+1}} f(x) dx = h \sum_{i=-\left[\frac{n-1}{2}\right]}^{\left[\frac{n}{2}\right]} A_i(m) f_i + R_n \quad *$$

(See Table 25.3 for $A_i(m)$.)

Newton-Cotes Formulas (Closed Type)

(For Trapezoidal and Simpson's Rules see 25.4.1-25.4.6.)

$$25.4.13 \quad (\text{Simpson's } \frac{3}{8} \text{ rule})$$

$$\int_{x_0}^{x_3} f(x) dx = \frac{3h}{8} (f_0 + 3f_1 + 3f_2 + f_3) - \frac{3f^{(4)}(\xi)h^5}{80}$$

25.4.14

(Bode's rule)

$$\begin{aligned}\int_{x_0}^{x_4} f(x) dx &= \frac{2h}{45} (7f_0 + 32f_1 + 12f_2 \\ &+ 32f_3 + 7f_4) - \frac{8f^{(6)}(\xi)h^7}{945}\end{aligned}$$

25.4.15

$$\begin{aligned}\int_{x_0}^{x_5} f(x) dx &= \frac{5h}{288} (19f_0 + 75f_1 + 50f_2 + 50f_3 \\ &+ 75f_4 + 19f_5) - \frac{275f^{(6)}(\xi)h^7}{12096}\end{aligned}$$

25.4.16

$$\begin{aligned}\int_{x_0}^{x_6} f(x) dx &= \frac{h}{140} (41f_0 + 216f_1 + 27f_2 + 272f_3 \\ &+ 27f_4 + 216f_5 + 41f_6) - \frac{9f^{(8)}(\xi)h^9}{1400}\end{aligned}$$

25.4.17

$$\begin{aligned}\int_{x_0}^{x_7} f(x) dx &= \frac{7h}{17280} (751f_0 + 3577f_1 + 1323f_2 \\ &+ 2989f_3 + 2989f_4 + 1323f_5 + 3577f_6 \\ &+ 751f_7) - \frac{8183f^{(8)}(\xi)h^9}{518400}\end{aligned}$$

25.4.18

$$\begin{aligned}\int_{x_0}^{x_8} f(x) dx &= \frac{4h}{14175} (989f_0 + 5888f_1 - 928f_2 \\ &+ 10496f_3 - 4540f_4 + 10496f_5 - 928f_6 + 5888f_7 \\ &+ 989f_8) - \frac{2368}{467775} f^{(10)}(\xi)h^{11}\end{aligned}$$

25.4.19

$$\begin{aligned}\int_{x_0}^{x_9} f(x) dx &= \frac{9h}{89600} \{ 2857(f_0 + f_9) \\ &+ 15741(f_1 + f_8) + 1080(f_2 + f_7) + 19344(f_3 + f_6) \\ &+ 5778(f_4 + f_5) \} - \frac{173}{14620} f^{(10)}(\xi)h^{11}\end{aligned}$$

*See page II.

25.4.20

$$\int_{x_0}^{x_{10}} f(x) dx = \frac{5h}{299376} \{ 16067(f_0 + f_{10}) \\ + 106300(f_1 + f_9) - 48525(f_2 + f_8) + 272400(f_3 + f_7) \\ - 260550(f_4 + f_6) + 427368f_5 \} \\ - \frac{1346350}{326918592} f^{(12)}(\xi) h^{13}$$

Newton-Cotes Formulas (Open Type)

25.4.21

$$\int_{x_0}^{x_3} f(x) dx = \frac{3h}{2} (f_1 + f_2) + \frac{f^{(2)}(\xi) h^3}{4}$$

25.4.22

$$\int_{x_0}^{x_4} f(x) dx = \frac{4h}{3} (2f_1 - f_2 + 2f_3) + \frac{28f^{(4)}(\xi) h^5}{90}$$

25.4.23

$$\int_{x_0}^{x_5} f(x) dx = \frac{5h}{24} (11f_1 + f_2 + f_3 + 11f_4) + \frac{95f^{(4)}(\xi) h^5}{144}$$

25.4.24

$$\int_{x_0}^{x_6} f(x) dx = \frac{6h}{20} (11f_1 - 14f_2 + 26f_3 - 14f_4 + 11f_5) \\ + \frac{41f^{(6)}(\xi) h^7}{140}$$

25.4.25

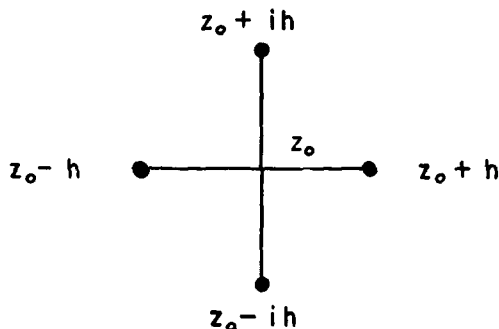
$$\int_{x_0}^{x_7} f(x) dx = \frac{7h}{1440} (611f_1 - 453f_2 + 562f_3 + 562f_4 \\ - 453f_5 + 611f_6) + \frac{5257}{8640} f^{(8)}(\xi) h^7$$

25.4.26

$$\int_{x_0}^{x_8} f(x) dx = \frac{8h}{945} (460f_1 - 954f_2 + 2196f_3 - 2459f_4 \\ + 2196f_5 - 954f_6 + 460f_7) + \frac{3956}{14175} f^{(8)}(\xi) h^9$$

Five Point Rule for Analytic Functions

25.4.27



$$\int_{z_0-h}^{z_0+h} f(z) dz = \frac{h}{15} \{ 24f(z_0) + 4[f(z_0+h) + f(z_0-h)] \\ - [f(z_0+ih) + f(z_0-ih)] \} + R$$

$|R| \leq \frac{|h|^7}{1890} \text{Max}_{z \in S} |f^{(6)}(z)|$, S designates the square with vertices $z_0 + i^k h$ ($k=0, 1, 2, 3$); h can be complex.

Chebyshev's Equal Weight Integration Formula

$$25.4.28 \quad \int_{-1}^1 f(x) dx = \frac{2}{n} \sum_{i=1}^n f(x_i) + R_n$$

Abscissas: x_i is the i^{th} zero of the polynomial part of

$$x^n \exp \left[\frac{-n}{2 \cdot 3x^2} - \frac{n}{4 \cdot 5x^3} - \frac{n}{6 \cdot 7x^4} - \dots \right]$$

(See Table 25.5 for x_i .)

For $n=8$ and $n \geq 10$ some of the zeros are complex.

Remainder:

$$R_n = \int_{-1}^{+1} \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\xi) dx \\ - \frac{2}{n(n+1)!} \sum_{i=1}^n x_i^{n+1} f^{(n+1)}(\xi_i)$$

where $\xi = \xi(x)$ satisfies $0 \leq \xi \leq x$ and $0 \leq \xi_i \leq x_i$

$$(i=1, \dots, n)$$

Integration Formulas of Gaussian Type

(For Orthogonal Polynomials see chapter 22)

Gauss' Formula

$$25.4.29 \quad \int_{-1}^1 f(x) dx = \sum_{i=1}^n w_i f(x_i) + R_n$$

Related orthogonal polynomials: Legendre polynomials $P_n(x)$, $P_n(1)=1$

Abscissas: x_i is the i^{th} zero of $P_n(x)$

Weights: $w_i = 2/(1-x_i^2) [P'_n(x_i)]^2$
(See Table 25.4 for x_i and w_i .)

$$R_n = \frac{2^{2n+1}(n!)^4}{(2n+1) [(2n)!]^3} f^{(2n)}(\xi) \quad (-1 < \xi < 1)$$

Gauss' Formula, Arbitrary Interval

$$25.4.30 \quad \int_a^b f(y) dy = \frac{b-a}{2} \sum_{i=1}^n w_i f(y_i) + R_n$$

$$y_i = \left(\frac{b-a}{2} \right) x_i + \left(\frac{b+a}{2} \right)$$

*See page II.

Related orthogonal polynomials: $P_n(x)$, $P_n(1)=1$

Abscissas: x_i is the i^{th} zero of $P_n(x)$

* Weights: $w_i = 2/(1-x_i) [P'_n(x_i)]^2$

$$R_n = \frac{(b-a)^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi)$$

Radau's Integration Formula

25.4.31

$$\int_{-1}^1 f(x) dx = \frac{2}{n^2} f_{-1} + \sum_{i=1}^{n-1} w_i f(x_i) + R_n$$

Related polynomials:

$$\frac{P_{n-1}(x) + P_n(x)}{x+1}$$

Abscissas: x_i is the i^{th} zero of

$$\frac{P_{n-1}(x) + P_n(x)}{x+1}$$

Weights:

$$w_i = \frac{1}{n^2} \frac{1-x_i}{[P_{n-1}(x_i)]^2} = \frac{1}{1-x_i} \frac{1}{[P'_{n-1}(x_i)]^2}$$

Remainder:

$$R_n = \frac{2^{2n-1} \cdot n}{[(2n-1)!]^3} [(n-1)!]^4 f^{(2n-1)}(\xi) \quad (-1 < \xi < 1)$$

Lobatto's Integration Formula

25.4.32

$$\int_{-1}^1 f(x) dx = \frac{2}{n(n-1)} [f(1) + f(-1)] + \sum_{i=2}^{n-1} w_i f(x_i) + R_n$$

Related polynomials: $P'_{n-1}(x)$

Abscissas: x_i is the $(i-1)^{\text{st}}$ zero of $P'_{n-1}(x)$

Weights:

$$w_i = \frac{2}{n(n-1)[P_{n-1}(x_i)]^2} \quad (x_i \neq \pm 1)$$

(See Table 25.6 for x_i and w_i .)

Remainder:

$$R_n = \frac{-n(n-1)^3 2^{2n-1} [(n-2)!]^4}{(2n-1)[(2n-2)!]^3} f^{(2n-2)}(\xi) \quad (-1 < \xi < 1)$$

$$25.4.33 \quad \int_0^1 x^k f(x) dx = \sum_{i=1}^n w_i f(x_i) + R_n$$

Related orthogonal polynomials:

$$q_n(x) = \sqrt{k+2n+1} P_n^{(k,0)}(1-2x)$$

(For the Jacobi polynomials $P_n^{(k,0)}$ see chapter 22.)

Abscissas:

x_i is the i^{th} zero of $q_n(x)$

Weights:

$$w_i = \left\{ \sum_{j=0}^{n-1} [q_j(x_i)]^2 \right\}^{-1} \quad *$$

(See Table 25.8 for x_i and w_i .)

Remainder:

$$R_n = \frac{f^{(2n)}(\xi)}{(k+2n+1)(2n)!} \left[\frac{n!(k+n)!}{(k+2n)!} \right]^2 \quad (0 < \xi < 1)$$

25.4.34

$$\int_0^1 f(x) \sqrt{1-x} dx = \sum_{i=1}^n w_i f(x_i) + R_n$$

Related orthogonal polynomials:

$$\frac{1}{\sqrt{1-x}} P_{2n+1}(\sqrt{1-x}), P_{2n+1}(1)=1$$

Abscissas: $x_i = 1 - \xi_i^2$ where ξ_i is the i^{th} positive zero of $P_{2n+1}(x)$.

Weights: $w_i = 2\xi_i^2 w_i^{(2n+1)}$ where $w_i^{(2n+1)}$ are the Gaussian weights of order $2n+1$.

Remainder:

$$R_n = \frac{2^{4n+3} [(2n+1)!]^4}{(2n)!(4n+3)[(4n+2)!]^2} f^{(2n)}(\xi) \quad (0 < \xi < 1)$$

25.4.35

$$\int_a^b f(y) \sqrt{b-y} dy = (b-a)^{3/2} \sum_{i=1}^n w_i f(y_i)$$

$$y_i = a + (b-a)x_i$$

Related orthogonal polynomials:

$$\frac{1}{\sqrt{1-x}} P_{2n+1}(\sqrt{1-x}), P_{2n+1}(1)=1$$

Abscissas: $x_i = 1 - \xi_i^2$ where ξ_i is the i^{th} positive zero of $P_{2n+1}(x)$.

Weights: $w_i = 2\xi_i^2 w_i^{(2n+1)}$ where $w_i^{(2n+1)}$ are the Gaussian weights of order $2n+1$.

$$25.4.36 \quad \int_0^1 \frac{f(x)}{\sqrt{1-x}} dx = \sum_{i=1}^n w_i f(x_i) + R_n$$

Related orthogonal polynomials:

$$P_{2n}(\sqrt{1-x}), P_{2n}(1)=1$$

Abscissas: $x_i = 1 - \xi_i^2$ where ξ_i is the i^{th} positive zero of $P_{2n}(x)$.

Weights: $w_i = 2w_i^{(2n)}$, $w_i^{(2n)}$ are the Gaussian weights of order $2n$.

Remainder:

$$R_n = \frac{2^{4n+1}}{4n+1} \frac{[(2n)!]^3}{[(4n)!]^2} f^{(2n)}(\xi) \quad (0 < \xi < 1)$$

$$25.4.37 \quad \int_a^b \frac{f(y)}{\sqrt{b-y}} dy = \sqrt{b-a} \sum_{i=1}^n w_i f(y_i) + R_n$$

$$y_i = a + (b-a)x_i$$

Related orthogonal polynomials:

$$P_{2n}(\sqrt{1-x}), P_{2n}(1)=1$$

Abscissas:

$x_i = 1 - \xi_i^2$ where ξ_i is the i^{th} positive zero of $P_{2n}(x)$.

Weights: $w_i = 2w_i^{(2n)}$, $w_i^{(2n)}$ are the Gaussian weights of order $2n$.

$$25.4.38 \quad \int_{-1}^{+1} \frac{f(x)}{\sqrt{1-x^2}} dx = \sum_{i=1}^n w_i f(x_i) + R_n$$

Related orthogonal polynomials: Chebyshev Polynomials of First Kind

$$T_n(x), T_n(1) = \frac{1}{2^{n-1}}$$

Abscissas:

$$x_i = \cos \frac{(2i-1)\pi}{2n}$$

Weights:

$$w_i = \frac{\pi}{n}$$

Remainder:

$$R_n = \frac{\pi}{(2n)! 2^{2n-1}} f^{(2n)}(\xi) \quad (-1 < \xi < 1)$$

25.4.39

$$\int_a^b \frac{f(y) dy}{\sqrt{(y-a)(b-y)}} = \sum_{i=1}^n w_i f(y_i) + R_n$$

$$y_i = \frac{b+a}{2} + \frac{b-a}{2} x_i$$

Related orthogonal polynomials:

$$T_n(x), T_n(1) = \frac{1}{2^{n-1}}$$

Abscissas:

$$x_i = \cos \frac{(2i-1)\pi}{2n}$$

Weights:

$$w_i = \frac{\pi}{n}$$

25.4.40

$$\int_{-1}^{+1} f(x) \sqrt{1-x^2} dx = \sum_{i=1}^n w_i f(x_i) + R_n$$

Related orthogonal polynomials: Chebyshev Polynomials of Second Kind

$$U_n(x) = \frac{\sin [(n+1) \arccos x]}{\sin (\arccos x)} \quad *$$

Abscissas:

$$x_i = \cos \frac{i}{n+1} \pi \quad *$$

Weights:

$$w_i = \frac{\pi}{n+1} \sin^2 \frac{i}{n+1} \pi \quad *$$

Remainder:

$$R_n = \frac{\pi}{(2n)! 2^{2n+1}} f^{(2n)}(\xi) \quad (-1 < \xi < 1)$$

25.4.41

$$\int_a^b \sqrt{(y-a)(b-y)} f(y) dy = \left(\frac{b-a}{2}\right)^2 \sum_{i=1}^n w_i f(y_i) + R_n$$

$$y_i = \frac{b+a}{2} + \frac{b-a}{2} x_i$$

Related orthogonal polynomials:

$$U_n(x) = \frac{\sin [(n+1) \arccos x]}{\sin (\arccos x)} \quad *$$

Abscissas:

$$x_i = \cos \frac{i}{n+1} \pi \quad *$$

Weights:

$$w_i = \frac{\pi}{n+1} \sin^2 \frac{i}{n+1} \pi \quad *$$

$$25.4.42 \quad \int_0^1 f(x) \sqrt{\frac{x}{1-x}} dx = \sum_{i=1}^n w_i f(x_i) + R_n$$

Related orthogonal polynomials:

$$\frac{1}{\sqrt{x}} T_{2n+1}(\sqrt{x})$$

Abscissas:

$$x_i = \cos^2 \frac{2i-1}{2n+1} \cdot \frac{\pi}{2}$$

Weights:

$$w_i = \frac{2\pi}{2n+1} x_i$$

Remainder:

$$R_n = \frac{\pi}{(2n)! 2^{4n+1}} f^{(2n)}(\xi) \quad (0 < \xi < 1)$$

25.4.43

$$\int_a^b f(x) \sqrt{\frac{x-a}{b-x}} dx = (b-a) \sum_{i=1}^n w_i f(y_i) + R_n$$

$$y_i = a + (b-a)x_i$$

Related orthogonal polynomials:

$$\frac{1}{\sqrt{x}} T_{2n+1}(\sqrt{x})$$

Abcissas:

$$x_i = \cos^2 \frac{2i-1}{2n+1} \frac{\pi}{2}$$

Weights:

$$w_i = \frac{2\pi}{2n+1} x_i$$

$$25.4.44 \quad \int_0^1 \ln x f(x) dx = \sum_{i=1}^n w_i f(x_i) + R_n$$

Related orthogonal polynomials: polynomials orthogonal with respect to the weight function $-\ln x$

Abcissas: See Table 25.7

Weights: See Table 25.7

25.4.45

$$\int_0^\infty e^{-x} f(x) dx = \sum_{i=1}^n w_i f(x_i) + R_n$$

Related orthogonal polynomials: Laguerre polynomials $L_n(x)$.Abcissas: x_i is the i^{th} zero of $L_n(x)$

Weights:

$$w_i = \frac{(n!)^2 x_i}{(n+1)^2 [L_{n+1}(x_i)]^2}$$

(See Table 25.9 for x_i and w_i .)

Remainder:

$$R_n = \frac{(n!)^2}{(2n)!} f^{(2n)}(\xi) \quad (0 < \xi < \infty)$$

25.4.46

$$\int_{-\infty}^\infty e^{-x^2} f(x) dx = \sum_{i=1}^n w_i f(x_i) + R_n$$

Related orthogonal polynomials: Hermite polynomials $H_n(x)$.Abcissas: x_i is the i^{th} zero of $H_n(x)$

Weights:

$$\frac{2^{n-1} n! \sqrt{\pi}}{n^2 [H_{n-1}(x_i)]^2}$$

(See Table 25.10 for x_i and w_i .)

Remainder:

$$R_n = \frac{n! \sqrt{\pi}}{2^n (2n)!} f^{(2n)}(\xi) \quad (-\infty < \xi < \infty)$$

Filon's Integration Formula³

25.4.47

$$\int_{x_0}^{x_n} f(x) \cos tx dx = h \left[\alpha(th) (f_{2n} \sin tx_{2n} - f_0 \sin tx_0) + \beta(th) \cdot C_{2n} + \gamma(th) \cdot C_{2n-1} + \frac{2}{45} th^4 S'_{2n-1} \right] - R_n$$

25.4.48

$$C_{2n} = \sum_{i=0}^n f_{2i} \cos(tx_{2i}) - \frac{1}{2} [f_{2n} \cos tx_{2n} + f_0 \cos tx_0]$$

25.4.49

$$C_{2n-1} = \sum_{i=1}^n f_{2i-1} \cos tx_{2i-1}$$

25.4.50

$$S'_{2n-1} = \sum_{i=1}^n f_{2i-1}^{(3)} \sin tx_{2i-1}$$

25.4.51

$$R_n = \frac{1}{90} n h^5 f^{(4)}(\xi) + O(th^7)$$

25.4.52

$$\alpha(\theta) = \frac{1}{\theta} + \frac{\sin 2\theta}{2\theta^2} - \frac{2 \sin^2 \theta}{\theta^3}$$

$$\beta(\theta) = 2 \left(\frac{1 + \cos^2 \theta}{\theta^2} - \frac{\sin 2\theta}{\theta^3} \right)$$

$$\gamma(\theta) = 4 \left(\frac{\sin \theta}{\theta^3} - \frac{\cos \theta}{\theta^2} \right)$$

For small θ we have

25.4.53

$$\alpha = \frac{2\theta^3}{45} - \frac{2\theta^5}{315} + \frac{2\theta^7}{4725} - \dots$$

$$\beta = \frac{2}{3} + \frac{2\theta^2}{15} - \frac{4\theta^4}{105} + \frac{2\theta^6}{567} - \dots$$

$$\gamma = \frac{4}{3} - \frac{2\theta^2}{15} + \frac{\theta^4}{210} - \frac{\theta^6}{11340} + \dots$$

25.4.54

$$\int_{x_0}^{x_{2n}} f(x) \sin tx dx = h \left[\alpha(th) (f_0 \cos tx_0 - f_{2n} \cos tx_{2n}) + \beta S_{2n} + \gamma S_{2n-1} + \frac{2}{45} th^4 C'_{2n-1} \right] - R_n$$

25.4.55

$$S_{2n} = \sum_{i=0}^n f_{2i} \sin(tx_{2i}) - \frac{1}{2} [f_{2n} \sin(tx_{2n}) + f_0 \sin(tx_0)]$$

³ For certain difficulties associated with this formula, see the article by J. W. Tukey, p. 400, "On Numerical Approximation," Ed. R. E. Langer, Madison, 1959.

$$25.4.56 \quad S_{2n-1} = \sum_{i=1}^n f_{2i-1} \sin (tx_{2i-1})$$

$$25.4.57 \quad C'_{2n-1} = \sum_{i=1}^n f_{2i-1}^{(3)} \cos (tx_{2i-1})$$

(See Table 25.11 for α, β, γ .)

Iterated Integrals

25.4.58

$$\begin{aligned} \int_0^x dt_n \int_0^{t_n} dt_{n-1} \dots \int_0^{t_3} dt_2 \int_0^{t_2} f(t_1) dt_1 \\ = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t) dt \end{aligned}$$

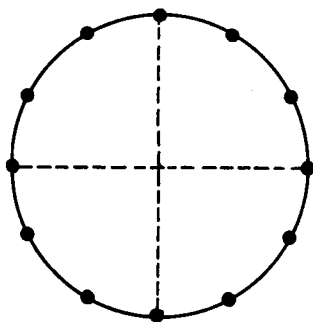
25.4.59

$$\begin{aligned} \int_a^x dt_n \int_a^{t_n} dt_{n-1} \dots \int_a^{t_3} dt_2 \int_a^{t_2} f(t_1) dt_1 \\ = \frac{(x-a)^n}{(n-1)!} \int_0^1 t^{n-1} f(x-(x-a)t) dt \end{aligned}$$

Multidimensional Integration

Circumference of Circle $\Gamma: x^2 + y^2 = h^2$.

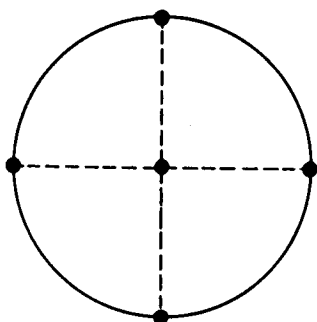
25.4.60



$$\begin{aligned} \frac{1}{2\pi h} \int_{\Gamma} f(x, y) ds = \frac{1}{2m} \sum_{n=1}^{2m} f\left(h \cos \frac{\pi n}{m}, h \sin \frac{\pi n}{m}\right) \\ + O(h^{2m-2}) \end{aligned}$$

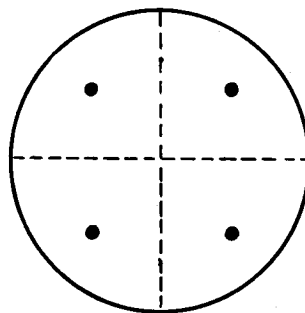
Circle $C: x^2 + y^2 \leq h^2$.

25.4.61

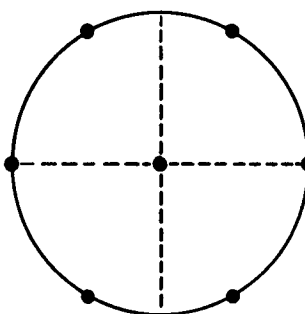


$$\frac{1}{\pi h^2} \iint_C f(x, y) dx dy = \sum_{i=1}^n w_i f(x_i, y_i) + R$$

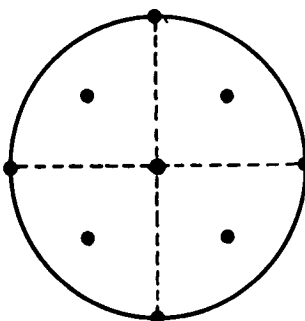
(x_i, y_i)	w_i	
$(0, 0)$	$1/2$	$R = O(h^4)$
$(\pm h, 0), (0, \pm h)$	$1/8$	



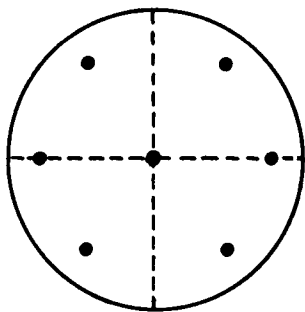
(x_i, y_i)	w_i	
$(\pm \frac{h}{2}, \pm \frac{h}{2})$	$1/4$	$R = O(h^4)$



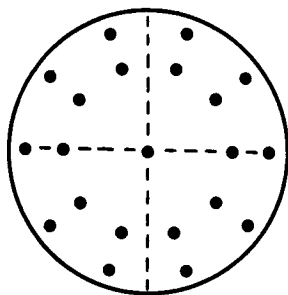
(x_i, y_i)	w_i	
$(0, 0)$	$1/2$	
$(\pm h, 0)$	$1/12$	$R = O(h^4)$
$(\pm \frac{h}{2}, \pm \frac{h}{2} \sqrt{3})$	$1/12$	



(x_i, y_i)	w_i
$(0, 0)$	$1/6$
$(\pm h, 0)$	$1/24 \quad R=O(h^8)$
$(0, \pm h)$	$1/24$
$(\pm \frac{h}{2}, \pm \frac{h}{2})$	$1/6$



(x_i, y_i)	w_i
$(0, 0)$	$1/4$
$(\pm \sqrt{\frac{2}{3}}h, 0)$	$1/8 \quad R=O(h^8)$
$(\pm \sqrt{\frac{1}{6}}h, \pm \frac{h}{2}\sqrt{2})$	$1/8$

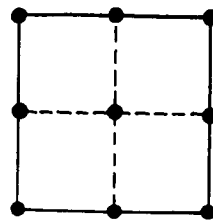


(x_i, y_i)	w_i
$(0, 0)$	$1/9$
$(\sqrt{\frac{6-\sqrt{6}}{10}}h \cos \frac{2\pi k}{10}, \sqrt{\frac{6-\sqrt{6}}{10}}h \sin \frac{2\pi k}{10})$	$\frac{16+\sqrt{6}}{360}$
	$(k=1, \dots, 10)$
$(\sqrt{\frac{6+\sqrt{6}}{10}}h \cos \frac{2\pi k}{10}, \sqrt{\frac{6+\sqrt{6}}{10}}h \sin \frac{2\pi k}{10})$	$\frac{16-\sqrt{6}}{360}$
	$R=O(h^{10})$

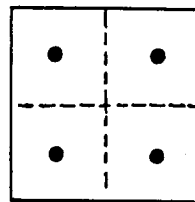
Square⁴ $S: |x| \leq h, |y| \leq h$

25.4.62

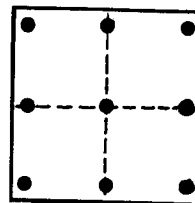
$$\frac{1}{4h^2} \iint_S f(x, y) dx dy = \sum_{i=1}^n w_i f(x_i, y_i) + R$$



(x_i, y_i)	w_i
$(0, 0)$	$4/9$
$(\pm h, \pm h)$	$1/36 \quad R=O(h^4)$
$(\pm h, 0)$	$1/9$
$(0, \pm h)$	$1/9$



(x_i, y_i)	w_i
$(\pm h\sqrt{\frac{1}{3}}, \pm h\sqrt{\frac{1}{3}})$	$1/4 \quad R=O(h^4)$



(x_i, y_i)	w_i
$(0, 0)$	$16/81$

⁴ For regions, such as the square, cube, cylinder, etc., which are the Cartesian products of lower dimensional regions, one may always develop integration rules by "multiplying together" the lower dimensional rules. Thus if

$$\int_0^1 f(x) dx \approx \sum_{i=1}^n w_i f(x_i)$$

is a one dimensional rule, then

$$\int_0^1 \int_0^1 f(x, y) dx dy \approx \sum_{i,j=1}^n w_i w_j f(x_i, y_j)$$

becomes a two dimensional rule. Such rules are not necessarily the most "economical".

$$\left(\pm\sqrt{\frac{3}{5}}h, \pm\sqrt{\frac{3}{5}}h\right) \quad 25/324 \quad R=O(h^6)$$

$$(0, \pm\sqrt{\frac{3}{5}}h) \quad 10/81$$

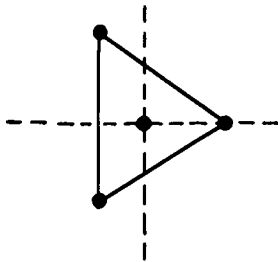
$$\left(\pm\sqrt{\frac{3}{5}}h, 0\right) \quad 10/81$$

Equilateral Triangle T

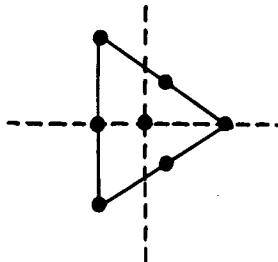
Radius of Circumscribed Circle= h

25.4.63

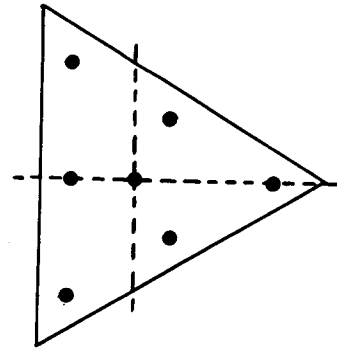
$$\frac{1}{\frac{3}{4}\sqrt{3}h^2} \iint_T f(x,y) dx dy = \sum_{i=1}^n w_i f(x_i, y_i) + R$$



(x_i, y_i)	w_i	
$(0, 0)$	$3/4$	$R=O(h^3)$
$(h, 0)$	$1/12$	
$\left(-\frac{h}{2}, \pm\frac{h}{2}\sqrt{3}\right)$	$1/12$	



(x_i, y_i)	w_i	
$(0, 0)$	$27/60$	$R=O(h^4)$
$(h, 0)$	$3/60$	
$\left(-\frac{h}{2}, \pm\frac{h}{2}\sqrt{3}\right)$	$3/60$	
$\left(-\frac{h}{2}, 0\right)$	$8/60$	
$\left(\frac{h}{4}, \pm\frac{h}{4}\sqrt{3}\right)$	$8/60$	



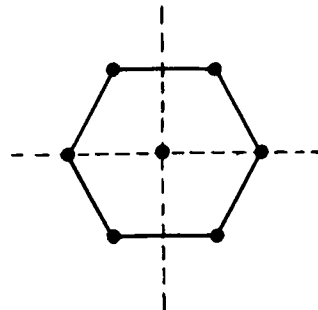
(x_i, y_i)	w_i	
$(0, 0)$	$270/1200$	$R=O(h^6)$
$\left(\left(\frac{\sqrt{15}+1}{7}\right)h, 0\right)$	$\frac{155-\sqrt{15}}{1200}$	
$\left(\left(\frac{-\sqrt{15}+1}{14}\right)h, \pm\left(\frac{\sqrt{15}+1}{14}\right)\sqrt{3}h\right)$	$\frac{155+\sqrt{15}}{1200}$	
$\left(\left(-\frac{\sqrt{15}-1}{7}\right)h, 0\right)$		
$\left(\left(\frac{\sqrt{15}-1}{14}\right)h, \pm\left(\frac{\sqrt{15}-1}{14}\right)\sqrt{3}h\right)$		

Regular Hexagon H

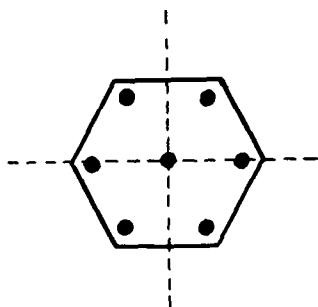
Radius of Circumscribed Circle= h

25.4.64

$$\frac{1}{\frac{3}{2}\sqrt{3}h^2} \iint_H f(x,y) dx dy = \sum_{i=1}^n w_i f(x_i, y_i) + R$$



(x_i, y_i)	w_i	
$(0, 0)$	$21/36$	$R=O(h^4)$
$\left(\pm\frac{h}{2}, \pm\frac{h}{2}\sqrt{3}\right)$	$5/72$	
$(\pm h, 0)$	$5/72$	

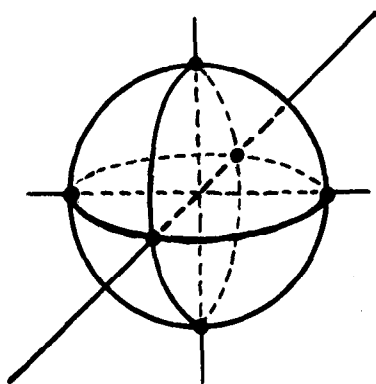


(x_i, y_i)	w_i	
$(0, 0)$	258/1008	
$(\pm \frac{h}{10} \sqrt{14}, \pm \frac{h}{10} \sqrt{42})$	125/1008	$R = O(h^6)$
$(\pm h \frac{\sqrt{14}}{5}, 0)$	125/1008	

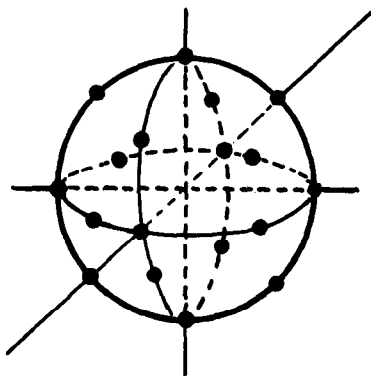
Surface of Sphere $\Sigma: x^2 + y^2 + z^2 = h^2$

25.4.65

$$\frac{1}{4\pi h^2} \int_{\Sigma} f(x, y, z) d\sigma = \sum_{i=1}^n w_i f(x_i, y_i, z_i) + R$$



(x_i, y_i, z_i)	w_i	
$(\pm h, 0, 0)$	1/6	$R = O(h^4)$
$(0, \pm h, 0)$	1/6	
$(0, 0, \pm h)$	1/6	



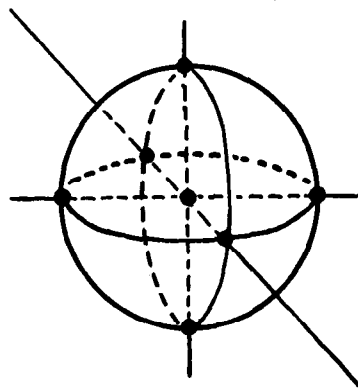
(x_i, y_i, z_i)	w_i
$(\pm \sqrt{\frac{1}{2}} h, \pm \sqrt{\frac{1}{2}} h, 0)$	
$(\pm \sqrt{\frac{1}{2}} h, 0, \pm \sqrt{\frac{1}{2}} h)$	1/15
$(0, \pm \sqrt{\frac{1}{2}} h, \pm \sqrt{\frac{1}{2}} h)$	
$(\pm h, 0, 0)$	$R = O(h^6)$
$(0, \pm h, 0)$	1/30
$(0, 0, \pm h)$	

(x_i, y_i, z_i)	w_i
$(\pm \sqrt{\frac{1}{3}} h, \pm \sqrt{\frac{1}{3}} h, \pm \sqrt{\frac{1}{3}} h)$	27/840
$(\pm \sqrt{\frac{1}{2}} h, \pm \sqrt{\frac{1}{2}} h, 0)$	
$(\pm \sqrt{\frac{1}{2}} h, 0, \pm \sqrt{\frac{1}{2}} h)$	32/840 $R = O(h^6)$
$(0, \pm \sqrt{\frac{1}{2}} h, \pm \sqrt{\frac{1}{2}} h)$	
$(\pm h, 0, 0)$	
$(0, \pm h, 0)$	40/840
$(0, 0, \pm h)$	

Sphere $S: x^2 + y^2 + z^2 \leq h^2$

25.4.66

$$\frac{1}{\frac{4}{3}\pi h^3} \iiint_S f(x, y, z) dx dy dz = \sum_{i=1}^n w_i f(x_i, y_i, z_i) + R$$



$$(x_i, y_i, z_i) \quad w_i$$

$$(0, 0, 0) \quad 2/5$$

$$(\pm h, 0, 0) \quad 1/10$$

$$R = O(h^4)$$

$$(0, \pm h, 0) \quad 1/10$$

$$(0, 0, \pm h) \quad 1/10$$

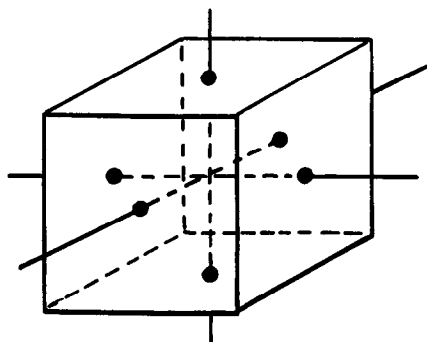
$$\text{Cube}^5 C: |x| \leq h$$

$$|y| \leq h$$

$$|z| \leq h$$

25.4.67

$$\frac{1}{8h^3} \iiint_C f(x, y, z) dx dy dz = \sum_{i=1}^n w_i f(x_i, y_i, z_i) + R$$



$$(x_i, y_i, z_i) \quad w_i$$

$$(\pm h, 0, 0) \quad 1/6$$

$$R = O(h^4)$$

$$(0, \pm h, 0) \quad 1/6$$

$$(0, 0, \pm h) \quad 1/6$$

25.4.68

$$\frac{1}{8h^3} \iiint_C f(x, y, z) dx dy dz$$

$$= \frac{1}{360} [-496f_m + 128\sum f_r + 8\sum f_f + 5\sum f_e] + O(h^6)$$

25.4.69

$$= \frac{1}{450} [91\sum f_r - 40\sum f_e + 16\sum f_d] + O(h^6)$$

 where $f_m = f(0, 0, 0)$.

⁵ See footnote to 25.4.62.

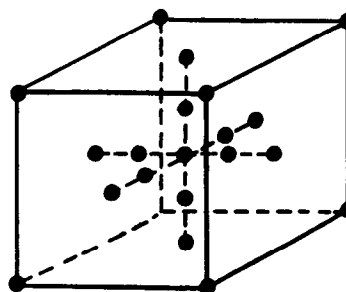
$\sum f_r$ = sum of values of f at the 6 points midway from the center of C to the 6 faces.

$\sum f_f$ = sum of values of f at the 6 centers of the faces of C .

$\sum f_e$ = sum of values of f at the 8 vertices of C .

$\sum f_d$ = sum of values of f at the 12 midpoints of edges of C .

$\sum f_a$ = sum of values of f at the 4 points on the diagonals of each face at a distance of $\frac{1}{2}\sqrt{5}h$ from the center of the face.


 Tetrahedron: $\mathcal{T}$

25.4.70

$$\begin{aligned} \frac{1}{V} \iiint_{\mathcal{T}} f(x, y, z) dx dy dz &= \frac{1}{40} \sum f_e + \frac{9}{40} \sum f_r \\ &\quad + \text{terms of 4th order} \\ &= \frac{32}{60} f_m + \frac{1}{60} \sum f_e + \frac{4}{60} \sum f_r \\ &\quad + \text{terms of 4th order} \end{aligned}$$

where

 V : Volume of $\mathcal{T}$

$\sum f_r$: Sum of values of the function at the vertices of $\mathcal{T}$.

$\sum f_e$: Sum of values of the function at midpoints of the edges of $\mathcal{T}$.

$\sum f_f$: Sum of values of the function at the center of gravity of the faces of $\mathcal{T}$.

f_m : Value of function at center of gravity of $\mathcal{T}$.

25.5. Ordinary Differential Equations⁶**First Order: $y' = f(x, y)$** **Point Slope Formula**

$$25.5.1 \quad y_{n+1} = y_n + h y'_n + O(h^2)$$

$$25.5.2 \quad y_{n+1} = y_{n-1} + 2h y'_n + O(h^3)$$

Trapezoidal Formula

$$25.5.3 \quad y_{n+1} = y_n + \frac{h}{2} (y'_{n+1} + y'_n) + O(h^3)$$

Adams' Extrapolation Formula

$$25.5.4 \quad y_{n+1} = y_n + \frac{h}{24} (55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}) + O(h^5)$$

Adams' Interpolation Formula

$$25.5.5 \quad y_{n+1} = y_n + \frac{h}{24} (9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}) + O(h^5)$$

Runge-Kutta Methods**Second Order**

$$25.5.6 \quad y_{n+1} = y_n + \frac{1}{2} (k_1 + k_2) + O(h^3)$$

$$k_1 = hf(x_n, y_n), k_2 = hf(x_n + h, y_n + k_1)$$

$$25.5.7 \quad y_{n+1} = y_n + k_2 + O(h^3)$$

$$k_1 = hf(x_n, y_n), k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$

Third Order

$$25.5.8 \quad y_{n+1} = y_n + \frac{1}{6}k_1 + \frac{2}{3}k_2 + \frac{1}{6}k_3 + O(h^4)$$

$$k_1 = hf(x_n, y_n), k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$

$$k_3 = hf(x_n + h, y_n - k_1 + 2k_2)$$

⁶The reader is cautioned against possible instabilities especially in formulas 25.5.2 and 25.5.13. See, e.g. [25.11], [25.12].

25.5.9

$$y_{n+1} = y_n + \frac{1}{4}k_1 + \frac{3}{4}k_3 + O(h^4)$$

$$k_1 = hf(x_n, y_n), k_2 = hf\left(x_n + \frac{1}{3}h, y_n + \frac{1}{3}k_1\right)$$

$$k_3 = hf\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}k_2\right)$$

Fourth Order**25.5.10**

$$y_{n+1} = y_n + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 + O(h^5)$$

$$k_1 = hf(x_n, y_n), k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right), k_4 = hf(x_n + h, y_n + k_3)$$

25.5.11

$$y_{n+1} = y_n + \frac{1}{8}k_1 + \frac{3}{8}k_2 + \frac{3}{8}k_3 + \frac{1}{8}k_4 + O(h^5)$$

$$k_1 = hf(x_n, y_n), k_2 = hf\left(x_n + \frac{1}{3}h, y_n + \frac{1}{3}k_1\right)$$

$$k_3 = hf\left(x_n + \frac{2}{3}h, y_n - \frac{1}{3}k_1 + k_2\right),$$

$$k_4 = hf(x_n + h, y_n + k_1 - k_2 + k_3)$$

Gill's Method**25.5.12**

$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2\left(1 - \sqrt{\frac{1}{2}}\right)k_2\right. \\ \left. + 2\left(1 + \sqrt{\frac{1}{2}}\right)k_3 + k_4\right) + O(h^5)$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \left(-\frac{1}{2} + \sqrt{\frac{1}{2}}\right)k_1\right. \\ \left. + \left(1 - \sqrt{\frac{1}{2}}\right)k_2\right)$$

$$k_4 = hf\left(x_n + h, y_n - \sqrt{\frac{1}{2}}k_2 + \left(1 + \sqrt{\frac{1}{2}}\right)k_3\right)$$

Predictor-Corrector Methods**Milne's Methods****25.5.13**

$$P: y_{n+1} = y_{n-3} + \frac{4h}{3}(2y'_n - y'_{n-1} + 2y'_{n-2}) + O(h^5)$$

$$C: y_{n+1} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1}) + O(h^5)$$

25.5.14

$$P: y_{n+1} = y_{n-5} + \frac{3h}{10} (11y'_n - 14y'_{n-1} + 26y'_{n-2} - 14y'_{n-3} + 11y'_{n-4}) + O(h^7)$$

$$C: y_{n+1} = y_{n-3} + \frac{2h}{45} (7y'_{n+1} + 32y'_n + 12y'_{n-1} + 32y'_{n-2} + 7y'_{n-3}) + O(h^7)$$

Formulas Using Higher Derivatives**25.5.15**

$$P: y_{n+1} = y_{n-2} + 3(y_n - y_{n-1}) + h^2(y''_n - y''_{n-1}) + O(h^5)$$

$$C: y_{n+1} = y_n + \frac{h}{2} (y'_{n+1} + y'_n) - \frac{h^2}{12} (y''_{n+1} - y''_n) + O(h^5)$$

25.5.16

$$P: y_{n+1} = y_{n-2} + 3(y_n - y_{n-1}) + \frac{h^3}{2} (y'''_n + y'''_{n-1}) + O(h^7)$$

$$C: y_{n+1} = y_n + \frac{h}{2} (y'_{n+1} + y'_n) - \frac{h^2}{10} (y''_{n+1} - y''_n) + \frac{h^3}{120} (y'''_{n+1} + y'''_n) + O(h^7)$$

Systems of Differential Equations

First Order: $y' = f(x, y, z)$, $z' = g(x, y, z)$.

Second Order Runge-Kutta**25.5.17**

$$y_{n+1} = y_n + \frac{1}{2} (k_1 + k_2) + O(h^3),$$

$$z_{n+1} = z_n + \frac{1}{2} (l_1 + l_2) + O(h^3)$$

$$k_1 = hf(x_n, y_n, z_n), \quad l_1 = hg(x_n, y_n, z_n)$$

$$k_2 = hf(x_n + h, y_n + k_1, z_n + l_1),$$

$$l_2 = hg(x_n + h, y_n + k_1, z_n + l_1)$$

Fourth Order Runge-Kutta**25.5.18**

$$y_{n+1} = y_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) + O(h^5),$$

$$z_{n+1} = z_n + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4) + O(h^5)$$

$$k_1 = hf(x_n, y_n, z_n) \quad l_1 = hg(x_n, y_n, z_n)$$

$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1, z_n + \frac{1}{2}l_1\right)$$

$$l_2 = hg\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1, z_n + \frac{1}{2}l_1\right)$$

$$k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2, z_n + \frac{1}{2}l_2\right)$$

$$l_3 = hg\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2, z_n + \frac{1}{2}l_2\right)$$

$$k_4 = hf(x_n + h, y_n + k_3, z_n + l_3)$$

$$l_4 = hg(x_n + h, y_n + k_3, z_n + l_3)$$

Second Order: $y'' = f(x, y, y')$

Milne's Method**25.5.19**

$$P: y'_{n+1} = y'_{n-3} + \frac{4h}{3} (2y''_{n-2} - y''_{n-1} + 2y''_n) + O(h^5)$$

$$C: y'_{n+1} = y'_{n-1} + \frac{h}{3} (y''_{n-1} + 4y''_n + y''_{n+1}) + O(h^5)$$

Runge-Kutta Method**25.5.20**

$$y_{n+1} = y_n + h \left[y'_n + \frac{1}{6} (k_1 + k_2 + k_3) \right] + O(h^5)$$

$$y'_{n+1} = y'_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_n, y_n, y'_n)$$

$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{h}{2}y'_n + \frac{h}{8}k_1, y'_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{h}{2}y'_n + \frac{h}{8}k_1, y'_n + \frac{k_2}{2}\right) \quad *$$

$$k_4 = hf\left(x_n + h, y_n + hy'_n + \frac{h}{2}k_3, y'_n + k_3\right)$$

Second Order: $y'' = f(x, y)$

Milne's Method**25.5.21**

$$P: y_{n+1} = y_n + y_{n-2} - y_{n-3}$$

$$+ \frac{h^2}{4} (5y''_n + 2y''_{n-1} + 5y''_{n-2}) + O(h^6)$$

$$C: y_n = 2y_{n-1} - y_{n-2} + \frac{h^2}{12} (y''_n + 10y''_{n-1} + y''_{n-2}) + O(h^6)$$

Runge-Kutta Method

$$25.5.22 \quad y_{n+1} = y_n + h \left(y'_n + \frac{1}{6} (k_1 + 2k_2) \right) + O(h^4)$$

$$y'_{n+1} = y'_n + \frac{1}{6} k_1 + \frac{2}{3} k_2 + \frac{1}{6} k_3$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}y'_n + \frac{h}{8}k_1\right)$$

$$k_3 = hf\left(x_n + h, y_n + hy'_n + \frac{h}{2}k_2\right).$$

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(For textbooks on numerical analysis, see texts in chapter 3)

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26. Probability Functions

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26. Probability Functions

Mathematical Properties³

26.1. Probability Functions: Definitions and Properties

Univariate Cumulative Distribution Functions

A real-valued function $F(x)$ is termed a (univariate) cumulative distribution function (c.d.f.) or simply distribution function if

- i) $F(x)$ is non-decreasing, i.e., $F(x_1) \leq F(x_2)$ for $x_1 \leq x_2$
- ii) $F(x)$ is everywhere continuous from the right, i.e., $F(x) = \lim_{\epsilon \rightarrow 0+} F(x + \epsilon)$
- iii) $F(-\infty) = 0, F(\infty) = 1$.

The function $F(x)$ signifies the probability of the event " $X \leq x$ " where X is a random variable, i.e., $Pr\{X \leq x\} = F(x)$, and thus describes the c.d.f. of X . The two principal types of distribution functions are termed *discrete* and *continuous*.

Discrete Distributions: Discrete distributions are characterized by the random variable X taking on an enumerable number of values . . . , $x_{-1}, x_0, x_1, \dots$ with point probabilities

$$p_n = Pr\{X = x_n\} \geq 0$$

which need only be subject to the restriction

$$\sum_n p_n = 1.$$

The corresponding distribution function can then be written

$$26.1.1 \quad F(x) = Pr\{X \leq x\} = \sum_{x_n \leq x} p_n$$

³ Comment on notation and conventions.

a. We follow the customary convention of denoting a random variable by a capital letter, i.e., X , and using the corresponding lower case letter, i.e., x , for a particular value that the random variable assumes.

b. For statistical applications it is often convenient to have tabulated the "upper tail area," $1 - F(x)$, or the c.d.f. for $|X|$, $F(x) - F(-x)$, instead of simply the c.d.f. $F(x)$. We use the notation P to indicate the c.d.f. of X , $Q = 1 - P$ to indicate the "upper tail area" and $A = P - Q$ to denote the c.d.f. of $|X|$. In particular we use $P(x)$, $Q(x)$, and $A(x)$ to denote the corresponding functions for the normal or Gaussian probability function, see 26.2.2-26.2.4. When these distributions depend on other parameters, say θ_1 and θ_2 , we indicate this by writing $P(x|\theta_1, \theta_2)$, $Q(x|\theta_1, \theta_2)$, or $A(x|\theta_1, \theta_2)$. For example the chi-square distribution 26.4 depends on the parameter ν and the tabulated function is written $Q(\chi^2|\nu)$.

where the summation is over all values of x for which $x_n \leq x$. The set $\{x_n\}$ of values for which $p_n > 0$ is termed the domain of the random variable X . A discrete distribution of a random variable is called a *lattice distribution* if there exist numbers a and $b \neq 0$ such that every possible value of X can be represented in the form $a + bn$ where n takes on only integral values. A summary of some properties of certain discrete distributions is presented in 26.1.19-26.1.24.

Continuous Distributions. Continuous distributions are characterized by $F(x)$ being absolutely continuous. Hence $F(x)$ possesses a derivative $F'(x) = f(x)$ and the c.d.f. can be written

$$26.1.2 \quad F(x) = Pr\{X \leq x\} = \int_{-\infty}^x f(t) dt.$$

The derivative $f(x)$ is termed the *probability density function* (p.d.f.) or *frequency function*, and the values of x for which $f(x) > 0$ make up the domain of the random variable X . A summary of some properties of certain selected continuous distributions is presented in 26.1.25-26.1.34.

Multivariate Probability Functions

The real-valued function $F(x_1, x_2, \dots, x_n)$ defines an n -variate cumulative distribution function if

- i) $F(x_1, x_2, \dots, x_n)$ is a non-decreasing function for each x_i
- ii) $F(x_1, x_2, \dots, x_n)$ is continuous from the right in each x_i ; i.e., $F(x_1, x_2, \dots, x_n) = \lim_{\epsilon \rightarrow 0+} F(x_1, \dots, x_i + \epsilon, \dots, x_n)$
- iii) $F(x_1, x_2, \dots, x_n) = 0$ when any $x_i = -\infty$; $F(\infty, \infty, \dots, \infty) = 1$.
- iv) $F(x_1, x_2, \dots, x_n)$ assigns nonnegative probability to the event $x_1 < X_1 \leq x_1 + h_1, x_2 < X_2 \leq x_2 + h_2, \dots, x_n < X_n \leq x_n + h_n$ for all $x_1, x_2, \dots, x_n$ and all nonnegative $h_1, h_2, \dots, h_n$, e.g., for $n=2$, $F(x_1 + h_1, x_2 + h_2) - F(x_1, x_2 + h_2) - F(x_1 + h_1, x_2) + F(x_1, x_2) \geq 0$ and in general for $x_i < X_i \leq x_i + h_i$ ($i=1, 2, \dots, n$), the k th order difference $\Delta_k F(x_1, x_2, \dots, x_n) > 0$ for $k=1, 2, \dots, n$.

*See page 11.

The joint probability of the event $X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n$ is $F(x_1, x_2, \dots, x_n)$. Analogous to the one-dimensional case, *discrete* distributions assign all probability to an enumerable set of vectors $(x_1, x_2, \dots, x_n)$ and *continuous* distributions are characterized by absolute continuity of $F(x_1, x_2, \dots, x_n)$.

Characteristics of distribution functions: Moments, characteristic functions, cumulants

		Continuous distributions	Discrete distributions
26.1.3	n^{th} moment about origin	$\mu'_n = \int_{-\infty}^{\infty} x^n f(x) dx$	$\mu'_n = \sum_i x_i^n p_i$
26.1.4	mean	$m = \mu'_1 = \int_{-\infty}^{\infty} x f(x) dx$	$m = \mu'_1 = \sum_i x_i p_i$
26.1.5	variance	$\sigma^2 = \mu'_2 - m^2 = \int_{-\infty}^{\infty} (x-m)^2 f(x) dx$	$\sigma^2 = \mu'_2 - m^2 = \sum_i (x_i - m)^2 p_i$
26.1.6	n^{th} central moment	$\mu_n = \int_{-\infty}^{\infty} (x-m)^n f(x) dx$	$\mu_n = \sum_i (x_i - m)^n p_i$
26.1.7	expected value operator for the function $g(x)$	$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$	$E[g(X)] = \sum_i g(x_i) p_i$
26.1.8	characteristic function of X	$\phi(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} f(x) dx$	$\phi(t) = E(e^{itX}) = \sum_i e^{itx_i} p_i$
26.1.9	characteristic function of $g(X)$	$\phi_g(t) = E(e^{itg(X)}) = \int_{-\infty}^{\infty} e^{itg(x)} f(x) dx$	$\phi_g(t) = E(e^{itg(X)}) = \sum_i e^{itg(x_i)} p_i$
26.1.10	inversion formula	$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t) dt$	$p_n = \frac{1}{2\pi} \int_{-\pi/b}^{\pi/b} e^{-itx_n} \phi(t) dt$ (lattice distributions only)

Relation of the Characteristic Function to Moments About the Origin

$$26.1.11 \quad \phi^{(n)}(0) = \left[\frac{d^n}{dt^n} \phi(t) \right]_{t=0} = i^n \mu'_n$$

Cumulant Function

$$26.1.12 \quad \ln \phi(t) = \sum_{n=0}^{\infty} \kappa_n \frac{(it)^n}{n!}$$

κ_n is called the n^{th} cumulant.

$$26.1.13 \quad \kappa_1 = m, \kappa_2 = \sigma^2, \kappa_3 = \mu_3, \kappa_4 = \mu_4 - 3\mu_2^2$$

Relation of Central Moments to Moments About the Origin

$$26.1.14 \quad \mu_n = \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} \mu'_j m^{n-j}$$

Coefficients of Skewness and Excess

$$26.1.15 \quad \gamma_1 = \frac{\kappa_3}{\kappa_2^{3/2}} = \frac{\mu_3}{\sigma^3} \quad (\text{skewness})$$

$$26.1.16 \quad \gamma_2 = \frac{\kappa_4}{\kappa_2^2} = \frac{\mu_4}{\sigma^4} - 3 \quad (\text{excess})$$

Occasionally coefficients of skewness and excess (or kurtosis) are given by

$$26.1.17 \quad \beta_1 = \gamma_1^2 = \left(\frac{\mu_3}{\sigma^3} \right)^2 \quad (\text{skewness})$$

$$26.1.18 \quad \beta_2 = \gamma_2 + 3 = \frac{\mu_4}{\sigma^4} \quad (\text{excess or kurtosis})$$

Some one-dimensional discrete distribution functions

Name	Domain	Point Probabilities	Restrictions on Parameters	Mean	Variance	Skewness γ_1	Excess γ_2	Characteristic function	Cumulants
26.1.19 Single point or degenerate	$x=c$ (c a constant)	$p=1$	$-\infty < c < +\infty$	c	0			$e^{ic\lambda}$	$\kappa_1=\lambda, \kappa_r=0$ for $r \geq 1$
26.1.20 Binomial	$x_s=t$, for $s=0, 1, 2, \dots, n$	$\binom{n}{s} p^s (1-p)^{n-s}$	$0 < p < 1$ ($q=1-p$)	np	npq	$\frac{q-p}{\sqrt{npq}}$	$\frac{1-6pq}{npq}$	$(q+pe^{it})^n$	$\kappa_1=np$ $\kappa_{r+1}=npq \frac{d^r}{dp}$ for $r \geq 1$
26.1.21 Hypergeometric	$x_s=t$, for $s=0, 1, \dots, \min(n, N_1)$	$\frac{\binom{N_1}{s} \binom{N_2}{n-s}}{\binom{N_1+N_2}{n}}$	N_1 and N_2 integers, and $n \leq N_1+N_2$ $(N_1-N_2)/N$ and $p=N_1/N$ and $q=1-p=N_2/N$	np	$npq \left(\frac{N-n}{N-1} \right)$	$\frac{q-p}{\sqrt{npq}} \left(\frac{N-1}{N-n} \right) \left(\frac{N-2n}{N-2} \right)$	Complicated	$\frac{\binom{N_1}{n}}{\binom{N_1+N_2}{n}} F(-n, -N_2; N_1+N_2-n+1; e^{it})$	Complicated
26.1.22 Poisson	$x_s=t$, for $s=0, 1, 2, \dots, \infty$	$\frac{e^{-m} m^s}{s!}$	$0 < m < \infty$	m	m	$m^{-\frac{1}{2}}$	m^{-1}	$e^{m(e^{it}-1)}$	$\kappa_r=m$ for $r=1, 2, \dots$
26.1.23 Negative binomial	$x_s=t$, for $s=0, 1, 2, \dots, \infty$	$\binom{n+s-1}{s} p^n (1-p)^s$	$n \geq 0$ and $0 < p < 1$ ($p=1/Q$ and $1-p=P/Q$)	nP	nPQ	$\frac{Q+P}{\sqrt{nPQ}}$	$\frac{1+6PQ}{nPQ}$	$(Q-Pe^{it})^{-n}$	$\kappa_1=nP$ $\kappa_{r+1}=PQ \frac{d^r}{dQ}$ for $r \geq 1$
26.1.24 Geometric	$x_s=t$, for $s=0, 1, 2, \dots, \infty$	$p(1-p)^s$	$0 < p < 1$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	$\frac{2-p}{\sqrt{1-p}}$	$6 + \frac{p^2}{1-p}$	$p[1-(1-p)e^{it}]^{-1}$	$\kappa_1=\frac{1-p}{p}$ $\kappa_{r+1}=-\frac{d^r}{dp} \frac{1-p}{p}$ for $r \geq 1$

Some one-dimensional continuous distribution functions

	Name	Domain	Probability Density Function $f(x)$	Restrictions on Parameters	Mean	Variance	Skewness γ_1	Excess γ_2	Characteristic function	Cumulants
26.1.25	Error function	$-\infty < x < \infty$	$\frac{h}{\sqrt{\pi}} e^{-h^2 x^2}$	$0 < h < \infty$	0	$\frac{1}{2h^3}$	0	0	$\frac{-t^2}{e^{i h^2 t^2}}$	$\kappa_1 = 0, \kappa_2 = \frac{1}{2h^2}$ $\kappa_n = 0$ for $n > 2$
26.1.26	Normal	$-\infty < x < \infty$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$	$-\infty < m < \infty$ $0 < \sigma < \infty$	m	σ^2	0	0	$e^{i m t - \frac{\sigma^2 t^2}{2}}$	$\kappa_1 = m, \kappa_2 = \sigma^2, \kappa_n = 0$ for $n > 2$
26.1.27	Cauchy	$-\infty < x < \infty$	$\frac{1}{\pi\beta} \frac{1}{1 + \left(\frac{x-\alpha}{\beta}\right)^2}$	$-\infty < \alpha < \infty$ $0 < \beta < \infty$	not defined	not defined	not defined	not defined	$e^{i\alpha t - \beta t }$	not defined
26.1.28	Exponential	$\alpha \leq x < \infty$	$\frac{1}{\beta} e^{-\left(\frac{x-\alpha}{\beta}\right)}$	$-\infty < \alpha < \infty$ $0 < \beta < \infty$	$\alpha + \beta$	β^2	2	6	$e^{i\alpha t(1-i\beta t)^{-1}}$	$\kappa_1 = \alpha + \beta, \kappa_n = \beta^n \Gamma(n)$ for $n > 1$
26.1.29	Laplace, or double exponential	$-\infty < x < \infty$	$\frac{1}{2\beta} e^{-\left \frac{x-\alpha}{\beta}\right }$	$-\infty < \alpha < \infty$ $0 < \beta < \infty$	α	$2\beta^2$	0	3	$e^{i\alpha t(1+\beta^2 t^2)^{-1}}$	$\kappa_1 = \alpha, \kappa_2 = 2\beta^2$ $\kappa_{2n+1} = 0, \kappa_{2n} = \frac{(2n)!}{n} \beta^{2n}$ for $n = 1, 2, \dots$
26.1.30	Extreme-Value, ⁴ (Fisher-Tippett Type I or doubly exponential)	$-\infty < x < \infty$	$\frac{1}{\beta} \exp(-y - e^{-y})$ with $y = \frac{x-\alpha}{\beta}$	$-\infty < \alpha < \infty$ $0 < \beta < \infty$	$\alpha + \gamma\beta$	$\frac{(\pi\beta)^2}{6}$	1.3	2.4	$\Gamma(1-i\beta t) e^{i\alpha t}$	$\kappa_1 = \gamma, \kappa_2 = \frac{(\pi\beta)^2}{6}$ $\kappa_n = \beta^n \Gamma(n) \sum_{r=1}^{\infty} \frac{1}{r^n}$ for $n > 2$
26.1.31	Pearson Type III	$\alpha \leq x < \infty$	$\frac{1}{\beta \Gamma(p)} y^{p-1} e^{-y}$ with $y = \frac{x-\alpha}{\beta}$	$-\infty < \alpha < \infty$ $0 < \beta < \infty$ $0 < p < \infty$	$\alpha + p\beta$	$p\beta^2$	$\frac{2}{\sqrt{p}}$	$6/p$	$e^{i\alpha t(1-i\beta t)^{-p}}$	$\kappa_1 = \alpha + p\beta, \kappa_n = \beta^n p \Gamma(n)$ for $n > 1$
26.1.32	Gamma distribution	$0 \leq x < \infty$	$\frac{1}{\Gamma(p)} x^{p-1} e^{-x}$	$0 < p < \infty$	p	p	$\frac{2}{\sqrt{p}}$	$6/p$	$(1-it)^{-p}$	$\kappa_1 = p, \kappa_n = p \Gamma(n)$ for $n > 1$
26.1.33	Beta distribution	$0 \leq x \leq 1$	$\frac{1}{B(a, b)} x^{a-1} (1-x)^{b-1}$	$1 \leq a < \infty$ $1 \leq b < \infty$	$\frac{a}{a+b}$	$\frac{ab}{(a+b)^2(a+b+1)}$	$\frac{2(a-b)}{(a+b+2)}$	See footnote 5.	$M(a, a+b, it)$	
26.1.34	Rectangular, or uniform	$m - \frac{h}{2} \leq x \leq m + \frac{h}{2}$	$\frac{1}{h}$	$-\infty < m < \infty$ $0 < h < \infty$	m	$\frac{h^3}{12}$	0	-1.2	$\frac{2}{ht} \sin\left(\frac{ht}{2}\right) e^{i m t}$	$\kappa_1 = m, \kappa_{2n+1} = 0$ $\kappa_{2n} = \frac{h^{2n} B_{2n}}{2n}$ B_{2n} (Bernoulli numbers), $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, \dots$

⁴ γ (Euler's constant) = .57721 56649⁵ $\gamma_2 = \sqrt{\frac{a+b+1}{ab}} \left\{ \frac{3(a+b+1)[2(a+b)^2 + ab(a+b-6)]}{ab(a+b+2)(a+b+3)} - 3 \right\}$.

* See page II.

Inequalities for distribution functions

($F(x)$ denotes the c.d.f. of the random variable X and t denotes a positive constant; further m is always assumed to be finite and all expectations are assumed to exist.)

Inequality	Conditions
26.1.35 $Pr\{g(X) \geq t\} \leq E[g(X)]/t$	(i) $g(X) \geq 0$
26.1.36 $Pr\{X \geq t\} \leq m/t$ $F(t) \geq 1 - \frac{m}{t}$	(i) $Pr\{X < 0\} = 0$ (ii) $E(X) = m$
26.1.37 $Pr\{ X - m \geq t\sigma\} \leq 1/t^2$ $F(m + t\sigma) - F(m - t\sigma) \geq 1 - \frac{1}{t^2}$	(i) $E(X) = m$ (ii) $E(X - m)^2 = \sigma^2$ *
26.1.38 $Pr\{ \bar{X} - \bar{m} \geq t\sigma\} \leq \frac{1}{nt^2}$	(i) $E(X_i) = m_i$ (ii) $E(X_i - m_i)^2 = \sigma_i^2$ (iii) $E([X_i - m_i][X_j - m_j]) = 0 (i \neq j)$ (iv) $\bar{X} = \sum_{i=1}^n \frac{X_i}{n}$ $\bar{m} = \sum_{i=1}^n \frac{m_i}{n}, \bar{\sigma} = \left[\sum_{i=1}^n \frac{\sigma_i^2}{n} \right]^{1/2}$
26.1.39 $Pr\{ X - m \geq t\sigma\} \leq \frac{4}{9} \left\{ \frac{1 + \left(\frac{m - x_0}{\sigma} \right)^2}{\left(t - \left \frac{m - x_0}{\sigma} \right \right)^2} \right\}$ $F(m + t\sigma) - F(m - t\sigma) \geq 1 - \frac{4}{9} \left\{ \frac{1 + \left(\frac{m - x_0}{\sigma} \right)^2}{\left(t - \left \frac{m - x_0}{\sigma} \right \right)^2} \right\}$	(i) $E(X - m)^2 = \sigma^2$ (ii) $F(x)$ is a continuous c.d.f. (iii) $F(x)$ is unimodal at x_0 †
26.1.40 $Pr\{ X - m \geq t\sigma\} \leq 4/9t^2$ $F(m + t\sigma) - F(m - t\sigma) \geq 1 - \frac{4}{9t^2}$	(i) $E(X - m)^2 = \sigma^2$ (ii) $F(x)$ is a continuous c.d.f. (iii) $F(x)$ is unimodal at x_0 † (iv) $m = x_0$
26.1.41 $Pr\{ X - m \geq t\sigma\} \leq \frac{\mu_4 - \sigma^4}{\mu_4 + t^4\sigma^4 - 2t^2\sigma^4}$ $F(m + t\sigma) - F(m - t\sigma) \geq 1 - \frac{\mu_4 - \sigma^4}{\mu_4 + t^4\sigma^4 - 2t^2\sigma^4}$	(i) $E(X - m)^2 = \sigma^2$ (ii) $E(X - m)^4 = \mu_4$

† x_0 is such that $F'(x_0) > F'(x)$ for $x \neq x_0$.

26.2. Normal or Gaussian Probability Function

$$26.2.1 \quad Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$26.2.2 \quad P(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt = \int_{-\infty}^x Z(t) dt$$

$$26.2.3 \quad Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt = \int_x^{\infty} Z(t) dt$$

$$26.2.4 \quad A(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^x e^{-t^2/2} dt = \int_{-x}^x Z(t) dt$$

$$26.2.5 \quad P(x) + Q(x) = 1$$

$$26.2.6 \quad P(-x) = Q(x)$$

$$26.2.7 \quad A(x) = 2P(x) - 1$$

Probability Integral with Mean m and Variance σ^2

A random variable X is said to be normally distributed with mean m and variance σ^2 if the probability that X is less than or equal to x is given by

26.2.8

$$Pr\{X \leq x\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{(t-m)^2}{2\sigma^2}} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-m)/\sigma} e^{-t^2/2} dt = P\left(\frac{x-m}{\sigma}\right).$$

The corresponding probability density function is

26.2.9

$$\frac{\partial}{\partial x} P\left(\frac{x-m}{\sigma}\right) = \frac{1}{\sigma} Z\left(\frac{x-m}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

and is symmetric around m , i.e.

$$Z\left(\frac{m+x}{\sigma}\right) = Z\left(\frac{m-x}{\sigma}\right).$$

The inflexion points of the probability density function are at $m \pm \sigma$.

Power Series ($x \geq 0$)

$$26.2.10 \quad P(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n! 2^n (2n+1)}$$

26.2.11

$$P(x) = \frac{1}{2} + Z(x) \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 \cdot 3 \cdot 5 \dots (2n+1)}$$

Asymptotic Expansions ($x > 0$)

26.2.12

$$Q(x) = \frac{Z(x)}{x} \left\{ 1 - \frac{1}{x^2} + \frac{1 \cdot 3}{x^4} + \dots + \frac{(-1)^n 1 \cdot 3 \dots (2n-1)}{x^{2n}} \right\} + R_n$$

where

$$R_n = (-1)^{n+1} 1 \cdot 3 \dots (2n+1) \int_x^{\infty} \frac{Z(t)}{t^{2n+2}} dt$$

which is less in absolute value than the first neglected term.

26.2.13

$$Q(x) \sim \frac{Z(x)}{x} \left\{ 1 - \frac{a_1}{x^2+2} + \frac{a_2}{(x^2+2)(x^2+4)} - \frac{a_3}{(x^2+2)(x^2+4)(x^2+6)} + \dots \right\}$$

where $a_1=1$, $a_2=1$, $a_3=5$, $a_4=9$, $a_5=129$ and the general term is

$$a_n = c_0 1 \cdot 3 \dots (2n-1) + 2c_1 1 \cdot 3 \dots (2n-3) + 2^2 c_2 1 \cdot 3 \dots (2n-5) + \dots + 2^{n-1} c_{n-1}$$

and c_s is the coefficient of t^{n-s} in the expansion of $t(t-1) \dots (t-n+1)$.

Continued Fraction Expansions

26.2.14

$$Q(x) = Z(x) \left\{ \frac{1}{x+} \frac{1}{x+} \frac{2}{x+} \frac{3}{x+} \frac{4}{x+} \dots \right\} \quad (x > 0)$$

26.2.15

$$Q(x) = \frac{1}{2} - Z(x) \left\{ \frac{x}{1-} \frac{x^2}{3+} \frac{2x^2}{5-} \frac{3x^2}{7+} \frac{4x^2}{9-} \dots \right\} \quad (x \geq 0)$$

Polynomial and Rational Approximations⁷ for $P(x)$ and $Z(x)$

$$0 \leq x < \infty$$

26.2.16

$$P(x) = 1 - Z(x)(a_1 t + a_2 t^2 + a_3 t^3) + \epsilon(x), \quad t = \frac{1}{1+px}$$

$$|\epsilon(x)| < 1 \times 10^{-5}$$

$$\begin{aligned} p &= .33267 & a_1 &= .43618 \quad 36 \\ & & a_2 &= -.12016 \quad 76 \\ & & a_3 &= .93729 \quad 80 \end{aligned}$$

26.2.17

$$P(x) = 1 - Z(x)(b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5) + \epsilon(x), \quad t = \frac{1}{1+px}$$

$$|\epsilon(x)| < 7.5 \times 10^{-8}$$

$$p = .23164 \quad 19$$

$$\begin{aligned} b_1 &= .31938 \quad 1530 & b_4 &= -1.82125 \quad 5978 \\ b_2 &= -.35656 \quad 3782 & b_5 &= 1.33027 \quad 4429 \\ b_3 &= 1.78147 \quad 7937 \end{aligned}$$

26.2.18

$$P(x) = 1 - \frac{1}{2} (1 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4)^{-4} + \epsilon(x)$$

$$|\epsilon(x)| < 2.5 \times 10^{-4}$$

$$\begin{aligned} c_1 &= .196854 & c_3 &= .000344 \\ c_2 &= .115194 & c_4 &= .019527 \end{aligned}$$

26.2.19

$$P(x) = 1 - \frac{1}{2} (1 + d_1 x + d_2 x^2 + d_3 x^3 + d_4 x^4 + d_5 x^5 + d_6 x^6)^{-16} + \epsilon(x)$$

$$|\epsilon(x)| < 1.5 \times 10^{-7}$$

$$\begin{aligned} d_1 &= .04986 \quad 73470 & d_4 &= .00003 \quad 80036 \\ d_2 &= .02114 \quad 10061 & d_5 &= .00004 \quad 88906 \\ d_3 &= .00327 \quad 76263 & d_6 &= .00000 \quad 53830 \end{aligned}$$

$$26.2.20 \quad Z(x) = (a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6)^{-1} + \epsilon(x)$$

$$|\epsilon(x)| < 2.7 \times 10^{-3}$$

$$\begin{aligned} a_0 &= 2.490895 & a_4 &= -.024393 \\ a_2 &= 1.466003 & a_6 &= .178257 \end{aligned}$$

⁷ Based on approximations in C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

26.2.21

$$Z(x) = (b_0 + b_2 x^2 + b_4 x^4 + b_6 x^6 + b_8 x^8 + b_{10} x^{10})^{-1} + \epsilon(x)$$

$$|\epsilon(x)| < 2.3 \times 10^{-4}$$

$$b_0 = 2.50523 \ 67 \quad b_6 = .13064 \ 69$$

$$b_2 = 1.28312 \ 04 \quad b_8 = -.02024 \ 90$$

$$b_4 = .22647 \ 18 \quad b_{10} = .00391 \ 32$$

Rational Approximations for x_p where $Q(x_p) = p$

$$0 < p \leq .5$$

26.2.22

$$x_p = t - \frac{a_0 + a_1 t}{1 + b_1 t + b_2 t^2} + \epsilon(p), \quad t = \sqrt{\ln \frac{1}{p^2}}$$

$$|\epsilon(p)| < 3 \times 10^{-3}$$

$$a_0 = 2.30753 \quad b_1 = .99229$$

$$a_1 = .27061 \quad b_2 = .04481$$

26.2.23

$$x_p = t - \frac{c_0 + c_1 t + c_2 t^2}{1 + d_1 t + d_2 t^2 + d_3 t^3} + \epsilon(p), \quad t = \sqrt{\ln \frac{1}{p^2}}$$

$$|\epsilon(p)| < 4.5 \times 10^{-4}$$

$$c_0 = 2.515517 \quad d_1 = 1.432788$$

$$c_1 = .802853 \quad d_2 = .189269$$

$$c_2 = .010328 \quad d_3 = .001308$$

Bounds Useful as Approximations to the Normal Distribution Function

26.2.24

$$P(x) \leq \begin{cases} P_1(x) = \frac{1}{2} + \frac{1}{2} (1 - e^{-2x^2/\pi})^{\frac{1}{2}} & (x > 0) \\ P_2(x) = 1 - \frac{(4+x^2)^{\frac{1}{2}} - x}{2} (2\pi)^{-\frac{1}{2}} e^{-x^2/2} & (x > 1.4) \end{cases}$$

26.2.25

$$P(x) \geq \begin{cases} P_3(x) = \frac{1}{2} + \frac{1}{2} \left(1 - e^{-2x^2/\pi} - \frac{2(\pi-3)}{3\pi^2} x^4 e^{-x^2/2} \right)^{\frac{1}{2}} & (x > 0) \\ P_4(x) = 1 - \frac{1}{x} (2\pi)^{-\frac{1}{2}} e^{-x^2/2} & (x > 2.2) \end{cases}$$

See Figure 26.1 for error curves.

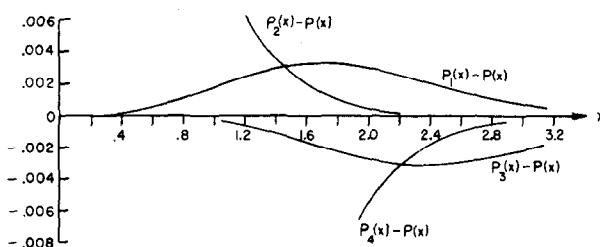


FIGURE 26.1. Error curves for bounds on normal distribution.

Derivatives of the Normal Probability Density Function

$$26.2.26 \quad Z^{(m)}(x) = \frac{d^m}{dx^m} Z(x)$$

Differential Equation

$$26.2.27 \quad Z^{(m+2)}(x) + xZ^{(m+1)}(x) + (m+1)Z^{(m)}(x) = 0$$

Value at $x=0$

26.2.28

$$Z^{(m)}(0) = \begin{cases} \frac{(-1)^{m/2} m!}{\sqrt{2\pi} 2^{m/2} \left(\frac{m}{2}\right)!} & \text{for } m=2r, r=0, 1, \dots \\ 0 & \text{for odd } m > 0 \end{cases}$$

Relation of $P(x)$ and $Z^{(m)}(x)$ to Other Functions

Function	Relation	
26.2.29 Error function	$\operatorname{erf} x = 2P(x\sqrt{2}) - 1$	$(x \geq 0)$
26.2.30 Incomplete gamma function (special case)	$\frac{\gamma(\frac{1}{2}, x)}{\Gamma(\frac{1}{2})} = [2P(\sqrt{2x}) - 1]$	$(x \geq 0)$
26.2.31 Hermite polynomial	$He_n(x) = (-1)^n \frac{Z^{(n)}(x)}{Z(x)}$	
26.2.32 “	$H_n(x) = (-1)^{n/2} \frac{Z^{(n)}(x\sqrt{2})}{Z(x\sqrt{2})}$	
26.2.33 Hh function	$Hh_{-n}(x) = (-1)^{n-1} \sqrt{2\pi} Z^{(n-1)}(x)$	$(n > 0)$
26.2.34 “	$Hh_n(x) = \frac{(-1)^n}{n!} Hh_{-1}(x) \frac{d^n}{dx^n} \left(\frac{Q(x)}{Z(x)} \right)$	$(n > 0)$
26.2.35 Tetrachoric function	$\tau_n(x) = \frac{(-1)^{n-1}}{\sqrt{n!}} Z^{(n-1)}(x)$	
26.2.36 Confluent hypergeometric function (special case)	$M\left(\frac{1}{2}, \frac{3}{2}, -\frac{x^2}{2}\right) = \frac{\sqrt{2\pi}}{x} \left\{ P(x) - \frac{1}{2} \right\}$	$(x > 0)$
26.2.37 “	$M\left(1, \frac{3}{2}, \frac{x^2}{2}\right) = \frac{1}{xZ(x)} \left\{ P(x) - \frac{1}{2} \right\}$	$(x > 0)$
26.2.38 “	$M\left(\frac{2m+1}{2}, \frac{1}{2}, -\frac{x^2}{2}\right) = \frac{Z^{(2m)}(x)}{Z^{(2m)}(0)}$	$(x \geq 0)$
26.2.39 “	$M\left(\frac{2m+2}{2}, \frac{3}{2}, -\frac{x^2}{2}\right) = \frac{Z^{(2m-1)}(x)}{xZ^{(2m)}(0)}$	$(x \geq 0)$
26.2.40 Parabolic cylinder function	$U\left(-n - \frac{1}{2}, x\right) = e^{-\frac{1}{2}x^2} (-1)^n \frac{Z^{(n)}(x)}{Z(x)}$	$(n > 0)$

Repeated Integrals of the Normal Probability Integral

$$26.2.41 \quad I_n(x) = \int_x^\infty I_{n-1}(t) dt \quad (n \geq 0)$$

where $I_{-1}(x) = Z(x)$

26.2.42

$$I_{-n}(x) = \left(-\frac{d}{dx}\right)^{n-1} Z(x) = (-1)^{n-1} Z^{(n-1)}(x) \quad (n \geq -1)$$

26.2.43

$$\left(\frac{d^2}{dx^2} + x \frac{dx}{dn} - n\right) I_n(x) = 0$$

26.2.44

$$(n+1)I_{n+1}(x) + xI_n(x) - I_{n-1}(x) = 0 \quad (n > -1)$$

26.2.45

$$I_n(x) = \int_x^\infty \frac{(t-x)^n}{n!} Z(t) dt = e^{-x^2/2} \int_0^\infty \frac{t^n}{n!} Z(t) dt \quad (n > -1)$$

$$26.2.46 \quad I_n(0) = I_{-n}(0) = \frac{1}{\left(\frac{n}{2}\right)! 2^{\frac{n+2}{2}}} \quad (n \text{ even})$$

Asymptotic Expansions of an Arbitrary Probability Density Function and Distribution Function

Let Y_i ($i=1, 2, \dots, n$) be n

independent random variables with mean m_i , variance σ_i^2 , and higher cumulants $\kappa_{r,i}$. Then asymptotic expansions with respect to n for the probability density and cumulative distribution function of

$$X = \frac{\sum_{i=1}^n (Y_i - m_i)}{\left(\sum_{i=1}^n \sigma_i^2\right)^{1/2}} \text{ are}$$

26.2.47

$$\begin{aligned} f(x) \sim Z(x) &- \left[\frac{\gamma_1}{6} Z^{(3)}(x) \right] + \left[\frac{\gamma_2}{24} Z^{(4)}(x) + \frac{\gamma_1^2}{72} Z^{(6)}(x) \right] \\ &- \left[\frac{\gamma_3}{120} Z^{(5)}(x) + \frac{\gamma_1 \gamma_2}{144} Z^{(7)}(x) + \frac{\gamma_1^3}{1296} Z^{(9)}(x) \right] \\ &+ \left[\frac{\gamma_4}{720} Z^{(6)}(x) + \frac{\gamma_2^2}{1152} Z^{(8)}(x) + \frac{\gamma_1 \gamma_3}{720} Z^{(8)}(x) \right. \\ &\quad \left. + \frac{\gamma_1^2 \gamma_2}{1728} Z^{(10)}(x) + \frac{\gamma_1^4}{31104} Z^{(12)}(x) \right] + \dots \end{aligned}$$

26.2.48

$$\begin{aligned} F(x) \sim P(x) &- \left[\frac{\gamma_1}{6} Z^{(2)}(x) \right] + \left[\frac{\gamma_2}{24} Z^{(3)}(x) + \frac{\gamma_1^2}{72} Z^{(5)}(x) \right] \\ &- \left[\frac{\gamma_3}{120} Z^{(4)}(x) + \frac{\gamma_1 \gamma_2}{144} Z^{(6)}(x) + \frac{\gamma_1^3}{1296} Z^{(8)}(x) \right] \\ &+ \left[\frac{\gamma_4}{720} Z^{(5)}(x) + \frac{\gamma_2^2}{1152} Z^{(7)}(x) + \frac{\gamma_1 \gamma_3}{720} Z^{(7)}(x) \right. \\ &\quad \left. + \frac{\gamma_1^2 \gamma_2}{1728} Z^{(9)}(x) + \frac{\gamma_1^4}{31104} Z^{(11)}(x) \right] + \dots \end{aligned}$$

where

$$\gamma_{r-2} = \frac{1}{n^{\frac{r}{2}-1}} \frac{\left(\frac{1}{n} \sum_{i=1}^n \kappa_{r,i}\right)}{\left(\frac{1}{n} \sum_{i=1}^n \sigma_i^2\right)^{r/2}}$$

Terms in brackets are terms of the same order with respect to n . When the Y_i have the same distribution, then $m_i = m$, $\sigma_i^2 = \sigma^2$, $\kappa_{r,i} = \kappa_r$ and

$$\gamma_{r-2} = \frac{1}{n^{\frac{r}{2}-1}} \left(\frac{\kappa_r}{\sigma^r} \right)$$

Asymptotic Expansion for the Inverse Function of an Arbitrary Distribution Function

Let the cumulative distribution function of $Y = \sum_{i=1}^n Y_i$ be denoted by $F(y)$. Then the (Cornish-Fisher) asymptotic expansion with respect to n for the value of y_p such that $F(y_p) = 1-p$ is

$$26.2.49 \quad y_p \sim m + \sigma w$$

where

$$\begin{aligned} w = x &+ [\gamma_1 h_1(x)] \\ &+ [\gamma_2 h_2(x) + \gamma_1^2 h_{11}(x)] \\ &+ [\gamma_3 h_3(x) + \gamma_1 \gamma_2 h_{12}(x) + \gamma_1^3 h_{111}(x)] \\ &+ [\gamma_4 h_4(x) + \gamma_2^2 h_{22}(x) + \gamma_1 \gamma_3 h_{13}(x) + \gamma_1^2 \gamma_2 h_{112}(x) \\ &\quad + \gamma_1^4 h_{1111}(x)] + \dots \end{aligned}$$

and

$$Q(x) = p, \quad \gamma_{r-2} = \frac{\kappa_r}{\kappa_2^{r/2}}, \quad r=3, 4, \dots$$

26.2.50

$$h_1(x) = \frac{1}{6} H e_2(x)$$

$$h_2(x) = \frac{1}{24} H e_3(x)$$

$$h_{11}(x) = -\frac{1}{36} [2H e_3(x) + H e_1(x)]$$

$$h_3(x) = \frac{1}{120} [H e_4(x)]$$

$$h_{12}(x) = -\frac{1}{24} [H e_4(x) + H e_2(x)]$$

$$h_{111}(x) = \frac{1}{324} [12H e_4(x) + 19H e_2(x)]$$

$$h_4(x) = \frac{1}{720} H e_5(x)$$

$$h_{22}(x) = -\frac{1}{384} [3H e_5(x) + 6H e_3(x) + 2H e_1(x)]$$

$$h_{13}(x) = -\frac{1}{180} [2H e_5(x) + 3H e_3(x)]$$

$$h_{112}(x) = \frac{1}{288} [14H e_5(x) + 37H e_3(x) + 8H e_1(x)]$$

$$h_{1111}(x) = -\frac{1}{7776} [252H e_5(x) + 832H e_3(x) + 227H e_1(x)]$$

Terms in brackets in 26.2.49 are terms of the same order with respect to n . The $H e_n(x)$ are the Hermite polynomials. (See chapter 22.)

$$26.2.51 \quad H e_n(x) = (-1)^n \frac{Z^{(n)}(x)}{Z(x)} = n! \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^m}{2^m m! (n-2m)!} x^{n-2m}$$

In the following auxiliary table, the polynomial functions $h_1(x), h_2(x) \dots h_{1111}(x)$ are tabulated for

$$p = .25, .1, .05, .025, .01, .005, .0025, .001, .0005.$$

Auxiliary coefficients^a for use with Cornish-Fisher asymptotic expansion. 26.2.49

	p								
	.25	.10	.05	.025	.01	.005	.0025	.001	.0005
x	.67449	1.28155	1.64485	1.95996	2.32635	2.57583	2.80703	3.09022	3.29053
$h_1(x)$	-.09084	.10706	.28426	.47358	.73532	.93915	1.14657	1.42491	1.63793
$h_2(x)$	-.07153	-.07249	-.02018	.06872	.23379	.39012	.57070	.84331	1.07320
$h_3(x)$	.07663	.06106	-.01878	-.14607	-.37634	-.59171	-.83890	-1.21025	-1.52234
$h_4(x)$	.00398	-.03464	-.04928	-.04410	-.00152	.06010	.14841	.30746	.46059
$h_5(x)$	.00282	.14644	.17532	.10210	-.17621	-.53531	-1.02868	-1.89355	-2.71243
$h_6(x)$	-.01428	-.11629	-.11900	-.02937	.25195	.59757	1.06301	1.86787	2.62337
$h_7(x)$	.00998	.00227	-.01082	-.02357	-.03176	-.02621	-.00666	.04591	.10950
$h_8(x)$	-.03285	.00776	.05985	.09659	.07888	-.01226	-.19116	-.59060	-1.03555
$h_9(x)$	-.05126	.01086	.09462	.16106	.16058	.05366	-.17498	-.70464	-1.30531
$h_{10}(x)$	.14764	-.10858	-.39517	-.55856	-.32621	.35696	1.60445	4.29304	7.23307
$h_{1111}(x)$	-.06898	.09585	.25623	.31624	.07286	-.46534	-1.39199	-3.32708	-5.40702

^a From R. A. Fisher, Contributions to mathematical statistics, Paper 30 (with E. A. Cornish) Extrait de la Revue de l'Institut International de Statistique 4, 1-14 (1937) (with permission).

26.3. Bivariate Normal Probability Function

26.3.1

$$g(x, y, \rho) = [2\pi\sqrt{1-\rho^2}]^{-1} \exp -\frac{1}{2} \left(\frac{x^2 - 2\rho xy + y^2}{1-\rho^2} \right)$$

$$26.3.2 \quad g(x, y, \rho) = (1-\rho^2)^{-1/2} Z(x) Z\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)$$

26.3.3

$$\begin{aligned} L(h, k, \rho) &= \int_h^\infty dx \int_k^\infty g(x, y, \rho) dy \\ &= \int_h^\infty Z(x) dx \int_w^\infty Z(w) dw, \quad w = \left(\frac{k - \rho x}{\sqrt{1-\rho^2}} \right) \end{aligned}$$

$$26.3.4 \quad L(-h, -k, \rho) = \int_{-\infty}^h dx \int_{-\infty}^k g(x, y, \rho) dy$$

$$26.3.5 \quad L(-h, k, -\rho) = \int_{-\infty}^h dx \int_k^\infty g(x, y, \rho) dy$$

$$26.3.6 \quad L(h, -k, -\rho) = \int_h^\infty dx \int_{-\infty}^k g(x, y, \rho) dy$$

$$26.3.7 \quad L(h, k, \rho) = L(k, h, \rho)$$

$$26.3.8 \quad L(-h, k, \rho) + L(h, k, -\rho) = Q(k)$$

$$26.3.9 \quad L(-h, -k, \rho) - L(h, k, \rho) = P(k) - Q(h)$$

26.3.10

$$\begin{aligned} * \quad 2[L(h, k, \rho) + L(h, k, -\rho) + P(h) - Q(k)] - 1 \\ = \int_{-h}^h dx \int_{-k}^k g(x, y, \rho) dy \end{aligned}$$

Probability Function With Means m_x, m_y , Variances σ_x^2, σ_y^2 , and Correlation ρ

The random variables X, Y are said to be distributed as a bivariate Normal distribution with

means and variances (m_x, m_y) and (σ_x^2, σ_y^2) and correlation ρ if the joint probability that X is less than or equal to h and Y less than or equal to k is given by

26.3.11

$$\begin{aligned} Pr\{X \leq h, Y \leq k\} &= \frac{1}{\sigma_x \sigma_y} \int_{-\infty}^{\frac{h-m_x}{\sigma_x}} \int_{-\infty}^{\frac{k-m_y}{\sigma_y}} g(s, t, \rho) ds dt \\ &= L\left(-\left(\frac{h-m_x}{\sigma_x}\right), -\left(\frac{k-m_y}{\sigma_y}\right), \rho\right) \end{aligned}$$

The probability density function is

26.3.12

$$\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp \frac{-Q}{2(1-\rho^2)} = \frac{1}{\sigma_x\sigma_y} g\left(\frac{x-m_x}{\sigma_x}, \frac{y-m_y}{\sigma_y}, \rho\right)$$

where

$$Q = \frac{(x-m_x)^2}{\sigma_x^2} - \frac{2\rho(x-m_x)(y-m_y)}{\sigma_x\sigma_y} + \frac{(y-m_y)^2}{\sigma_y^2}$$

Circular Normal Probability Density Function

26.3.13

$$\frac{1}{\sigma^2} g\left(\frac{x-m_x}{\sigma}, \frac{y-m_y}{\sigma}, 0\right) =$$

$$\frac{1}{2\pi\sigma^2} \exp -\frac{(x-m_x)^2 + (y-m_y)^2}{2\sigma^2}$$

*See page 11.

Special Values of $L(h, k, \rho)$

$$26.3.14 \quad L(h, k, 0) = Q(h)Q(k)$$

$$26.3.15 \quad L(h, k, -1) = 0 \quad (h+k \geq 0)$$

$$26.3.16 \quad L(h, k, -1) = P(h) - Q(k) \quad (h+k \leq 0)$$

$$26.3.17 \quad L(h, k, 1) = Q(h) \quad (k \leq h)$$

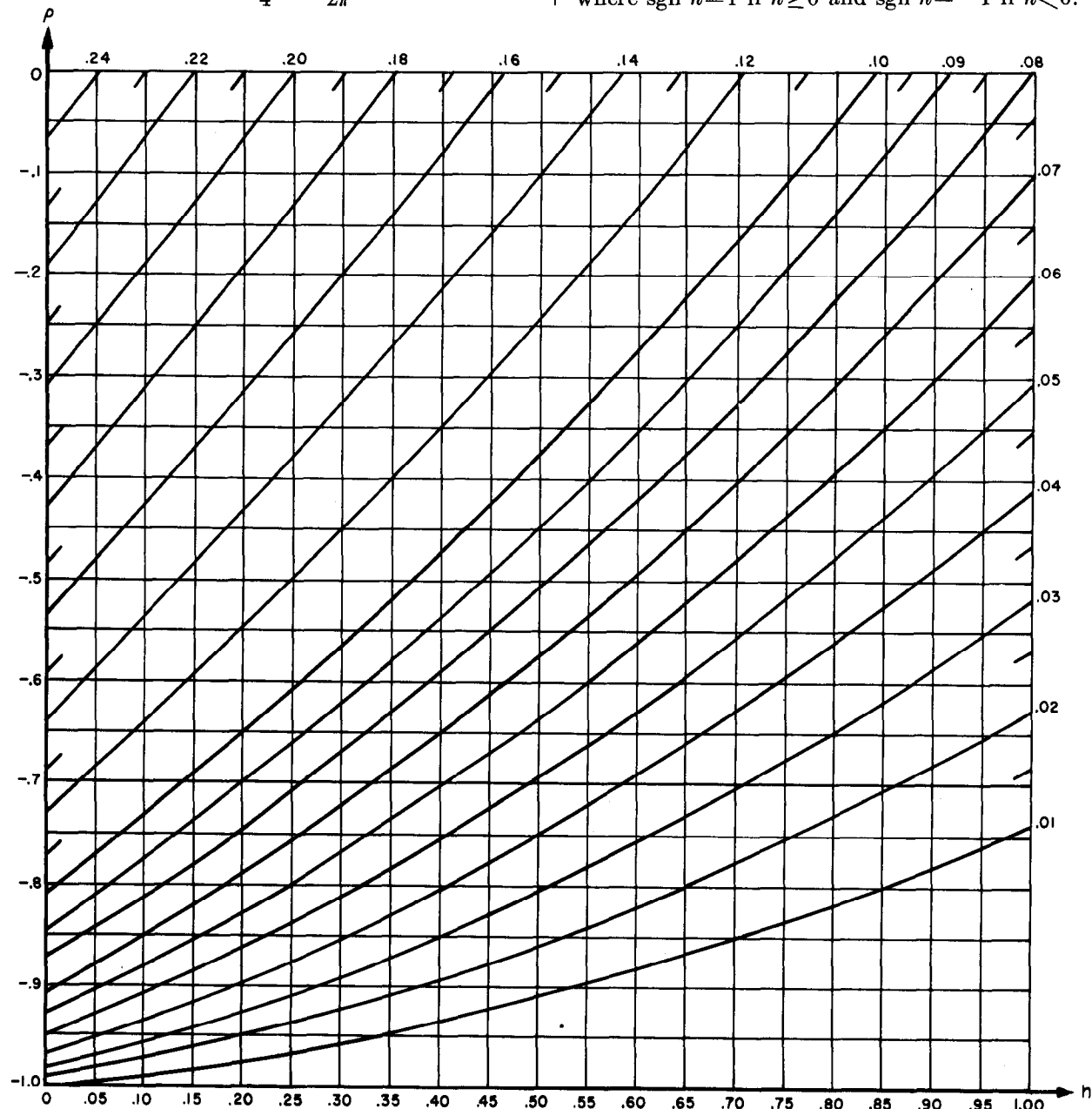
$$26.3.18 \quad L(h, k, 1) = Q(k) \quad (k \geq h)$$

$$26.3.19 \quad L(0, 0, \rho) = \frac{1}{4} + \frac{\arcsin \rho}{2\pi}$$

 $L(h, k, \rho)$ as a Function of $L(h, 0, \rho)$

26.3.20

$$L(h, k, \rho) = L\left(h, 0, \frac{(\rho h - k)(\operatorname{sgn} h)}{\sqrt{h^2 - 2\rho h k + k^2}}\right) + L\left(k, 0, \frac{(\rho k - h)(\operatorname{sgn} k)}{\sqrt{h^2 - 2\rho h k + k^2}}\right) - \begin{cases} 0 & \text{if } h k > 0 \text{ or } h k = 0 \\ & \text{and } h + k \geq 0 \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

 where $\operatorname{sgn} h = 1$ if $h \geq 0$ and $\operatorname{sgn} h = -1$ if $h < 0$.

 FIGURE 26.2. $L(h, 0, \rho)$ for $0 \leq h \leq 1$ and $-1 \leq \rho \leq 0$.

 Values for $h < 0$ can be obtained using $L(h, 0, -\rho) = \frac{1}{2} - L(-h, 0, \rho)$.

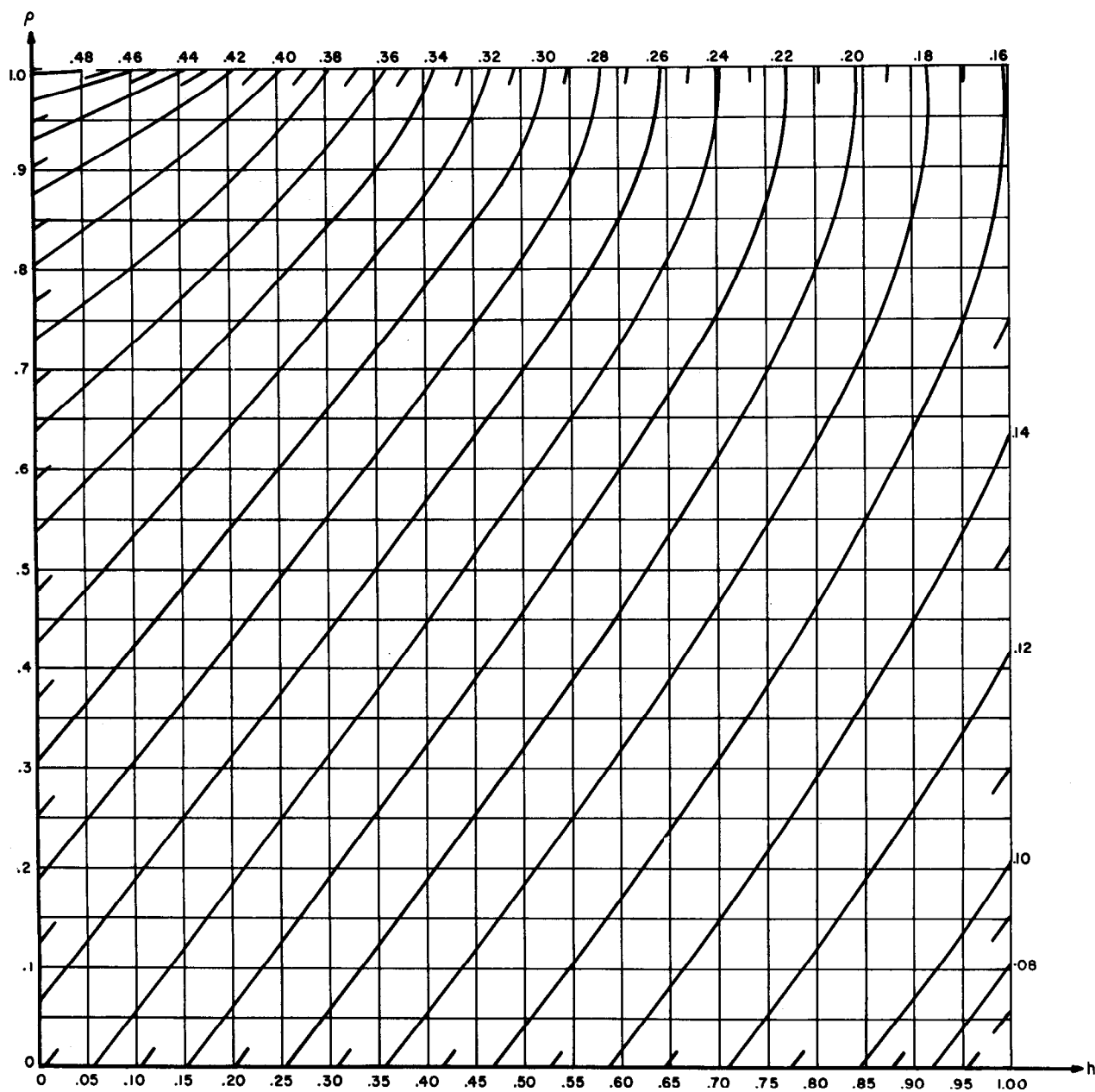
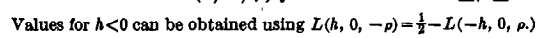


FIGURE 26.3. $L(h, 0, \rho)$ for $0 \leq h \leq 1$ and $0 \leq \rho \leq 1$.

Values for $h < 0$ can be obtained using $L(h, 0, -\rho) = \frac{1}{2} - L(-h, 0, \rho)$.



Integral Over an Ellipse With Center at (m_x, m_y)

26.3.21

$$\iint_A (\sigma_x \sigma_y)^{-1} g\left(\frac{x-m_x}{\sigma_x}, \frac{y-m_y}{\sigma_y}, \rho\right) dx dy = 1 - e^{-a^2/2}$$

where A is the area enclosed by the ellipse

$$\left(\frac{x-m_x}{\sigma_x}\right)^2 - \frac{2\rho(x-m_x)(y-m_y)}{\sigma_x \sigma_y} + \left(\frac{y-m_y}{\sigma_y}\right)^2 = a^2(1-\rho^2)$$

Integral Over an Arbitrary Region

26.3.22

$$\begin{aligned} \iint_{A(x,y)} (\sigma_x \sigma_y)^{-1} g\left(\frac{x-m_x}{\sigma_x}, \frac{y-m_y}{\sigma_y}, \rho\right) dx dy \\ = \iint_{A^*(s,t)} g(s, t, \rho) ds dt \end{aligned}$$

where $A^*(s, t)$ is the transformed region obtained from the transformation

$$s = \frac{1}{\sqrt{2+2\rho}} \left(\frac{x-m_x}{\sigma_x} + \frac{y-m_y}{\sigma_y} \right)$$

$$t = \frac{-1}{\sqrt{2-2\rho}} \left(\frac{x-m_x}{\sigma_x} - \frac{y-m_y}{\sigma_y} \right)$$

Integral of the Circular Normal Probability Function With Parameters $m_x=m_y=0$, $\sigma=1$ Over the Triangle Bounded by $y=0$, $y=ax$, $x=h$

26.3.23

$$\begin{aligned} V(h, ah) &= \frac{1}{2\pi} \int_0^h \int_0^{ah} e^{-\frac{1}{2}(x^2+y^2)} dx dy \\ &= \frac{1}{4} + L(h, 0, \rho) - L(0, 0, \rho) - \frac{1}{2} Q(h) \end{aligned}$$

where

$$\rho = -\frac{a}{\sqrt{1+a^2}}$$

Integral of Circular Normal Distribution Over an Offset Circle With Radius $R\sigma$ and Center a Distance $r\sigma$ From (m_x, m_y)

26.3.24

$$\int_A \int \sigma^{-2} g\left(\frac{x-m_x}{\sigma}, \frac{y-m_y}{\sigma}, 0\right) dx dy = P(R^2|2, r^2)$$

where $P(R^2|2, r^2)$ is the c.d.f. of the non-central χ^2 distribution (see 26.4.25) with $\nu=2$ degrees of freedom and noncentrality parameter r^2 .

Approximation to $P(R^2|2, r^2)$

26.3.25

Approximation	Condition
$\frac{2R^2}{4+R^2} \exp - \frac{2r^2}{4+R^2}$	$R < 1$

26.3.26 $P(x_1)$	$R > 1$
------------------	---------

26.3.27 $P(x_2)$	$R > 5$
------------------	---------

$$x_1 = \frac{[R^2/(2+r^2)]^{1/3} - \left[1 - \frac{2}{9} \frac{2+2r^2}{(2+r^2)^2}\right]}{\left[\frac{2}{9} \frac{2+2r^2}{(2+r^2)^2}\right]^{1/3}}$$

$$x_2 = R - \sqrt{r^2 - 1} \quad R, r \text{ both large} \quad *$$

Inequality

26.3.28

$$Q(h) - \frac{1-\rho^2}{\rho h - k} Z(k) \left[Q\left(\frac{h-\rho k}{\sqrt{1-\rho^2}}\right) \right] < L(h, k, \rho) < Q(h)$$

where

$$\rho h - k > 0, \quad 0 < \rho < 1.$$

Series Expansion

26.3.29

$$L(h, k, \rho) = Q(h)Q(k) + \sum_{n=0}^{\infty} \frac{Z^{(n)}(h)Z^{(n)}(k)}{(n+1)!} \rho^{n+1}$$

26.4. Chi-Square Probability Function

26.4.1

$$P(\chi^2|\nu) = \left[2^{\nu/2} \Gamma\left(\frac{\nu}{2}\right) \right]^{-1} \int_0^{\chi^2} (t)^{\frac{\nu}{2}-1} e^{-\frac{t}{2}} dt \quad (0 \leq \chi^2 < \infty)$$

26.4.2

$$\begin{aligned} Q(\chi^2|\nu) &= 1 - P(\chi^2|\nu) \\ &= \left[2^{\nu/2} \Gamma\left(\frac{\nu}{2}\right) \right]^{-1} \int_{\chi^2}^{\infty} (t)^{\frac{\nu}{2}-1} e^{-\frac{t}{2}} dt \end{aligned} \quad (0 \leq \chi^2 < \infty)$$

Relation to Normal Distribution

Let $X_1, X_2, \dots, X_\nu$ be independent and identically distributed random variables each following a normal distribution with mean zero and unit variance. Then $X^2 = \sum_{i=1}^{\nu} X_i^2$ is said to follow the chi-square distribution with ν degrees of freedom and the probability that $X^2 \leq \chi^2$ is given by $P(\chi^2|\nu)$.

Cumulants

$$26.4.3 \quad \kappa_{n+1} = 2^n n! \nu \quad (n=0, 1, \dots)$$

*See page II.

Series Expansions

26.4.4

$$Q(\chi^2|\nu) = 2Q(\chi) + 2Z(\chi) \sum_{r=1}^{\nu-1} \frac{\chi^{2r-1}}{1 \cdot 3 \cdot 5 \dots (2r-1)}$$

 $(\nu \text{ odd}) \text{ and } \chi = \sqrt{\chi^2}$

26.4.5

$$Q(\chi^2|\nu) = \sqrt{2\pi} Z(\chi) \left\{ 1 + \sum_{r=1}^{\frac{\nu-2}{2}} \frac{\chi^{2r}}{2 \cdot 4 \dots (2r)} \right\}$$

(ν even)

26.4.6

$$P(\chi^2|\nu) = \left(\frac{1}{2}\chi^2\right)^{\nu/2} \frac{e^{-\chi^2/2}}{\Gamma\left(\frac{\nu+2}{2}\right)}$$

$$* \left\{ 1 + \sum_{r=1}^{\infty} \frac{\chi^{2r}}{(\nu+2)(\nu+4) \dots (\nu+2r)} \right\}$$

$$26.4.7 \quad P(\chi^2|\nu) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} \sum_{n=0}^{\infty} \frac{(-1)^n (\chi^2/2)^{\frac{\nu}{2}+n}}{n! \left(\frac{\nu}{2}+n\right)}$$

Recurrence and Differential Relations

$$26.4.8 \quad Q(\chi^2|\nu+2) = Q(\chi^2|\nu) + \frac{(\chi^2/2)^{\nu/2} e^{-\chi^2/2}}{\Gamma\left(\frac{\nu}{2}+1\right)}$$

$$26.4.9 \quad \frac{\partial^m Q(\chi^2|\nu)}{\partial (\chi^2)^m} = \frac{1}{2^m} \sum_{j=0}^m \binom{m}{j} (-1)^{m+j} Q(\chi^2|\nu-2j)$$

Continued Fraction

$$26.4.10 \quad *Q(\chi^2|\nu) = \frac{(\chi^2)^{\nu/2} e^{-\chi^2/2}}{2^{\nu/2} \Gamma(\nu/2)}$$

$$\left\{ \frac{1}{\chi^2/2+} \frac{1-\nu/2}{1+} \frac{1}{\chi^2/2+} \frac{2-\nu/2}{1+} \frac{2}{\chi^2/2+} \dots \right\}$$

Asymptotic Distribution for Large ν

$$26.4.11 \quad P(\chi^2|\nu) \sim P(x) \quad \text{where } x = \frac{\chi^2 - \nu}{\sqrt{2\nu}}$$

Asymptotic Expansions for Large χ^2

26.4.12

$$Q(\chi^2|\nu) \sim \frac{(\chi^2)^{\frac{\nu}{2}-1} e^{-\chi^2/2}}{2^{\nu/2} \Gamma(\nu/2)} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma\left(1-\frac{\nu}{2}+j\right)}{\Gamma\left(1-\frac{\nu}{2}\right)} \frac{2^{j+1}}{(\chi^2)^j}$$

*See page II.

Approximations to the Chi-Square Distribution for Large ν

26.4.13

<i>Approximation</i>	<i>Condition</i>
$Q(\chi^2 \nu) \approx Q(x_1), \quad x_1 = \sqrt{2\chi^2} - \sqrt{2\nu-1}$	$(\nu > 100)$

26.4.14

$Q(\chi^2 \nu) \approx Q(x_2), \quad x_2 = \frac{(\chi^2/\nu)^{1/3} - \left(1 - \frac{2}{9\nu}\right)}{\sqrt{2/9\nu}}$	$(\nu > 30)$
--	--------------

26.4.15

$Q(\chi^2 \nu) \approx Q(x_2 + h_\nu), \quad h_\nu = \frac{60}{\nu} h_{80}$	$(\nu > 30)$
---	--------------

Values of h_{80}

x	h_{80}	x	h_{80}	x	h_{80}
-3.5	-.0118	-1.0	+.0006	+1.5	-.0005
-3.0	-.0087	-.5	+.0006	2.0	+.0002
-2.5	-.0063	0	+.0002	2.5	.0017
-2.0	-.0040	+.5	+.0003	3.0	.0043
-1.5	+.0001	1.0	-.0006	3.5	.0082

Approximations for the Inverse Function for Large ν

If $Q(\chi_p^2|\nu) = p$ and $Q(x_p) = 1 - P(x_p) = p$, then

<i>Approximation</i>	<i>Condition</i>
$26.4.16 \quad \chi_p^2 \approx \frac{1}{2} \left\{ x_p + \sqrt{2\nu-1} \right\}^2$	$(\nu > 100)$

$26.4.17 \quad \chi_p^2 \approx \nu \left\{ 1 - \frac{2}{9\nu} + x_p \sqrt{\frac{2}{9\nu}} \right\}^3$	$(\nu > 30)$
--	--------------

$26.4.18 \quad \chi_p^2 \approx \nu \left\{ 1 - \frac{2}{9\nu} + (x_p - h_\nu) \sqrt{\frac{2}{9\nu}} \right\}^3$	$(\nu > 30)$
--	--------------

where h_ν is given by 26.4.15.

Relation to Other Functions

26.4.19 Incomplete gamma function

$$\frac{\gamma(a, x)}{\Gamma(a)} = P(\chi^2|\nu), \quad \nu = 2a, \chi^2 = 2x$$

$$\frac{\Gamma(a, x)}{\Gamma(a)} = Q(\chi^2|\nu)$$

26.4.20 Pearson's incomplete gamma function

$$I(u, p) = \frac{1}{\Gamma(p+1)} \int_0^{u\sqrt{p+1}} t^p e^{-t} dt = P(\chi^2|\nu)$$

$$\nu = 2(p+1), \chi^2 = 2u\sqrt{p+1}$$

26.4.21 Poisson distribution

$$Q(\chi^2|\nu) = \sum_{j=0}^{c-1} e^{-m} \frac{m^j}{j!}, \quad c = \frac{\nu}{2}, m = \frac{\chi^2}{2}, (\nu \text{ even})$$

$$Q(\chi^2|\nu) - Q(\chi^2|\nu-2) = e^{-m} \frac{m^{c-1}}{(c-1)!}$$

26.4.22 Pearson Type III

$$\left[\frac{ab}{e}\right]^{ab} \int_{-a}^x \left(1+\frac{t}{a}\right)^{ab} e^{-bt} dt = P(\chi^2|\nu)$$

$$\nu = 2ab + 2, \chi^2 = 2b(x+a)$$

26.4.23 Incomplete moments of Normal distribution

$$\int_0^x t^n Z(t) dt = \begin{cases} (n-1)!! \frac{P(\chi^2|\nu)}{2} & (n \text{ even}) \\ \frac{(n-1)!!}{\sqrt{2\pi}} P(\chi^2|\nu) & (n \text{ odd}) \end{cases}$$

$$\chi^2 = x^2, \nu = n+1$$

26.4.24 Generalized Laguerre Polynomials

$$n! L_n^{(\alpha)}(x) = \frac{\sum_{j=0}^{n+1} (-1)^{n+j} \binom{n+1}{j} Q(\chi^2|\nu+2-2j)}{2^n [Q(\chi^2|\nu+2) - Q(\chi^2|\nu)]}$$

$$x = \chi^2/2, \alpha = \nu/2$$

Non-Central χ^2 Distribution Function

26.4.25

$$P(\chi'^2|\nu, \lambda) = \sum_{j=0}^{\infty} e^{-\lambda/2} \frac{(\lambda/2)^j}{j!} P(\chi'^2|\nu+2j)$$

where $\lambda \geq 0$ is termed the non-centrality parameter.

Relation of Non-Central χ^2 Distribution With $\nu=2$ to the Integral of Circular Normal Distribution ($\sigma^2=1$) Over an Offset Circle Having Radius R and Center a Distance $r=\sqrt{\lambda}$ From the Origin. (See 26.3.24-26.3.27.)

26.4.26

$$\iint_A g(x, y, 0) dx dy = P(\chi^2 = R^2|\nu=2, \lambda)$$

$$= 1 - \sum_{j=0}^{\infty} \frac{e^{-\lambda/2} \lambda^j}{2^j j!} Q(R^2|2+2j)$$

Approximations to the Non-Central χ^2 Distribution

$$a = \nu + \lambda \quad b = \frac{\lambda}{\nu + \lambda}$$

Approximating Function

Approximation

26.4.27 χ^2 distribution $P(\chi'^2|\nu, \lambda) \approx P\left(\frac{\chi^2}{1+b} \middle| \nu^*\right), \quad \nu^* = \frac{a}{1+b}$

26.4.28 Normal distribution $P(\chi'^2|\nu, \lambda) \approx P(x), \quad x = \frac{(\chi'^2/a)^{1/3} - \left[1 - \frac{2}{9} \left(\frac{1+b}{a}\right)\right]}{\sqrt{\frac{2}{9} \left(\frac{1+b}{a}\right)}}$

26.4.29 Normal distribution $P(\chi'^2|\nu, \lambda) \approx P(x), \quad x = \left[\frac{2\chi'^2}{1+b}\right]^{\frac{1}{2}} - \left[\frac{2a}{1+b} - 1\right]^{\frac{1}{2}}$

Approximations to the Inverse Function of Non-Central χ^2 Distribution

If $Q(\chi_p'^2|\nu, \lambda) = p$, $Q(\chi_p^2|\nu^*) = p$, and $Q(x_p) = p$ then

Approximating Variable

Approximation to the Inverse Function

26.4.30 $\chi^2 \quad \chi_p'^2 \approx (1+b)\chi_p^2$

26.4.31 Normal $\chi_p'^2 \approx \frac{1+b}{2} \left[x_p + \sqrt{\frac{2a}{1+b} - 1} \right]^2$

26.4.32 Normal $\chi_p'^2 \approx a \left[x_p \sqrt{\frac{2(1+b)}{9a}} + 1 - \frac{2}{9} \left(\frac{1+b}{a}\right) \right]^3$

Properties of Chi-Square, Non-Central Chi-Square, and Related Quantities

		$a=r+\lambda$ $b=\frac{\lambda}{r+\lambda}$			
		$\psi(z)=\frac{d}{dz} \ln \Gamma(z)$ $\psi'(z)=\frac{d^2}{dz^2} \psi(z)$			
Variable	Mean	Variance	Coefficient of skewness (γ_1)	Coefficient of excess (γ_2)	
26.4.33 χ^2	ν	2ν	$\frac{2^{3/2}}{\sqrt{\nu}}$	$12\nu^{-1}$	
26.4.34 $\sqrt{2}\chi^2$	$(2\nu-1)^{1/2} \{1 + [16\nu(\nu-1)]^{-1}\} + O(\nu^{-1/2})$	$1 - \frac{1}{4\nu} + \frac{1}{8\nu^2} + \frac{5}{64\nu^3} - O(\nu^{-4})$	$\frac{1}{\sqrt{2\nu}} \left[1 + \frac{5}{8\nu} - \frac{1}{128\nu^2} \right] + O(\nu^{-3/2})$	$\frac{3}{2^2} \frac{1}{\nu^2} \left[1 + \frac{3}{2\nu} \right] + O(\nu^{-3})$	
26.4.35 $(\chi^2/\nu)^{1/2}$	$1 - \frac{2}{32\nu} + \frac{80}{3^2\nu^2} + O(\nu^{-3})$	$\frac{2}{3^2\nu} - \frac{104}{3^3\nu^2} + O(\nu^{-3})$	$\frac{2^{7/2}}{3^2\nu^{3/2}} \left[1 + \frac{8}{3^2\nu} \right] + O(\nu^{-1/2})$	$-\frac{4}{9\nu} \left[1 + \frac{16}{9\nu} \right] + O(\nu^{-2})$	
26.4.36 $\ln (\chi^2/\nu)$	$\psi\left(\frac{\nu}{2}\right) - \ln\left(\frac{\nu}{2}\right) = -\frac{1}{\nu} - \frac{1}{3\nu^2} + O(\nu^{-3})$	$\psi'\left(\frac{\nu}{2}\right) = \frac{2}{\nu-1} \left[1 - \frac{1}{3(\nu-1)^2} \right] + O((\nu-1)^{-3})$	$\frac{\psi''\left(\frac{\nu}{2}\right)}{\psi'\left(\frac{\nu}{2}\right)^{3/2}} = -\sqrt{\frac{2}{\nu-1}} \left[1 - \frac{1}{2(\nu-1)^2} \right] + O((\nu-1)^{-3/2})$	$\frac{\psi^{(3)}\left(\frac{\nu}{2}\right)}{\psi'\left(\frac{\nu}{2}\right)^{5/2}} = \frac{4}{\nu-1} \left[1 + \frac{4}{3(\nu-1)^2} \right] + O((\nu-1)^{-3})$	
26.4.37 χ'^2	a	$2a(1+b)$	$\left(\frac{2}{1+b}\right)^{1/2} (1+2b)a^{-1/2}$	$\frac{12}{a} \frac{(1+3b)}{(1+b)^2}$	
26.4.38 $\sqrt{2}\chi'^2$	$[2a - (1+b)]^{1/2} + O(a^{-1/2})$	$(1+b) - \frac{\sigma^2}{4} [8b + (1+b)(1-7b)] + O(a^{-1})$	$\frac{a^{-1/2}(1-b)(1+3b)}{2^{3/2}(1+b)^{3/2}} + O(a^{-1})$	$\frac{3b(b+2)}{(1+b)^2} + O(a^{-1})$	
26.4.39 $(\chi'^2/a)^{1/2}$	$1 - \frac{2}{3^2} \frac{1+b}{a} - \frac{40}{3^4} \frac{b^2}{a^2} + O(a^{-3})$	$\frac{2}{9} a^{-1}(1+b) + \frac{16}{27} a^{-2}b + O(a^{-3})$	$\left(\frac{2}{1+b}\right)^{1/2} b a^{-1/2} + O(a^{-3/2})$	$-\frac{4}{3^3} \frac{(1+3b+12b^2-44b^3)}{a(1+b)^3} - O(a^{-3})$	

26.5. Incomplete Beta Function

26.5.1

$$I_x(a, b) = \frac{1}{B(a, b)} \int_0^x t^{a-1} (1-t)^{b-1} dt \quad (0 \leq x \leq 1)$$

26.5.2

$$I_x(a, b) = 1 - I_{1-x}(b, a)$$

Relation to the Chi-Square Distribution

If X_1^2 and X_2^2 are independent random variables following chi-square distributions 26.4.1 with ν_1 and ν_2 degrees of freedom respectively, then $\frac{X_1^2}{X_1^2 + X_2^2}$ is said to follow a beta distribution with ν_1 and ν_2 degrees of freedom and has the distribution function

26.5.3

$$P\left\{\frac{X_1^2}{X_1^2 + X_2^2} \leq x\right\} = \frac{1}{B(a, b)} \int_0^x t^{a-1} (1-t)^{b-1} dt = I_x(a, b) \quad a = \frac{\nu_1}{2}, b = \frac{\nu_2}{2}$$

Series Expansions ($0 < x < 1$)

26.5.4

$$* I_x(a, b) = \frac{x^a (1-x)^b}{a B(a, b)} \left\{ 1 + \sum_{n=0}^{\infty} \frac{B(a+1, n+1)}{B(a+b, n+1)} x^{n+1} \right\}$$

26.5.5

$$I_x(a, b) = \frac{x^a (1-x)^{b-1}}{a B(a, b)} \left\{ 1 + \sum_{n=0}^{\infty} \frac{B(a+1, n+1)}{B(b-n-1, n+1)} \left(\frac{x}{1-x}\right)^{n+1} \right\} = \frac{x^a (1-x)^{b-1}}{a B(a, b)} \left\{ 1 + \sum_{n=0}^{s-2} \frac{B(a+1, n+1)}{B(b-n-1, n+1)} \left(\frac{x}{1-x}\right)^{n+1} \right\} + I_x(a+s, b-s)$$

26.5.6

$$1 - I_x(a, b) = I_{1-x}(b, a) = \frac{(1-x)^b}{B(a, b)} \sum_{i=0}^{a-1} (-1)^i \binom{a-1}{i} \frac{(1-x)^i}{b+i} \quad (\text{integer } a)$$

26.5.7

$$1 - I_x(a, b) = I_{1-x}(b, a) = (1-x)^{a+b-1} \sum_{i=0}^{a-1} \binom{a+b-1}{i} \left(\frac{x}{1-x}\right)^i \quad (\text{integer } a)$$

Continued Fractions

26.5.8

$$I_x(a, b) = \frac{x^a (1-x)^b}{a B(a, b)} \left\{ \frac{1}{1+} \frac{d_1}{1+} \frac{d_2}{1+} \dots \right\}^*$$

$$d_{2m+1} = -\frac{(a+m)(a+b+m)}{(a+2m)(a+2m+1)} x$$

$$d_{2m} = \frac{m(b-m)}{(a+2m-1)(a+2m)} x$$

Best results are obtained when $x < \frac{a-1}{a+b-2}$.

Also the $4m$ and $4m+1$ convergents are less than $I_x(a, b)$ and the $4m+2$, $4m+3$ convergents are greater than $I_x(a, b)$.

26.5.9

$$I_x(a, b) = \frac{x^a (1-x)^{b-1}}{a B(a, b)} \left[\frac{e_1}{1+} \frac{e_2}{1+} \frac{e_3}{1+} \dots \right]$$

$$* \quad x < 1 \quad e_1 = 1$$

$$e_{2m} = -\frac{(a+m-1)(b-m)}{(a+2m-2)(a+2m-1)} \frac{x}{1-x}$$

$$e_{2m+1} = \frac{m(a+b-1+m)}{(a+2m-1)(a+2m)} \frac{x}{1-x}$$

Recurrence Relations

26.5.10

$$I_x(a, b) = x I_x(a-1, b) + (1-x) I_x(a, b-1)$$

26.5.11

$$I_x(a, b) = \frac{1}{x} \{ I_x(a+1, b) - (1-x) I_x(a+1, b-1) \}$$

26.5.12

$$[I_x(a, b)] = \frac{1}{a(1-x)+b} \{ b I_x(a, b+1) + a(1-x) I_x(a+1, b-1) \}^*$$

26.5.13

$$I_x(a, b) = \frac{1}{a+b} \{ a I_x(a+1, b) + b I_x(a, b+1) \}$$

26.5.14

$$I_x(a, a) = \frac{1}{2} I_{1-x'}\left(a, \frac{1}{2}\right), \quad x' = 4 \left(x - \frac{1}{2}\right)^2 \left[x \leq \frac{1}{2}\right]^*$$

26.5.15

$$I_x(a, b) = \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)} x^a (1-x)^{b-1} + I_x(a+1, b-1)$$

26.5.16

$$I_x(a, b) = \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)} x^a (1-x)^b + I_x(a+1, b)$$

* See page II.

Asymptotic Expansions

26.5.17

$$1 - I_x(a, b) = I_{1-x}(b, a) \sim \frac{\Gamma(b, y)}{\Gamma(b)} - \frac{1}{24N^2} \left\{ \frac{y^b e^{-y}}{(b-2)!} (b+1+y) \right\} + \frac{1}{5760N^4} \left\{ \frac{y^b e^{-y}}{(b-2)!} [(b-3)(b-2)(5b+7)(b+1+y) - (5b-7)(b+3+y)y^2] \right\}$$

$$y = -N \ln x, \quad N = a + \frac{b}{2} - \frac{1}{2}$$

26.5.18

$$I_x(a, b) \sim \frac{\Gamma(a, w)}{\Gamma(a)} + \frac{e^{-w} w^a}{\Gamma(a)} \left\{ \frac{(a-1-w)}{2b} + \frac{1}{(2b)^2} \left(\frac{a^3}{2} - \frac{5}{3} a^2 + \frac{3}{2} a - \frac{1}{3} - w \left[\frac{3}{2} a^2 - \frac{11}{6} a + \frac{1}{3} \right] + w^2 \left(\frac{3}{2} a - \frac{1}{6} \right) - \frac{1}{2} w^3 \right) \right\}$$

$$w = b \left(\frac{x}{1-x} \right)$$

26.5.19

$$I_x(a, b) \sim P(y) - Z(y) \left[a_1 + \frac{a_2(y-a_1)}{1+a_2} + \frac{a_3(1+y^2/2)}{1+a_2} + \dots \right]$$

$$a_1 = \frac{2}{3} (b-a) [(a+b-2)(a-1)(b-1)]^{-1/2}$$

$$a_2 = \frac{1}{12} \left[\frac{1}{a-1} + \frac{1}{b-1} - \frac{13}{a+b-1} \right]$$

$$a_3 = -\frac{8}{15} \left[a_1 \left(a_2 + \frac{3}{a+b-2} \right) \right]$$

$$y^2 = 2 \left[(a+b-1) \ln \frac{a+b-1}{a+b-2} + (a-1) \ln \frac{a-1}{(a+b-1)x} + (b-1) \ln \frac{b-1}{(a+b-1)(1-x)} \right]$$

and y is taken negative when $x < \frac{a-1}{a+b-2}$

Approximations

26.5.20 If $(a+b-1)(1-x) \leq .8$

$$I_x(a, b) = Q(x^2 | \nu) + \epsilon,$$

$$|\epsilon| < 5 \times 10^{-3} \text{ if } a+b > 6$$

$$x^2 = (a+b-1)(1-x)(3-x) - (1-x)(b-1),$$

$$\nu = 2b$$

26.5.21 If $(a+b-1)(1-x) \geq .8$

$$I_x(a, b) = P(y) + \epsilon,$$

$$|\epsilon| < 5 \times 10^{-3} \text{ if } a+b > 6$$

$$y = \frac{3 \left[w_1 \left(1 - \frac{1}{9b} \right) - w_2 \left(1 - \frac{1}{9a} \right) \right]}{\left[\frac{w_1^2}{b} + \frac{w_2^2}{a} \right]^{1/2}},$$

$$w_1 = (bx)^{1/3}, \quad w_2 = [a(1-x)]^{1/3}$$

Approximation to the Inverse Function

26.5.22 If $I_{x_p}(a, b) = p$ and $Q(y_p) = p$ then

$$x_p \approx \frac{a}{a + b e^{2w}}$$

$$w = \frac{y_p(h+\lambda)^{1/2}}{h} - \left(\frac{1}{2b-1} - \frac{1}{2a-1} \right) \left(\lambda + \frac{5}{6} - \frac{2}{3h} \right)$$

$$h = 2 \left(\frac{1}{2a-1} + \frac{1}{2b-1} \right)^{-1}, \quad \lambda = \frac{y_p^2 - 3}{6}$$

Relations to Other Functions and Distributions

Function

Relation

26.5.23 Hypergeometric function

$$\frac{1}{B(a, b)} \frac{x^a}{a} F(a, 1-b; a+1; x) = I_x(a, b)$$

26.5.24 Binomial distribution

$$\sum_{s=a}^n \binom{n}{s} p^s (1-p)^{n-s} = I_p(a, n-a+1)$$

26.5.25 "

$$\binom{n}{a} p^a (1-p)^{n-a} = I_p(a, n-a+1) - I_p(a+1, n-a) *$$

26.5.26 Negative binomial distribution

$$\sum_{s=a}^n \binom{n+s-1}{s} p^s q^n = I_q(a, n)$$

26.5.27 Student's distribution

$$\frac{1}{2} [1 - A(t|\nu)] = \frac{1}{2} I_x \left(\frac{\nu}{2}, \frac{1}{2} \right), \quad x = \frac{\nu}{\nu + t^2} *$$

26.5.28 F -(variance-ratio) distribution

$$Q(F|\nu_1, \nu_2) = I_x \left(\frac{\nu_2}{2}, \frac{\nu_1}{2} \right), \quad x = \frac{\nu_2}{\nu_2 + \nu_1 F}$$

*See page 11.

26.6. *F*-(Variance-Ratio) Distribution Function

26.6.1

$$P(F|v_1, v_2) = \frac{v_1^{1/2} v_2^{1/2}}{B\left(\frac{1}{2}v_1, \frac{1}{2}v_2\right)} \int_0^F t^{1/2(v_1-2)} (v_2 + v_1 t)^{-1/2(v_1+v_2)} dt \quad (F \geq 0)$$

26.6.2

$$Q(F|v_1, v_2) = 1 - P(F|v_1, v_2) = I_x\left(\frac{v_2}{2}, \frac{v_1}{2}\right)$$

where

$$x = \frac{v_2}{v_2 + v_1 F}$$

Relation to the Chi-Square Distribution

If X_1^2 and X_2^2 are independent random variables following chi-square distributions 26.4.1 with v_1 and v_2 degrees of freedom respectively, then the distribution of $F = \frac{X_1^2/v_1}{X_2^2/v_2}$ is said to follow the variance ratio or *F*-distribution with v_1 and v_2 degrees of freedom. The corresponding distribution function is $P(F|v_1, v_2)$.

Statistical Properties

26.6.3

mean: $m = \frac{v_2}{v_2 - 2} \quad (v_2 > 2)$

variance: $\sigma^2 = \frac{2v_2^2(v_1 + v_2 - 2)}{v_1(v_2 - 2)^2(v_2 - 4)} \quad (v_2 > 4)$

third central moment:

$$\mu_3 = \left(\frac{v_2}{v_1}\right)^3 \frac{8v_1(v_1 + v_2 - 2)(2v_1 + v_2 - 2)}{(v_2 - 2)^3(v_2 - 4)(v_2 - 6)} \quad (v_2 > 6)$$

moments about the origin:

$$\mu'_n = \left(\frac{v_2}{v_1}\right)^n \frac{\Gamma\left(\frac{v_1 + 2n}{2}\right) \Gamma\left(\frac{v_1 - 2n}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right) \Gamma\left(\frac{v_2}{2}\right)} \quad (v_2 > 2n)$$

characteristic function:

$$\phi(t) = E(e^{itF}) = M\left(\frac{v_1}{2}, -\frac{v_2}{2}, -\frac{v_2}{v_1} it\right)$$

Series Expansions

$$x = \frac{v_2}{v_2 + v_1 F}$$

26.6.4

$$* Q(F|v_1, v_2) = x^{v_2/2} \left[1 + \frac{v_2}{2}(1-x) + \frac{v_2(v_2+2)}{2 \cdot 4}(1-x)^2 + \dots + \frac{v_2(v_2+2) \dots (v_2+v_1-4)}{2 \cdot 4 \dots (v_1-2)} (1-x)^{v_1/2} \right] \quad (v_1 \text{ even})$$

26.6.5

$$Q(F|v_1, v_2) = 1 - (1-x)^{v_1/2} \left[1 + \frac{v_1}{2}x + \frac{v_1(v_1+2)}{2 \cdot 4}x^2 + \dots + \frac{v_1(v_1+2) \dots (v_2+v_1-4)}{2 \cdot 4 \dots (v_2-2)} x^{v_2/2} \right] \quad (v_2 \text{ even})$$

26.6.6

$$Q(F|v_1, v_2) = x^{\frac{v_1+v_2-2}{2}} \left[1 + \frac{v_1+v_2-2}{2} \left(\frac{1-x}{x}\right) + \frac{(v_1+v_2-2)(v_1+v_2-4)}{2 \cdot 4} \left(\frac{1-x}{x}\right)^2 + \dots + \frac{(v_1+v_2-2) \dots (v_2+2)}{2 \cdot 4 \dots (v_1-2)} \left(\frac{1-x}{x}\right)^{\frac{v_1-2}{2}} \right] \quad (v_1 \text{ even})$$

26.6.7

$$Q(F|v_1, v_2) = 1 - (1-x)^{\frac{v_1+v_2-2}{2}} \left[1 + \frac{v_1+v_2-2}{2} \left(\frac{x}{1-x}\right) + \dots + \frac{(v_1+v_2-2) \dots (v_1+2)}{2 \cdot 4 \dots (v_2-2)} \left(\frac{x}{1-x}\right)^{\frac{v_2-2}{2}} \right] \quad (v_2 \text{ even})$$

26.6.8

$$Q(F|v_1, v_2) = 1 - A(t|v_2) + \beta(v_1, v_2) \quad (v_1, v_2 \text{ odd})$$

$$A(t|v_2) = \begin{cases} \frac{2}{\pi} \left\{ \theta + \sin \theta \left[\cos \theta + \frac{2}{3} \cos^3 \theta + \dots + \frac{2 \cdot 4 \dots (v_2-3)}{3 \cdot 5 \dots (v_2-2)} \cos^{v_2-2} \theta \right] \right\} & \text{for } v_2 > 1 \\ \frac{2\theta}{\pi} & \text{for } v_2 = 1 \end{cases}$$

$$\beta(v_1, v_2) = \begin{cases} \frac{2 \left(\frac{v_2-1}{2}\right)!}{\sqrt{\pi} \left(\frac{v_2-2}{2}\right)!} \sin \theta \cos^{v_2} \theta \left\{ 1 + \frac{v_2+1}{3} \sin^2 \theta + \dots + \frac{(v_2+1)(v_2+3) \dots (v_1+v_2-4)}{3 \cdot 5 \dots (v_1-2)} \sin^{v_1-3} \theta \right\} & \text{for } v_2 > 1 \\ 0 & \text{for } v_1 = 1 \end{cases} *$$

where

$$\theta = \arctan \sqrt{\frac{v_1}{v_2} F}$$

Reflexive Relation

If $F_p(v_1, v_2)$ and $F_{1-p}(v_2, v_1)$ satisfy

$$Q(F_p(v_1, v_2)|v_1, v_2) = p$$

$$Q(F_{1-p}(v_2, v_1)|v_2, v_1) = 1 - p$$

26.6.9 then

$$F_p(\nu_1, \nu_2) = \frac{1}{F_{1-p}(\nu_2, \nu_1)}$$

Relation to Student's *t*-Distribution Function (See 26.7)

$$26.6.10 \quad Q(F|\nu_1=1, \nu_2)=1-A(t|\nu_2) \quad t=\sqrt{F}$$

Limiting Forms

26.6.11

$$\lim_{\nu_1 \rightarrow \infty} Q(F|\nu_1, \nu_2) = Q(\chi^2|\nu_1), \quad \chi^2 = \nu_1 F$$

26.6.12

$$\lim_{\nu_1 \rightarrow \infty} Q(F|\nu_1, \nu_2) = P(\chi^2|\nu_2), \quad \chi^2 = \frac{\nu_2}{F}$$

Approximations

26.6.13

$$Q(F|\nu_1, \nu_2) \approx Q(x), \quad x = \frac{F - \frac{\nu_2}{\nu_2 - 2}}{\frac{\nu_2}{\nu_2 - 2} \sqrt{\frac{2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 4)}}}$$

(ν_1 and ν_2 large)

26.6.14

$$Q(F|\nu_1, \nu_2) \approx Q(x), \quad x = \frac{\sqrt{(2\nu_2 - 1) \frac{\nu_1}{\nu_2} F - \sqrt{2\nu_1 - 1}}}{\sqrt{1 + \frac{\nu_1}{\nu_2} F}}$$

Relation of Non-Central *F*-Distribution Function to Other Functions

Function

26.6.18 *F*-distribution

26.6.19 Non-central *t*-distribution

26.6.20 Incomplete Beta function

26.6.21 Confluent hypergeometric function

26.6.15

$$Q(F|\nu_1, \nu_2) \approx Q(x), \quad x = \frac{F^{1/3} \left(1 - \frac{2}{9\nu_2}\right) - \left(1 - \frac{2}{9\nu_1}\right)}{\sqrt{\frac{2}{9\nu_1} + F^{2/3} \frac{2}{9\nu_2}}}$$

Approximation to the Inverse Function

26.6.16 If $Q(F_p|\nu_1, \nu_2) = p$, then

$$F_p \approx e^{2w} \text{ where } w \text{ is given by 26.5.22, with } \nu_1 = 2b, \nu_2 = 2a$$

Non-Central *F*-Distribution Function

26.6.17

$$P(F'|\nu_1, \nu_2, \lambda) = \int_0^{F'} p(t|\nu_1, \nu_2, \lambda) dt = 1 - Q(F'|\nu_1, \nu_2, \lambda)$$

where

$$p(t|\nu_1, \nu_2, \lambda) = \sum_{j=0}^{\infty} e^{-\lambda/2} \frac{(\lambda/2)^j}{j!} \frac{\frac{\nu_1+2j}{2} \frac{\nu_2}{2}}{B\left(\frac{\nu_1+2j}{2}, \frac{\nu_2}{2}\right)} \times t^{\frac{\nu_1+2j-2}{2}} [\nu_2 + (\nu_1+2j)t]^{-(\nu_1+2j+\nu_2)/2}$$

and $\lambda \geq 0$ is termed the non-centrality parameter.

Relation

$$P(F'|\nu_1, \nu_2, \lambda) = \sum_{j=0}^{\infty} e^{-\lambda/2} \frac{(\lambda/2)^j}{j!} P(F'|\nu_1+2j, \nu_2)$$

$$P(F'|\nu_1, \nu_2, \lambda=0) = P(F'|\nu_1, \nu_2)$$

$$P(F'|\nu_1=1, \nu_2, \lambda) = P(t'|\nu, \delta), \quad t' = \sqrt{F'}, \nu = \nu_2, \delta = \sqrt{\lambda}$$

$$P(F'|\nu_1, \nu_2) = \sum_{j=0}^{\infty} e^{-\lambda/2} \frac{(\lambda/2)^j}{j!} I_z\left(\frac{\nu_1}{2} + j, \frac{\nu_2}{2}\right),$$

$$x = \frac{\nu_1 F'}{\nu_1 F' + \nu_2} *$$

$$P(F'|\nu_1, \nu_2, \lambda) = \sum_{i=0}^{\frac{\nu_2}{2}-1} \frac{2e^{-\lambda/2}}{(\nu_1 + \nu_2) B\left(\frac{\nu_1}{2} + i + 1, \frac{\nu_2}{2} - i\right)} \times$$

$$x^{\frac{\nu_1}{2}+1} (1-x)^{\frac{\nu_2}{2}-i-1} M\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1}{2} + i + 1, \frac{\lambda x}{2}\right)$$

$$\left(\nu_2 \text{ even and } x = \frac{\nu_2}{\nu_1 F' + \nu_2} \right)$$

*See page II.

Series Expansion

26.6.22

$$P(F'|v_1, v_2, \lambda) = e^{-\frac{\lambda}{2}(1-x)} x^{\frac{1}{2}(\nu_1 + \nu_2 - 2)} \sum_{i=0}^{\frac{\nu_2-1}{2}} T_i \quad (\nu_2 \text{ even})$$

where

$$T_0 = 1$$

$$T_1 = \frac{1}{2} (\nu_1 + \nu_2 - 2 + \lambda x) \frac{1-x}{x}$$

$$T_i = \frac{1-x}{2i} [(\nu_1 + \nu_2 - 2i + \lambda x) T_{i-1} + \lambda(1-x) T_{i-2}]$$

$$x = \frac{\nu_2}{\nu_1 F' + \nu_2}$$

Limiting Forms

26.6.23

$$\lim_{\nu_1 \rightarrow \infty} P(F'|v_1, v_2, \lambda) = P(\chi'^2|\nu, \lambda), \quad \chi'^2 = \nu_1 F', \quad \nu = \nu_1$$

26.6.24

$$\lim_{\nu_1 \rightarrow \infty} P(F'|v_1, v_2, \lambda) = Q(\chi^2|\nu), \quad \chi^2 = \frac{\nu_2(1+c^2)}{F'}$$

where $\lambda/\nu_1 \rightarrow c^2$ as $\nu_1 \rightarrow \infty$.

Approximations to the Non-Central F-Distribution

$$26.6.25 \quad P(F'|v_1, v_2, \lambda) \approx P(x_1), \quad (\nu_1 \text{ and } \nu_2 \text{ large})$$

where

$$x_1 = \frac{F' - \frac{\nu_2(\nu_1 + \lambda)}{\nu_1(\nu_2 - 2)}}{\frac{\nu_2}{\nu_1} \left[\frac{2}{(\nu_2 - 2)(\nu_2 - 4)} \left\{ \frac{(\nu_1 + \lambda)^2}{\nu_2 - 2} + \nu_1 + 2\lambda \right\} \right]^{\frac{1}{2}}}$$

26.6.26

$$P(F'|v_1, v_2, \lambda) \approx P(F|v_1^*, v_2),$$

$$F = \frac{\nu_1}{\nu_1 + \lambda} F', \quad v_1^* = \frac{(\nu_1 + \lambda)^2}{\nu_1 + 2\lambda}$$

26.6.27

$$P(F'|v_1, v_2, \lambda) \approx P(x_2),$$

$$x_2 = \frac{\left[\frac{\nu_1 F'}{(\nu_1 + \lambda)} \right]^{1/3} \left[1 - \frac{2}{9\nu_2} \right] - \left[1 - \frac{2(\nu_1 + 2\lambda)}{9(\nu_1 + \lambda)^2} \right]}{\left[\frac{2}{9} \frac{\nu_1 + 2\lambda}{(\nu_1 + \lambda)^2} + \frac{2}{9\nu_2} \left(\frac{\nu_1}{\nu_1 + \lambda} F' \right)^{2/3} \right]^{\frac{1}{2}}}$$

26.7. Student's *t*-Distribution

If X is a random variable following a normal distribution with mean zero and variance unity, and χ^2 is a random variable following an independent chi-square distribution with ν degrees of freedom, then the distribution of the ratio $\frac{X}{\sqrt{\chi^2/\nu}}$

is called Student's *t*-distribution with ν degrees of freedom. The probability that $\frac{X}{\sqrt{\chi^2/\nu}}$ will be less in absolute value than a fixed constant t is

26.7.1

$$A(t|\nu) = P_r \left\{ \left| \frac{X}{\sqrt{\chi^2/\nu}} \right| \leq t \right\} \\ = \left[\sqrt{\nu} B \left(\frac{1}{2}, \frac{\nu}{2} \right) \right]^{-1} \int_{-t}^t \left(1 + \frac{x^2}{\nu} \right)^{-\frac{\nu+1}{2}} dx \\ = 1 - I_x \left(\frac{\nu}{2}, \frac{1}{2} \right), \quad (0 \leq t < \infty) \quad *$$

where

$$x = \frac{\nu}{\nu + t^2}$$

Statistical Properties

26.7.2

$$\text{mean:} \quad m = 0$$

$$\text{variance:} \quad \sigma^2 = \frac{\nu}{\nu - 2} \quad (\nu > 2)$$

$$\text{skewness:} \quad \gamma_1 = 0$$

$$\text{excess:} \quad \gamma_2 = \frac{6}{\nu - 4} \quad (\nu > 4)$$

moments:

$$\mu_{2n} = \frac{1 \cdot 3 \dots (2n-1) \nu^n}{(\nu-2)(\nu-4) \dots (\nu-2n)} \quad (\nu > 2n)$$

$$\mu_{2n+1} = 0$$

characteristic function:

$$\phi(t) = E \left[\exp \left(it \frac{X}{\sqrt{\chi^2/\nu}} \right) \right] = \frac{\left(\frac{|t|}{2\sqrt{\nu}} \right)^{\nu/2}}{\pi \Gamma(\nu/2)} Y_{\frac{\nu}{2}} \left(\frac{|t|}{\sqrt{\nu}} \right)$$

Series Expansions

$$\left(\theta = \arctan \frac{t}{\sqrt{\nu}} \right)$$

26.7.3

$$A(t|\nu) = \begin{cases} \frac{2}{\pi} \left\{ \theta + \sin \theta \left[\cos \theta + \frac{2}{3} \cos^3 \theta + \dots \right. \right. \\ \quad \left. \left. + \frac{2 \cdot 4 \dots (\nu-3)}{1 \cdot 3 \dots (\nu-2)} \cos^{\nu-2} \theta \right] \right\} & (\nu > 1 \text{ and odd}) \\ \frac{2}{\pi} \theta & (\nu = 1) \end{cases} \quad *$$

26.7.4

$$A(t|\nu) = \sin \theta \left\{ 1 + \frac{1}{2} \cos^2 \theta + \frac{1 \cdot 3}{2 \cdot 4} \cos^4 \theta + \dots \right. \\ \left. + \frac{1 \cdot 3 \cdot 5 \dots (\nu-3)}{2 \cdot 4 \cdot 6 \dots (\nu-2)} \cos^{\nu-2} \theta \right\} \quad (\nu \text{ even}) \quad *$$

*See page 11.

Asymptotic Expansion for the Inverse Function

If $A(t_p|v) = 1 - 2p$ and $Q(x_p) = p$, then

26.7.5

$$t_p \sim x_p + \frac{g_1(x_p)}{v} + \frac{g_2(x_p)}{v^2} + \frac{g_3(x_p)}{v^3} + \dots$$

$$g_1(x) = \frac{1}{4} (x^3 + x)$$

$$g_2(x) = \frac{1}{96} (5x^5 + 16x^3 + 3x)$$

$$g_3(x) = \frac{1}{384} (3x^7 + 19x^5 + 17x^3 - 15x)$$

$$g_4(x) = \frac{1}{92160} (79x^9 + 776x^7 + 1482x^5 - 1920x^3 - 945x)$$

Limiting Distribution**26.7.6**

$$\lim_{v \rightarrow \infty} A(t|v) = \frac{1}{\sqrt{2\pi}} \int_{-t}^t e^{-x^2/2} dx = A(t)$$

Approximation for Large Values of t and $v \leq 5$

$$26.7.7 \quad A(t|v) \approx 1 - 2 \left\{ \frac{a_v}{t^v} + \frac{b_v}{t^{v+1}} \right\}$$

v	1	2	3	4	5
a_v	.3183	.4991	1.1094	3.0941	9.948
b_v	.0000	.0518	-.0460	-2.756	-14.05

Approximation for Large v

$$26.7.8 \quad A(t|v) \approx 2P(x) - 1, \quad x = \frac{t \left(1 - \frac{1}{4v}\right)}{\sqrt{1 + \frac{t^2}{2v}}}$$

Non-Central t-Distribution**26.7.9**

$$P(t'|v, \delta) = \frac{1}{\sqrt{v} B\left(\frac{1}{2}, \frac{v}{2}\right)} \int_{-\infty}^{t'} \left(\frac{v}{v+x^2}\right)^{\frac{v+1}{2}} e^{-\frac{1}{2} \frac{x^2}{v+x^2}} H_h \left(\frac{-\delta x}{\sqrt{v+x^2}}\right) dx$$

$$= 1 - \sum_{j=0}^{\infty} e^{-\delta^2/2} \frac{(\delta^2/2)^j}{2j!} I_x \left(\frac{v}{2}, \frac{1}{2} + j\right), \quad x = \frac{v}{v+t'^2} *$$

where δ is termed the non-centrality parameter.

Approximation to the Non-Central t-Distribution**26.7.10**

$$P(t'|v, \delta) \approx P(x) \quad \text{where } x = \frac{t' \left(1 - \frac{1}{4v}\right) - \delta}{\left(1 + \frac{t'^2}{2v}\right)^{1/2}}$$

Numerical Methods**26.8. Methods of Generating Random Numbers and Their Applications ***

Random digits are digits generated by repeated independent drawings from the population 0, 1, 2, . . . , 9 where the probability of selecting any digit is one-tenth. This is equivalent to putting 10 balls, numbered from 0 to 9, into an urn and drawing one ball at a time, replacing the ball after each drawing. The recorded set of numbers forms a collection of random digits. Any group of n successive random digits is known as a *random number*.

Several lengthy tables of random digits are available (see references). However, the use of random numbers in electronic computers has resulted in a need for random numbers to be generated in a completely deterministic way. The numbers so generated are termed pseudo-random numbers. The quality of pseudo-random numbers is determined by subjecting the numbers to several statistical tests, see [26.55], [26.56]. The purpose of these statistical tests is to detect any properties of the pseudo-random numbers which are different from the (conceptual) properties of random numbers.

* The authors wish to express their appreciation to Professor J. W. Tukey who made many penetrating and helpful suggestions in this section.

Experience has shown that the congruence method is the most preferable device for generating random numbers on a computer. Let the sequence of pseudo-random numbers be denoted by $\{X_n\}$, $n=0, 1, 2, \dots$. Then the congruence method of generating pseudo-random numbers is

$$X_{n+1} = aX_n + b \pmod{T}$$

where b and T are relatively prime. The choice of T is determined by the capacity and base of the computer; a and b are chosen so that: (1) the resulting sequence $\{X_n\}$ possesses the desired statistical properties of random numbers, (2) the period of the sequence is as long as possible, and (3) the speed of generation is fast. A guide for choosing a and b is to make the correlation between the numbers be near zero, e.g., the correlation between X_n and X_{n+s} is

$$\rho_s = \frac{1 - 6 \frac{b_s}{T} \left(1 - \frac{b_s}{T}\right)}{a_s} + e$$

where

$$a_s = a^s \pmod{T}$$

$$b_s = (1 + a + a^2 + \dots + a^{s-1})b \pmod{T}$$

$$|e| < a_s/T$$

*See page 11.

which occur in

$$X_{n+s} = a_s X_n + b_s \pmod{T}$$

When a is chosen so that $a \approx T^{1/2}$, the correlation $\rho_1 \approx T^{-1/2}$.

The sequence defined by the multiplicative congruence method will have a full period of T numbers if

- (i) b is relatively prime to T
- (ii) $a \equiv 1 \pmod{p}$ if p is a prime factor of T
- (iii) $a \equiv 1 \pmod{4}$ if 4 is a factor of T .

Consequently if $T=2^q$, b need only be odd, and

$a \equiv 1 \pmod{4}$. When $T=10^q$, b need only be not divisible by 2 or 5, and $a \equiv 1 \pmod{20}$. The most convenient choices for a are of the form $a=2^s+1$ (for binary computers) and $a=10^s+1$ (for decimal computers). This results in the fastest generation of random numbers as the operations only require a shift operation plus two additions. Also any number can serve as the starting point to generate a sequence of random digits. A good summary of generating pseudo-random numbers is [26.51].

Below are listed various congruence schemes and their properties.

Congruence methods for generating random numbers

$$X_{n+1} = aX_n + b \pmod{T}, \quad T \text{ and } b \text{ relatively prime}$$

	a	b	T	Period	X_0	Special cases for which random numbers have passed statistical tests for randomness ¹⁰
26.8.1	$1+t^s$	odd	$T=t^q$	t^q	$0 \leq X_0 < T$	$T=2^{24}$, X_0 unknown; $a=2^7+1$, $b=1$; $T=2^{47}$, $a=2^8+1$, $b=29741\ 09625\ 8473$, $X_0=76293\ 94531\ 25$.
26.8.2	$r2^s \pm 1$ (r odd, $s \geq 2$)	0	$T=t^q$	t^{q-s}	relatively prime to T	$T=2^{46}$, 2^{42} , $X_0=1$; $a=5^{17}(s=2)$
26.8.3	$r2^s \pm 1$ (r odd, $s \geq 2$)	0	$T=t^q \pm 1$	(varies)	relatively prime to T	$T=2^{26}$, $X_0=1$; $T=2^{26}$, $X_0=1-2^{-26}$, .5478126193; $a=5^{13}(s=2)$
26.8.4	74^{s+1}	0	$T=10^q$	$5 \cdot 10^{q-2}$	relatively prime to T	$T=2^{28}$, $X_0=1$; $a=5^{15}(s=2)$
26.8.5	34^{s+1} ($s=0, 2, 3, 4$)	0	$T=10^q$	$5 \cdot 10^{q-2}$	relatively prime to T	$T=2^{24}+1$, $X_0=10,987,654,321$; $a=23$; period $\approx 10^6$ $T=10^8+1$, $X_0=47,594,118$; $a=23$; period $\approx 5.8 \times 10^6$
						$T=10^{10}$, $X_0=1$; $a=7$ $T=10^{11}$, $X_0=1$; $a=7^{13}$

¹⁰ X_0 given is the starting point for random numbers when statistical tests were made.

When the numbers are generated using a congruence scheme, the least significant digits have short periods. Hence the entire word length cannot be used. If one desired random numbers with as many digits as possible, one would have to modify the congruence schemes. One way is to generate the numbers mod $T \pm 1$. This unfortunately reduces the period.

Generation of Random Deviates

Let $\{X\}$ be a generated sequence of independent random numbers having the domain $(0, T)$. Then $\{U\} = \{T^{-1}X\}$ is a sequence of random deviates (numbers) from a uniform distribution on the interval $(0, 1)$. This is usually a necessary preliminary step in the generation of random deviates having a given cumulative distribution function $F(y)$ or probability density function $f(y)$. Below are summarized some general techniques

for producing arbitrary random deviates. (In what follows $\{U\}$ will always denote a sequence of random deviates from a uniform distribution on the interval $(0, 1)$.)

1. Inverse Method

The solutions $\{y\}$ of the equations $\{u=F(y)\}$ form a sequence of independent random deviates with cumulative distribution function $F(y)$. (If $F(y)$ has a discontinuity at $y=y_0$, then whenever u is such that $F(y_0-0) < u < F(y_0)$, select y_0 as the corresponding deviate.) Generally the inverse method is not practical unless the inverse function $y=F^{-1}(u)$ can be obtained explicitly or can be conveniently approximated.

2. Generating a Discrete Random Variable

Let Y be a discrete random variable with point probabilities $p_i = \text{Pr}\{Y=y_i\}$ for $i=1, 2, \dots$.

*See page II.

The direct way to generate Y is to generate $\{U\}$ and put $Y=y_1$ if

$$p_1 + p_2 + \dots + p_{i-1} < U < p_1 + p_2 + \dots + p_i.$$

However, this method requires complicated machine programs that take too long.

An alternative way due to Marsaglia [26.53] is simple, fast, and seems to be well suited to high-speed computations. Let p_i for $i=1, 2, \dots, n$ be expressed by k decimal digits as $p_i = .\delta_{i1}\delta_{i2}\dots\delta_{ik}$ where the δ 's are the decimal digits. (If the domain of the random variable is infinite, it is necessary to truncate the probability distribution at p_n .) Define

$$P_0=0, P_r=10^{-r} \sum_{i=1}^n \delta_{ri} \text{ for } r=1, 2, \dots, k, \text{ and}$$

$$\Pi_s = \sum_{r=0}^k 10^r P_r, s=1, 2, \dots, k.$$

Number the computer memory locations by 0, 1, 2, $\dots$, Π_k-1 . The memory locations are divided into k mutually exclusive sets such that the s th set consists of memory locations $\Pi_{s-1}, \Pi_{s-1}+1, \dots, \Pi_s-1$. The information stored in the memory locations of the s th set consists of y_1 in δ_{s1} locations, y_2 in δ_{s2} locations, $\dots$, y_n in δ_{sn} locations.

Denote the decimal expansion of the uniform deviates generated by the computer by $u = .d_1d_2d_3\dots$ and finally let $\sigma\{m\}$ be the contents of memory location m . Then if

$$\sum_{i=0}^{s-1} P_i \leq U < \sum_{i=0}^s P_i$$

put

$$y = a \left\{ d_1 d_2 \dots d_s + \Pi_{s-1} - 10^s \sum_{i=1}^{s-1} P_i \right\}.$$

This method is perhaps the best all-around method for generating random deviates from a discrete distribution. In order to illustrate this method consider the problem of generating deviates from the binomial distribution with point probabilities

$$p_i = \binom{n}{i} p^i (1-p)^{n-i}$$

for $n=5$ and $p=.20$. The point probabilities to 4 D are

Value of Random Variable	Point Probabilities
0	$p_0=.3277$
1	$p_1=.4096$
2	$p_2=.2048$
3	$p_3=.0512$
4	$p_4=.0064$
5	$p_5=.0003$

and thus $P_0=0, P_1=.9, P_2=.07, P_3=.027, P_4=.0030$ from which $\Pi_0=0, \Pi_1=9, \Pi_2=16, \Pi_3=43, \Pi_4=73$. The 73 memory locations are divided into 4 mutually exclusive sets such that

Set	Memory Locations
1	0, 1, $\dots$, 8
2	9, 10, $\dots$, 15
3	16, $\dots$, 42
4	43, $\dots$, 72

Among the nine memory locations of set 1, zero is stored $\delta_{10}=3$ times, 1 is stored $\delta_{11}=4$ times, 2 is stored $\delta_{12}=2$ times; the seven locations of set 2 store 0 $\delta_{20}=2$ times and 3 $\delta_{23}=5$ times; etc. A summary of the memory locations is set out below:

	Value of Random Variable					
	0	1	2	3	4	5
Frequency (set 1)	3	4	2	0	0	0
Frequency (set 2)	2	0	0	5	0	0
Frequency (set 3)	7	9	4	1	6	0
Frequency (set 4)	7	6	8	2	4	3

Then to generate the random variables if

$0 \leq u < .9$	put	$y = a\{d_1\}$
$.9 \leq u < .97$		$y = a\{d_1 d_2 - 81\}$
$.97 \leq u < .997$		$y = a\{d_1 d_2 d_3 - 954\}$
$.997 \leq u < 1.000$		$y = a\{d_1 d_2 d_3 d_4 - 9927\}$

3. Generating a Continuous Random Variable

The method for generating deviates from a discrete distribution can be adapted to random variables having a continuous distribution. Let $F(y)$ be the cumulative distribution function and assume that the domain of the random variable is (a, b) where the interval is finite. (If the domain is infinite, it must be truncated at (say) the points a and b .) Divide the interval $(b-a)$ into n sub-intervals of length Δ ($n\Delta=b-a$) such that the boundary of the i th interval is (y_{i-1}, y_i) where $y_i = a + i\Delta$ for $i=0, 1, \dots, n$. Now define a discrete distribution having domain

$$\left\{ z_i = \frac{y_i + y_{i-1}}{2} \right\}$$

with point probabilities $p_i = F(y_i) - F(y_{i-1})$. Finally, let W be a random variable having a uniform distribution on $\left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right)$. This can be done by setting $W = \Delta\left(U - \frac{1}{2}\right)$. Then random

deviates from the distribution function $F(y)$, can be generated (approximately) by setting $y = z + w = z + \Delta \left(u - \frac{1}{2}\right)$. This is simply an approximate decomposition of the continuous random variable into the sum of a discrete and continuous random variable. The discrete variable can be generated quickly by the method described previously. The smaller the value of Δ the better will be the approximation. Each number can be generated by using the leading digits of U to generate the discrete random variable Z and the remaining digits forming a uniformly distributed deviate having $(0,1)$ domain.

4. Acceptance-Rejection Methods

In what follows the random variable Y will be assumed to have finite domain (a, b) . If the domain is infinite, it must be truncated for computational purposes at (say) the points a and b . Then the resulting random deviates will only have this truncated domain.

a) Let f be the maximum of $f(y)$. Then the procedure for generating random deviates is: (1) generate a pair of uniform deviates U_1, U_2 ; (2) compute a point $y = a + (b-a)u_2$ in (a, b) ; (3) if $u_1 < f(y)/f$ accept y as the random deviate, otherwise reject the pair (u_1, u_2) and start again. The acceptance ratio of deviates actually produced is $[(b-a)f]^{-1}$. Hence the acceptance ratio decreases as the domain increases. One way to increase the acceptance ratio is to divide the interval (a, b) into mutually exclusive sub-intervals and then carry out the acceptance-rejection process. For this purpose let the interval (a, b) be divided into k sub-intervals such that the end points of the j th interval are (ξ_{j-1}, ξ_j) with $\xi_0 = a, \xi_k = b$ and $\int_{\xi_{j-1}}^{\xi_j} f(y) dy = p_j$; further let the maximum of $f(y)$ in the j th interval be f_j . Then to generate random deviates from $f(y)$, generate n pairs of deviates $(u_{1s}, u_{2s}) s = 1, 2, \dots, n$. Assign $[np_j]$ such pairs to the j th interval and compute $y_j = \xi_{j-1} + (\xi_j - \xi_{j-1})u_{2s}$. If $u_{1s} < f(y_j)/f_j$ accept y_j as a deviate. The acceptance ratio of this method is

$$\sum_{j=1}^k p_j [(\xi_j - \xi_{j-1})f_j]^{-1}$$

b) Let $F(y)$ be such that $f(y) = f_1(y)f_2(y)$ where the domain of y is (a, b) . Let f_1 and f_2 be the maximum of $f_1(y)$ and $f_2(y)$ respectively. Then the procedure for generating random de-

viates having the probability density function $f(y)$ is: (1) generate U_1, U_2, U_3 ; (2) define $z = a + (b-a)u_3$; (3) if both $u_1 < \frac{f_1(z)}{f_1}$ and $u_2 < \frac{f_2(z)}{f_2}$, take z as the random deviate; otherwise take another sample of three uniform deviates. The acceptance ratio of this method is $[(b-a)f_1f_2]^{-1}$ and can be increased by dividing (a, b) into sub-intervals as in the previous case.

c) Let the probability density function of Y be

$$f(y) = \int_{\alpha}^{\beta} g(y, t) dt, (\alpha \leq t \leq \beta), (a \leq y \leq b).$$

Let g be the maximum of $g(y, t)$. Then the procedure for generating random deviates having the probability density function $f(y)$ is: (1) generate U_1, U_2, U_3 ; (2) define $s = \alpha + (\beta - \alpha)u_2$; $z = a + (b-a)u_3$; (3) if $u_1 < \frac{g(z, s)}{g}$, take z as the random deviate; otherwise take another sample of three. The acceptance ratio for this method is $[(b-a)g]^{-1}$ and can be increased by dividing the domain of t and y into sub-domains.

5. Composition Method

Let $g_z(y)$ be a probability density function which depends on the parameter z ; further let $H(z)$ be the cumulative distribution function for z . In order to generate random deviates Y having the frequency function

$$f(y) = \int_{-\infty}^{\infty} g_z(y) dH(z)$$

one draws a deviate having the cumulative distribution function $H(z)$; then draws a second sample having the probability density function $g_z(y)$.

6. Generation of Random Deviates From Well Known Distributions

a. Normal distribution

(1) *Inverse method*: The inverse method depends on having a convenient approximation to the inverse function $x = P^{-1}(u)$ where

$$u = (2\pi)^{-1/2} \int_{-\infty}^x e^{-t^2/2} dt.$$

Two ways of performing this operation are to (i) use 26.2.23 with $t = \left(\ln \frac{1}{u^2}\right)^{1/2}$ or (ii) approximate $x = P^{-1}(u)$ piecewise using Chebyshev polynomials, see [26.54].

(2) *Sum of uniform deviates*: Let $U_1, U_2, \dots, U_n$ be a sequence of n uniform deviates. Then

$$X_n = \left(\sum_{i=1}^n U_i - \frac{n}{2} \right) \left(\frac{n}{12} \right)^{-1/2}$$

will be distributed asymptotically as a normal random deviate. When $n=12$, the maximum errors made in the normal deviate are 9×10^{-3} for $|X| < 2$, 9×10^{-1} for $2 < |X| < 3$. An improvement can be made by taking a polynomial function of X_n (say)

$$X_n^* = X_n \sum_{s=0}^k a_{2s} X_n^{2s}$$

as the normal deviate where a_{2s} are suitable coefficients. These coefficients may be calculated using (say) Chebyshev polynomials or simply by making the asymptotic random deviate agree with the correct normal deviate at certain specified points. When $n=12$, the maximum error in the normal deviate is 8×10^{-4} using the coefficients

$$\begin{array}{ll} * a_0 = 9.8746 & * a_6 = (-7) - 5.102 \\ * a_2 = (-3)3.9439 & * a_8 = (-7)1.141 \\ * a_4 = (-5)7.474 & \end{array}$$

(3) *Direct method*: Generate a pair of uniform deviates (U_1, U_2) . Then

$$X_1 = (-2 \ln U_1)^{1/2} \cos 2\pi U_2,$$

$X_2 = (-2 \ln U_1)^{1/2} \sin 2\pi U_2$ will be a pair of independent normal random deviates with mean zero and unit variance. This procedure can be modified by calculating $\cos 2\pi U$ and $\sin 2\pi U$ using an acceptance rejection method; e.g., (1) generate (U_1, U_2) ; (2) if $(2U_1 - 1)^2 + (2U_2 - 1)^2 \leq 1$ generate a third uniform deviate U_3 , otherwise reject the pair and start over; (3) calculate $y_1 = (-\ln u_3)^{1/2} \frac{u_1^2 - u_2^2}{u_1^2 + u_2^2}$, $y_2 = \pm 2(-\ln u_3)^{1/2} \frac{u_1 u_2}{u_1^2 + u_2^2}$ ($\pm$ random). Both y_1 and y_2 are the desired random deviates.

(4) *Acceptance-rejection method*: 1) Generate a pair of uniform deviates (U_1, U_2) ; 2) compute $x = -\ln u_1$; 3) if $e^{-\frac{1}{2}(x-1)^2} \geq u_2$ (or equivalently $(x-1)^2 \leq -2(\ln u_2)$) accept x , otherwise reject the

pair and start over. The quantity will be the required normal deviate with mean zero and unit variance.

b. Bivariate normal distribution

Let $\{X_1, X_2\}$ be a pair of independent normal deviates with mean zero and unit variance. Then $\{X_1, \rho X_1 + (1 - \rho^2)^{1/2} X_2\}$ represent a pair of deviates from a bivariate normal distribution with zero means, unit variances, and correlation coefficient ρ .

c. Exponential distribution

(1) *Inverse method*: Since $F(x) = e^{-x/\theta}$, $X = -\theta \ln U$ will be a deviate from the exponential distribution with parameter θ .

(2) *Acceptance-rejection method*: 1) Generate a pair of independent uniform deviates (U_0, U_1) ; 2) if $U_1 < U_0$ generate a third value U_2 ; 3) if $U_1 + U_2 < U_0$ generate a fourth value U_3 , etc.; 4) continue generating uniform deviates until an n is obtained such that $U_1 + U_2 + \dots + U_{n-1} < U_0 < U_1 + \dots + U_n$; 5) if n is even reject the procedure and start a fresh trial with a new value of U_0 , otherwise if n is odd take $X = \theta U_0$ as the desired deviate; 6) in general if t is the number of trials until an acceptable sequence is obtained $X = \theta(t + U_0)$. The random deviates produced in this way follow an exponential distribution with parameter θ . One can expect to generate approximately six uniform deviates for every exponential deviate.

(3) *Discrete Distribution Method*: Let Y and n be discrete random variables with point probabilities

$$* \Pr\{Y=r\} = (e-1)e^{-(r+1)} \quad r=0, 1, 2, \dots$$

$$\Pr\{n=s\} = [s!(e-1)]^{-1} \quad s=1, 2, 3, \dots$$

Then $X = Y + \min(U_1, U_2, \dots, U_n)$ will follow an exponential distribution. The average value of n is 1.58 so that one needs, on the average, only 1.58 u 's from which the minimum is selected.

26.9. Use and Extension of the Tables

Use of Probability Function Inequalities

Example 1. Let X be a random variable with finite mean and variance equal to m and σ^2 , respectively. Use the inequalities for probability functions 26.1.37, 40, 41 to place lower bounds on

$$A(t) = F(t) - F(-t) = P\left\{\frac{|X-m|}{\sigma} \leq t\right\}$$

for $t=1(1)4$.

*See page 11.

Lower bounds on $A(t) = F(t) - F(-t)$					Remarks
$t=1$	2	3	4		
0	.7500	.8889	.9375		no knowledge of $F(t)$; 26.1.37
.5556	.8889	.9506	.9722		$F(t)$ is unimodal and continuous; 26.1.40
0	.8182	.9697	.9912		$F(t)$ is such that $\mu_4=3$; 26.1.41

It is of interest to note that the standard normal distribution is unimodal, has mean zero, unit variance $\mu_4=3$, is continuous, and such that

$$A(t) = P(t) - P(-t) \\ = .6827, .9545, .9973, \text{ and } .9999$$

for $t=1, 2, 3$ and 4 respectively.

Interpolation for $P(x)$ in Table 26.1

Example 2. Compute $P(x)$ for $x=2.576$ to fifteen decimal places using a Taylor expansion.

Writing $x=x_0+\theta$ we have

$$P(x) = P(x_0) + Z(x_0)\theta + Z^{(1)}(x_0)\frac{\theta^2}{2!} \\ + Z^{(2)}(x_0)\frac{\theta^3}{3!} + Z^{(3)}(x_0)\frac{\theta^4}{4!} + \dots$$

Taking $x_0=2.58$ and $\theta=-4\times 10^{-3}$ we calculate the successive terms to 16D

+.99505	99842	42230	
—	5	72204	35976 6
—		2952	57449 6
—		8	63097 8
—			1439 4
—			9
.99500	24676	84265	7

The result correct to 17D is

$$P(2.576) = .99500 \quad 24676 \quad 84264 \quad 98$$

Calculation for Arbitrary Mean and Variance

Example 3. Find the value to 5D of

$$P\{X \leq .50\} = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{.5} e^{-1/2(\frac{t-1}{2})^2} dt$$

using 26.2.8 and Table 26.1.

This represents the probability of the random variable being less than or equal to .5 for a normal distribution with mean $m=1$ and variance $\sigma^2=4$. Using 26.2.8 we have

$$P\{X \leq .5\} = P\left(\frac{.5-1}{2}\right) = P(-.25)$$

Since $P(-x) = 1 - P(x)$, we have

$$P(-.25) = 1 - P(.25) = 1 - .59871 = .40129$$

where a two-term Taylor series was used for interpolation. Note that when interpolating for $P(x)$ for a value of x midway between the tabulated

values we can write $x=x_0+.01$ and a two-term Taylor series is $P(x) = P(x_0) + Z(x_0)10^{-2}$. Thus one need only multiply $Z(x_0)$ by 10^{-2} and add the result to $P(x_0)$.

Calculation of $P(x)$ for x Approximate

Example 4. Using Table 26.1, find $P(x)$ for $x=1.96$, when there is a possible error in x of $\pm 5 \times 10^{-3}$.

This is an example where the argument is only known approximately. The question arises as to how many decimal places one should retain in $P(x)$. If Δx and $\Delta P(x)$ denote the error in x and the resulting error in $P(x)$, respectively, then

$$\Delta P(x) \approx Z(x)\Delta x$$

Hence $\Delta P(1.960) = 3 \times 10^{-4}$ which indicates that $P(1.960)$ need only be calculated to 4D. Therefore $P(1.960) = .9750$.

Inverse Interpolation for $P(x)$

Example 5. Find the value of x for which $P(x) = .97500 \quad 00000 \quad 00000$ using Table 26.1 and determining as many decimal places as is consistent with the tabulated function.

For inverse interpolation the tabulated function $P(x)$ may be regarded as having a possible error of $.5 \times 10^{-15}$. Hence

$$\Delta x \approx \frac{\Delta P(x)}{Z(x)} = \frac{.5 \times 10^{-15}}{Z(x)}$$

Let $P(x_0)$ correspond to the closest tabulated value of $P(x)$. Then a convenient formula for inverse interpolation is

$$x = x_0 + t + \frac{x_0 t^2}{2} + \frac{2x_0^2 + 1}{6} t^3$$

where

$$t = \frac{P(x) - P(x_0)}{Z(x_0)}$$

If only the first two terms (i.e., $x=x_0+t$) are used, the error in x will be bounded by $\frac{x}{8} \times 10^{-4}$ and the true value will always be greater than the value thus calculated.

With respect to this example, $\Delta x \approx 10^{-14}$ and thus the interpolated value of x may be in error by one unit in the fourteenth place. The closest value to $P(x) = .97500 \quad 00000 \quad 00000$ is $P(x_0) = .97500 \quad 21048 \quad 51780$ with $x_0=1.96$. Hence using the preceding inverse interpolation formulas with

$$t = -.00003 \ 60167 \ 31129$$

and carrying fifteen decimals we have the successive terms

$$\begin{array}{r} +1.96000 \ 00000 \ 00000 \\ -.00003 \ 60167 \ 31129 \\ + \quad \quad \quad 12 \ 71261 \\ - \quad \quad \quad \quad \quad 68 \\ \quad \quad \quad \quad \quad \quad 0 \\ \hline +1.95996 \ 39845 \ 40064 \end{array}$$

Edgeworth Asymptotic Expansion

Example 6. Find the Edgeworth asymptotic expansion 26.2.49 for the c.d.f. of chi-square.

Method 1. Expansion for χ^2

Let

$$Q(\chi^2|\nu) = 1 - F(t)$$

where

$$t = \frac{\chi^2 - \nu}{(2\nu)^{\frac{1}{2}}}$$

Since the values of γ_1 and γ_2 26.4.33 are

$$\gamma_1 = 2\sqrt{2/\nu^{\frac{1}{2}}}$$

$$\gamma_2 = 12/\nu,$$

we obtain, by using the first two bracketed terms of 26.2.49

$$F(t) \sim P(t) - \frac{1}{\nu^{\frac{1}{2}}} \left[\frac{\sqrt{2}}{3} Z^{(2)}(t) \right] + \frac{1}{\nu} \left[\frac{1}{2} Z^{(3)}(t) + \frac{1}{9} Z^{(5)}(t) \right]$$

The Edgeworth expansion is an asymptotic expansion in terms of derivatives of the normal distribution function. It is often possible to transform a random variable so that the distribution of the transformed random variable more closely approximates the normal distribution function than does the distribution of the original random variable. Hence for the same number of terms, greater accuracy may be achieved by using the transformed variable in the expansion. Since the distribution of $\sqrt{2\chi^2}$ is more closely approximated by a normal distribution than χ^2 itself (as judged by a comparison of the values of γ_1 and γ_2), we would expect that the Edgeworth asymptotic expansion of $\sqrt{2\chi^2}$ would be superior to that of χ^2 .

Method 2. Expansion for $\sqrt{2\chi^2}$. Let

$$Q(\chi^2|\nu) = 1 - F(t) = 1 - F\left(\frac{\sqrt{2\chi^2} - (2\nu-1)^{\frac{1}{2}}}{\left(1 - \frac{1}{4\nu}\right)^{\frac{1}{2}}}\right)$$

where $(2\nu-1)^{\frac{1}{2}}$ and $1 - \frac{1}{4\nu}$ are the mean and variance to terms of order ν^{-2} of $\sqrt{2\chi^2}$ (see 26.4.34). The values of γ_1 and γ_2 for $\sqrt{2\chi^2}$ are

$$\gamma_1 \approx \frac{1}{\sqrt{2\nu}} \left[1 + \frac{5}{8\nu} \right] \quad \gamma_2 \approx \frac{3}{4\nu^2}$$

Thus we obtain

$$F(t) \sim P(t) - \frac{1}{\nu^{\frac{1}{2}}} \left[\frac{\sqrt{2}}{12} \left(1 + \frac{5}{8\nu} \right) Z^{(2)}(t) \right] + \frac{1}{\nu} \left[\frac{1}{32\nu} Z^{(3)}(t) + \frac{1}{144} \left(1 + \frac{5}{8\nu} \right)^2 Z^{(5)}(t) \right]$$

For numerical examples using these expansions see **Example 12.**

Calculation of $L(h, k, \rho)$

Example 7. Find $L(.5, .4, .8)$. Using 26.3.20

$$\sqrt{h^2 - 2\rho hk + k^2} = \sqrt{.09} = .3$$

$$L(.5, .4, .8) = L(.5, 0, 0) + L(.4, 0, -.6)$$

Reference to **Figure 26.2** yields

$$L(.5, 0, 0) + L(.4, 0, -.6) = .16 + .08 = .24$$

The answer to 3D is $L(.5, .4, .8) = .250$.

Calculation of the Bivariate Normal Probability Function

Example 8. Let X and Y follow a bivariate normal distribution with parameters $m_x=3$, $m_y=2$, $\sigma_x=4$, $\sigma_y=2$, and $\rho=-.125$. Find the value of $P_r\{X \geq 2, Y \geq 4\}$ using 26.3.20 and **Figures 26.2, 26.3.**

Since $P_r\{X \geq h, Y \geq k\} = L\left(\frac{h-m_x}{\sigma_x}, \frac{k-m_y}{\sigma_y}, \rho\right)$ we have $P\{X \geq 2, Y \geq 4\} = L(-.25, 1, -.125)$. Using 26.3.20

$$L(-.25, 1, -.125) = L(-.25, 0, .969) + L(1, 0, .125) - \frac{1}{2}$$

Figure 26.2 only gives values for $h > 0$, however, using the relationship 26.3.3 with $k=0$, $L(-h, 0, \rho) = \frac{1}{2} - L(h, 0, -\rho)$ and thus $L(-.25, 0, .969) = \frac{1}{2} - L(.25, 0, -.969)$. Therefore $L(-.25, 1, -.125) = -L(.25, 0, -.969) + L(1, 0, .125) = -.01 + .09 = .08$.

The answer to 3D is $L(-.25, 1, -.125) = .080$.

Integral of a Bivariate Normal Distribution Over a Polygon

Example 9. Let the random variables X and Y have a bivariate normal distribution with parameters $m_x=5$, $\sigma_x=2$, $m_y=9$, $\sigma_y=4$, and $\rho=.5$. Find the probability that the point (X, Y) be inside the triangle whose vertices are $A=(7, 8)$, $B=(9, 13)$, and $C=(2, 9)$.

When obtaining the integral of a bivariate normal distribution over a polygon, it is first necessary to use 26.3.22 in order to transform the variates so that one deals with a circular normal distribution. The polygon in the region of the transformed variables is then divided into configurations such that the integral over any selected configuration can be easily obtained. Below are listed some of the most useful configurations.

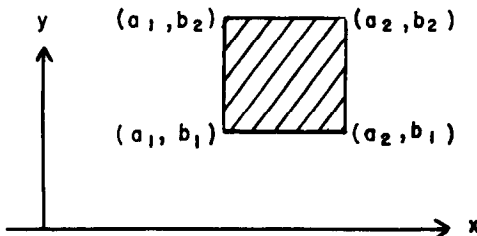


FIGURE 26.5

$$\int_{a_1}^{a_2} \int_{b_1}^{b_2} g(x, y, 0) dx dy = [P(a_2) - P(a_1)] [P(b_2) - P(b_1)]$$

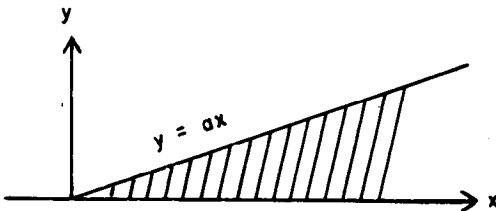


FIGURE 26.6

$$\int_0^\infty \int_0^{ax} g(x, y, 0) dx dy = \frac{\arctan a}{2\pi}$$

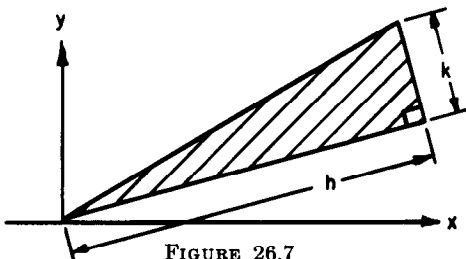


FIGURE 26.7

$$\int_0^h \int_0^{\frac{k}{h}x} g(x, y, 0) dx dy = V(h, k)^{11}$$

¹¹ See 26.3.23 for definition of $V(h, k)$.

For the following two configurations we define

$$h = \frac{|t_2 s_1 - t_1 s_2|}{[(s_2 - s_1)^2 + (t_2 - t_1)^2]^{\frac{1}{2}}}$$

$$k_1 = \frac{|s_1(s_2 - s_1) + t_1(t_2 - t_1)|}{[(s_2 - s_1)^2 + (t_2 - t_1)^2]^{\frac{1}{2}}}$$

$$k_2 = \frac{|s_2(s_2 - s_1) + t_2(t_2 - t_1)|}{[(s_2 - s_1)^2 + (t_2 - t_1)^2]^{\frac{1}{2}}}$$

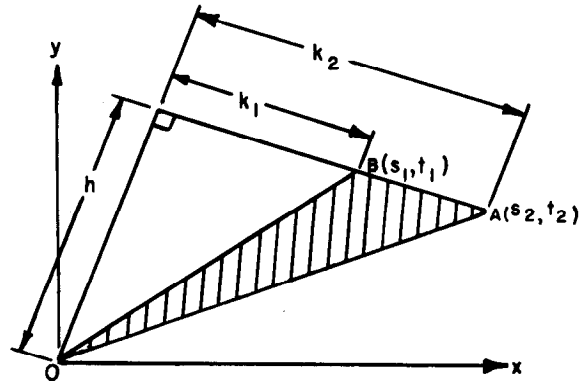


FIGURE 26.8

$$\iint_{\Delta AOB} g(x, y, 0) dx dy = V(h, k_2) - V(h, k_1)$$

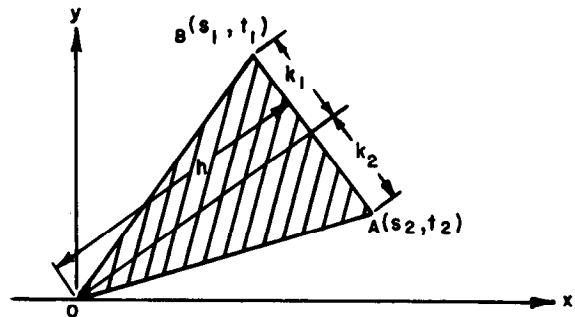


FIGURE 26.9

$$\iint_{\Delta AOB} g(x, y, 0) dx dy = V(h, k_2) + V(h, k_1)$$

Using the circularizing transformation 26.3.22 for our example results in

$$s = \frac{1}{\sqrt{3}} \left(\frac{x-5}{2} + \frac{y-9}{4} \right)$$

$$t = -\frac{1}{1} \left(\frac{x-5}{2} - \frac{y-9}{4} \right)$$

The vertices of the triangle in the (s, t) coordinates become $A = (\sqrt{3}/4, -5/4)$, $B = (\sqrt{3}, -1)$ and $C = (-\frac{\sqrt{3}}{2}, \frac{3}{2})$. These points are plotted below. From the figure it is seen that the desired probability is the sum of the probabilities that the point having the transformed variables as coordinates is inside the triangles AOB , AOC , and BOC .

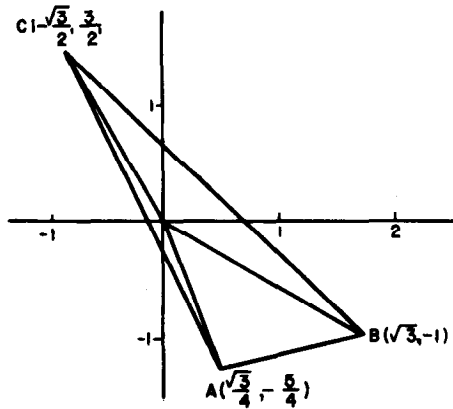


FIGURE 26.10

For these three triangles we have

	h	k_1	k_2
ΔAOB	$\frac{2}{7}\sqrt{21}$	$\sqrt{7}/14$	$\frac{4}{7}\sqrt{7}$
ΔAOC	$\frac{1}{74}\sqrt{111}$	$\frac{8}{37}\sqrt{37}$	$\frac{21}{74}\sqrt{37}$
ΔBOC	$\frac{1}{13}\sqrt{39}$	$\frac{7}{13}\sqrt{13}$	$\frac{6}{13}\sqrt{13}$

From the graph it is seen that the probability over AOB may be found in the same manner as that over Figure 26.8, and over AOC and BOC the probabilities may be found as that over Figure 26.9.

Hence

$$\begin{aligned} \iint_{\Delta} g(x, y, .5) dx dy &= \iint_{\Delta ABC} g(s, t, 0) ds dt \\ &= \iint_{\Delta AOB} g(s, t, 0) ds dt + \iint_{\Delta AOC} g(s, t, 0) ds dt \\ &\quad + \iint_{\Delta BOC} g(s, t, 0) ds dt \end{aligned}$$

and consequently using 26.3.23 and Figure 26.2

$$\begin{aligned} \iint_{\Delta AOB} g(s, t, 0) ds dt &= V\left(\frac{2}{7}\sqrt{21}, \frac{4\sqrt{7}}{7}\right) - V\left(\frac{2}{7}\sqrt{21}, \frac{\sqrt{7}}{14}\right) \\ &= \left[\frac{1}{4} + L(1.31, 0, -.76) - L(0, 0, -.76) - \frac{1}{2} Q(1.31)\right] \\ &\quad - \left[\frac{1}{4} + L(1.31, 0, -.14) - L(0, 0, -.14) - \frac{1}{2} Q(1.31)\right] \\ &= L(1.31, 0, -.76) - L(0, 0, -.76) \\ &\quad - L(1.31, 0, -.14) + L(0, 0, -.14) \\ &= .00 - .11 - .04 + .23 = .08 \end{aligned}$$

$$\begin{aligned} \iint_{\Delta AOC} g(s, t, 0) ds dt &= V\left(\frac{\sqrt{111}}{74}, \frac{8\sqrt{37}}{37}\right) + V\left(\frac{\sqrt{111}}{74}, \frac{21\sqrt{37}}{74}\right) \\ &= \left[\frac{1}{4} + L(.14, 0, -.99) - L(0, 0, -.99) - \frac{1}{2} Q(.14)\right] \\ &\quad + \left[\frac{1}{4} + L(.14, 0, -1) - L(0, 0, -1) - \frac{1}{2} Q(.14)\right] \\ &= .01 + .02 = .03 \end{aligned}$$

$$\begin{aligned} \iint_{\Delta BOC} g(s, t, 0) ds dt &= V\left(\frac{\sqrt{39}}{13}, \frac{7\sqrt{13}}{13}\right) + V\left(\frac{\sqrt{39}}{13}, \frac{6\sqrt{13}}{13}\right) \\ &= \left[\frac{1}{4} + L(.48, 0, -.97) - L(0, 0, -.97) - \frac{1}{2} Q(.48)\right] \\ &\quad + \left[\frac{1}{4} + L(.48, 0, -.96) - L(0, 0, -.96) - \frac{1}{2} Q(.48)\right] \\ &= .05 + .04 = .09 \end{aligned}$$

Thus adding all parts, the probability that X and Y are in triangle ABC is $.08 + .03 + .09 = .20$. The answer to 3D is .211.

Calculation of a Circular Normal Distribution Over an Offset Circle

Example 10. Let X and Y have a circular normal distribution with $\sigma = 1000$. Find the probability that the point (X, Y) falls within a circle having a radius equal to 540 whose center is displaced 1210 from the mean of the circular normal distribution.

In units of σ , the radius and displacement from the center are, respectively, $R = \frac{540}{1000} = .54$ and $r = \frac{1210}{1000} = 1.21$. The problem is thus reduced to finding the probability of X and Y falling in a circle of radius $R = .54$ displaced $r = 1.21$ from the center of the distribution where $\sigma = 1$.

Since $R < 1$, the approximation 26.3.25 is used. This results in

$$P(R^2|2, r^2) = \frac{2(.54)^2}{4 + (.54)^2} \exp \frac{-2(1.21)^2}{4 + (.54)^2} \\ = (.1359)e^{-.6823} = .06869$$

The answer to 5D is .06870.

Interpolation for $Q(x^2|\nu)$

Example 11. Find $Q(25.298|20)$ using the interpolation formula given with Table 26.7.

Taking $x^2 = 25$, $\theta = .298$ and applying the interpolation formula results in

$$Q(25.298|20) = \frac{1}{8} \{ Q(25|16)\theta^2 + Q(25|18)(4\theta - 2\theta^2) \\ + Q(25|20)(8 - 4\theta + \theta^2) \} \\ = \frac{1}{8} \{ (.06982)(.088804) \\ + (.12492)(1.014392) \\ + (.20143)(6.896804) \} \\ = .19027$$

A less accurate interpolate may be obtained by setting θ^2 equal to zero in the above formula. This results in the value .19003. The correct value to 6D is $Q(25.298|20) = .190259$.

On the other hand if $x^2 = 25.298$ is assumed to have an error of $\pm 5 \times 10^{-4}$, then how large an error arises in $Q(x^2|\nu)$? Denoting the error in x^2 by Δx^2 and the resulting error in $Q(x^2|\nu)$ by $\Delta Q(x^2|\nu)$, we then have the approximate relationship

$$\Delta Q(x^2|\nu) \approx \frac{\partial Q(x^2|\nu)}{\partial x^2} \Delta x^2$$

Using 26.4.8 we can write

$$\frac{\partial Q(x^2|\nu)}{\partial x^2} = \frac{1}{2} [Q(x^2|\nu-2) - Q(x^2|\nu)]$$

and

$$\Delta Q(x^2|\nu) \approx \frac{1}{2} [Q(x^2|\nu-2) - Q(x^2|\nu)] \Delta x^2$$

For practical purposes it is sufficient to evaluate the derivative to one or two significant figures. Consequently we can write

$$\frac{\partial Q(x^2|\nu)}{\partial x^2} \approx \frac{\partial Q(x_0^2|\nu)}{\partial x^2}$$

where x_0^2 is the closest value to x^2 for which Q is tabulated. Hence

$$\Delta Q(x^2|\nu) \approx \frac{1}{2} [Q(x_0^2|\nu-2) - Q(x_0^2|\nu)] \Delta x^2$$

For this example $\Delta x^2 = \pm 5 \times 10^{-4}$ and $x_0^2 = 25$. This results in

$$\Delta Q(x^2|\nu) = \frac{1}{2} (-.076)(\pm 5)10^{-4} = \pm 2 \times 10^{-5}$$

as the possible error in $Q(x^2|\nu)$.

Calculation of $Q(x^2|\nu)$ Outside the Range of Table 26.7

Example 12. Find the value of $Q(84|72)$.

Since this value is outside the range of Table 26.7 we can approximate $Q(84|72)$ by (1) using the Edgeworth expansion for $Q(x^2|\nu)$ given in Example 6, (2) the cube root approximation 26.4.14, (3) the improved cube root approximation 26.4.15 or (4) the square root approximation 26.4.13. The results of using all four methods are presented below:

1. Edgeworth expansion

The successive terms of the Edgeworth expansion for the distribution of chi-square result in

$$1 - Q(84|72) = .841345 \\ .000000 \\ .001120 \\ \hline .842465$$

Hence $Q(84|72) = .15754$.

The successive terms of the Edgeworth expansion for the distribution of $\sqrt{2}x^2$ result in

$$1 - Q(84|72) = .842544 \\ -.000034 \\ -.000138 \\ \hline .842372$$

Hence $Q(84|72) = .15764$.

2. Cube root approximation 26.4.14

Using the cube root approximation we have

$$Q(84|72) = Q(x)$$

where

$$x = \frac{\left(\frac{84}{72}\right)^{1/3} \left[1 - \frac{2}{9(72)}\right]}{\left[\frac{2}{9(72)}\right]^{1/2}} = 1.0046$$

This results in $Q(84|72) = Q(1.0046) = 1 - P(1.0046) = .15754$.

3. Improved cube root approximation 26.4.15

The improved cube root approximation involves calculating a correction factor h_r to x . Linearly interpolating for h_{80} (which appears below 26.4.15) with $x = 1.0046$ results in $h_{80} = -.0006$ and hence

$$h_{72} = \frac{60}{72}(-.0006) = -.00049$$

Thus

$$Q(84|72) = Q(1.0046 - .0005) = Q(1.0041) \\ = 1 - P(1.0041) = .15766$$

4. Square root approximation 26.4.13

Using the square root approximation we have $Q(84|72) = Q(x)$ where

$$x = \sqrt{2(84)} - \sqrt{2(72)} - 1 = 1.0032.$$

This results in

$$Q(84|72) = Q(1.0032) = 1 - P(1.0032) = .15788$$

The value correct to 6D is $Q(84|72) = .157653$. Generally the improved cube root approximation will be correct with a maximum error of a few units in the fifth decimal and is recommended for calculations which are outside the range of **Table 26.7**.

Calculation of x^2 for $Q(x^2|\nu)$ Outside the Range of **Table 26.8**

Example 13. Find the value of x^2 for which $Q(x^2|144) = .01$.

Since $\nu = 144$ is outside the range of **Table 26.8**, we can compute it by using (1) the Cornish-Fisher asymptotic expansion 26.2.50, for x^2 , (2) the cube approximation 26.4.17, (3) the improved cube approximation 26.4.18, or (4) the square approximation 26.4.16. We shall compute the value by all four methods.

1. Cornish-Fisher asymptotic expansion 26.2.50

The Cornish-Fisher asymptotic expansion for x^2 with $\nu = 144$ can be written as

$$x^2 \sim 144 + 12\sqrt{2}x + 4h_1(x) + \frac{4\sqrt{2}}{12}[3h_2(x) + 2h_{11}(x)] \\ + \frac{8}{12^2}[6h_3(x) + 3h_{12}(x) + 2h_{111}(x)] + \frac{16\sqrt{2}}{12^3}[30h_4(x) \\ + 9h_{22}(x) + 12h_{13}(x) + 6h_{112}(x) + 4h_{1111}(x)]$$

Hence using the auxiliary table following 26.2.51 with $p = .01$ we have

144.0000
39.4794
2.9413
-.0242
-.0019
+.0002
218.395

$$x^2 = 186.395$$

2. Cube approximation 26.4.17

Taking $x_{.01} = 2.32635$ we have

$$x^2 = 144 \left\{ \left[1 - \frac{2}{9(144)} \right] + (2.32635) \sqrt{\frac{2}{9(144)}} \right\}^3 = 186.405$$

3. Improved cube approximation 26.4.18

From the table for h_{60} we obtain using linear interpolation with $x = 2.33$ (approximately)

$$h_{60} = .0012 \text{ and thus } h_{144} = \frac{60}{144}(.0012) = .00049$$

Hence

$$x^2 = 144 \left[1 - \frac{2}{9(144)} + (2.32635 - .00049) \sqrt{\frac{2}{9(144)}} \right]^3 = 186.394$$

4. Square approximation 26.4.16

$$x^2 = \frac{1}{2} [2.32635 + \sqrt{2(144)} - 1]^2 = 185.616$$

The correct answer to 3D is $x^2 = 186.394$. Generally the improved cube approximation will give results correct in the second or third decimal for $\nu > 30$.

Calculation of the Incomplete Gamma Function

Example 14. Find the value of

$$\gamma(2.5, .9) = \int_0^9 t^{1.5} e^{-t} dt$$

making use of 26.4.19 and **Table 26.7**.

Using 26.4.19 we have

$$\gamma(2.5, .9) = \Gamma(2.5)P(1.8|5) = \Gamma(2.5)[1 - Q(1.8|5)]$$

$$\gamma(2.5, .9) = \frac{3}{4}\sqrt{\pi}[1 - .87607] = .16475$$

Poisson Distribution

Example 15. Find the value of m for which

$$\sum_{i=0}^3 e^{-m} \frac{m^i}{i!} = .99$$

using 26.4.21 and **Table 26.8**.

From **Table 26.8** with $\nu = 2c = 8$ and $Q = .99$ we have $x^2 = 1.646482$. Hence $m = x^2/2 = .823241$.

Inverse of the Incomplete Beta Function

Example 16. Find the value of x for which $I_x(10, 6) = .10$ using **Table 26.9** and 26.5.28.* Using 26.5.28 we have

*See page II.

$$I_x(10, 6) = Q(F|12, 20) = .10 \text{ where } x = \frac{20}{20+12F}$$

From **Table 26.9** the upper 10 percent point of F with 12 and 20 degrees of freedom is $F=1.89$. Hence

$$x = \frac{20}{20+12(1.89)} = .469$$

The correct value to 4D is $x=.4683$.

Calculation of $I_x(a, b)$ for a or b Small Integers

Example 17. Calculate $I_{.10}(3, 20)$.

Values of $I_x(a, b)$ for small integral a or b can conveniently be calculated using **26.5.6** or **26.5.7**. Using **26.5.6** we have

$$\begin{aligned} 1 - I_{.90}(20, 3) &= \frac{(.9)^{20}}{B(3, 20)} \left\{ \sum_{i=0}^2 (-1)^i \binom{2}{i} \frac{.9^i}{20+i} \right\} \\ &= \frac{.121576}{.216450 \times 10^{-3}} (.110390 \times 10^{-2}) = .620040 \end{aligned}$$

Binomial Distribution

Example 18. Find the value of p which satisfies

$$\sum_{s=0}^{20} \binom{50}{s} p^s q^{50-s} = .95, \quad q = 1 - p$$

using **26.5.24** and **Table 26.9**.

* Combining **26.5.24** and **26.5.28** we have

$$\sum_{s=0}^n \binom{n}{s} p^s q^{n-s} = Q(F|v_1, v_2)$$

where

$$v_1 = 2(n-a+1), v_2 = 2(a), \text{ and } p = \frac{a}{a+(n-a+1)F}$$

Hence

$$\begin{aligned} \sum_{s=0}^{20} \binom{50}{s} p^s q^{50-s} &= 1 - \sum_{s=21}^{50} \binom{50}{s} p^s q^{50-s} \\ &= 1 - Q(F|60, 42) = .95 \end{aligned}$$

Harmonic interpolation on v_2 in the table for which $Q(F|v_1, v_2) = .05$ results in $F=1.624$ for $v_1=60, v_2=42$, and thus $p = \frac{42}{42+60(1.624)} = .301$.

The correct answer to 4D is $p=.3003$.

Approximating the Incomplete Beta Function

Example 19. Find $I_{.60}(16, 10.5)$ using **26.5.21**.

Values of $I_x(a, b)$ can conveniently be calculated with good accuracy using the approximation given by **26.5.20** or **26.5.21**. For this example $(a+b-1)(1-x) = 10.20$ which is greater than .8 and hence **26.5.21** will be used. Thus

$$w_1 = [(10.5)(.60)]^{1/3} = 1.8469, w_2 = [16(.4)]^{1/3} = 1.8566$$

$$y = \frac{3[(1.8469)(.98942) - (1.8566)(.99306)]}{\left[\frac{(1.8469)^2}{10.5} + \frac{(1.8566)^2}{16} \right]^{1/2}} = -.0668$$

and interpolating in **Table 26.1** gives

$$P(-.0668) = 1 - P(.0668) = .47336$$

The answer correct to 5D is $I_{.60}(16, 10.5) = .47332$.

Interpolation for F in Table 26.9

Example 20. Find the value of F for which

$$Q(F|7, 20) = .05 \text{ using Table 26.9.}$$

Interpolation in **Table 26.9** is approximately linear when the reciprocals of the degrees of freedom (v_1, v_2) are used as the interpolating variable. For this example it is only necessary to interpolate with respect to $1/v_1$. Thus linear interpolation on $1/v_1$ results in $F=2.51$ which is the correct interpolate.

Calculation of F for $Q(F|v_1, v_2) > .50$

Example 21. Find the value of F for which $Q(F|4, 8) = .90$ using **26.6.9** and **Table 26.9**.

Table 26.9 only tabulates values of F for which $Q(F|v_1, v_2) = p$ where $p = .500, .250, .100, .050, .025, .010, .005, .001$. However making use of **Table 26.9** we can find the values of F_p for which $p = .75, .9, .95, .975, .99, .995, .999$. For this example we have

$$F_{.90}(4, 8) = \frac{1}{F_{.10}(8, 4)}$$

and referring to the table for which $Q(F|v_1, v_2) = .10$ gives $F_{.10}(8, 4) = 3.95$ and thus $F_{.90}(4, 8) = \frac{1}{3.95} = .253$.

Calculation of $Q(F|v_1, v_2)$ for Small Integral v_1 or v_2

Example 22. Compute $Q(2.5|4, 15)$ using **26.6.4**.

Values of $Q(F|v_1, v_2)$ can be readily computed for small v_1 or v_2 using the expansions **26.6.4** to **26.6.8** inclusive. We have using **26.6.4**

$$x = \frac{15}{15+4(2.50)} = .60$$

and

$$Q(2.50|4, 15) = (.6)^{7.5} \left[1 + \frac{15}{2} (.4) \right] = .086735$$

*See page 11.

Approximating $Q(F|\nu_1, \nu_2)$

Example 23. Calculate $Q(1.714|10, 40)$ using 26.6.15.

The approximation given by 26.6.15 will result in a maximum error of .0005. For this example we have

$$x = \frac{(1.714)^{1/3} \left(1 - \frac{2}{9(40)}\right) - \left(1 - \frac{2}{9(10)}\right)}{\left[\frac{2}{9(10)} + (1.714)^{2/3} \frac{2}{9(40)}\right]^{1/2}} = 1.2222$$

Interpolating in Table 26.1 results in

$$Q(1.714|10, 40) \approx Q(1.2222) = 1 - P(1.2222) = .1108$$

The correct value to 5D is $Q(1.714|10, 40) = .11108$.

On the other hand the approximation given by 26.6.14 which is usually less accurate results in

$$x = \frac{\sqrt{[2(40) - 1] \left(\frac{10}{40}\right) (1.714) - \sqrt{2(10) - 1}}}{\sqrt{1 + \frac{10}{40} (1.714)}} = 1.2210$$

and interpolating in Table 26.1 gives

$$Q(1.714|10, 40) \approx Q(1.2210) = 1 - P(1.2210) = .1112$$

Calculation of F Outside the Range of Table 26.9

Example 24. Find the value of F for which $Q(F|10, 20) \approx .0001$ using 26.6.16 and 26.5.22.

For this problem we have $a = \frac{\nu_2}{2} = 10$, $b = \frac{\nu_1}{2} = 5$, $p = .0001$. The value of the normal deviate which cuts off .0001 in the tail of the distribution is

$y = 3.7190$ (i.e., $Q(3.7190) = .0001$). Hence substituting in 26.5.22 gives

$$h = 2 \left[\frac{1}{19} + \frac{1}{9} \right]^{-1} = 12.2143$$

$$\lambda = \frac{3.7190^2 - 3}{6} = 1.8052$$

$$w = 3.7190 \frac{(12.2143 + 1.8052)^{1/2}}{12.2143}$$

$$- \left(\frac{1}{9} - \frac{1}{19} \right) \left[1.8052 + .8333 - \frac{2}{3(12.2143)} \right]$$

$$w = .9889$$

and thus $F \approx e^{2w} = 7.23$. The correct answer is $F = 7.180$.

Approximating the Non-Central F -Distribution

Example 25. Compute $P(3.71|3, 10, 4)$ using the approximation 26.6.27 to the non-central F -distribution.

Using 26.6.27 with $\nu_1 = 3$, $\nu_2 = 10$, $\lambda = 4$, $F' = 3.71$ we have

$$x = \frac{\left[\left(\frac{3}{3+4} \right) (3.71) \right]^{1/3} \left[1 - \frac{2}{9(10)} \right] - \left[1 - \frac{2}{9} \frac{(3+8)}{(3+4)^2} \right]}{\left[\frac{2}{9} \frac{3+8}{(3+4)^2} + \frac{2}{9(10)} \left[\left(\frac{3}{3+4} \right) (3.71) \right]^{2/3} \right]^{1/2}} = .675$$

and interpolating in Table 26.1 gives

$$P(3.71|3, 10, 4) \approx P(.675) = .750$$

The exact answer is $P(3.71|3, 10, 4) = .745$.

References

Texts

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Chi-Square, Non-Central Chi-Square, Probability Integral, Incomplete Gamma Function, Poisson Distribution

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F (Variance-Ratio) and Non-Central F Distribution

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- $$\nu_1 = 1(1)6, 8, 12, 24, \infty; \nu_2 = 1(1)30, 40, 60, 120, \infty, 2D \text{ for } F, 4D \text{ for } Z.$$
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27. Miscellaneous Functions

IRENE A. STEGUN¹

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¹ National Bureau of Standards.

27. Miscellaneous Functions

27.1. Debye Functions

Series Representations

27.1.1

$$\int_0^x \frac{t^n dt}{e^t - 1} = x^n \left[\frac{1}{n} - \frac{x}{2(n+1)} + \sum_{k=1}^{\infty} \frac{B_{2k} x^{2k}}{(2k+n)(2k)!} \right] \\ (|x| < 2\pi, n \geq 1)$$

(For Bernoulli numbers B_{2k} , see chapter 23.)

27.1.2

$$\int_x^{\infty} \frac{t^n dt}{e^t - 1} = \sum_{k=1}^{\infty} e^{-kx} \left[\frac{x^n}{k} + \frac{n x^{n-1}}{k^2} + \frac{(n)(n-1)x^{n-2}}{k^3} \right. \\ \left. + \dots + \frac{n!}{k^{n+1}} \right] (x > 0, n \geq 1)$$

Relation to Riemann Zeta Function (see chapter 23)

27.1.3 $\int_0^{\infty} \frac{t^n dt}{e^t - 1} = n! \zeta(n+1).$

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$$\frac{3}{x^2} \int_0^x \frac{y^3 dy}{e^y - 1}, \frac{12}{x^3} \left[\int_0^x \frac{y^3 dy}{e^y - 1} - \frac{3x}{e^x - 1} \right], x = 0(.01)24, \quad 6S.$$

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$$\frac{20}{x^4} \int_0^x \frac{t^4 dt}{e^t - 1} - \frac{4x}{e^x - 1}, \\ x = 0(.1)13(.2)18(1)20(2)52(4)80, \quad 4S.$$

Table 27.1

Debye Functions

x	$\frac{1}{x} \int_0^x \frac{t dt}{e^t - 1}$	$\frac{2}{x^2} \int_0^x \frac{t^2 dt}{e^t - 1}$	$\frac{3}{x^3} \int_0^x \frac{t^3 dt}{e^t - 1}$	$\frac{4}{x^4} \int_0^x \frac{t^4 dt}{e^t - 1}$
0.0	1.000000	1.000000	1.000000	1.000000
0.1	0.975278	0.967083	0.963000	0.960555
0.2	0.951111	0.934999	0.926999	0.922221
0.3	0.927498	0.903746	0.891995	0.884994
0.4	0.904437	0.873322	0.857985	0.848871
0.5	0.881927	0.843721	0.824963	0.813846
0.6	0.859964	0.814940	0.792924	0.779911
0.7	0.838545	0.786973	0.761859	0.747057
0.8	0.817665	0.759813	0.731759	0.715275
0.9	0.797320	0.733451	0.702615	0.684551
1.0	0.777505	0.707878	0.674416	0.654874
1.1	0.758213	0.683086	0.647148	0.626228
1.2	0.739438	0.659064	0.620798	0.598598
1.3	0.721173	0.635800	0.595351	0.571967
1.4	0.703412	0.613281	0.570793	0.546317
1.6	0.669366	0.570431	0.524275	0.497882
1.8	0.637235	0.530404	0.481103	0.453131
2.0	0.606947	0.493083	0.441129	0.411893
2.2	0.578427	0.458343	0.404194	0.373984
2.4	0.551596	0.426057	0.370137	0.339218
2.6	0.526375	0.396095	0.338793	0.307405
2.8	0.502682	0.368324	0.309995	0.278355
3.0	0.480435	0.342614	0.283580	0.251879
3.2	0.459555	0.318834	0.259385	0.227792
3.4	0.439962	0.296859	0.237252	0.205915
3.6	0.421580	0.276565	0.217030	0.186075
3.8	0.404332	0.257835	0.198571	0.168107
4.0	0.388148	0.240554	0.181737	0.151855
4.2	0.372958	0.224615	0.166396	0.137169
4.4	0.358696	0.209916	0.152424	0.123913
4.6	0.345301	0.196361	0.139704	0.111957
4.8	0.332713	0.183860	0.128129	0.101180
5.0	0.320876	0.172329	0.117597	0.091471
5.5	0.294240	0.147243	0.095241	0.071228
6.0	0.271260	0.126669	0.077581	0.055677
6.5	0.251331	0.109727	0.063604	0.043730
7.0	0.233948	0.095707	0.052506	0.034541
7.5	0.218698	0.084039	0.043655	0.027453
8.0	0.205239	0.074269	0.036560	0.021968
8.5	0.193294	0.066036	0.030840	0.017702
9.0	0.182633	0.059053	0.026200	0.014368
9.5	0.173068	0.053092	0.022411	0.011747
10.0	0.164443	0.047971	0.019296	0.009674

$$\left[\begin{smallmatrix} (-4)5 \\ 5 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4)6 \\ 5 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4)6 \\ 5 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4)6 \\ 5 \end{smallmatrix} \right]$$

Planck's Radiation Function

Table 27.2

$$f(x) = x^{-5}(e^{1/x} - 1)^{-1}$$

x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$
0.050	0.007	0.10	4.540	0.20	21.199	0.40	8.733	0.9	0.831
0.055	0.025	0.11	6.998	0.22	20.819	0.45	6.586	1.0	0.582
0.060	0.074	0.12	9.662	0.24	19.777	0.50	5.009	1.1	0.419
0.065	0.179	0.13	12.296	0.26	18.372	0.55	3.850	1.2	0.309
0.070	0.372	0.14	14.710	0.28	16.809	0.60	2.995	1.3	0.233
0.075	0.682	0.15	16.780	0.30	15.224	0.65	2.356	1.4	0.178
0.080	1.137	0.16	18.446	0.32	13.696	0.70	1.875	1.5	0.139
0.085	1.752	0.17	19.692	0.34	12.270	0.75	1.508	2.0	0.048
0.090	2.531	0.18	20.539	0.36	10.965	0.80	1.225	2.5	0.021
0.095	3.466	0.19	21.025	0.38	9.787	0.85	1.005	3.0	0.010
0.100	4.540	0.20	21.199	0.40	8.733	0.90	0.831	3.5	0.006

$$\left[\begin{matrix} (-2)2 \\ 4 \end{matrix} \right] \quad \left[\begin{matrix} (-2)5 \\ 5 \end{matrix} \right] \quad \left[\begin{matrix} (-2)8 \\ 5 \end{matrix} \right] \quad \left[\begin{matrix} (-2)7 \\ 5 \end{matrix} \right] \quad \left[\begin{matrix} (-2)1 \\ 4 \end{matrix} \right]$$

$$x_{\max} = .20140\ 52353 \quad f(x_{\max}) = 21.20143\ 58.$$

[27.3] Miscellaneous Physical Tables, Planck's radiation functions and electronic functions, MT 17 (U.S. Government Printing Office, Washington, D.C., 1941).

$$R_{\lambda} = c_1 \lambda^{-5} (e^{c_2/\lambda T} - 1)^{-1}, \quad R_{0-\lambda} = \int_0^{\lambda} R_{\lambda} d\lambda,$$

$$N_{\lambda} = 2\pi c \lambda^{-4} (e^{c_2/\lambda T} - 1)^{-1}, \quad N_{0-\lambda} = \int_0^{\lambda} N_{\lambda} d\lambda$$

Table I: $\frac{R_{\lambda}}{R_{\lambda \max}}, \frac{R_{0-\lambda}}{R_{0-\infty}}, \frac{N_{\lambda}}{N_{\lambda \max}}, \frac{N_{0-\lambda}}{N_{0-\infty}}$ for $\lambda T = [.05(.001).1(.005).4(.01).6(.02)1(.05)2] \text{ cm K}^{\circ}$.

Table II: $R_{\lambda}, R_{0-\lambda}, N_{\lambda}, N_{0-\lambda}$ ($T = 1000^{\circ} \text{ K}$) for $\lambda = [.5(.01)1(.05)4(.1)6(.2)10(.5)20] \text{ microns}$.

Table III: N_{λ} for $\lambda = [.25(.05)1.6(.2)3(1)10] \text{ microns}$, $T = [1000^{\circ}(500^{\circ})3500^{\circ} \text{ K and } 6000^{\circ} \text{ K}]$.

Einstein Functions

Table 27.3

x	$\frac{x^2 e^x}{(e^x - 1)^2}$	$\frac{x}{e^x - 1}$	$\ln(1 - e^{-x})$	$\frac{x}{e^x - 1} - \ln(1 - e^{-x})$
0.00	1.00000	1.00000	$-\infty$	∞
0.05	0.99979	0.97521	-3.02063	3.99584
0.10	0.99917	0.95083	-2.35217	3.30300
0.15	0.99813	0.92687	-1.97118	2.89806
0.20	0.99667	0.90333	-1.70777	2.61110
0.25	0.99481	0.88020	-1.50869	2.38888
0.30	0.99253	0.85749	-1.35023	2.20771
0.35	0.98985	0.83519	-1.21972	2.05491
0.40	0.98677	0.81330	-1.10963	1.92293
0.45	0.98329	0.79182	-1.01508	1.80690
0.50	0.97942	0.77075	-0.93275	1.70350
0.55	0.97517	0.75008	-0.86026	1.61035
0.60	0.97053	0.72982	-0.79587	1.52569
0.65	0.96552	0.70996	-0.73824	1.44820
0.70	0.96015	0.69050	-0.68634	1.37684
0.75	0.95441	0.67144	-0.63935	1.31079
0.80	0.94833	0.65277	-0.59662	1.24939
0.85	0.94191	0.63450	-0.55759	1.19209
0.90	0.93515	0.61661	-0.52184	1.13844
0.95	0.92807	0.59910	-0.48897	1.08809
1.00	0.92067	0.58198	-0.45868	1.04065
1.05	0.91298	0.56523	-0.43069	0.99592
1.10	0.90499	0.54886	-0.40477	0.95363
1.15	0.89671	0.53285	-0.38073	0.91358
1.20	0.88817	0.51722	-0.35838	0.87560
1.25	0.87937	0.50194	-0.33758	0.83952
1.30	0.87031	0.48702	-0.31818	0.80520
1.35	0.86102	0.47245	-0.30008	0.77253
1.40	0.85151	0.45824	-0.28315	0.74139
1.45	0.84178	0.44436	-0.26732	0.71168
1.50	0.83185	0.43083	-0.25248	0.68331

$$\left[\begin{matrix} (-5)5 \\ 3 \end{matrix} \right] \quad \left[\begin{matrix} (-5)5 \\ 3 \end{matrix} \right]$$

Table 27.3

Einstein Functions

x	$\frac{x^2 e^x}{(e^x - 1)^2}$	$\frac{x}{e^x - 1}$	$\ln(1 - e^{-x})$	$\frac{x}{e^x - 1} - \ln(1 - e^{-x})$
1.6	0.81143	0.40475	-0.22552	0.63027
1.7	0.79035	0.37998	-0.20173	0.58171
1.8	0.76869	0.35646	-0.18068	0.53714
1.9	0.74657	0.33416	-0.16201	0.49617
2.0	0.72406	0.31304	-0.14541	0.45845
2.1	0.70127	0.29304	-0.13063	0.42367
2.2	0.67827	0.27414	-0.11744	0.39158
2.3	0.65515	0.25629	-0.10565	0.36194
2.4	0.63200	0.23945	-0.09510	0.33455
2.5	0.60889	0.22356	-0.08565	0.30921
2.6	0.58589	0.20861	-0.07718	0.28578
2.7	0.56307	0.19453	-0.06957	0.26410
2.8	0.54049	0.18129	-0.06274	0.24403
2.9	0.51820	0.16886	-0.05659	0.22545
3.0	0.49627	0.15719	-0.05107	0.20826
3.2	0.45363	0.13598	-0.04162	0.17760
3.4	0.41289	0.11739	-0.03394	0.15133
3.6	0.37429	0.10113	-0.02770	0.12883
3.8	0.33799	0.08695	-0.02262	0.10958
4.0	0.30409	0.07463	-0.01849	0.09311
4.2	0.27264	0.06394	-0.01511	0.07905
4.4	0.24363	0.05469	-0.01235	0.06705
4.6	0.21704	0.04671	-0.01010	0.05681
4.8	0.19277	0.03983	-0.00826	0.04809
5.0	0.17074	0.03392	-0.00676	0.04068
5.2	0.15083	0.02885	-0.00553	0.03438
5.4	0.13290	0.02450	-0.00453	0.02903
5.6	0.11683	0.02078	-0.00370	0.02449
5.8	0.10247	0.01761	-0.00303	0.02065
6.0	0.08968	0.01491	-0.00248	0.01739

$$\left[\begin{matrix} (-4)3 \\ 4 \end{matrix} \right]$$

$$\left[\begin{matrix} (-4)3 \\ 4 \end{matrix} \right]$$

$$\left[\begin{matrix} (-4)4 \\ 4 \end{matrix} \right]$$

$$\left[\begin{matrix} (-4)6 \\ 4 \end{matrix} \right]$$

[27.4] H. L. Johnston, L. Savedoff and J. Belzer, Contributions to the thermodynamic functions by a Planck-Einstein oscillator in one degree of freedom, NAVEXOS p. 646, Office of Naval Research, Department of the Navy, Washington, D.C. (1949). Values of $x^2 e^x (e^x - 1)^{-2}$, $x(e^x - 1)^{-1}$, $-\ln(1 - e^{-x})$ and $x(e^x - 1)^{-1} - \ln(1 - e^{-x})$ for $x = 0(.001)3(.01)14.99$, 5D with first differences.

27.4. Sievert Integral

$$\int_0^\theta e^{-x \sec \phi} d\phi$$

Relation to the Error Function

27.4.1

$$\int_0^\theta e^{-x \sec \phi} d\phi \sim \sqrt{\frac{\pi}{2x}} e^{-x} \operatorname{erf}\left(\sqrt{\frac{x}{2}} \theta\right) \quad (x \rightarrow \infty)$$

(For erf, see chapter 7.)

Representation in Terms of Exponential Integrals

27.4.2

$$\int_0^\theta e^{-x \sec \phi} d\phi = \int_0^{\frac{\pi}{2}} e^{-x \sec \phi} d\phi - \sum_{k=0}^{\infty} \alpha_k (\cos \theta)^{2k+1} E_{2k+2} \left(\frac{x}{\cos \theta} \right) \quad \left(x \geq 0, 0 < \theta < \frac{\pi}{2} \right)$$

$$\alpha_0 = 1, \alpha_k = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k)}$$

(For $E_{2k+2}(x)$, see chapter 5.)

Relation to the Integral of the Bessel Function $K_0(x)$

27.4.3

$$\int_0^{\frac{\pi}{2}} e^{-x \sec \phi} d\phi = \operatorname{Ki}_1(x) = \int_x^\infty K_0(t) dt \text{ where}$$

$$x^{\frac{1}{2}} e^x \operatorname{Ki}_1(x) \sim \left(\frac{1}{2} \pi \right)^{\frac{1}{2}} \left\{ 1 - \frac{5}{8x} + \frac{129}{128x^2} - \frac{2655}{1024x^3} + \frac{301035}{32768x^4} - \cdots \right\}$$

(For $\operatorname{Ki}_1(x)$, see chapter 11.)

[27.5] National Bureau of Standards, Table of the Sievert integral, Applied Math. Series— (U.S. Government Printing Office, Washington, D.C. In press).

$$x=0(.01)2(.02)5(.05)10, \theta=0^\circ(1^\circ)90^\circ, \quad 9D.$$

[27.6] R. M. Sievert, Die v-Strahlungsintensität an der Oberfläche und in der nächsten Umgebung von Radiumnadeln, Acta Radiologica 11, 239-301 (1930).

$$\int_0^A e^{-A \sec \phi} d\phi, \phi=30^\circ(1^\circ)90^\circ, A=0(.01).5, \quad 3D.$$

Sievert Integral $\int_0^\theta e^{-x \sec \phi} d\phi$

Table 27.4

$x \backslash \theta$	10°	20°	30°	40°	50°	60°	75°	90°
0.0	0.174533	0.349066	0.523599	0.698132	0.872665	1.047198	1.308997	1.570796
0.1	0.157843	0.315187	0.471456	0.625886	0.777323	0.923778	1.123611	1.228632
0.2	0.142749	0.284598	0.424515	0.561159	0.692565	0.815477	0.968414	1.023680
0.3	0.129099	0.256978	0.382255	0.503165	0.617194	0.720366	0.837712	0.868832
0.4	0.116754	0.232040	0.344209	0.451198	0.550154	0.636769	0.727031	0.745203
0.5	0.105589	0.209522	0.309957	0.404629	0.490508	0.563236	0.632830	0.643694
0.6	0.095492	0.189191	0.279118	0.362893	0.437428	0.498504	0.552287	0.558890
0.7	0.086361	0.170833	0.251353	0.325486	0.390178	0.441478	0.483134	0.487198
0.8	0.078103	0.154256	0.226354	0.291957	0.348109	0.391204	0.423535	0.426062
0.9	0.070634	0.139289	0.203845	0.261901	0.310642	0.346851	0.371996	0.373579
1.0	0.063880	0.125775	0.183579	0.234956	0.277267	0.307694	0.327288	0.328286
1.2	0.052247	0.102553	0.148899	0.189138	0.221027	0.242523	0.254485	0.254889
1.4	0.042733	0.083620	0.120780	0.152298	0.176336	0.191533	0.198885	0.199051
1.6	0.034951	0.068183	0.097979	0.122667	0.140792	0.151541	0.156087	0.156156
1.8	0.028587	0.055597	0.079488	0.098829	0.112497	0.120105	0.122932	0.122961
2.0	0.023381	0.045335	0.064492	0.079644	0.089954	0.095342	0.097108	0.097121
2.2	0.019123	0.036967	0.052329	0.064201	0.071979	0.075797	0.076905	0.076911
2.4	0.015641	0.030145	0.042463	0.051766	0.057635	0.060342	0.061040	0.061043
2.6	0.012793	0.024582	0.034460	0.041750	0.046179	0.048100	0.048541	0.048542
2.8	0.010463	0.020045	0.027968	0.033680	0.037024	0.038387	0.038667	0.038668
3.0	0.008558	0.016347	0.022700	0.027177	0.029702	0.030670	0.030848	0.030848
3.5	0.005178	0.009817	0.013477	0.015912	0.017164	0.017576	0.017634	0.017634
4.0	0.003132	0.005896	0.008005	0.009330	0.009951	0.010128	0.010147	0.010147
4.5	0.001895	0.003542	0.004756	0.005478	0.005787	0.005862	0.005869	0.005869
5.0	0.001147	0.002127	0.002828	0.003221	0.003374	0.003407	0.003409	0.003409
5.5	0.000694	0.001278	0.001682	0.001896	0.001972	0.001986	0.001987	0.001987
6.0	0.000420	0.000768	0.001001	0.001117	0.001155	0.001162	0.001162	0.001162
6.5	0.000254	0.000461	0.000596	0.000659	0.000678	0.000681	0.000681	0.000681
7.0	0.000154	0.000277	0.000355	0.000389	0.000399	0.000400	0.000400	0.000400
7.5	0.000093	0.000167	0.000211	0.000230	0.000235	0.000235	0.000235	0.000235
8.0	0.000056	0.000100	0.000126	0.000136	0.000139	0.000139	0.000139	0.000139
8.5	0.000034	0.000060	0.000075	0.000081	0.000082	0.000082	0.000082	0.000082
9.0	0.000021	0.000036	0.000045	0.000048	0.000048	0.000048	0.000048	0.000048
9.5	0.000012	0.000022	0.000027	0.000028	0.000029	0.000029	0.000029	0.000029
10.0	0.000008	0.000013	0.000016	0.000017	0.000017	0.000017	0.000017	0.000017
	$\left[\begin{smallmatrix} (-3)2 \\ 6 \end{smallmatrix} \right]$	$\left[\begin{smallmatrix} (-4)5 \\ 6 \end{smallmatrix} \right]$	$\left[\begin{smallmatrix} (-4)8 \\ 6 \end{smallmatrix} \right]$	$\left[\begin{smallmatrix} (-3)1 \\ 7 \end{smallmatrix} \right]$	$\left[\begin{smallmatrix} (-3)1 \\ 7 \end{smallmatrix} \right]$	$\left[\begin{smallmatrix} (-3)2 \\ 7 \end{smallmatrix} \right]$	$\left[\begin{smallmatrix} (-3)4 \\ 7 \end{smallmatrix} \right]$	$\left[\begin{smallmatrix} (-2)2 \\ 11 \end{smallmatrix} \right]$

$$27.5. f_m(x) = \int_0^\infty t^m e^{-t^2 - \frac{x}{t}} dt \text{ and}$$

Related Integrals

$$m=0, 1, 2 \dots$$

Differential Equations

$$27.5.1 \quad x f_m'' - (m-1) f_m' + 2 f_m = 0$$

$$27.5.2 \quad f_m' = -f_{m-1} \quad (m=1, 2, \dots)$$

Recurrence Relation

$$27.5.3 \quad 2 f_m = (m-1) f_{m-2} + x f_{m-3} \quad (m \geq 3)$$

Power Series Representations

$$27.5.4 \quad 2 f_1(x) = \sum_{k=0}^{\infty} (a_k \ln x + b_k) x^k$$

$$a_k = \frac{-2a_{k-2}}{k(k-1)(k-2)}$$

$$a_0 = a_1 = 0$$

$$b_0 = 1$$

$$b_k = \frac{-2b_{k-2} - (3k^2 - 6k + 2)a_k}{k(k-1)(k-2)}$$

$$a_2 = -b_0$$

$$b_1 = -\sqrt{\pi} \quad b_2 = \frac{3}{2}(1-\gamma)$$

(For γ , see chapter 6.)

27.5.5

$$2f_1(x) = 1 - \sqrt{\pi}x + .6342x^2 + .5908x^3 - .1431x^4 \\ - .01968x^5 + .00324x^6 + .000188x^7 \dots \\ - x^2 \ln x (1 - .08333x^2 + .001389x^4 - .0000083x^6 + \dots)$$

27.5.6

$$2f_2(x) = \frac{\sqrt{\pi}}{2} - x + \frac{\sqrt{\pi}}{2}x^2 - .3225x^3 - .1477x^4 + .03195x^5 \\ + .00328x^6 - .000491x^7 - .0000235x^8 \dots \\ + x^3 \ln x (\frac{1}{3} - .01667x^2 + .000198x^4 - \dots)$$

27.5.7

$$2f_3(x) = 1 - \frac{\sqrt{\pi}}{2}x + \frac{x^2}{2} - .2954x^3 + .1014x^4 + .02954x^5 \\ - .00578x^6 - .00047x^7 + .000064x^8 \dots \\ - x^4 \ln x (.0833 - .00278x^2 + .000025x^4 - \dots)$$

Asymptotic Representation

27.5.8

$$f_m(x) \sim \sqrt{\frac{\pi}{3}} 3^{-\frac{m}{2}} v^{\frac{m}{2}} e^{-v} \left(a_0 + \frac{a_1}{v} + \frac{a_2}{v^2} + \dots + \frac{a_k}{v^k} + \dots \right) \\ (x \rightarrow \infty)$$

$$v = 3 \left(\frac{x}{2} \right)^{2/3}$$

$$a_0 = 1, a_1 = \frac{1}{12} (3m^2 + 3m - 1)$$

$$12(k+2)a_{k+2} = -(12k^2 + 36k - 3m^2 - 3m + 25)a_{k+1} \\ + \frac{1}{2}(m-2k)(2k+3-m)(2k+3+2m)a_k \\ (k=0, 1, 2 \dots)$$

$$27.5.9 \quad g_1(x) + ig_2(x) = \int_0^\infty t^3 e^{-t^2 + it \frac{x}{i}} dt$$

27.5.10

$$g_1(x) = \mathcal{R}f_3(ix) \quad g_2(x) = -\mathcal{I}f_3(ix)$$

Asymptotic Representation

27.5.11

$$g_1(x) = \left(\frac{\pi}{3} \right)^{1/2} \frac{x}{2} \exp \left[-\frac{3}{2} \left(\frac{x}{2} \right)^{2/3} \right] (A \sin \theta + B \cos \theta)$$

27.5.12

$$g_2(x) = -\left(\frac{\pi}{3} \right)^{1/2} \frac{x}{2} \exp \left[-\frac{3}{2} \left(\frac{x}{2} \right)^{2/3} \right] (A \cos \theta - B \sin \theta)$$

$$\theta = \frac{3}{2} \sqrt{3} \left(\frac{x}{2} \right)^{2/3}$$

$$A \sim a_0 - a_3 \left(\frac{2}{x} \right)^2 + \frac{1}{2} \left[a_1 \left(\frac{2}{x} \right)^{2/3} - a_2 \left(\frac{2}{x} \right)^{4/3} \right. \\ \left. - a_4 \left(\frac{2}{x} \right)^{8/3} + a_5 \left(\frac{2}{x} \right)^{10/3} - \dots \right] \quad (x \rightarrow \infty)$$

$$B \sim \frac{\sqrt{3}}{2} \left[a_1 \left(\frac{2}{x} \right)^{2/3} + a_2 \left(\frac{2}{x} \right)^{4/3} - a_4 \left(\frac{2}{x} \right)^{8/3} \right. \\ \left. - a_5 \left(\frac{2}{x} \right)^{10/3} + \dots \right] \quad (x \rightarrow \infty)$$

$$a_0 = 1 \quad a_1 = .972222 \quad a_2 = .148534$$

$$a_3 = -.017879 \quad a_4 = .004594 \quad a_5 = -.000762$$

[27.7] M. Abramowitz, Evaluation of the integral $\int_0^\infty e^{-u^2 - x/u} du$, J. Math. Phys. **32**, 188-192 (1953).

[27.8] H. Faxén, Expansion in series of the integral $\int_y^\infty \exp[-x(t \pm t^{-n})] t^2 dt$, Ark. Mat., Astr., Fys. **15**, 13, 1-57 (1921).

[27.9] J. E. Kilpatrick and M. F. Kilpatrick, Discrete energy levels associated with the Lennard-Jones potential, J. Chem. Phys. **19**, 7, 930-933 (1951).

[27.10] U. E. Kruse and N. F. Ramsey, The integral $\int_0^\infty y^3 \exp\left(-y^2 + i \frac{x}{y}\right) dy$, J. Math. Phys. **30**, 40 (1951).

[27.11] O. Laporte, Absorption coefficients for thermal neutrons, Phys. Rev. **52**, 72-74 (1937).

[27.12] H. C. Torrey, Notes on intensities of radio frequency spectra, Phys. Rev. **59**, 293 (1941).

[27.13] C. T. Zahn, Absorption coefficients for thermal neutrons, Phys. Rev. **52**, 67-71 (1937).

$$\int_0^\infty y^n e^{-y - x/\sqrt{y}} dy \text{ for } n=0, \frac{1}{2}, 1; x=0(.01).1(.1)1.$$

$$f_m(x) = \int_0^{\infty} t^m e^{-t^2 - \frac{x}{t}} dt$$

Table 27.5

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	x	$f_1(x)$	$f_2(x)$	$f_3(x)$	x	$f_1(x)$	$f_2(x)$	$f_3(x)$
0.00	0.5000	0.4431	0.5000	0.1	0.4263	0.3970	0.4580	0.6	0.2255	0.2415	0.3025
0.01	0.4914	0.4382	0.4956	0.2	0.3697	0.3573	0.4204	0.7	0.2015	0.2202	0.2793
0.02	0.4832	0.4333	0.4912	0.3	0.3238	0.3227	0.3864	0.8	0.1807	0.2011	0.2584
0.03	0.4753	0.4285	0.4869	0.4	0.2855	0.2923	0.3557	0.9	0.1626	0.1839	0.2392
0.04	0.4676	0.4238	0.4826	0.5	0.2531	0.2654	0.3278	1.0	0.1466	0.1685	0.2215
0.05	0.4602	0.4191	0.4784								
$\left[\begin{smallmatrix} (-5)5 \\ 2 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-5)5 \\ 2 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-5)5 \\ 2 \end{smallmatrix} \right]$			$\left[\begin{smallmatrix} (-3)1 \\ 4 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-4)7 \\ 3 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-4)5 \\ 3 \end{smallmatrix} \right]$			$\left[\begin{smallmatrix} (-4)6 \\ 3 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-4)4 \\ 3 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-4)4 \\ 3 \end{smallmatrix} \right]$					

x	$\mathcal{R}_{f_3}(ix)$	$-\mathcal{I}_{f_3}(ix)$	x	$\mathcal{R}_{f_3}(ix)$	$-\mathcal{I}_{f_3}(ix)$	x	$\mathcal{R}_{f_3}(ix)$	$-\mathcal{I}_{f_3}(ix)$
0.0	0.50000	0.00000	4.0	-0.2626	0.0430	8.0	0.06078	-0.09808
0.2	0.49019	0.08754	4.2	-0.2552	+0.0094	8.5	0.07562	-0.07131
0.4	0.46229	0.16933	4.4	-0.2441	-0.0214	9.0	0.08221	-0.04496
0.6	0.41950	0.24139	4.6	-0.2299	-0.0490	9.5	0.08191	-0.02082
0.8	0.36543	0.30136	4.8	-0.2132	-0.0734	10.0	0.07626	-0.00010
1.0	0.30366	0.34805	5.0	-0.1945	-0.0944	10.5	0.06684	+0.01654
1.2	0.23746	0.38122	5.2	-0.1745	-0.1120	11.0	0.05507	0.02889
1.4	0.16972	0.40127	5.4	-0.1536	-0.1263	11.5	0.04224	0.03707
1.6	0.10288	0.40910	5.6	-0.1322	-0.1374	12.0	0.02937	0.04146
1.8	+0.03892	0.40592	5.8	-0.1108	-0.1455	12.5	0.01727	0.04259
2.0	-0.02062	0.39314	6.0	-0.0896	-0.1507	13.0	+0.00650	0.04109
2.2	-0.0746	0.3722	6.2	-0.0691	-0.1533	13.5	-0.00259	0.03758
2.4	-0.1221	0.3448	6.4	-0.0493	-0.1535	14.0	-0.00982	0.03268
2.6	-0.1629	0.3122	6.6	-0.0307	-0.1515	14.5	-0.01517	0.02696
2.8	-0.1966	0.2759	6.8	-0.0132	-0.1476	15.0	-0.01872	0.02089
3.0	-0.2233	0.2371	7.0	+0.00286	-0.14211	16.0	-0.02118	+0.00921
3.2	-0.2432	0.1971	7.2	0.01749	-0.13518	17.0	-0.01906	-0.00022
3.4	-0.2565	0.1569	7.4	0.03061	-0.12709	18.0	-0.01435	-0.00650
3.6	-0.2639	0.1173	7.6	0.04220	-0.11805	19.0	-0.00879	-0.00965
3.8	-0.2657	0.0792	7.8	0.05224	-0.10830	20.0	-0.00360	-0.01021

$$\left[\begin{smallmatrix} (-3)2 \\ 6 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-3)2 \\ 5 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4)5 \\ 3 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-4)4 \\ 4 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-3)1 \\ 5 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-4)7 \\ 5 \end{smallmatrix} \right]$$

Compiled from U. E. Kruse and N. F. Ramsey, The integral $\int_0^{\infty} y^3 \exp\left(-y^2 + i\frac{x}{y}\right) dy$, J. Math. Phys. 30, 40 (1951) (with permission).

$$27.6. f(x) = \int_0^{\infty} \frac{e^{-t^2}}{t+x} dt$$

Power Series Representation

27.6.1

$$f(x) = -e^{-x^2} \ln x + e^{-x^2} \left[\sqrt{\pi} \sum_{k=0}^{\infty} \frac{x^{2k+1}}{k!(2k+1)} - \sum_{k=1}^{\infty} \frac{x^{2k}}{k! 2k} \frac{\gamma}{2} \right]$$

27.6.2

$$= -e^{-x^2} \ln x + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k \psi(k+1) x^{2k}}{k!} + \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-2)^k x^{2k+1}}{1 \cdot 3 \cdot 5 \cdots (2k+1)}$$

(For γ and the digamma function $\psi(x)$, see chapter 6.)

Relation to the Exponential Integral

$$27.6.3 f(x) = -\frac{1}{2} e^{-x^2} \text{Ei}(x^2) + \sqrt{\pi} e^{-x^2} \int_0^x e^{t^2} dt$$

(For $\text{Ei}(x)$ see chapter 5; $e^{-x^2} \int_0^x e^{t^2} dt$, see chapter 7.

Asymptotic Representation

27.6.4

$$f(x) \sim \frac{\sqrt{\pi}}{2} \left[\frac{1}{x} + \frac{1}{2x^3} + \frac{1 \cdot 3}{4x^5} + \frac{1 \cdot 3 \cdot 5}{8x^7} + \cdots \right] - \frac{1}{2} \left[\frac{1}{x^2} + \frac{1}{x^4} + \frac{2!}{x^6} + \frac{3!}{x^8} + \cdots \right] \quad (x \rightarrow \infty)$$

[27.14] A. Erdélyi, Note on the paper "On a definite integral" by R. H. Ritchie, Math. Tables Aids Comp. 4, 31, 179 (1950).

[27.15] E. T. Goodwin and J. Staton, Table of $\int_0^{\infty} \frac{e^{-u^2}}{u+x} du$, Quart. J. Mech. Appl. Math. 1, 319 (1948). $x=0(.02)2(.05)3(.1)10$. Auxiliary function for $x=0(.01)1$.

[27.16] R. H. Ritchie, On a definite integral, Math. Tables Aids Comp. 4, 30, 75 (1950).

Table 27.6

$$f(x) = \int_0^{\infty} \frac{e^{-t^2}}{t+x} dt$$

x	$f(x) + \ln x$	x	$f(x) + \ln x$	x	$f(x)$	x	$f(x)$	x	$f(x)$
0.00	-0.2886	0.50	0.2704	1.0	0.6051	2.0	0.3543	3.0	0.2519
0.05	-0.2081	0.55	0.3100	1.1	0.5644	2.1	0.3404	3.5	0.2203
0.10	-0.1375	0.60	0.3479	1.2	0.5291	2.2	0.3276	4.0	0.1958
0.15	-0.0735	0.65	0.3842	1.3	0.4980	2.3	0.3157	4.5	0.1762
0.20	-0.0146	0.70	0.4192	1.4	0.4705	2.4	0.3046	5.0	0.1602
0.25	+0.0402	0.75	0.4529	1.5	0.4460	2.5	0.2944	5.5	0.1468
0.30	0.0915	0.80	0.4854	1.6	0.4239	2.6	0.2848	6.0	0.1356
0.35	0.1398	0.85	0.5168	1.7	0.4040	2.7	0.2758	6.5	0.1259
0.40	0.1856	0.90	0.5472	1.8	0.3860	2.8	0.2673	7.0	0.1175
0.45	0.2290	0.95	0.5766	1.9	0.3695	2.9	0.2594	7.5	0.1102
0.50	0.2704	1.00	0.6051	2.0	0.3543	3.0	0.2519	8.0	0.1037
$\left[\begin{smallmatrix} (-3)1 \\ 4 \end{smallmatrix} \right]$		$\left[\begin{smallmatrix} (-4)2 \\ 3 \end{smallmatrix} \right]$		$\left[\begin{smallmatrix} (-4)7 \\ 4 \end{smallmatrix} \right]$		$\left[\begin{smallmatrix} (-4)1 \\ 3 \end{smallmatrix} \right]$		$\left[\begin{smallmatrix} (-4)9 \\ 4 \end{smallmatrix} \right]$	

Compiled from E. T. Goodwin and J. Staton, Table of $\int_0^{\infty} \frac{e^{-u^2}}{u+x} du$, Quart. J. Mech. Appl. Math. 1, 319 (1948) (with permission).

27.7. Dilogarithm

(Spence's Integral for $n=2$)

$$27.7.1 \quad f(x) = -\int_1^x \frac{\ln t}{t-1} dt$$

Series Expansion

$$27.7.2 \quad f(x) = \sum_{k=1}^{\infty} (-1)^k \frac{(x-1)^k}{k^2} \quad (2 \geq x \geq 0)$$

Functional Relationships

27.7.3

$$f(x) + f(1-x) = -\ln x \ln(1-x) + \frac{\pi^2}{6} \quad (1 \geq x \geq 0)$$

27.7.4

$$f(1-x) + f(1+x) = \frac{1}{2} f(1-x^2) \quad (1 \geq x > 0)$$

$$27.7.5 \quad f(x) + f\left(\frac{1}{x}\right) = -\frac{1}{2} (\ln x)^2 \quad (0 \leq x \leq 1)$$

27.7.6

$$f(x+1) - f(x) = -\ln x \ln(x+1) - \frac{\pi^2}{12} - \frac{1}{2} f(x^2) \quad (2 \geq x \geq 0)$$

Relation to Debye Functions

$$27.7.7 \quad f(e^{-t}) = -f(e^t) - \frac{t^2}{2} = \int_0^t \frac{tdt}{e^t - 1}$$

[27.17] L. Lewin, Dilogarithms and associated functions (Macdonald, London, England, 1958).

[27.18] K. Mitchell, Tables of the function $\int_0^z \frac{-\log|1-y|}{y} dy$, with an account of some properties of this and related functions, Phil. Mag. 40, 351-368 (1949). $x = -1(.01)1$; $x = 0(.001).5$, 9D.

[27.19] E. O. Powell, An integral related to the radiation integrals, Phil. Mag. 7, 34, 600-607 (1943). $\int_1^x \frac{\log y}{y-1} dy$, $x = 0(.01)2(.02)6$, 7D.

[27.20] A. van Wijngaarden, Polylogarithms, by the Staff of the Computation Department, Report R24, Mathematisch Centrum, Amsterdam, Holland (1954). $F_n(z) = \sum_{h=1}^{\infty} h^{-n} z^h$ for $z = x = -1(.01)1$; $z = ix$, for $x = 0(.01)1$; $z = e^{i\alpha/2}$ for $\alpha = 0(.01)2$, 10D.

Dilogarithm

Table 27.7

$$f(x) = -\int_1^x \frac{\ln t}{t-1} dt$$

x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$
0.00	1.64493 4067	0.10	1.29971 4723	0.20	1.07479 4600	0.30	0.88937 7624	0.40	0.72758 6308
0.01	1.58862 5448	0.11	1.27452 9160	0.21	1.05485 9830	0.31	0.87229 1733	0.41	0.71239 5042
0.02	1.54579 9712	0.12	1.25008 7584	0.22	1.03527 7934	0.32	0.85542 7404	0.42	0.69736 1058
0.03	1.50789 9041	0.13	1.22632 0101	0.23	1.01603 0062	0.33	0.83877 6261	0.43	0.68247 9725
0.04	1.47312 5860	0.14	1.20316 7961	0.24	0.99709 9088	0.34	0.82233 0471	0.44	0.66774 6644
0.05	1.44063 3797	0.15	1.18058 1124	0.25	0.97846 9393	0.35	0.80608 2689	0.45	0.65315 7631
0.06	1.40992 8300	0.16	1.15851 6487	0.26	0.96012 6675	0.36	0.79002 6024	0.46	0.63870 8705
0.07	1.38068 5041	0.17	1.13693 6560	0.27	0.94205 7798	0.37	0.77415 3992	0.47	0.62439 6071
0.08	1.35267 5161	0.18	1.11580 8451	0.28	0.92425 0654	0.38	0.75846 0483	0.48	0.61021 6108
0.09	1.32572 8728	0.19	1.09510 3088	0.29	0.90669 4053	0.39	0.74293 9737	0.49	0.59616 5361
0.10	1.29971 4723	0.20	1.07479 4600	0.30	0.88937 7624	0.40	0.72758 6308	0.50	0.58224 0526
[(-3)2]		[(-4)1]		[(-5)5]		[(-5)3]		[(-5)2]	

From K. Mitchell, Tables of the function $\int_0^x \frac{-\log |1-y|}{y} dy$, with an account of some properties of this and related functions, Phil. Mag. 40, 351-368 (1949) (with permission).

27.8. Clausen's Integral and Related Summations

27.8.1

$$f(\theta) = -\int_0^\theta \ln \left(2 \sin \frac{t}{2} \right) dt = \sum_{k=1}^{\infty} \frac{\sin k\theta}{k^2} \quad (0 \leq \theta \leq \pi)$$

Series Representation

27.8.2

$$f(\theta) = -\theta \ln |\theta| + \theta + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k)!} B_{2k} \frac{\theta^{2k+1}}{2k(2k+1)} \quad \left(0 \leq \theta < \frac{\pi}{2} \right)$$

27.8.3

$$f(\pi - \theta) = \theta \ln 2 - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k)!} B_{2k} (2^{2k} - 1) \frac{\theta^{2k+1}}{2k(2k+1)} \quad (\pi/2 < \theta < \pi)$$

Functional Relationship

$$27.8.4 \quad f(\pi - \theta) = f(\theta) - \frac{1}{2} f(2\theta) \quad \left(0 \leq \theta \leq \frac{\pi}{2} \right)$$

Relation to Spence's Integral

27.8.5

$$if(\theta) = g(e^{i\theta}) + \frac{\theta^2}{4} \text{ where } g(x) = \int_1^x \frac{dt}{t} \ln |1+t|$$

Summable Series

27.8.6

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} = -\ln \left(2 \sin \frac{\theta}{2} \right) \quad (0 < \theta < 2\pi)$$

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2} = \frac{\pi^2}{6} - \frac{\pi\theta}{2} + \frac{\theta^2}{4} \quad (0 \leq \theta \leq 2\pi)$$

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n^4} = \frac{\pi^4}{90} - \frac{\pi^2\theta^2}{12} + \frac{\pi\theta^3}{12} - \frac{\theta^4}{48} \quad (0 \leq \theta \leq 2\pi)$$

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \frac{1}{2} (\pi - \theta) \quad (0 < \theta < 2\pi)$$

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n^3} = \frac{\pi^2\theta}{6} - \frac{\pi\theta^2}{4} + \frac{\theta^3}{12} \quad (0 \leq \theta \leq 2\pi)$$

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n^5} = \frac{\pi^4\theta}{90} - \frac{\pi^2\theta^3}{36} + \frac{\pi\theta^4}{48} - \frac{\theta^5}{240} \quad (0 \leq \theta \leq 2\pi)$$

[27.21] A. Ashour and A. Sabri, Tabulation of the function

$$\psi(\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}, \text{ Math. Tables Aids Comp. 10, 54, 57-65 (1956).}$$

[27.22] T. Clausen, Über die Zerlegung reeller gebrochener Funktionen, J. Reine Angew. Math. 8, 298-300 (1832). $x=0^\circ(1^\circ)180^\circ$, 16D.

[27.23] L. B. W. Jolley, Summation of series (Chapman Publishing Co., London, England, 1925).

[27.24] A. D. Wheelon, A short table of summable series, Report No. SM-14642, Douglas Aircraft Co., Inc., Santa Monica, Calif. (1953).

Table 27.8

Clausen's Integral

$$f(\theta) = -\int_0^\theta \ln \left(2 \sin \frac{t}{2} \right) dt$$

θ°	$f(\theta) + \theta \ln \theta$	θ°	$f(\theta)$	θ°	$f(\theta)$	θ°	$f(\theta)$	θ°	$f(\theta)$
0	0.000000	15	0.612906	30	0.864379	60	1.014942	90	0.915966
1	0.017453	16	0.635781	32	0.886253	62	1.014421	95	0.883872
2	0.034908	17	0.657571	34	0.906001	64	1.012886	100	0.848287
3	0.052362	18	0.678341	36	0.923755	66	1.010376	105	0.809505
4	0.069818	19	0.698149	38	0.939633	68	1.006928	110	0.767800
5	0.087276	20	0.717047	40	0.953741	70	1.002576	115	0.723427
6	0.104735	21	0.735080	42	0.966174	72	0.997355	120	0.676628
7	0.122199	22	0.752292	44	0.977020	74	0.991294	125	0.627629
8	0.139664	23	0.768719	46	0.986357	76	0.984425	130	0.576647
9	0.157133	24	0.784398	48	0.994258	78	0.976776	135	0.523889
10	0.174607	25	0.799360	50	1.000791	80	0.968375	140	0.469554
11	0.192084	26	0.813635	52	1.006016	82	0.959247	145	0.413831
12	0.209567	27	0.827249	54	1.009992	84	0.949419	150	0.356908
13	0.227055	28	0.840230	56	1.012773	86	0.938914	160	0.240176
14	0.244549	29	0.852599	58	1.014407	88	0.927755	170	0.120755
15	0.262049	30	0.864379	60	1.014942	90	0.915966	180	0.000000

$$\left[\begin{smallmatrix} (-7) & 8 \\ & 3 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4) & 1 \\ & 4 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4) & 3 \\ & 4 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4) & 1 \\ & 4 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4) & 4 \\ & 6 \end{smallmatrix} \right]$$

Compiled from A. Ashour and A. Sabri, Tabulation of the function $\psi(\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}$, Math. Tables Aids Comp. **10**, 54, 57-65 (1956) (with permission).

27.9. Vector-Addition Coefficients

(Wigner coefficients or Clebsch-Gordan coefficients)

Definition

27.9.1

$$(j_1 j_2 m_1 m_2 | j_1 j_2 j m) = \delta(m, m_1 + m_2) \cdot \sqrt{\frac{(j_1 + j_2 - j)!(j + j_1 - j_2)!(j + j_2 - j_1)!(2j + 1)}{(j + j_1 + j_2 + 1)!}}$$

$$\cdot \sum_k \frac{(-1)^k \sqrt{(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!(j + m)!(j - m)!}}{k!(j_1 + j_2 - j - k)!(j_1 - m_1 - k)!(j_2 + m_2 - k)!(j - j_2 + m_1 + k)!(j - j_1 - m_2 + k)!}$$

$$\delta(i, k) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$$

Conditions

27.9.2 $j_1, j_2, j = +n \text{ or } +\frac{n}{2} \quad (n = \text{integer})$

27.9.3 $j_1 + j_2 + j = n$

27.9.4 $j_1 + j_2 - j$
 27.9.5 $j_1 - j_2 + j$
 27.9.6 $-j_1 + j_2 + j$ $\left. \begin{matrix} \\ \\ \end{matrix} \right\} \geq 0$

27.9.7 $m_1, m_2, m = \pm n \text{ or } \pm \frac{n}{2}$

27.9.8 $|m_1| \leq j_1, |m_2| \leq j_2, |m| \leq j$

27.9.9 $(j_1 j_2 m_1 m_2 | j_1 j_2 j m) = 0 \quad m_1 + m_2 \neq m$

Special Values

27.9.10 $(j_1 0 m_1 0 | j_1 0 j m) = \delta(j_1, j) \delta(m_1, m)$

27.9.11 $(j_1 j_2 0 0 | j_1 j_2 j 0) = 0 \quad j_1 + j_2 + j = 2n + 1$

27.9.12 $(j_1 j_1 m_1 m_1 | j_1 j_1 j m) = 0 \quad 2j_1 + j = 2n + 1$

Symmetry Relations

27.9.13

$$(j_1 j_2 m_1 m_2 | j_1 j_2 j m)$$

$$= (-1)^{j_1+j_2-j} (j_1 j_2 - m_1 - m_2 | j_1 j_2 j - m)$$

27.9.14

$$= (j_2 j_1 - m_2 - m_1 | j_2 j_1 j - m)$$

27.9.15

$$= (-1)^{j_1+j_2-j} (j_2 j_1 m_1 m_2 | j_2 j_1 j m)$$

27.9.16

$$= \sqrt{\frac{2j+1}{2j_1+1}} (-1)^{j_2+m_2} (j j_2 - m m_2 | j j_2 j_1 - m_1)$$

27.9.17

$$= \sqrt{\frac{2j+1}{2j_1+1}} (-1)^{j_1-m_1+j-m} (j j_2 m - m_2 | j j_2 j_1 m_1)$$

27.9.18

$$= \sqrt{\frac{2j+1}{2j_1+1}} (-1)^{j-m+j_1-m_1} (j_2 j m_2 - m | j_2 j j_1 - m_1)$$

27.9.19

$$= \sqrt{\frac{2j+1}{2j_2+1}} (-1)^{j_1-m_1} (j_1 j m_1 - m | j_1 j j_2 - m_2)$$

27.9.20

$$= \sqrt{\frac{2j+1}{2j_2+1}} (-1)^{j_1-m_1} (j j_1 m - m_1 | j j_1 j_2 m_2)$$

$$(j_1 \frac{1}{2} m_1 m_2 | j_1 \frac{1}{2} j m)$$

Table 27.9.1

$j =$	$m_2 = \frac{1}{2}$	$m_2 = -\frac{1}{2}$
$j_1 + \frac{1}{2}$	$\sqrt{\frac{j_1 + m + \frac{1}{2}}{2j_1 + 1}}$	$\sqrt{\frac{j_1 - m + \frac{1}{2}}{2j_1 + 1}}$
$j_1 - \frac{1}{2}$	$-\sqrt{\frac{j_1 - m + \frac{1}{2}}{2j_1 + 1}}$	$\sqrt{\frac{j_1 + m + \frac{1}{2}}{2j_1 + 1}}$

$$(j_1 1 m_1 m_2 | j_1 1 j m)$$

Table 27.9.2

$j =$	$m_2 = 1$	$m_2 = 0$	$m_2 = -1$
$j_1 + 1$	$\sqrt{\frac{(j_1 + m)(j_1 + m + 1)}{(2j_1 + 1)(2j_1 + 2)}}$	$\sqrt{\frac{(j_1 - m + 1)(j_1 + m + 1)}{(2j_1 + 1)(j_1 + 1)}}$	$\sqrt{\frac{(j_1 - m)(j_1 - m + 1)}{(2j_1 + 1)(2j_1 + 2)}}$
j_1	$-\sqrt{\frac{(j_1 + m)(j_1 - m + 1)}{2j_1(j_1 + 1)}}$	$\frac{m}{\sqrt{j_1(j_1 + 1)}}$	$\sqrt{\frac{(j_1 - m)(j_1 + m + 1)}{2j_1(j_1 + 1)}}$
$j_1 - 1$	$\sqrt{\frac{(j_1 - m)(j_1 - m + 1)}{2j_1(2j_1 + 1)}}$	$-\sqrt{\frac{(j_1 - m)(j_1 + m)}{j_1(2j_1 + 1)}}$	$\sqrt{\frac{(j_1 + m + 1)(j_1 + m)}{2j_1(2j_1 + 1)}}$

Table 27.9.3

 $(j_1 \frac{1}{2} m_1 m_2 | j_1 \frac{1}{2} j m)$

$j =$	$m_2 = \frac{1}{2}$	$m_2 = \frac{1}{2}$
$j_1 + \frac{1}{2}$	$\sqrt{\frac{(j_1 + m - \frac{1}{2})(j_1 + m + \frac{1}{2})(j_1 + m + \frac{3}{2})}{(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)}}$	$\sqrt{\frac{3(j_1 + m + \frac{1}{2})(j_1 + m + \frac{3}{2})(j_1 - m + \frac{3}{2})}{(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)}}$
$j_1 + \frac{1}{2}$	$-\sqrt{\frac{3(j_1 + m - \frac{1}{2})(j_1 + m + \frac{1}{2})(j_1 - m + \frac{3}{2})}{2j_1(2j_1 + 1)(2j_1 + 3)}}$	$-(j_1 - 3m + \frac{3}{2})\sqrt{\frac{j_1 + m + \frac{1}{2}}{2j_1(2j_1 + 1)(2j_1 + 3)}}$
$j_1 - \frac{1}{2}$	$\sqrt{\frac{3(j_1 + m - \frac{1}{2})(j_1 - m + \frac{1}{2})(j_1 - m + \frac{3}{2})}{(2j_1 - 1)(2j_1 + 1)(2j_1 + 2)}}$	$-(j_1 + 3m - \frac{1}{2})\sqrt{\frac{j_1 - m + \frac{1}{2}}{(2j_1 - 1)(2j_1 + 1)(2j_1 + 2)}}$
$j_1 - \frac{3}{2}$	$-\sqrt{\frac{(j_1 - m - \frac{1}{2})(j_1 - m + \frac{1}{2})(j_1 - m + \frac{3}{2})}{2j_1(2j_1 - 1)(2j_1 + 1)}}$	$\sqrt{\frac{3(j_1 + m - \frac{1}{2})(j_1 - m - \frac{1}{2})(j_1 - m + \frac{1}{2})}{2j_1(2j_1 - 1)(2j_1 + 1)}}$
$j =$	$m_2 = -\frac{1}{2}$	$m_2 = -\frac{1}{2}$
$j_1 + \frac{3}{2}$	$\sqrt{\frac{3(j_1 + m + \frac{3}{2})(j_1 - m + \frac{1}{2})(j_1 - m + \frac{3}{2})}{(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)}}$	$\sqrt{\frac{(j_1 - m - \frac{1}{2})(j_1 - m + \frac{1}{2})(j_1 - m + \frac{3}{2})}{(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)}}$
$j_1 + \frac{1}{2}$	$(j_1 + 3m + \frac{3}{2})\sqrt{\frac{j_1 - m + \frac{1}{2}}{2j_1(2j_1 + 1)(2j_1 + 3)}}$	$\sqrt{\frac{3(j_1 + m + \frac{3}{2})(j_1 - m - \frac{1}{2})(j_1 - m + \frac{1}{2})}{2j_1(2j_1 + 1)(2j_1 + 3)}}$
$j_1 - \frac{1}{2}$	$-(j_1 - 3m - \frac{1}{2})\sqrt{\frac{j_1 + m + \frac{1}{2}}{(2j_1 - 1)(2j_1 + 1)(2j_1 + 2)}}$	$\sqrt{\frac{3(j_1 + m + \frac{1}{2})(j_1 + m + \frac{3}{2})(j_1 - m - \frac{1}{2})}{(2j_1 - 1)(2j_1 + 1)(2j_1 + 2)}}$
$j_1 - \frac{3}{2}$	$-\sqrt{\frac{3(j_1 + m - \frac{1}{2})(j_1 + m + \frac{1}{2})(j_1 - m - \frac{1}{2})}{2j_1(2j_1 - 1)(2j_1 + 1)}}$	$\sqrt{\frac{(j_1 + m - \frac{1}{2})(j_1 + m + \frac{1}{2})(j_1 + m + \frac{3}{2})}{2j_1(2j_1 - 1)(2j_1 + 1)}}$

Table 27.9.4

 $(j_1 \ 2 \ m_1 \ m_2 | j_1 \ 2 \ j \ m)$

$j =$	$m_2 = 2$	$m_2 = 1$	$m_2 = 0$
$j_1 + 2$	$\sqrt{\frac{(j_1 + m - 1)(j_1 + m)(j_1 + m + 1)(j_1 + m + 2)}{(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)(2j_1 + 4)}}$	$\sqrt{\frac{(j_1 - m + 2)(j_1 + m + 2)(j_1 + m + 1)(j_1 + m)}{(2j_1 + 1)(j_1 + 1)(2j_1 + 3)(j_1 + 2)}}$	$\sqrt{\frac{3(j_1 - m + 2)(j_1 - m + 1)(j_1 + m + 2)(j_1 + m + 1)}{(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)(j_1 + 2)}}$
$j_1 + 1$	$-\sqrt{\frac{(j_1 + m - 1)(j_1 + m)(j_1 + m + 1)(j_1 - m + 2)}{2j_1(j_1 + 1)(j_1 + 2)(2j_1 + 1)}}$	$-(j_1 - 2m + 2)\sqrt{\frac{(j_1 + m + 1)(j_1 + m)}{2j_1(2j_1 + 1)(j_1 + 1)(j_1 + 2)}}$	$m\sqrt{\frac{3(j_1 - m + 1)(j_1 + m + 1)}{j_1(2j_1 + 1)(j_1 + 1)(j_1 + 2)}}$
j_1	$\sqrt{\frac{3(j_1 + m - 1)(j_1 + m)(j_1 - m + 1)(j_1 - m + 2)}{(2j_1 - 1)2j_1(j_1 + 1)(2j_1 + 3)}}$	$(1 - 2m)\sqrt{\frac{3(j_1 - m + 1)(j_1 + m)}{(2j_1 - 1)j_1(2j_1 + 2)(2j_1 + 3)}}$	$\frac{3m^2 - j_1(j_1 + 1)}{\sqrt{(2j_1 - 1)j_1(j_1 + 1)(2j_1 + 3)}}$
$j_1 - 1$	$-\sqrt{\frac{(j_1 + m - 1)(j_1 - m)(j_1 - m + 1)(j_1 - m + 2)}{2(j_1 - 1)j_1(j_1 + 1)(2j_1 + 1)}}$	$(j_1 + 2m - 1)\sqrt{\frac{(j_1 - m + 1)(j_1 - m)}{(j_1 - 1)j_1(2j_1 + 1)(2j_1 + 2)}}$	$-m\sqrt{\frac{3(j_1 - m)(j_1 + m)}{(j_1 - 1)j_1(2j_1 + 1)(j_1 + 1)}}$
$j_1 - 2$	$\sqrt{\frac{(j_1 - m - 1)(j_1 - m)(j_1 - m + 1)(j_1 - m + 2)}{(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1)}}$	$-\sqrt{\frac{(j_1 - m + 1)(j_1 - m)(j_1 - m - 1)(j_1 + m - 1)}{(j_1 - 1)(2j_1 - 1)j_1(2j_1 + 1)}}$	$\sqrt{\frac{3(j_1 - m)(j_1 - m - 1)(j_1 + m)(j_1 + m - 1)}{(2j_1 - 2)(2j_1 - 1)j_1(2j_1 + 1)}}$
$j =$	$m_2 = -1$	$m_2 = -2$	
$j_1 + 2$	$\sqrt{\frac{(j_1 - m + 2)(j_1 - m + 1)(j_1 - m)(j_1 + m + 2)}{(2j_1 + 1)(j_1 + 1)(2j_1 + 3)(j_1 + 2)}}$	$\sqrt{\frac{(j_1 - m - 1)(j_1 - m)(j_1 - m + 1)(j_1 - m + 2)}{(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)(2j_1 + 4)}}$	
$j_1 + 1$	$(j_1 + 2m + 2)\sqrt{\frac{(j_1 - m + 1)(j_1 - m)}{j_1(2j_1 + 1)(2j_1 + 2)(j_1 + 2)}}$	$\sqrt{\frac{(j_1 - m - 1)(j_1 - m)(j_1 - m + 1)(j_1 + m + 2)}{j_1(2j_1 + 1)(j_1 + 1)(2j_1 + 4)}}$	
j_1	$(2m + 1)\sqrt{\frac{3(j_1 - m)(j_1 + m + 1)}{(2j_1 - 1)j_1(2j_1 + 2)(2j_1 + 3)}}$	$\sqrt{\frac{3(j_1 - m - 1)(j_1 - m)(j_1 + m + 1)(j_1 + m + 2)}{(2j_1 - 1)j_1(2j_1 + 2)(2j_1 + 3)}}$	
$j_1 - 1$	$-(j_1 - 2m - 1)\sqrt{\frac{(j_1 + m + 1)(j_1 + m)}{(j_1 - 1)j_1(2j_1 + 1)(2j_1 + 2)}}$	$\sqrt{\frac{(j_1 - m - 1)(j_1 + m)(j_1 + m + 1)(j_1 + m + 2)}{(j_1 - 1)j_1(2j_1 + 1)(2j_1 + 2)}}$	
$j_1 - 2$	$-\sqrt{\frac{(j_1 - m - 1)(j_1 + m + 1)(j_1 + m)(j_1 + m - 1)}{(j_1 - 1)(2j_1 - 1)j_1(2j_1 + 1)}}$	$\sqrt{\frac{(j_1 + m - 1)(j_1 + m)(j_1 + m + 1)(j_1 + m + 2)}{(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1)}}$	

Table 27.9.5 [By use of symmetry relations, coefficients may be put in standard form $j_1 \leq j_2 \leq j$ and $m \geq 0$]

m_2	m	j_1	j	$(j_1 j_2 m_1 m_2 j_1 j_2 j m)$	
$j_2 = \frac{1}{2}$					
$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\sqrt{\frac{1}{2}}$	0. 70711
$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\sqrt{\frac{1}{2}}$	0. 70711
$\frac{1}{2}$	1	$\frac{1}{2}$	1		1. 00000
$j_2 = 1$					
-1	0	1	1	$\sqrt{\frac{1}{2}}$	0. 70711
0	0	1	1		0. 00000
1	0	1	1	$-\sqrt{\frac{1}{2}}$	-0. 70711
0	1	1	1	$\sqrt{\frac{1}{2}}$	0. 70711
1	1	1	1	$-\sqrt{\frac{1}{2}}$	-0. 70711
0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\sqrt{\frac{3}{2}}$	0. 81650
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\sqrt{\frac{1}{2}}$	0. 57735
1	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$		1. 00000
-1	0	1	2	$\sqrt{\frac{1}{2}}$	0. 40825
0	0	1	2	$\sqrt{\frac{3}{2}}$	0. 81650
1	0	1	2	$\sqrt{\frac{1}{2}}$	0. 40825
0	1	1	2	$\sqrt{\frac{1}{2}}$	0. 70711
1	1	1	2	$\sqrt{\frac{1}{2}}$	0. 70711
1	2	1	2		1. 00000
$j_2 = \frac{3}{2}$					
$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$\sqrt{\frac{8}{15}}$	0. 73030
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$-\sqrt{\frac{1}{15}}$	-0. 25820
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$-\sqrt{\frac{2}{5}}$	-0. 63246
$\frac{1}{2}$	$\frac{3}{2}$	1	$\frac{3}{2}$	$\sqrt{\frac{3}{5}}$	0. 63246
$\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$-\sqrt{\frac{3}{5}}$	-0. 77460
$-\frac{1}{2}$	0	$\frac{1}{2}$	2	$\sqrt{\frac{1}{2}}$	0. 70711
$\frac{1}{2}$	0	$\frac{1}{2}$	2	$\sqrt{\frac{1}{2}}$	0. 70711
$\frac{1}{2}$	1	$\frac{1}{2}$	2	$\frac{1}{2}\sqrt{3}$	0. 86603
$\frac{3}{2}$	1	$\frac{1}{2}$	2		0. 50000
$\frac{3}{2}$	2	$\frac{1}{2}$	2		1. 00000
$-\frac{1}{2}$	0	$\frac{3}{2}$	2		0. 50000
$-\frac{1}{2}$	0	$\frac{3}{2}$	2		0. 50000
$\frac{1}{2}$	0	$\frac{3}{2}$	2		-0. 50000
$\frac{1}{2}$	0	$\frac{3}{2}$	2		-0. 50000
$-\frac{1}{2}$	1	$\frac{3}{2}$	2	$\sqrt{\frac{1}{2}}$	0. 70711
$\frac{1}{2}$	1	$\frac{3}{2}$	2		0. 00000
$\frac{1}{2}$	1	$\frac{3}{2}$	2	$-\sqrt{\frac{1}{2}}$	-0. 70711
$\frac{1}{2}$	2	$\frac{3}{2}$	2	$\sqrt{\frac{1}{2}}$	0. 70711
$\frac{3}{2}$	2	$\frac{3}{2}$	2	$-\sqrt{\frac{1}{2}}$	-0. 70711
$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{5}{2}$	$\sqrt{\frac{3}{10}}$	0. 54772
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{5}{2}$	$\sqrt{\frac{5}{6}}$	0. 77460
$\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{5}{2}$	$\sqrt{\frac{1}{10}}$	0. 31623
$\frac{1}{2}$	$\frac{3}{2}$	1	$\frac{5}{2}$	$\sqrt{\frac{5}{6}}$	0. 77460
$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{5}{2}$	$\sqrt{\frac{5}{6}}$	0. 63246
$\frac{3}{2}$	$\frac{5}{2}$	1	$\frac{5}{2}$		1. 00000

Compiled from A. Simon, Numerical tables of the Clebsch-Gordan coefficients, Oak Ridge National Laboratory Report 1718, Oak Ridge, Tenn. (1954) (with permission).

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 $C(j_1 j_2 j; m_1 m_2 m)$ for all angular moments $< \frac{1}{2}$, 10D.

28. Scales of Notation

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28. Scales of Notation

Representation of Numbers

Any positive real number x can be uniquely represented in the scale of some integer $b > 1$ as

$$x = (A_m \dots A_1 A_0 \cdot a_{-1} a_{-2} \dots)_{(b)},$$

where every A_i and a_{-j} is one of the integers $0, 1, \dots, b-1$, not all A_i, a_{-j} are zero, and $A_m > 0$ if $x \geq 1$. There is a one-to-one correspondence between the number and the sequence

$$x = A_m b^m + \dots + A_1 b + A_0 + \sum_{j=1}^{\infty} a_{-j} b^{-j}$$

where the infinite series converges. The integer b is called the base or radix of the scale.

The sequence for x in the scale of b may terminate, i.e., $a_{-n-1} = a_{-n-2} = \dots = 0$ for some $n \geq 1$ so that

$$x = (A_m \dots A_1 A_0 \cdot a_{-1} a_{-2} \dots a_{-n})_{(b)};$$

then x is said to be a finite b -adic number.

A sequence which does not terminate may have the property that the infinite sequence $a_{-1}, a_{-2}, \dots$ becomes periodic from a certain digit $a_{-n} (n \geq 1)$ on; according as $n=1$ or $n>1$ the sequence is then said to be pure or mixed recurring.

A sequence which neither terminates nor recurs represents an irrational number.

Names of Scales

Base	Scale	Base	Scale
2	Binary	8	Octal
3	Ternary	9	Nonary
4	Quaternary	10	Decimal
5	Quinary	11	Undenary
6	Senary	12	Duodenary
7	Septenary	16	Hexadecimal

General Conversion Methods

Any number can be converted from the scale of b to the scale of some integer $\bar{b} \neq b, \bar{b} > 1$, by using arithmetic operations in either the b -scale or the $\bar{b}$ -scale. Accordingly, there are four methods of conversion, depending on whether the number to be converted is an integer or a proper fraction.

Integers $X = (A_m \dots A_1 A_0)_{(b)}$

(I) $\bar{b}$ -scale arithmetic. Convert $\bar{b}$ to the b -scale and define

$$X/\bar{b} = X_1 + \bar{A}'_0/\bar{b},$$

$$X_1/\bar{b} = X_2 + \bar{A}'_1/\bar{b},$$

.

.

$$X_m/\bar{b} = 0 + \bar{A}'_m/\bar{b},$$

where $\bar{A}'_0, \bar{A}'_1, \dots, \bar{A}'_m$ are the remainders and $X_1, X_2, \dots, X_m$ the quotients (in the b -scale) where $X, X_1, \dots, X_{m-1}$, respectively are divided by $\bar{b}$ in the b -scale. Then convert the remainders to the $\bar{b}$ -scale,

$$(\bar{A}'_0)_{(\bar{b})} = \bar{A}_0, (\bar{A}'_1)_{(\bar{b})} = \bar{A}_1, \dots, (\bar{A}'_m)_{(\bar{b})} = \bar{A}_m$$

and obtain

$$X = (\bar{A}_m \dots \bar{A}_1 \bar{A}_0)_{(\bar{b})}.$$

(II) $\bar{b}$ -scale arithmetic. Convert b and $A_0, A_1, \dots, A_m$ to the $\bar{b}$ -scale and define, using arithmetic operations in the $\bar{b}$ -scale,

$$X_{m-1} = A_m b + A_{m-1},$$

$$X_{m-2} = X_{m-1} b + A_{m-2},$$

$$X_1 = X_2 b + A_1,$$

then

$$X = X_1 b + A_0.$$

Proper fractions $x = (0.a_{-1}a_{-2}\dots)_{(b)}$

To convert a proper fraction x , given to n digits in the b -scale, to the scale of $\bar{b} \neq b$ such that inverse conversion from the $\bar{b}$ -scale may yield the same n rounded digits in the b -scale, the representation of x in the $\bar{b}$ -scale must be obtained to $\bar{n}$ rounded digits where $\bar{n}$ satisfies $\bar{b}^{\bar{n}} > b^n$.

(III) $\bar{b}$ -scale arithmetic. Convert $\bar{b}$ to the b -scale and define

$$x\bar{b} = x_1 + \bar{a}'_{-1}$$

$$x_1\bar{b} = x_2 + \bar{a}'_{-2}$$

$$x_{\bar{n}-1}\bar{b} = x_{\bar{n}} + \bar{a}'_{-\bar{n}}$$

where $\bar{a}_{-1}, \bar{a}'_{-2}, \dots, \bar{a}_{-n}$ are the integral parts and $x_1, x_2, \dots, x_n$ the fractional parts (in the b -scale) of the products $x\bar{b}, x\bar{b}, \dots, x_{n-1}\bar{b}$, respectively. Then convert the integral parts to the $\bar{b}$ -scale,

$$(\bar{a}'_{-1})_{(\bar{b})} = \bar{a}_{-1}, (\bar{a}'_{-2})_{(\bar{b})} = \bar{a}_{-2}, \dots, (\bar{a}'_{-n})_{(\bar{b})} = \bar{a}_{-n},$$

and obtain

$$x = (0.\bar{a}_{-1}\bar{a}_{-2} \dots \bar{a}_{-n})_{(\bar{b})}.$$

(IV) $\bar{b}$ -scale arithmetic. Convert b and $a_{-1}, a_{-2}, \dots, a_{-n}$ to the $\bar{b}$ -scale and define, using arithmetic operations in the $\bar{b}$ -scale,

$$x_{-n+1} = a_{-n}/b + a_{-n+1},$$

$$x_{-n+2} = x_{-n+1}/b + a_{-n+2},$$

$$x_{-1} = x_2/b + a_{-1};$$

then

$$x = x_{-1}/b.$$

Numerical Methods

The examples are restricted to the scales of 2, 8, 10 because of their importance to electronic computers.

Note that the octal scale is a power of the binary scale. In fact, an octal digit corresponds to a triplet of binary digits. Then, binary arithmetic may be used whenever a number either is to be converted to the octal scale or is given in the octal scale and is to be converted to some other scale.

Decimal	1	2	3	4	5	6	7	8	9	10
Octal	1	2	3	4	5	6	7	10	11	12
Binary	1	10	11	100	101	110	111	1 000	1 001	1 010

Example 1. Convert $X = (1369)_{(10)}$ to the octal scale. By (I) we have $b = 10$, $\bar{b} = 8_{(10)}$ and so, using decimal arithmetic,

$$1369/8 = 171 + 1/8,$$

$$171/8 = 21 + 3/8,$$

$$21/8 = 2 + 5/8,$$

$$2/8 = 0 + 2/8;$$

then

$$X = (2531)_{(8)}.$$

By (II) we have $b = (12)_{(8)}$ and $A_3 = 1_{(8)}$, $A_2 = 3_{(8)}$, $A_1 = 6_{(8)}$, $A_0 = (11)_{(8)}$. Hence, using octal arithmetic,

$$X_2 = 1 \cdot 12 + 3 = (15)_{(8)},$$

$$X_1 = 15 \cdot 12 + 6 = (210)_{(8)},$$

$$X = 210 \cdot 12 + 11 = (2531)_{(8)}.$$

Using binary arithmetic we have, by (II), $b = (1010)_{(2)}$ and $A_3 = 1_{(2)}$, $A_2 = (11)_{(2)}$, $A_1 = (110)_{(2)}$, $A_0 = (1001)_{(2)}$. Thus

$$X_2 = 1 \cdot 1010 + 11 = (1101)_{(2)},$$

$$X_1 = 1101 \cdot 1010 + 110 = (10 001 000)_{(2)},$$

$$X = 10 001 000 \cdot 1010 + 1001 = (10 101 011 001)_{(2)},$$

whence, on converting to the octal scale,

$$X = (2531)_{(8)}.$$

Example 2. Convert $X = (2531)_{(8)}$ to the decimal scale. By (I) we have $\bar{b} = 10 = (12)_{(8)}$ and hence, using octal arithmetic,

$$2531/12 = 210 + 11/12$$

$$210/12 = 15 + 6/12$$

$$15/12 = 1 + 3/12$$

$$1/12 = 0 + 1/12$$

Thus, converting to the decimal scale,

$$\bar{A}_0 = (11)_{(8)} = 9, \bar{A}_1 = 6_{(8)} = 6, \bar{A}_2 = 3_{(8)} = 3, \bar{A}_3 = 1,$$

and so

$$X = (1369)_{(10)}.$$

By (II) we have $\bar{b} = 10$, and the octal digits of X are unchanged in the decimal scale. Hence, using decimal arithmetic,

$$X_2 = 2 \cdot 8 + 5 = (21)_{(10)},$$

$$X_1 = 21 \cdot 8 + 3 = (171)_{(10)},$$

$$X = 171 \cdot 8 + 1 = (1369)_{(10)}.$$

Using binary arithmetic we have, by (II), $b = 8 = (1000)_{(2)}$ and $A_0 = 1$, $A_1 = (11)_{(2)}$, $A_2 = (101)_{(2)}$, $A_3 = (10)_{(2)}$. Then,

$$X_2 = 10 \cdot 1000 + 101 = (10 101)_{(2)},$$

$$X_1 = 10 101 \cdot 1000 + 11 = (10 101 011)_{(2)},$$

$$X = 10 101 011 \cdot 1000 + 1 = (10 101 011 001)_{(2)},$$

whence, on converting to the decimal scale,

$$X = (1369)_{(10)}.$$

Observe that in both examples above, octal arithmetic is used as an intermediate step to convert, according to (II), the given number to the binary scale. If, instead, the given number is first converted to the binary scale, then binary arithmetic may be applied directly to convert, according to (I), the given number from the binary scale to the scale desired.

For example, in converting $X=(2531)_{(8)}$ to the decimal scale, we find first $X=(10101011001)_{(2)}$ and then obtain, using (I) with $\bar{b}=10=(1010)_{(2)}$,

$$\begin{aligned} 10 \ 101 \ 011 \ 001/1010 &= 10 \ 001 \ 000 + 1001/1010, \\ 10 \ 001 \ 000/1010 &= 1101 + 110/1010, \\ 1101/1010 &= 1 + 11/1010, \\ 1/1010 &= 0 + 1/1010. \end{aligned}$$

Thus, on converting to the decimal scale,

$$\begin{aligned} A_0 &= (1001)_{(2)} = 9, \quad A_1 = (110)_{(2)} = 6, \\ A_2 &= (11)_{(2)} = 3, \quad A_3 = 1, \\ \text{whence} \quad X &= (1369)_{(10)}. \end{aligned}$$

Example 3. Convert $x=(0.355)_{(10)}$ to the binary scale.

We first convert to the octal scale, using decimal arithmetic. By (III), we find with $\bar{b}=8$

$$\begin{aligned} (0.355) \cdot 8 &= 2 + 0.840, & (0.080) \cdot 8 &= 0 + 0.640 \\ (0.840) \cdot 8 &= 6 + 0.720, & (0.640) \cdot 8 &= 5 + 0.120 \\ (0.720) \cdot 8 &= 5 + 0.760, & (0.120) \cdot 8 &= 0 + 0.960 \\ (0.760) \cdot 8 &= 6 + 0.080, & (0.960) \cdot 8 &= 7 + 0.680 \end{aligned}$$

whence $x=(0.26560507 \dots)_{(8)}$. Thus, on converting to the binary scale,

$$x=(0.010 \ 110 \ 101 \ 110 \ 000 \ 101 \ 000 \ 111 \ \dots)_{(2)}.$$

In order that inverse conversion of x from the binary to the decimal scale yield again x to the given number n of decimal digits, we must round x in the binary scale to at least $\bar{n}$ digits where $\bar{n}$ is chosen such that $2^{\bar{n}} > 10^n$. As a working rule, we may take $\bar{n} \geq \frac{10}{3}n$. Hence, to obtain $x=(0.355)_{(10)}$ by inverse conversion, x must be rounded in the binary scale to $\bar{n} \geq \frac{10}{3}3 = 10$ digits.

Thus,

$$x=(0.010 \ 110 \ 110 \ 0)_{(2)}.$$

To carry out the inverse conversion we can first convert to the octal scale,

$$x=(0.266)_{(8)},$$

and then apply (IV) with $b=8$, using decimal arithmetic:

$$\begin{aligned} x_{-2} &= 6/8 + 6 = 6.75, \\ x_{-1} &= 6.75/8 + 2 = 2.84375, \\ x &= 2.84375/8 = 0.355 \ 46875. \end{aligned}$$

Alternatively, we can apply (III) with $\bar{b}=(1010)_{(2)}$, using binary arithmetic:

$$\begin{aligned} (0.010 \ 110 \ 11) \cdot 1010 &= 11 + (0.100 \ 011 \ 1), \\ (0.100 \ 011 \ 1) \cdot 1010 &= 101 + (0.100 \ 011), \\ (0.100 \ 011) \cdot 1010 &= 101 + (0.011 \ 11), \\ (0.011 \ 11) \cdot 1010 &= 100 + (0.101 \ 1). \end{aligned}$$

Converting the integral parts to the decimal scale, we find

$$\begin{aligned} \bar{a}_{-1} &= (11)_{(2)} = 3, \quad \bar{a}_{-2} = \bar{a}_{-3} = (101)_{(2)} = 5, \\ \bar{a}_{-4} &= (100)_{(2)} = 4, \end{aligned}$$

and thus

$$x=(0.3554)_{(10)}$$

Note that the fractional part in any step is the unconverted remainder. Thus, to round at any step, it is only necessary to ascertain whether the unconverted portion to be neglected is greater or less than $\frac{1}{2}$; i.e., whether, in the binary scale, the first neglected digit is 1 or 0.

Example 4. Convert $x=(3.141593)_{(10)} \cdot 10^{-9}$ to the binary scale.

The desired representation is

$$x=(1.a_{-1}a_{-2} \dots a_{-n})_{(2)} \cdot 2^{-k}$$

where n and k are such that inverse conversion from the binary scale to the decimal scale will produce x to the same given 15 decimal digits. Accordingly, by the rule stated in **Example 3**, n and k are to be chosen so as to satisfy $n+k \geq \frac{10}{3} \cdot 15 = 50$.

From **Table 28.1** we find

$$2^{-29} < (3.141593)_{(10)} \cdot 10^{-9} < 2^{-28}$$

Thus, we must take $k=29$ and, consequently, choose $n > 21$. The conversion on a desk calculator thus proceeds as follows. First, we obtain by use of **Table 28.1**

$$2^{29}x=(1.686 \ 629 \ 899)_{(10)}$$

Then, for convenience's sake, we convert this number to the octal scale, using the method of **Example 3** and rounding as required, to at least 7 octal (=21 binary) digits. We find

$$2^{29}x=(1.537 \ 4337)_{(8)}.$$

Hence

$$x=(1.537 \ 433 \ 7)_{(8)} \cdot 2^{-29}$$

and, consequently,

$$x=(1. \ 101 \ 011 \ 111 \ 100 \ 011 \ 011 \ 111)_{(2)} \cdot 2^{-29}.$$

To convert x back to the decimal scale we only need to obtain from **Table 28.1** the various powers of 2 which appear in the above representation and sum them. However, since $2^{-m} = 2^{-m+1} - 2^{-m}$ for any real constant m , it is more convenient to reduce first the binary representation of x to the form

$$x = 2^{-28} - 2^{-31} - 2^{-33} - 2^{-39} + 2^{-42} - 2^{-45} - 2^{-50}$$

and then sum these powers of 2. (Note that the number of summands is thereby decreased from 16 to 7.) From **Table 28.1** we have

$+2^{-28} = +$	3.725	290	298	$\cdot 10^{-9}$
$-2^{-31} = -$	.465	661	287	$\cdot 10^{-9}$
$-2^{-33} = -$	.116	415	322	$\cdot 10^{-9}$
$-2^{-39} = -$	.001	818	989	$\cdot 10^{-9}$
$+2^{-42} = +$	.000	227	374	$\cdot 10^{-9}$
$-2^{-45} = -$	.000	028	422	$\cdot 10^{-9}$
$-2^{-50} = -$	.000	000	888	$\cdot 10^{-9}$
$x =$	3.141	592	764	$\cdot 10^{-9}$

Nine decimal digits are used for sufficient accuracy reserve. Hence, rounding to seven significant figures, we find

$$x = (3.141593)_{(10)} \cdot 10^{-9}.$$

To convert a number such as

$$x = (\xi)_{(10)} \cdot 10^k$$

to the binary scale, where k is a positive integer so large that **Table 28.1** cannot be used, apply the following device: Compute

$$\log_2 x = \frac{\log_{10} x}{\log_{10} 2} = k + \frac{x_1}{\log_{10} 2}$$

where k is the quotient and x_1 the remainder, the division being carried out in the decimal scale. Then find $\eta = 10^{x_1}$, i.e., $x_1 = \log_{10} \eta$, so that

$$\log_2 x = k + \frac{\log_{10} \eta}{\log_{10} 2} = k + \log_2 \eta$$

whence

$$x = (\eta)_{(10)} 2^k.$$

Now convert $(\eta)_{(10)}$ to the binary scale by any of the methods described above.

A similar device may be used to convert to the decimal scale a binary number that is outside the range of **Table 28.1**.

Example 5. Convert $x = (2.773)_{(10)} \cdot 10^{83}$ to the binary scale.

We first compute, using **4.1.19** and **Table 4.1**,

$$\log_2 x = \frac{\log_{10} x}{\log_{10} 2} = \frac{83.44295}{.30103} = 277 + \frac{.05764}{.30103},$$

and find from **Table 4.1**, $.05764 = \log_{10} 1.1419$. Hence

$$\log_2 x = 277 + \frac{\log_{10} 1.1419}{\log_{10} 2} = 277 + \log_2 1.1419$$

and so

$$x = (1.1419)_{(10)} \cdot 2^{277}.$$

Now we apply the methods of **Example 3** to obtain $(1.1419)_{(10)} = (1.110516)_{(8)}$ where octal notation is used for the sake of convenience.

To round such that inverse conversion will yield the same decimal digits of x , observe that the last non-zero decimal digit of x is $3 \cdot 10^{80}$. **Table 28.4** shows that $2^{265} < 10^{80} < 2^{266}$. Hence, in the binary scale, x must be a binary integer times 2^{265} ; i.e., $(1.110516)_{(8)}$ must be rounded to 4 octal (=12 binary) digits. As a result,

$$\begin{aligned} x &= (1.1105)_{(8)} \cdot 2^{277} = (11105)_{(8)} \cdot 2^{265} \\ &= (1\ 001\ 001\ 000\ 101)_{(2)} 2^{265} \end{aligned}$$

Conversion back to the decimal scale proceeds as follows, we write

$$\begin{aligned} \log_{10} x &= \log_{10} 2 \log_2 x \\ &= \log_{10} 2 \{ 265 + \log_2 (11105)_{(8)} \} \\ &= \log_{10} 2 \left\{ 265 + \frac{\log_{10} (11105)_{(8)}}{\log_{10} 2} \right\} \\ &= 265 \log_{10} 2 + \log_{10} (11105)_{(8)}. \end{aligned}$$

Hence, converting $(11105)_{(8)}$ to the decimal scale by any of the methods of **Example 2**, we obtain

$$\log_{10} x = 265 \log_{10} 2 + \log_{10} 4677$$

which yields, using **Table 4.1**

$$\log_{10} x = 83.44292$$

Thus, by **Table 4.1**, we find, rounded to four significant figures,

$$x = (2.773)_{(10)} \cdot 10^{83}.$$

References

- [28.1] J. Malengreau, *Étude des écritures binaires*, Bibliothèque Sci. 32 Mathématique. Édition Griffon, Neuchâtel, Suisse (1958).
- [28.2] D. D. McCracken, *Digital computer programming* (John Wiley & Sons, Inc., New York, N.Y., 1957).
- [28.3] R. K. Richards, *Arithmetic operation in digital computers* (D. Van Nostrand Co., Inc., New York, N.Y., 1955).

29. Laplace Transforms

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29. Laplace Transforms

29.1. Definition of the Laplace Transform

One-dimensional Laplace Transform

$$29.1.1 \quad f(s) = \mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t) dt$$

$F(t)$ is a function of the real variable t and s is a complex variable. $F(t)$ is called the original function and $f(s)$ is called the image function. If the integral in 29.1.1 converges for a real $s=s_0$, i.e.,

$$\lim_{\substack{A \rightarrow 0 \\ B \rightarrow \infty}} \int_A^B e^{-s_0 t} F(t) dt$$

exists, then it converges for all s with $\Re s > s_0$, and the image function is a single valued analytic

function of s in the half-plane $\Re s > s_0$.

Two-dimensional Laplace Transform

29.1.2

$$f(u, v) = \mathcal{L}\{F(x, y)\} = \int_0^{\infty} \int_0^{\infty} e^{-ux-vy} F(x, y) dx dy$$

Definition of the Unit Step Function

$$29.1.3 \quad u(t) = \begin{cases} 0 & (t < 0) \\ \frac{1}{2} & (t = 0) \\ 1 & (t > 0) \end{cases}$$

In the following tables the factor $u(t)$ is to be understood as multiplying the original function $F(t)$.

29.2. Operations for the Laplace Transform¹

	Original Function $F(t)$	Image Function $f(s)$
29.2.1	$F(t)$	$\int_0^{\infty} e^{-st} F(t) dt$
	Inversion Formula	
29.2.2	$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts} f(s) ds$	$f(s)$
	Linearity Property	
29.2.3	$AF(t) + BG(t)$	$Af(s) + Bg(s)$
	Differentiation	
29.2.4	$F'(t)$	$sf(s) - F(+0)$
29.2.5	$F^{(n)}(t)$	$s^n f(s) - s^{n-1} F(+0) - s^{n-2} F'(+0) - \dots - F^{(n-1)}(+0)$
	Integration	
29.2.6	$\int_0^t F(\tau) d\tau$	$\frac{1}{s} f(s)$
29.2.7	$\int_0^t \int_0^{\tau} F(\lambda) d\lambda d\tau$	$\frac{1}{s^2} f(s)$
	Convolution (Faltung) Theorem	
29.2.8	$\int_0^t F_1(t-\tau) F_2(\tau) d\tau = F_1 * F_2$	$f_1(s) f_2(s)$
		Differentiation
29.2.9	$-tF(t)$	$f'(s)$
29.2.10	$(-1)^n t^n F(t)$	$f^{(n)}(s)$

¹ Adapted by permission from R. V. Churchill, Operational mathematics, 2d ed., McGraw-Hill Book Co., Inc., New York, N.Y., 1958.

<i>Original Function $F(t)$</i>		<i>Image Function $f(s)$</i>
		Integration
29.2.11	$\frac{1}{t} F(t)$	$\int_s^\infty f(x) dx$
		Linear Transformation
29.2.12	$e^{at} F(t)$	$f(s-a)$
29.2.13	$\frac{1}{c} F\left(\frac{t}{c}\right) \quad (c > 0)$	$f(cs)$
29.2.14	$\frac{1}{c} e^{(b/c)t} F\left(\frac{t}{c}\right) \quad (c > 0)$	$f(cs-b)$
		Translation
29.2.15	$F(t-b)u(t-b) \quad (b > 0)$	$e^{-bs}f(s)$
		Periodic Functions
29.2.16	$F(t+a)=F(t)$	$\frac{\int_0^a e^{-st} F(t) dt}{1-e^{-as}}$
29.2.17	$F(t+a)=-F(t)$	$\frac{\int_0^a e^{-st} F(t) dt}{1+e^{-as}}$
		Half-Wave Rectification of $F(t)$ in 29.2.17
29.2.18	$F(t) \sum_{n=0}^{\infty} (-1)^n u(t-na)$	$\frac{f(s)}{1-e^{-as}}$
		Full-Wave Rectification of $F(t)$ in 29.2.17
29.2.19	$ F(t) $	$f(s) \coth \frac{as}{2}$
		Heaviside Expansion Theorem
29.2.20	$\sum_{n=1}^m \frac{p(a_n)}{q'(a_n)} e^{a_n t}$	$\frac{p(s)}{q(s)}, q(s) = (s-a_1)(s-a_2) \dots (s-a_m)$
		$p(s)$ a polynomial of degree $< m$
29.2.21	$e^{at} \sum_{n=1}^r \frac{p^{(r-n)}(a)}{(r-n)!} \frac{t^{n-1}}{(n-1)!}$	$\frac{p(s)}{(s-a)^r}$
		$p(s)$ a polynomial of degree $< r$

29.3. Table of Laplace Transforms^{2, 3}

For a comprehensive table of Laplace and other integral transforms see [29.9]. For a table of two-dimensional Laplace transforms see [29.11].

	$f(s)$	$F(t)$
29.3.1	$\frac{1}{s}$	1
29.3.2	$\frac{1}{s^2}$	t

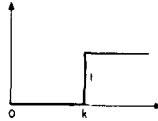
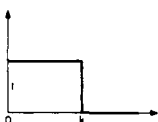
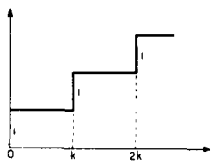
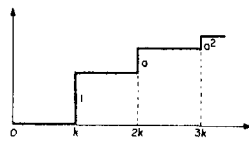
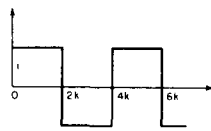
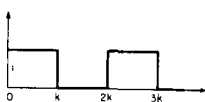
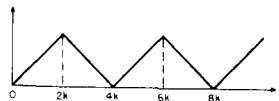
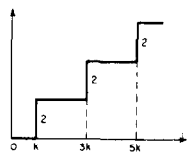
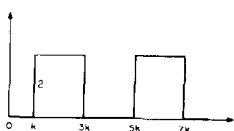
² The numbers in bold type in the $f(s)$ and $F(t)$ columns indicate the chapters in which the properties of the respective higher mathematical functions are given.

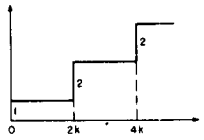
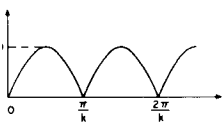
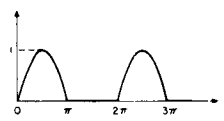
³ Adapted by permission from R. V. Churchill, Operational mathematics, 2d. ed., McGraw-Hill Book Co., Inc., New York, N. Y., 1958.

	$f(s)$		$F(t)$
29.3.3	$\frac{1}{s^n} \quad (n=1, 2, 3, \dots)$		$\frac{t^{n-1}}{(n-1)!}$
29.3.4	$\frac{1}{\sqrt{s}}$		$\frac{1}{\sqrt{\pi t}}$
29.3.5	$s^{-3/2}$		$2\sqrt{t/\pi}$
29.3.6	$s^{-(n+\frac{1}{2})} \quad (n=1, 2, 3, \dots)$		$\frac{2^n t^{n-\frac{1}{2}}}{1 \cdot 3 \cdot 5 \dots (2n-1) \sqrt{\pi}}$
29.3.7	$\frac{\Gamma(k)}{s^k} \quad (k>0)$	6	t^{k-1}
29.3.8	$\frac{1}{s+a}$		e^{-at}
29.3.9	$\frac{1}{(s+a)^2}$		te^{-at}
29.3.10	$\frac{1}{(s+a)^n} \quad (n=1, 2, 3, \dots)$		$\frac{t^{n-1} e^{-at}}{(n-1)!}$
29.3.11	$\frac{\Gamma(k)}{(s+a)^k} \quad (k>0)$	6	$t^{k-1} e^{-at}$
29.3.12	$\frac{1}{(s+a)(s+b)} \quad (a \neq b)$		$\frac{e^{-at} - e^{-bt}}{b-a}$
29.3.13	$\frac{s}{(s+a)(s+b)} \quad (a \neq b)$		$\frac{ae^{-at} - be^{-bt}}{a-b}$
29.3.14	$\frac{1}{(s+a)(s+b)(s+c)}$ (a, b, c distinct constants)		$-\frac{(b-c)e^{-at} + (c-a)e^{-bt} + (a-b)e^{-ct}}{(a-b)(b-c)(c-a)}$
29.3.15	$\frac{1}{s^2+a^2}$		$\frac{1}{a} \sin at$
29.3.16	$\frac{s}{s^2+a^2}$		$\cos at$
29.3.17	$\frac{1}{s^2-a^2}$		$\frac{1}{a} \sinh at$
29.3.18	$\frac{s}{s^2-a^2}$		$\cosh at$
29.3.19	$\frac{1}{s(s^2+a^2)}$		$\frac{1}{a^2} (1 - \cos at)$
29.3.20	$\frac{1}{s^2(s^2+a^2)}$		$\frac{1}{a^3} (at - \sin at)$
29.3.21	$\frac{1}{(s^2+a^2)^2}$		$\frac{1}{2a^3} (\sin at - at \cos at)$

	$f(s)$	$F(t)$	
29.3.22	$\frac{s}{(s^2+a^2)^2}$	$\frac{t}{2a} \sin at$	
29.3.23	$\frac{s^2}{(s^2+a^2)^2}$	$\frac{1}{2a} (\sin at + at \cos at)$	
29.3.24	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t \cos at$	
29.3.25	$\frac{s}{(s^2+a^2)(s^2+b^2)} \quad (a^2 \neq b^2)$	$\frac{\cos at - \cos bt}{b^2 - a^2}$	
29.3.26	$\frac{1}{(s+a)^2 + b^2}$	$\frac{1}{b} e^{-at} \sin bt$	
29.3.27	$\frac{s+a}{(s+a)^2 + b^2}$	$e^{-at} \cos bt$	
29.3.28	$\frac{3a^2}{s^3+a^3}$	$e^{-at} - e^{iat} \left(\cos \frac{at\sqrt{3}}{2} - \sqrt{3} \sin \frac{at\sqrt{3}}{2} \right)$	
29.3.29	$\frac{4a^3}{s^4+4a^4}$	$\sin at \cosh at - \cos at \sinh at$	
29.3.30	$\frac{s}{s^4+4a^4}$	$\frac{1}{2a^2} \sin at \sinh at$	
29.3.31	$\frac{1}{s^4-a^4}$	$\frac{1}{2a^3} (\sinh at - \sin at)$	
29.3.32	$\frac{s}{s^4-a^4}$	$\frac{1}{2a^2} (\cosh at - \cos at)$	
29.3.33	$\frac{8a^3 s^2}{(s^2+a^2)^3}$	$(1+a^2 t^2) \sin at - at \cos at$	
29.3.34	$\frac{1}{s} \left(\frac{s-1}{s} \right)^n$	$L_n(t)$	22
29.3.35	$\frac{s}{(s+a)^{\frac{1}{2}}}$	$\frac{1}{\sqrt{\pi t}} e^{-at} (1-2at)$	
29.3.36	$\sqrt{s+a} - \sqrt{s+b}$	$\frac{1}{2\sqrt{\pi t^3}} (e^{-bt} - e^{-at})$	
29.3.37	$\frac{1}{\sqrt{s+a}}$	$\frac{1}{\sqrt{\pi t}} - a e^{at} \operatorname{erfc} a\sqrt{t}$	7
29.3.38	$\frac{\sqrt{s}}{s-a^2}$	$\frac{1}{\sqrt{\pi t}} + a e^{at} \operatorname{erf} a\sqrt{t}$	7
29.3.39	$\frac{\sqrt{s}}{s+a^2}$	$\frac{1}{\sqrt{\pi t}} - \frac{2a}{\sqrt{\pi}} e^{-a^2 t} \int_0^{a\sqrt{t}} e^{\lambda^2} d\lambda$	7
29.3.40	$\frac{1}{\sqrt{s}(s-a^2)}$	$\frac{1}{a} e^{at} \operatorname{erf} a\sqrt{t}$	7

	$f(s)$	$F(t)$	
29.3.41	$\frac{1}{\sqrt{s}(s+a^2)}$	$\frac{2}{a\sqrt{\pi}} e^{-a^2 t} \int_0^{a\sqrt{t}} e^{\lambda^2} d\lambda$	7
29.3.42	$\frac{b^2-a^2}{(s-a^2)(b+\sqrt{s})}$	$e^{a^2 t} [b-a \operatorname{erf} a\sqrt{t}] - b e^{b^2 t} \operatorname{erfc} b\sqrt{t}$	7
29.3.43	$\frac{1}{\sqrt{s}(\sqrt{s}+a)}$	$e^{a^2 t} \operatorname{erfc} a\sqrt{t}$	7
29.3.44	$\frac{1}{(s+a)\sqrt{s+b}}$	$\frac{1}{\sqrt{b-a}} e^{-at} \operatorname{erf}(\sqrt{b-a}\sqrt{t})$	7
29.3.45	$\frac{b^2-a^2}{\sqrt{s}(s-a^2)(\sqrt{s}+b)}$	$e^{a^2 t} \left[\frac{b}{a} \operatorname{erf}(a\sqrt{t}) - 1 \right] + e^{b^2 t} \operatorname{erfc} b\sqrt{t}$	7
29.3.46	$\frac{(1-s)^n}{s^{n+\frac{1}{2}}}$	$\frac{n!}{(2n)!\sqrt{\pi t}} H_{2n}(\sqrt{t})$	22
29.3.47	$\frac{(1-s)^n}{s^{n+\frac{1}{2}}}$	$\frac{n!}{(2n+1)!\sqrt{\pi}} H_{2n+1}(\sqrt{t})$	22
29.3.48	$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s}} - 1$	$ae^{-at} [I_1(at) + I_0(at)]$	9
29.3.49	$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-\frac{1}{2}(a+b)t} I_0\left(\frac{a-b}{2} t\right)$	9
29.3.50	$\frac{\Gamma(k)}{(s+a)^k(s+b)^k} \quad (k>0) \quad 6$	$\sqrt{\pi} \left(\frac{t}{a-b}\right)^{k-\frac{1}{2}} e^{-\frac{1}{2}(a+b)t} I_{k-\frac{1}{2}}\left(\frac{a-b}{2} t\right)$	10
29.3.51	$\frac{1}{(s+a)^{\frac{1}{2}}(s+b)^{\frac{1}{2}}}$	$t e^{-\frac{1}{2}(a+b)t} \left[I_0\left(\frac{a-b}{2} t\right) + I_1\left(\frac{a-b}{2} t\right) \right]$	9
29.3.52	$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s+2a}+\sqrt{s}}$	$\frac{1}{t} e^{-at} I_1(at)$	9
29.3.53	$\frac{(a-b)^k}{(\sqrt{s+a}+\sqrt{s+b})^{2k}} \quad (k>0)$	$\frac{k}{t} e^{-\frac{1}{2}(a+b)t} I_k\left(\frac{a-b}{2} t\right)$	9
29.3.54	$\frac{(\sqrt{s+a}+\sqrt{s})^{-2\nu}}{\sqrt{s}\sqrt{s+a}} \quad (\nu>-1)$	$\frac{1}{a^\nu} e^{-\frac{1}{2}at} I_\nu\left(\frac{1}{2}at\right)$	9
29.3.55	$\frac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$	9
29.3.56	$\frac{(\sqrt{s^2+a^2}-s)^\nu}{\sqrt{s^2+a^2}} \quad (\nu>-1)$	$a^\nu J_\nu(at)$	9
29.3.57	$\frac{1}{(s^2+a^2)^k} \quad (k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)} \left(\frac{t}{2a}\right)^{k-\frac{1}{2}} J_{k-\frac{1}{2}}(at)$	6, 10

	$f(s)$	$F(t)$	
29.3.58	$(\sqrt{s^2+a^2}-s)^k \quad (k>0)$	$\frac{ka^k}{t} J_k(at)$	9
29.3.59	$\frac{(s-\sqrt{s^2-a^2})^\nu}{\sqrt{s^2-a^2}} \quad (\nu>-1)$	$a^\nu I_\nu(at)$	9
29.3.60	$\frac{1}{(s^2-a^2)^k} \quad (k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)} \left(\frac{t}{2a}\right)^{k-\frac{1}{2}} I_{k-\frac{1}{2}}(at)$	6, 10
29.3.61	$\frac{1}{s} e^{-ks}$	$u(t-k)$	
29.3.62	$\frac{1}{s^2} e^{-ks}$	$(t-k)u(t-k)$	
29.3.63	$\frac{1}{s^\mu} e^{-ks} \quad (\mu>0)$	$\frac{(t-k)^{\mu-1}}{\Gamma(\mu)} u(t-k)$	 6
29.3.64	$\frac{1-e^{-ks}}{s}$	$u(t)-u(t-k)$	
29.3.65	$\frac{1}{s(1-e^{-ks})} = \frac{1+\coth \frac{1}{2}ks}{2s}$	$\sum_{n=0}^{\infty} u(t-nk)$	
29.3.66	$\frac{1}{s(e^{ks}-a)}$	$\sum_{n=1}^{\infty} a^{n-1} u(t-nk)$	
29.3.67	$\frac{1}{s} \tanh ks$	$u(t)+2 \sum_{n=1}^{\infty} (-1)^n u(t-2nk)$	
29.3.68	$\frac{1}{s(1+e^{-ks})}$	$\sum_{n=0}^{\infty} (-1)^n u(t-nk)$	
29.3.69	$\frac{1}{s^2} \tanh ks$	$t u(t)+2 \sum_{n=1}^{\infty} (-1)^n (t-2nk) u(t-2nk)$	
29.3.70	$\frac{1}{s \sinh ks}$	$2 \sum_{n=0}^{\infty} u[t-(2n+1)k]$	
29.3.71	$\frac{1}{s \cosh ks}$	$2 \sum_{n=0}^{\infty} (-1)^n u[t-(2n+1)k]$	

	$f(s)$	$F(t)$	
29.3.72	$\frac{1}{s} \coth ks$	$u(t) + 2 \sum_{n=1}^{\infty} u(t-2nk)$	
29.3.73	$\frac{k}{s^2+k^2} \coth \frac{\pi s}{2k}$	$ \sin kt $	
29.3.74	$\frac{1}{(s^2+1)(1-e^{-\pi s})}$	$\sum_{n=0}^{\infty} (-1)^n u(t-n\pi) \sin t$	
29.3.75	$\frac{1}{s} e^{-\frac{k}{s}}$	$J_0(2\sqrt{kt})$	9
29.3.76	$\frac{1}{\sqrt{s}} e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}} \cos 2\sqrt{kt}$	
29.3.77	$\frac{1}{\sqrt{s}} e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}} \cosh 2\sqrt{kt}$	
29.3.78	$\frac{1}{s^{3/2}} e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi k}} \sin 2\sqrt{kt}$	
29.3.79	$\frac{1}{s^{3/2}} e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi k}} \sinh 2\sqrt{kt}$	
29.3.80	$\frac{1}{s^\mu} e^{-\frac{k}{s}} \quad (\mu > 0)$	$\left(\frac{t}{k}\right)^{\frac{\mu-1}{2}} J_{\mu-1}(2\sqrt{kt})$	9
29.3.81	$\frac{1}{s^\mu} e^{\frac{k}{s}} \quad (\mu > 0)$	$\left(\frac{t}{k}\right)^{\frac{\mu-1}{2}} I_{\mu-1}(2\sqrt{kt})$	9
29.3.82	$e^{-k\sqrt{s}} \quad (k > 0)$	$\frac{k}{2\sqrt{\pi t^3}} \exp\left(-\frac{k^2}{4t}\right)$	
29.3.83	$\frac{1}{s} e^{-k\sqrt{s}} \quad (k \geq 0)$	$\operatorname{erfc} \frac{k}{2\sqrt{t}}$	7
29.3.84	$\frac{1}{\sqrt{s}} e^{-k\sqrt{s}} \quad (k \geq 0)$	$\frac{1}{\sqrt{\pi t}} \exp\left(-\frac{k^2}{4t}\right)$	
29.3.85	$\frac{1}{s^{3/2}} e^{-k\sqrt{s}} \quad (k \geq 0)$	$2\sqrt{\frac{t}{\pi}} \exp\left(-\frac{k^2}{4t}\right) - k \operatorname{erfc} \frac{k}{2\sqrt{t}} = 2\sqrt{t} \operatorname{erfc} \frac{k}{2\sqrt{t}}$	7
29.3.86	$\frac{1}{s^{1+\frac{1}{2}n}} e^{-k\sqrt{s}} \quad (n=0, 1, 2, \dots; k \geq 0)$	$(4t)^{\frac{1}{2}n} \operatorname{erfc} \frac{k}{2\sqrt{t}}$	7
29.3.87	$\frac{n-1}{s^{\frac{n}{2}}} e^{-k\sqrt{s}} \quad (n=0, 1, 2, \dots; k > 0)$	$\frac{\exp\left(-\frac{k^2}{4t}\right)}{2^n \sqrt{\pi t^{n+1}}} H_n\left(\frac{k}{2\sqrt{t}}\right)$	22
29.3.88	$\frac{e^{-k\sqrt{s}}}{a+\sqrt{s}} \quad (k \geq 0)$	$\frac{1}{\sqrt{\pi t}} \exp\left(-\frac{k^2}{4t}\right) - a e^{a^2 t} \operatorname{erfc}\left(a\sqrt{t} + \frac{k}{2\sqrt{t}}\right)$	7

*See page II.

	$f(s)$	$F(t)$	
29.3.89	$\frac{ae^{-k\sqrt{s}}}{s(a+\sqrt{s})} \quad (k \geq 0)$	$-e^{ak}e^{a^2t} \operatorname{erfc}\left(a\sqrt{t} + \frac{k}{2\sqrt{t}}\right) + \operatorname{erfc} \frac{k}{2\sqrt{t}}$	7
29.3.90	$\frac{e^{-k\sqrt{s}}}{\sqrt{s}(a+\sqrt{s})} \quad (k \geq 0)$	$e^{ak}e^{a^2t} \operatorname{erfc}\left(a\sqrt{t} + \frac{k}{2\sqrt{t}}\right)$	7
29.3.91	$\frac{e^{-k\sqrt{s(s+a)}}}{\sqrt{s(s+a)}} \quad (k \geq 0)$	$e^{-\frac{1}{2}at} I_0\left(\frac{1}{2}a\sqrt{t^2-k^2}\right)u(t-k)$	9
29.3.92	$\frac{e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}} \quad (k \geq 0)$	$J_0(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.93	$\frac{e^{-k\sqrt{s^2-a^2}}}{\sqrt{s^2-a^2}} \quad (k \geq 0)$	$I_0(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.94	$\frac{e^{-k(\sqrt{s^2+a^2}-s)}}{\sqrt{s^2+a^2}} \quad (k \geq 0)$	$J_0(a\sqrt{t^2+2kt})$	9
29.3.95	$e^{-ks} - e^{-k\sqrt{s^2+a^2}} \quad (k > 0)$	$\frac{ak}{\sqrt{t^2-k^2}} J_1(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.96	$e^{-k\sqrt{s^2-a^2}} - e^{-ks} \quad (k > 0)$	$\frac{ak}{\sqrt{t^2-k^2}} I_1(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.97	$\frac{a^\nu e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}(\sqrt{s^2+a^2}+s)^\nu} \quad (\nu > -1, k \geq 0)$	$\left(\frac{t-k}{t+k}\right)^{\frac{1}{2}\nu} J_\nu(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.98	$\frac{1}{s} \ln s$	$-\gamma - \ln t \quad (\gamma = .57721\ 56649 \dots \text{Euler's constant})$	
29.3.99	$\frac{1}{s^k} \ln s \quad (k > 0)$	$\frac{t^{k-1}}{\Gamma(k)} [\psi(k) - \ln t]$	6
29.3.100	$\frac{\ln s}{s-a} \quad (a > 0)$	$e^{at} [\ln a + E_1(at)]$	5
29.3.101	$\frac{\ln s}{s^2+1}$	$\cos t \operatorname{Si}(t) - \sin t \operatorname{Ci}(t)$	5
29.3.102	$\frac{s \ln s}{s^2+1}$	$-\sin t \operatorname{Si}(t) - \cos t \operatorname{Ci}(t)$	5
29.3.103	$\frac{1}{s} \ln(1+ks) \quad (k > 0)$	$E_1\left(\frac{t}{k}\right)$	5
29.3.104	$\ln \frac{s+a}{s+b}$	$\frac{1}{t} (e^{-bt} - e^{-at})$	
29.3.105	$\frac{1}{s} \ln(1+k^2s^2) \quad (k > 0)$	$-2 \operatorname{Ci}\left(\frac{t}{k}\right)$	5
29.3.106	$\frac{1}{s} \ln(s^2+a^2) \quad (a > 0)$	$2 \ln a - 2 \operatorname{Ci}(at)$	5

	$f(s)$		$F(t)$	
29.3.107	$\frac{1}{s^2} \ln(s^2 + a^2) \quad (a > 0)$		$\frac{2}{a} [at \ln a + \sin at - at \operatorname{Ci}(at)]$	5
29.3.108	$\ln \frac{s^2 + a^2}{s^2}$		$\frac{2}{t} (1 - \cos at)$	
29.3.109	$\ln \frac{s^2 - a^2}{s^2}$		$\frac{2}{t} (1 - \cosh at)$	
29.3.110	$\arctan \frac{k}{s}$		$\frac{1}{t} \sin kt$	
29.3.111	$\frac{1}{s} \arctan \frac{k}{s}$		$\operatorname{Si}(kt)$	5
29.3.112	$e^{ks^2} \operatorname{erfc} ks \quad (k > 0)$	7	$\frac{1}{k\sqrt{\pi}} \exp\left(-\frac{t^2}{4k^2}\right)$	
29.3.113	$\frac{1}{s} e^{ks^2} \operatorname{erfc} ks \quad (k > 0)$	7	$\operatorname{erf} \frac{t}{2k}$	7
29.3.114	$e^{ks} \operatorname{erfc} \sqrt{ks} \quad (k > 0)$	7	$\frac{\sqrt{k}}{\pi \sqrt{t(t+k)}}$	
29.3.115	$\frac{1}{\sqrt{s}} \operatorname{erfc} \sqrt{ks} \quad (k \geq 0)$	7	$\frac{1}{\sqrt{\pi t}} u(t-k)$	
29.3.116	$\frac{1}{\sqrt{s}} e^{ks} \operatorname{erfc} \sqrt{ks} \quad (k \geq 0)$	7	$\frac{1}{\sqrt{\pi(t+k)}}$	
29.3.117	$\operatorname{erf} \frac{k}{\sqrt{s}}$	7	$\frac{1}{\pi t} \sin 2k\sqrt{t}$	
29.3.118	$\frac{1}{\sqrt{s}} e^{\frac{k^2}{s}} \operatorname{erfc} \frac{k}{\sqrt{s}}$	7	$\frac{1}{\sqrt{\pi t}} e^{-2k\sqrt{t}}$	
29.3.119	$K_0(ks) \quad (k > 0)$	9	$\frac{1}{\sqrt{t^2 - k^2}} u(t-k)$	
29.3.120	$K_0(k\sqrt{s}) \quad (k > 0)$	9	$\frac{1}{2t} \exp\left(-\frac{k^2}{4t}\right)$	
29.3.121	$\frac{1}{s} e^{ks} K_1(ks) \quad (k > 0)$	9	$\frac{1}{k} \sqrt{t(t+2k)}$	
29.3.122	$\frac{1}{\sqrt{s}} K_1(k\sqrt{s}) \quad (k > 0)$	9	$\frac{1}{k} \exp\left(-\frac{k^2}{4t}\right)$	
29.3.123	$\frac{1}{\sqrt{s}} e^{\frac{k}{s}} K_0\left(\frac{k}{s}\right) \quad (k > 0)$	9	$\frac{2}{\sqrt{\pi t}} K_0(2\sqrt{2kt})$	9
29.3.124	$\pi e^{-ks} I_0(ks) \quad (k > 0)$	9	$\frac{1}{\sqrt{t(2k-t)}} [u(t) - u(t-2k)]$	
29.3.125	$e^{-ks} I_1(ks) \quad (k > 0)$	9	$\frac{k-t}{\pi k \sqrt{t(2k-t)}} [u(t) - u(t-2k)]$	

	$f(s)$		$F(t)$
29.3.126	$e^{as}E_1(as) \quad (a>0)$	5	$\frac{1}{t+a}$
29.3.127	$\frac{1}{a}-se^{as}E_1(as) \quad (a>0)$	5	$\frac{1}{(t+a)^2}$
29.3.128	$a^{1-n}e^{as}E_n(as) \quad (a>0; n=0, 1, 2, \dots)$	5	$\frac{1}{(t+a)^n}$
29.3.129	$\left[\frac{\pi}{2}-\text{Si}(s)\right] \cos s + \text{Ci}(s) \sin s$	5	$\frac{1}{t^2+1}$

29.4. Table of Laplace-Stieltjes Transforms ⁴

	$\phi(s)$	$\Phi(t)$
29.4.1	$\int_0^\infty e^{-st}d\Phi(t)$	$\Phi(t)$
29.4.2	$e^{-ks} \quad (k>0)$	$u(t-k)$
29.4.3	$\frac{1}{1-e^{-ks}} \quad (k>0)$	$\sum_{n=0}^\infty u(t-nk)$
29.4.4	$\frac{1}{1+e^{-ks}} \quad (k>0)$	$\sum_{n=0}^\infty (-1)^n u(t-nk)$
29.4.5	$\frac{1}{\sinh ks} \quad (k>0)$	$2 \sum_{n=0}^\infty u[t-(2n+1)k]$
29.4.6	$\frac{1}{\cosh ks} \quad (k>0)$	$2 \sum_{n=0}^\infty (-1)^n u[t-(2n+1)k]$
29.4.7	$\tanh ks \quad (k>0)$	$u(t) + 2 \sum_{n=1}^\infty (-1)^n u(t-2nk)$
29.4.8	$\frac{1}{\sinh (ks+a)} \quad (k>0)$	$2 \sum_{n=0}^\infty e^{-(2n+1)a} u[t-(2n+1)k]$
29.4.9	$\frac{e^{-hs}}{\sinh (ks+a)} \quad (k>0, h>0)$	$2 \sum_{n=0}^\infty e^{-(2n+1)a} u[t-h-(2n+1)k]$
29.4.10	$\frac{\sinh (hs+b)}{\sinh (ks+a)} \quad (0<h<k)$	$\sum_{n=0}^\infty e^{-(2n+1)a} \{ e^b u[t+h-(2n+1)k] - e^{-b} u[t-h-(2n+1)k] \}$
29.4.11	$\sum_{n=0}^\infty a_n e^{-k_n s} \quad (0<k_0<k_1<\dots)$	$\sum_{n=0}^\infty a_n u(t-k_n)$

For the definition of the Laplace-Stieltjes transform see [29.7]. In practice, Laplace-Stieltjes transforms are often written as ordinary Laplace transforms involving Dirac's delta function $\delta(t)$. This "function" may formally be considered as

the derivative of the unit step function, $du(t)=\delta(t)dt$, so that $\int_{-\infty}^x du(t) = \int_{-\infty}^x \delta(t)dt = \begin{cases} 0 & (x<0) \\ 1 & (x>0). \end{cases}$ The correspondence 29.4.2, for instance, then assumes the form $e^{-ks} = \int_0^\infty e^{-st}\delta(t-k)dt$.

⁴ Adapted by permission from P. M. Morse and H. Feshbach, *Methods of theoretical physics*, vols. 1, 2, McGraw-Hill Book Co., Inc., New York, N.Y., 1953.

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$P_n^*(x)$ shifted Legendre polynomial.....	774	$W_{\kappa, \mu}(z)$ Whittaker function.....	505
$P_n^{(\alpha, \beta)}(x)$ Jacobi polynomial.....	774	$W\{f(x), g(x)\} (= f(x)g'(x) - f'(x)g(x))$ Wronskian re- lation.....	505
$\operatorname{Pr}\{X \leq x\}$ probability of the event $X \leq x$	927	$[x_0, x_1, \dots, x_k]$ divided difference.....	877
q nome.....	591	$y_n(z)$ spherical Bessel function of the second kind.....	437
$Q(x) = 1 - P(x)$ normal probability function (tail area).....	931	$Y_n(z)$ Bessel function of the second kind.....	358
		$Y_n^m(\theta, \varphi)$ surface harmonic of the first kind.....	332
		$Z(x)$ normal probability density function.....	931

Notation — Greek Letters

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α modular angle (elliptic function).....	590	$\Theta(u m)$ Jacobi's theta function.....	577
$\alpha_n(z) = \int_1^\infty t^n e^{-st} dt$	228	κ_n nth cumulant.....	928
$\beta_n(z) = \int_{-1}^1 t^n e^{-st} dt$	228	$\kappa_{mn}^{(p)}$ joining factor for spheroidal wave functions..	757
$\beta(n) = \sum_{k=0}^\infty (-1)^k (2k+1)^{-n}$	807	$\lambda(n) = \sum_{k=0}^\infty (2k+1)^{-n}$	807
$B_x(a, b)$ incomplete beta function.....	263	λ_{mn} characteristic value of the spheroidal wave equation.....	753
$B(z, w)$ beta function.....	258	$\Lambda_0(\varphi \backslash \alpha)$ Heuman's lambda function.....	595
γ Euler's constant.....	255	$\mu(f_n)$ mean difference.....	877
$\gamma(a, x)$ incomplete gamma function (normalized) ..	260	$\mu(n)$ Möbius function.....	826
$\gamma_1 = \frac{\mu_3}{\sigma_3}$ coefficient of skewness.....	928	μ_n nth central moment.....	928
$\gamma_2 = \frac{\mu_4}{\sigma^4} - 3$ coefficient of excess.....	928	μ'_n nth moment about the origin.....	928
$\Gamma(z)$ gamma function.....	255	$\pi(x)$ number of primes $\leq x$	231
$\Gamma(a, x)$ incomplete gamma function.....	260	$\pi_n(x) = (x-x_0)(x-x_1) \dots (x-x_n)$	878
δ_{ij} Kronecker delta ($=0$ if $i \neq j$; $=1$ if $i=j$).....	822	$\Pi(n; \varphi \backslash \alpha)$ elliptic integral of the third kind.....	590
$\delta_k^*(f_n)$ central difference.....	877	$\Pi(z)$ factorial function.....	255
Δ difference operator.....	822	ρ correlation coefficient.....	936
Δ discriminant of Weierstrass' canonical form.....	629	$\rho_n(x_0, x_1, \dots, x_n)$ reciprocal difference.....	878
$\Delta(f_n)$ forward difference.....	877	$\rho_n(\nu, x)$ Poisson-Charlier function.....	509
Δx absolute error.....	14	σ standard deviation.....	298
$\zeta(x)$ Riemann zeta function.....	807	σ^2 variance.....	928
$\zeta'(z)$ Weierstrass zeta function.....	629	$\sigma(z)$ Weierstrass sigma function.....	629
$Z(u m)$ Jacobi's zeta function.....	578	$\sigma_k(n)$ divisor function.....	827
$\eta(n) = \sum_{k=1}^\infty (-1)^{k-1} k^{-n}$	807	$\tau_n(x)$ tetrachoric function.....	934
$\eta_a = \zeta(\omega_a)$ Weierstrass elliptic function.....	631	$\varphi = \text{am } u$, amplitude.....	569
$H(u), H_1(u)$ Jacobi's eta function.....	577	$\varphi(n)$ Euler-Totient function.....	826
$\vartheta_n(z)$ theta function.....	576	$\varphi(t) = E(e^{itX})$ characteristic function of X	928
$\vartheta_2(\epsilon \backslash \alpha), \vartheta_3(\epsilon \backslash \alpha)$, Neville's notation.....	578	$\Phi(a; b; z)$ confluent hypergeometric function.....	504
for $\vartheta_n(\epsilon \backslash \alpha), \vartheta_s(\epsilon \backslash \alpha)$ theta functions		$\psi(z)$ logarithmic derivative of the gamma function.....	258
		$\Psi(a; c; z)$ confluent hypergeometric function.....	504
		ω_a period of Weierstrass elliptic functions.....	629
		$\omega_{\epsilon, \mu}(x)$ Cunningham function.....	510

Miscellaneous Notations

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$[a_{ik}]$ determinant.....	19	$\langle x \rangle$ nearest integer to x	222
$[a_i]$ column matrix.....	19	$\bar{z}$ complex conjugate of $z (=x-iy)$	16
∇^n Laplacian operator.....	752	$z=x+iy$ complex number (Cartesian form).....	16
Δ_n^k forward difference operator.....	877	$=re^{i\theta}$ (polar form).....	16
$\frac{\partial}{\partial z}$ partial derivative.....	883	$ z $ absolute value or modulus of z	16
i ($=\sqrt{-1}$).....	70	Σ overall summation.....	822
$\binom{n}{k}$ binomial coefficient.....	10	Σ' restricted summation.....	755
$n!$ factorial function.....	255	$\Sigma \Pi$ sum or product taken over all prime numbers p	807
$(2n)!! = 2 \cdot 4 \cdot 6 \dots (2n) = 2^n n!$	258	$\Sigma \Pi$ sum or product over all positive divisors d of n	826
(m, n) greatest common divisor.....	822	$\int$ Cauchy's principal value of the integral.....	228
$(n, k) = \frac{\Gamma(\frac{1}{2}+n+k)}{k! \Gamma(\frac{1}{2}+n-k)}$ (Hankel's symbol).....	437	$\approx$ approximately equal.....	14
$(n; n_1, n_2, \dots, n_m)$ multinomial coefficient.....	823	$\sim$ asymptotically equal.....	15
$[x]$ largest integer $\leq x$	66	$<, >, \leq, \geq$ inequality, inclusion.....	10
		$\neq$ unequal.....	12